Lecture 1-a

Catenary Risers: Global Analysis

Static Problem
general equations

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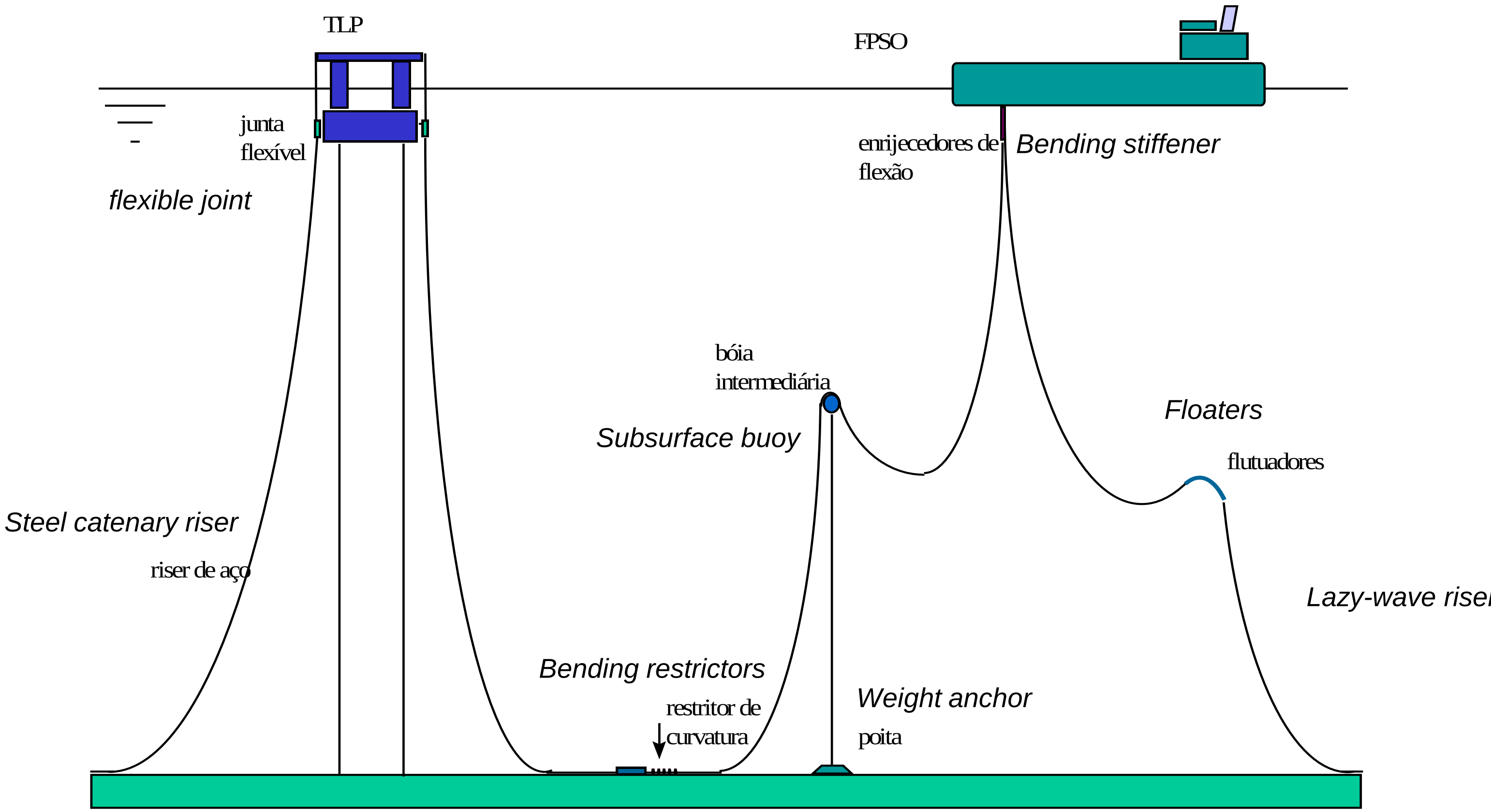
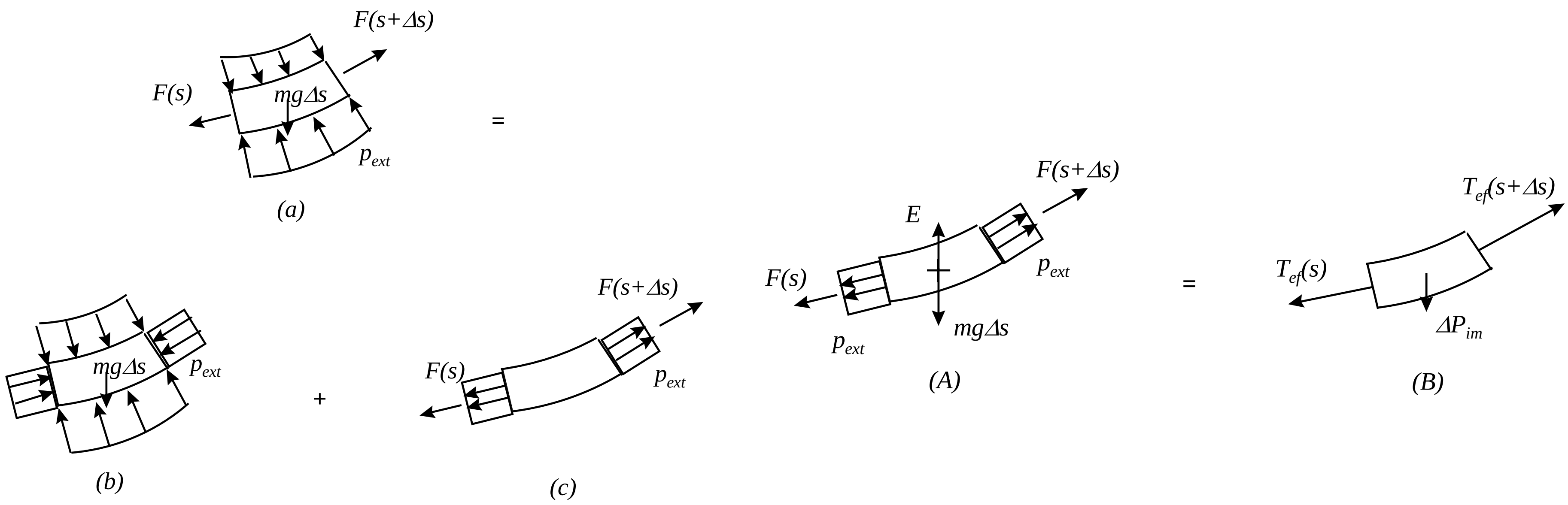
Catenary lines



Effective Tension Concept



$$T_{ef}(s) = F(s) + p_{ext}(s)S_o(s)$$

$$p_{ext}(s) = \rho_a g\left(H - y(s)\right)$$

General Kirshoff-Clebsh-Love Equations
(KCL equations)

Curved bars – large displacements, small deformations

$$\frac{dT}{ds} - Q_v\kappa_w + Q_w\kappa_v + f_u = 0$$

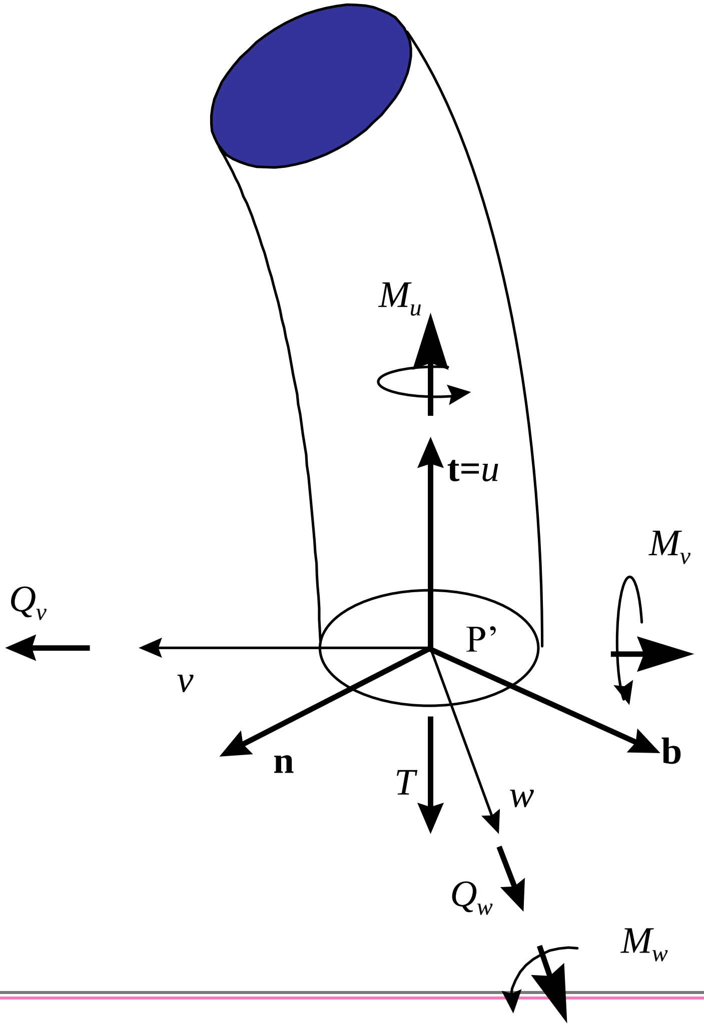
$$\frac{dQ_v}{ds} - Q_w\kappa_u + T\kappa_w + f_v = 0$$

$$\frac{dQ_w}{ds} - T\kappa_v + Q_v\kappa_u + f_w = 0$$

$$\frac{dM_u}{ds} - M_v\kappa_w + M_w\kappa_v + K_u = 0$$

$$\frac{dM_v}{ds} - M_w\kappa_u + M_u\kappa_w - Q_w + K_v = 0$$

$$\frac{dM_w}{ds} - M_u\kappa_v + M_v\kappa_u + Q_v + K_w = 0$$

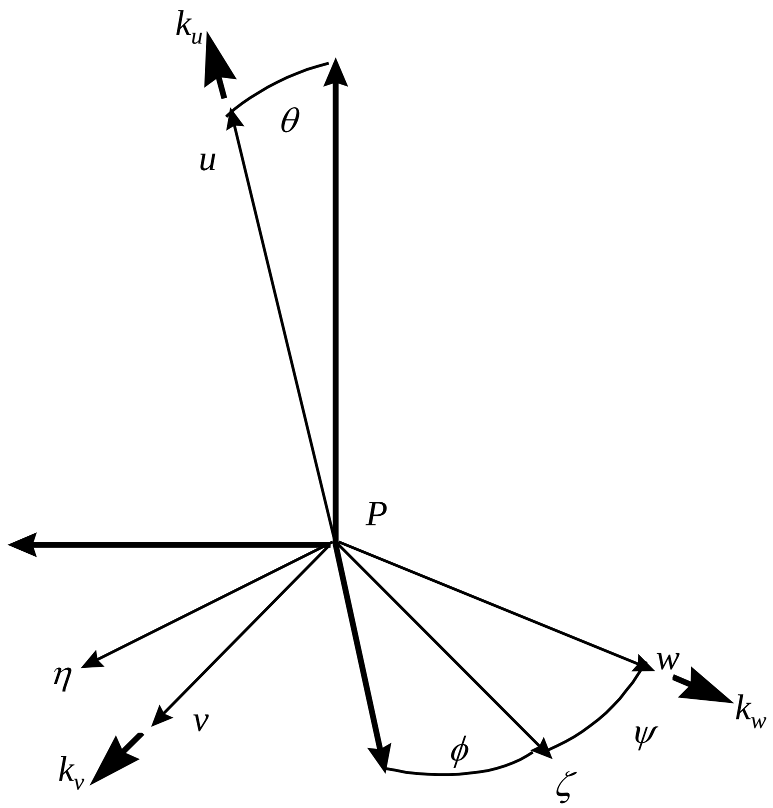


Euler angles and Curvature analogy

$$\kappa_u = \frac{d\psi}{ds} + \frac{d\phi}{ds}\cos\theta$$

$$\kappa_v = \frac{d\theta}{ds}\operatorname{sen}\psi - \frac{d\phi}{ds}\operatorname{sen}\theta\cos\psi$$

$$\kappa_w = \frac{d\theta}{ds}\cos\psi + \frac{d\phi}{ds}\operatorname{sen}\theta\operatorname{sen}\psi$$



Rigid Body Dynamic Equations

$$\frac{d\mathbf{Q}}{ds} + \mathbf{c} \wedge \mathbf{Q} + \mathbf{f} = \mathbf{0}$$

$$\frac{d\mathbf{M}}{ds} + \mathbf{c} \wedge \mathbf{M} + \mathbf{u} \wedge \mathbf{Q} + \mathbf{K} = \mathbf{0}$$

$$\mathbf{c} = \left(\kappa_u, \kappa_v, \kappa_w\right)$$

$$\mathbf{Q} = \left(T, Q_v, Q_w\right)$$
$$\mathbf{M} = \left(M_u, M_v, M_w\right)$$
$$\mathbf{f} = \left(f_u, f_v, f_w\right)$$
$$\mathbf{K} = \left(K_u, K_v, K_w\right)$$

Mathematically analogous

Q momentum

M angular momentum

c angular velocity vector

K Moment of external forces wrt an arbitrary point O

u velocity of point O

Back to KCL Equations:

$$M_u = B_u \kappa_u$$
$$M_v = B_v \kappa_v$$
$$M_w = B_w \kappa_w$$

$\rightarrow$

$$\frac{dT}{ds} = Q_v \kappa_w - Q_w \kappa_v - f_u$$
$$\frac{dQ_v}{ds} = Q_w \kappa_u - T \kappa_w - f_v$$
$$\frac{dQ_w}{ds} = T \kappa_v - Q_v \kappa_u - f_w$$
$$\frac{d}{ds}(B_u \kappa_u) - (B_v - B_w)\kappa_v \kappa_w + K_u = 0$$
$$\frac{d}{ds}(B_v \kappa_v) - (B_w - B_u)\kappa_w \kappa_u - Q_w + K_v = 0$$
$$\frac{d}{ds}(B_w \kappa_w) - (B_u - B_v)\kappa_u \kappa_v + Q_v + K_w = 0$$

$\downarrow$

$$\frac{dT}{ds} = Q_v \kappa_w - Q_w \kappa_v - f_u$$
$$\frac{d}{ds}(B_u \kappa_u) - (B_v - B_w)\kappa_v \kappa_w + K_u = 0$$
$$\frac{d^2}{ds^2}(B_v \kappa_v) - \frac{d}{ds}\left((B_w - B_u)\kappa_w \kappa_u\right) - T\kappa_v + Q_v \kappa_u + f_w + \frac{dK_v}{ds} = 0$$
$$\frac{d^2}{ds^2}(B_w \kappa_w) - \frac{d}{ds}\left((B_u - B_v)\kappa_u \kappa_v\right) + Q_w \kappa_u - T\kappa_w - f_v + \frac{dK_w}{ds} = 0$$

If stiffness independent on s: →

$$\frac{dT}{ds} = Q_v\kappa_w - Q_w\kappa_v - f_u$$

$$B_u \frac{d\kappa_u}{ds} - \left(B_v - B_w\right)\kappa_v\kappa_w + K_u = 0$$

$$B_v \frac{d^2\kappa_v}{ds^2} - \left(B_w - B_u\right)\frac{d}{ds}\left(\kappa_w\kappa_u\right) - T\kappa_v + Q_v\kappa_u + f_w + \frac{dK_v}{ds} = 0$$

$$B_w \frac{d^2\kappa_w}{ds^2} - \left(B_u - B_v\right)\frac{d}{ds}\left(\kappa_u\kappa_v\right) + Q_w\kappa_u - T\kappa_w - f_v + \frac{dK_w}{ds} = 0$$

If plane case, no twisting: $\kappa_v = 0;\ K_u = 0$ → $\boxed{\kappa_u = \tau_0, \text{constant}}$

If distributed loads absent:

$$\frac{dT}{ds} = Q_v\kappa_w$$

$$Q_v = \left(B_w - B_u\right)\frac{d\kappa_w}{ds}$$

$$\kappa_u = \tau_0$$

$$B_w \frac{d^2\kappa_w}{ds^2} + Q_w\tau_0 - T\kappa_w = 0$$

If a constant pitch helix case: $\kappa_w = \kappa_b = \chi_0$ →

$$T(s) = T_0$$

$$Q_v = 0$$

$$\kappa_u = \tau_0$$

$$Q_w\tau_0 = T_0\chi_0$$

If distributed moment loading absent: $(K_u; K_v; K_w) = (0;0;0)$

$$\frac{dT}{ds} = Q_v\kappa_w - Q_w\kappa_v - f_u$$

$$B_u \frac{d\kappa_u}{ds} - (B_v - B_w)\kappa_v\kappa_w = 0$$

$$B_v \frac{d^2\kappa_v}{ds^2} - (B_w - B_u)\frac{d}{ds}(\kappa_w\kappa_u) - T\kappa_v + Q_v\kappa_u + f_w = 0$$

$$B_w \frac{d^2\kappa_w}{ds^2} - (B_u - B_v)\frac{d}{ds}(\kappa_u\kappa_v) + Q_w\kappa_u - T\kappa_w - f_v = 0$$

If symmetric section: $B_v = B_w = B_f$ → $\kappa_u = \tau_0$, constant

$$\frac{dT}{ds} = Q_v\kappa_w - Q_w\kappa_v - f_u$$

$$\kappa_u = \tau_0$$

$$B_f \frac{d^2\kappa_v}{ds^2} - (B_f - B_u)\tau_0 \frac{d\kappa_w}{ds} - T\kappa_v + Q_v\tau_0 + f_w = 0$$

$$B_f \frac{d^2\kappa_w}{ds^2} - (B_u - B_f)\tau_0 \frac{d\kappa_v}{ds} + Q_w\tau_0 - T\kappa_w - f_v = 0$$

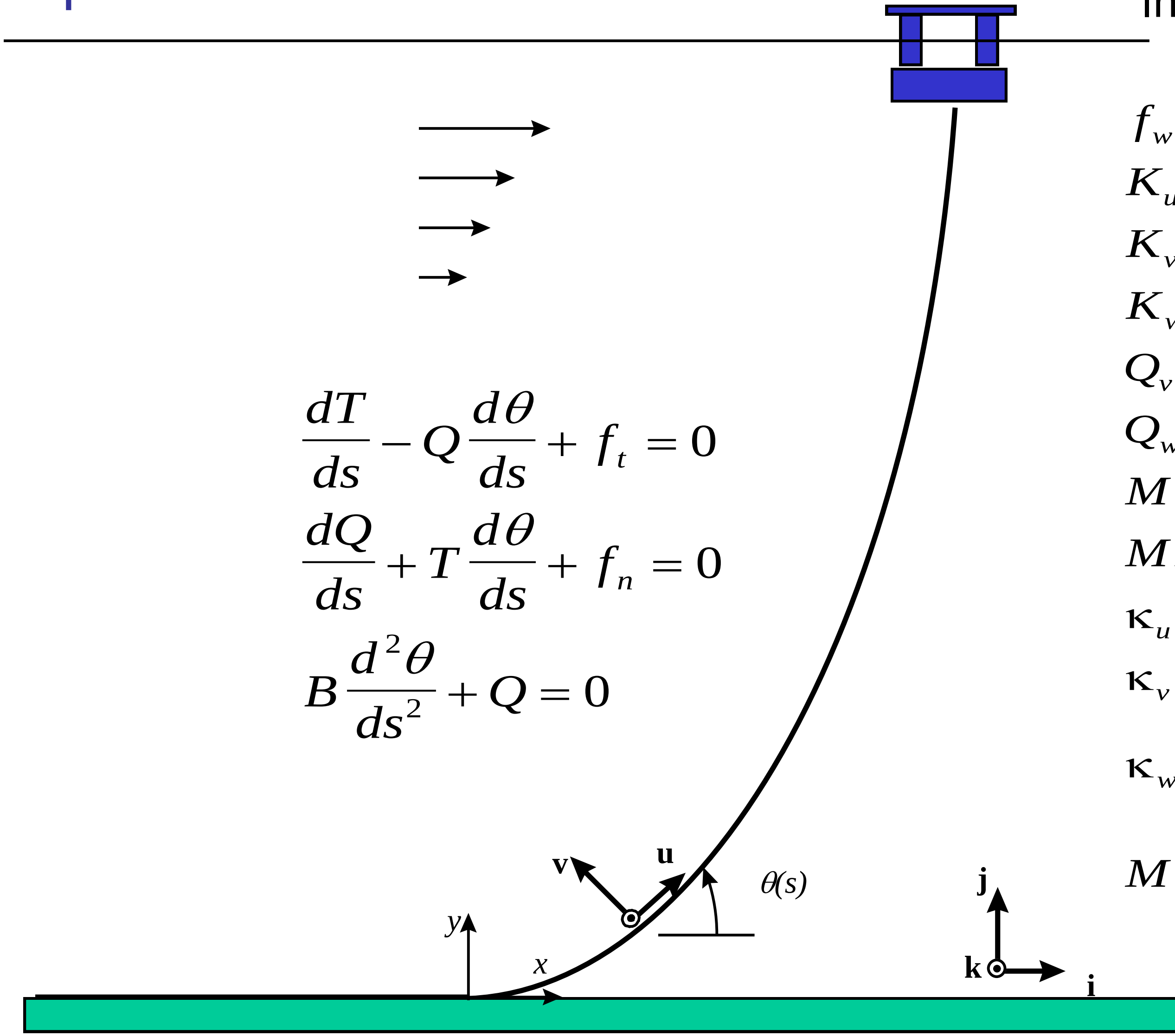
Symmetric sections:

necessary (though not suficient) condition for null twist is the the applied twisting load be null. In this case:

$$\boxed{\kappa_u = \tau_0 = 0} \quad \rightarrow \quad \boxed{\begin{aligned} &\frac{dT}{ds} = Q_v\kappa_w - Q_w\kappa_v - f_u \\ &\kappa_u = 0 \\ &B_f \frac{d^2\kappa_v}{ds^2} - T\kappa_v + f_w = 0 \\ &B_f \frac{d^2\kappa_w}{ds^2} - T\kappa_w - f_v = 0 \end{aligned}}$$

Though relevant to instability analysis under twist (see Ramos & Pesce, 2003; Gay neto & Martins, 2011), torsion will be left aside from now on !

The static problem in the vertical plane



$$\frac{dT}{ds} - Q\frac{d\theta}{ds} + f_t = 0$$

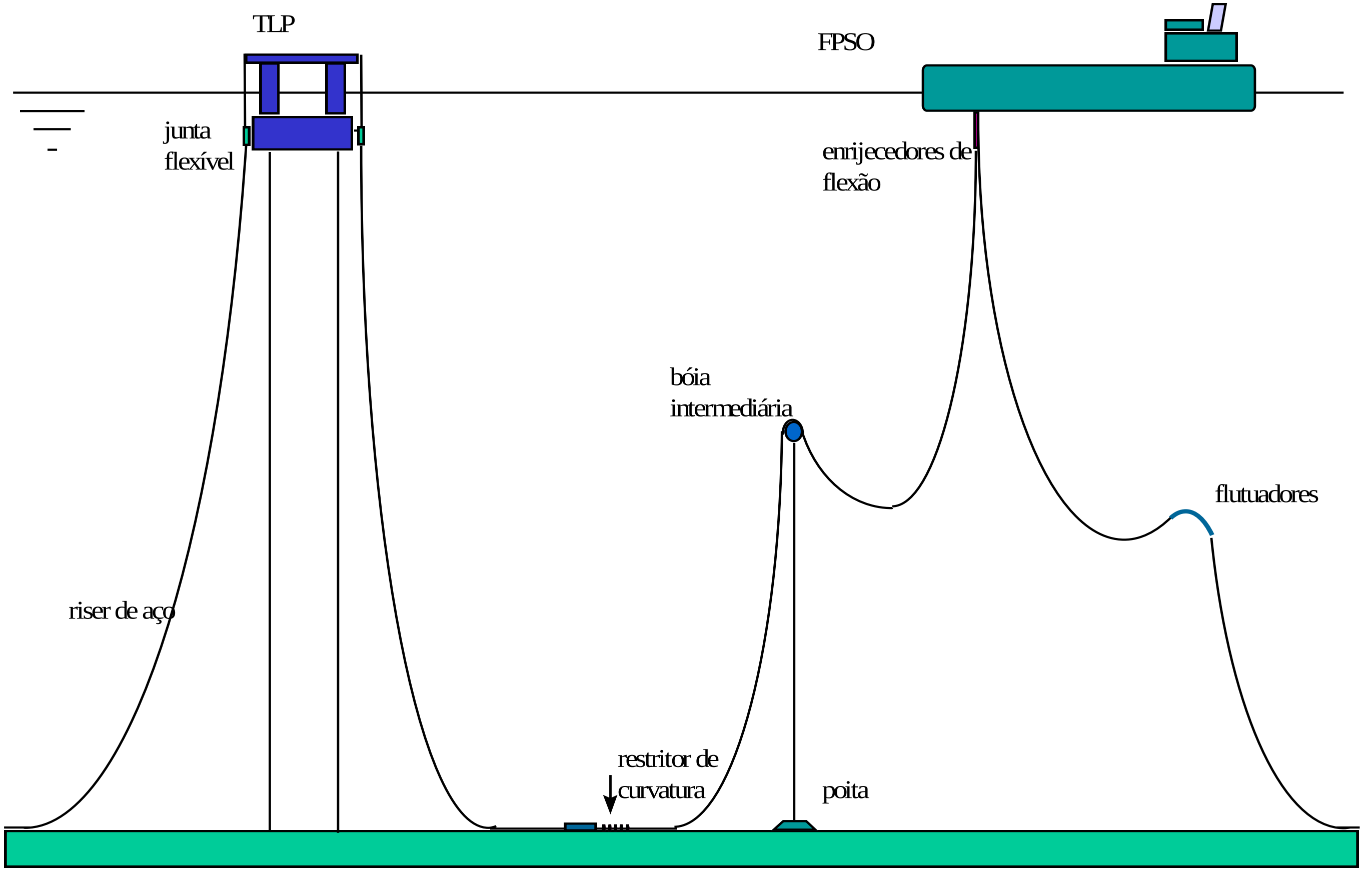
$$\frac{dQ}{ds} + T\frac{d\theta}{ds} + f_n = 0$$

$$B\frac{d^2\theta}{ds^2} + Q = 0$$

In fact:

$$f_w = f_b = 0$$
$$K_u = K_t = 0$$
$$K_v = K_n = 0$$
$$K_w = K_b = 0$$
$$Q_v = Q_n = Q(s)$$
$$Q_w = Q_b = 0$$
$$M_u = M_t = 0$$
$$M_v = M_n = 0$$
$$\kappa_u = \kappa_t = 0$$
$$\kappa_v = \kappa_n = 0$$
$$\kappa_w = \kappa_b = \frac{d\theta}{ds}$$
$$M_w = M_b = M(s) = B\kappa = B\frac{d\theta}{ds}$$

Catenary lines



The static equilibrium equations can be reduced to a single nonlinear ordinary differential equation:

Taking

$$f_v = f_n = -q\cos\theta + h_n(s)$$
$$f_u = f_t = -q\,\mathrm{sen}\,\theta + h_t(s)$$

Where

h stands for hydrodynamic forces and

q is the immersed weight

It follows, after a long algebraic effort (Love; see also Pesce, 1997),

$$B\frac{d^2\theta}{ds^2}\sec\theta + qs - \int_s \left(h_n \sec\theta + \sec^2\theta \left(\frac{d\theta}{ds}\right) \int_s \left(h_n sen\theta - h_t\cos\theta\right) d\xi \right) ds = \\ = T_0 \tan\theta - Q_0$$

Absence of ocean current

In this case, the integral term is identically null:

$$B\frac{d^2\theta}{ds^2}\sec\theta + qs = T_0\tan\theta - Q_0$$

If bending stiffness effect is neglected leads to the Catenary Equation

$$\tan\theta_c(s_c) = \frac{qs_c}{T_{0c}}$$

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The catenary curvature is then given by:

$$\chi_c(s_c) = \frac{d\theta_c}{ds_c} = \frac{q}{T_{0c}} \cos^2 \theta_c(s_c) = \frac{q}{T_{0c}} \frac{1}{1+\left(\frac{qs_c}{T_{0c}}\right)^2}$$

Tension is:

$$\frac{dT_c}{ds_c} = q \,\text{sen}\, \theta_c \quad \rightarrow \quad T_c(s_c) = T_{0c} \sec \theta_c$$

$$T_y = T_c(s_c)\,\text{sen}\,\theta_c = T_{0c} \tan \theta_c = qs_c$$

$$T_x = T_c(s_c) \cos \theta_c = T_{0c}; \quad \text{constante}$$

The horizontal component of tension is constant along the catenary line!

Cartesian coordinates

From: $$\frac{dx}{ds} = \cos\theta; \quad \frac{dy}{ds} = sen\theta; \quad \frac{dy}{dx} = \tan\theta$$

and

$$\frac{d}{ds_c}\left(T_c(s_c)\,\text{sen}\,\theta_c\right) = \frac{dT_c}{ds_c}\text{sen}\,\theta_c + T_c(s_c)\cos\theta_c\frac{d\theta_c}{ds_c} = q\,\text{sen}^2\,\theta_c + q\cos^2\theta_c = q$$

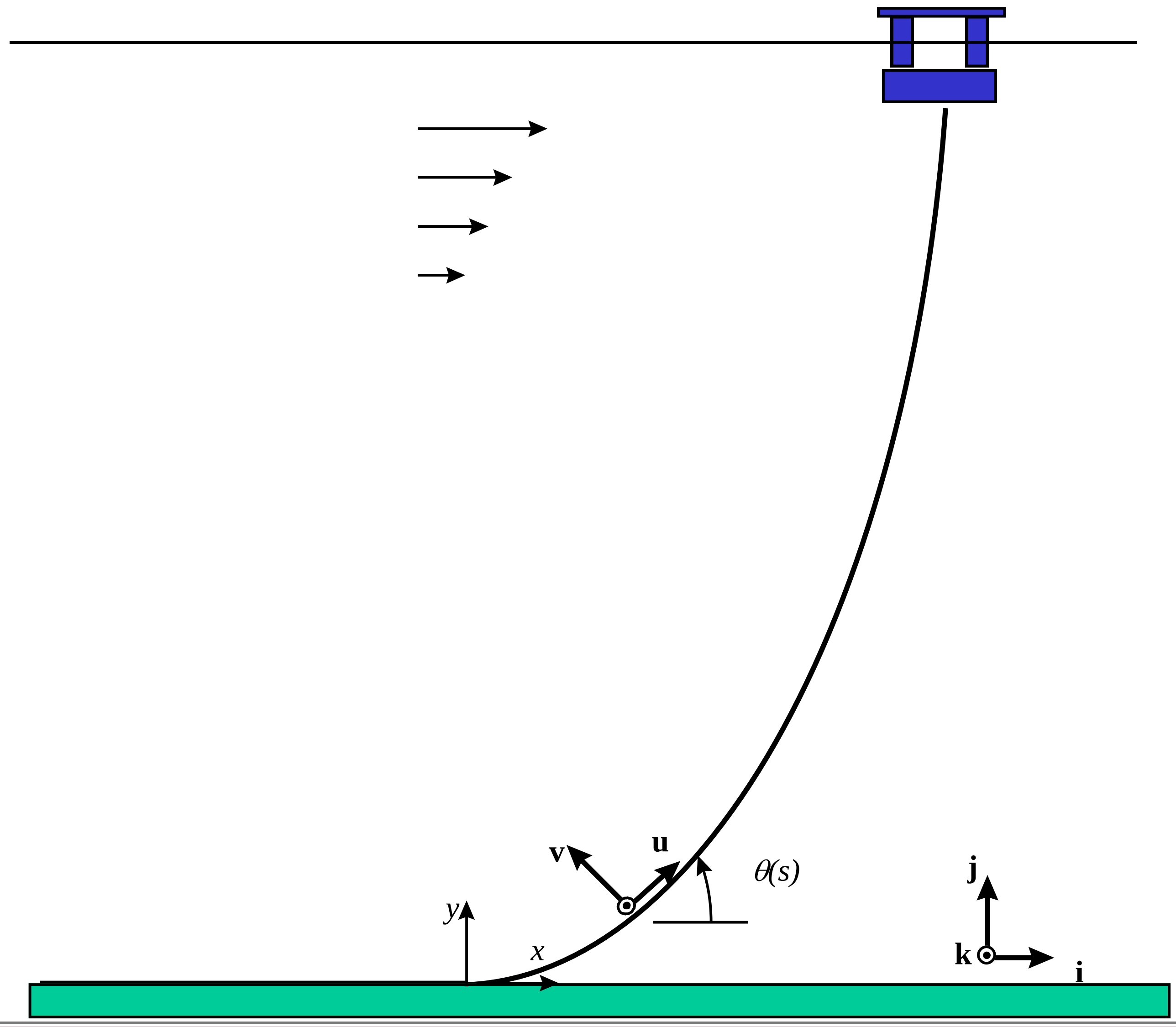
follows:

$$\boxed{\frac{d^2y_c}{dx_c^{\,2}} - \frac{q}{T_{0c}}\left(1 + \left(\frac{dy_c}{dx_c}\right)^2\right)^{1/2} = 0}$$

Whose solution is the wellknown catenary equation:

$$\boxed{y_c(x) = \frac{T_{0c}}{q}\cosh\left(\frac{q}{T_0}x_c + C_1\right) + C_2}$$

Special interest: existence of a TDP, ou “touch down point”:



Special interest: existence of a TDP, ou “touch down point”:

$$y_c = dy_c / dx_c = 0 \text{ at } x_c = 0$$

Leading to:

$$\boxed{y_c(x) = \frac{T_{0c}}{q}\left\{\cosh\left(\frac{q}{T_0}x_c\right) - 1\right\}}$$

$$\frac{dx}{ds} = \cos\theta = \frac{T_0}{T_c(s)} = \frac{T_0}{\left(T_0^{\,2} + (qs)^2\right)^{1/2}}$$

$$\frac{dy}{ds} = \operatorname{sen}\theta = \frac{qs}{T_c(s)} = \frac{qs}{\left(T_0^{\,2} + (qs)^2\right)^{1/2}}$$

$\longrightarrow$

$$x_c(s) = \frac{T_{0c}}{q}\operatorname{arcsenh}\left(\frac{qs_c}{T_0}\right)$$

$$y_c(s) = \frac{T_{0c}}{q}\left(\left(1 + \left(\frac{qs_c}{T_{0c}}\right)^2\right)^{1/2} - 1\right)$$

Parametric Equations

Back to the catenary curvature function:

$$\chi_c(s_c) = \frac{d\theta_c}{ds_c} = \frac{q}{T_{0c}} \cos^2\theta_c(s_c) = \frac{q}{T_{0c}} \frac{1}{1+\left(\frac{qs_c}{T_{0c}}\right)^2}$$

And observing that $\chi_c(0) = \chi_{0c} = \frac{q}{T_{0c}}$ is the curvature at TDP:

$$\chi_c(s) = \chi_{0c} \cos^2\theta_c(s_c) = \chi_{0c} \frac{1}{1+\left(\chi_{0c}s_c\right)^2}$$

$$\chi_c(s) \cong \chi_{0c}\left(1-\left(\chi_{0c}s_c\right)^2\right); \quad \chi_{0c}s_c << 1$$

Other useful relations:

$$L_c = \frac{T_{0c}}{q} \tan \theta_c^t$$

$$H = \int_0^H dy_c = \frac{T_{0c}}{q} \int_0^{\theta_c^t} \operatorname{sen} \theta_c \sec^2 \theta_c d\theta_c = \frac{T_{0c}}{q} \left(\sec \theta_c^t - 1 \right)$$

$$\cos \theta_c^t = \left(\frac{qH}{T_{0c}} + 1 \right)^{-1}$$

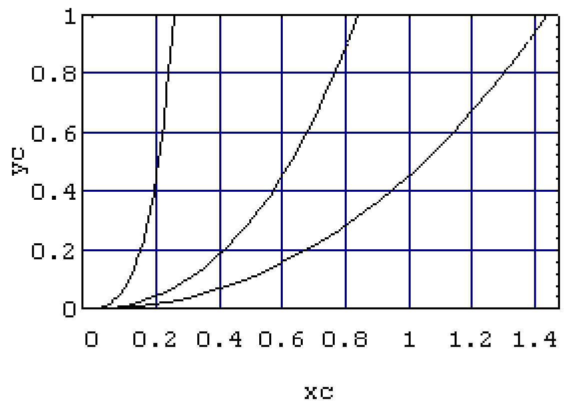
$$\operatorname{sen} \theta_c^t = \left(\left(1 + \frac{qH}{T_{0c}} \right)^2 - 1 \right)^{1/2} \left(\frac{qH}{T_{0c}} + 1 \right)^{-1}$$

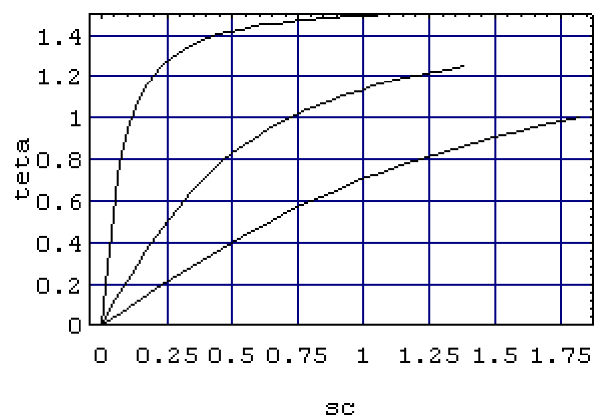
$$L_c = \frac{T_{0c}}{q} \left(\left(1 + \frac{qH}{T_{0c}} \right)^2 - 1 \right)^{1/2}$$

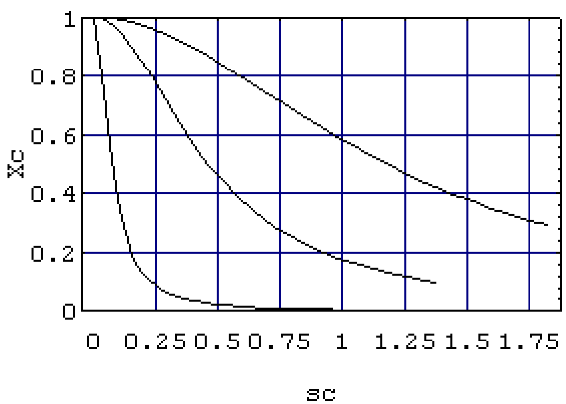
Catenary with a TDP on a horizontal bottom
No current

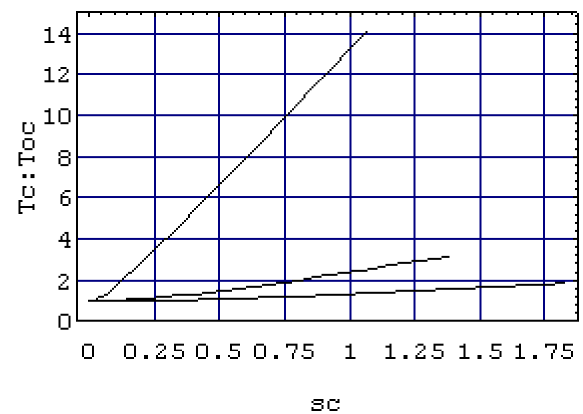
Nondimensional curves; parameterized wrt angle at upper end

$$\theta_{cL} = 1.0;1.25;1.5\,\text{rad}$$









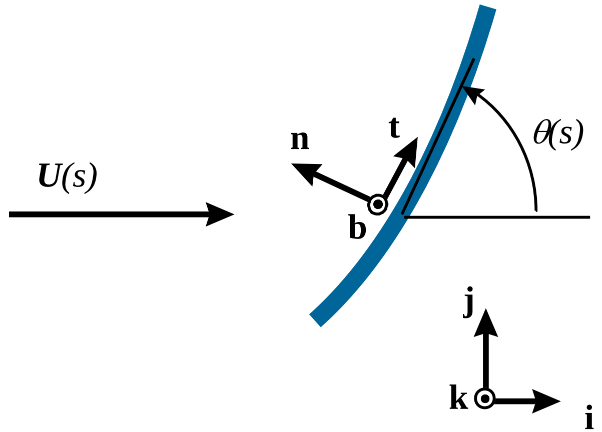
Summary

- Effective tension is a fundamental concept in submerged lines.
- *The classic Kirschoff-Clebsh-Love equilibrium equations are a essential tool in catenary riser analysis.*
- In the planar problem Love equations can be reduced to a single second-order ODE in $\theta(s)$.
- *This equation must be solved iteratively, since the hydrodynamic forces depend on the sought equilibrium configuration.*
- In the absence of current *KCL equations reduce to the well known catenary equation.*

Appendix:
The planar static problem under current

$$\mathbf{h}(s)=\frac{1}{2}\rho_a D U(s)\left|U(s)\right|\left(C_T\cos^2\theta(s)\mathbf{t}-\text{sinal}(\chi(s))C_D\,\text{sen}^2\theta(s)\mathbf{n}\right)$$

$$\text{sinal}(\chi)=\begin{cases}-1;\ \chi<0\\ 1;\ \chi\geq 0\end{cases}$$



$$T_0\tan\theta+$$

$$-\frac{1}{2}\rho_a D U_0^{\ 2}\left(C_D\int_s \text{sen}\,\theta\tan\theta ds+\int_s\sec^2\theta\frac{d\theta}{ds}\int_s\left(C_D\,\text{sen}^3\theta(\xi)+C_T\cos^3\theta(\xi)\right)d\xi ds\right)=qs$$

$$\frac{d\theta}{ds}=\cos^2\theta\left\{\frac{q+\alpha C_D\,\text{sen}\,\theta\tan\theta}{(T_0-F_x(s))}\right\}$$

$$\alpha=\frac{1}{2}\rho_a D U_0^{\ 2}$$

Curvature at TDP

- Under or not current forces curvature at TDP is given by:

$$\chi_0 = \left.\frac{d\theta}{ds}\right|_{s=0} = \frac{q}{T_0}$$

- Current effect is implicit to the tension at TDP: T_0

First-order approximate solution under constant current profile

In first-order, around the catenary solution:

Horizontal coordinates of upper end point and center of mass

$$\frac{x_{Lc}}{H}=\frac{\ln\left(\tan\theta_{cL}+\sec\theta_{cL}\right)}{\sec\theta_{cL}-1} \qquad \frac{x_{Gc}}{H}=\frac{\cot\theta_{cL}}{\sec\theta_{cL}-1}\left\{1-\sec\theta_{cL}+\tan\theta_{cL}\ln\left(\tan\theta_{cL}+\sec\theta_{cL}\right)\right\}$$

Hydrodynamic forces center coordinates:

$$\frac{x_{Y_C}}{H}=\frac{1}{\sec\theta_{cL}-1}\left\{\frac{a_Y(\theta_{cL})+\eta b_Y(\theta_{cL})}{c_Y(\theta_{cL})+\eta d_Y(\theta_{cL})}\right\} \qquad \frac{y_{X_C}}{H}=\frac{1}{\sec\theta_{cL}-1}\left\{\frac{a_X(\theta_{cL})+\eta b_X(\theta_{cL})}{c_X(\theta_{cL})+\eta d_X(\theta_{cL})}\right\}$$

Angle at upper end:

$$\tan\tilde{\theta}_L=\frac{\tan\theta_{cL}+Y_c/T_{0c}}{T_0/T_{0c}-X_c/T_{0c}}$$

Tension atTDP:

$$\frac{\tilde{T}_0}{T_{0c}}=\frac{1}{H}\left[\left(x_{Lc}-x_{Gc}\right)\tan\theta_{cL}+\left(x_{Lc}-x_{Yc}\right)\frac{Y_c}{T_{0c}}+\left(H-y_{Xc}\right)\frac{X_c}{T_{0c}}\right]$$

First-order approximate solution under constant current profile

where:

$$a_Y(\theta) = -\left(1/2\right)\ln^2(\sec\theta + \tan\theta) - \mathrm{sen}\,\theta\ln(\sec\theta + \tan\theta) - \ln(\cos\theta)$$

$$b_Y(\theta) = \theta - \cos\theta\ln(\sec\theta + \tan\theta)$$

$$c_Y(\theta) = \mathrm{sen}\,\theta - \ln(\sec\theta + \tan\theta)$$

$$d_Y(\theta) = 1 - \cos\theta$$

and:

$$a_X(\theta) = 2 - \cos\theta - \sec\theta + \frac{1}{2}\tan^2\theta + \ln(\cos\theta)$$

$$b_X(\theta) = \theta - sen\theta$$

$$c_X(\theta) = \cos\theta + \sec\theta - 2$$

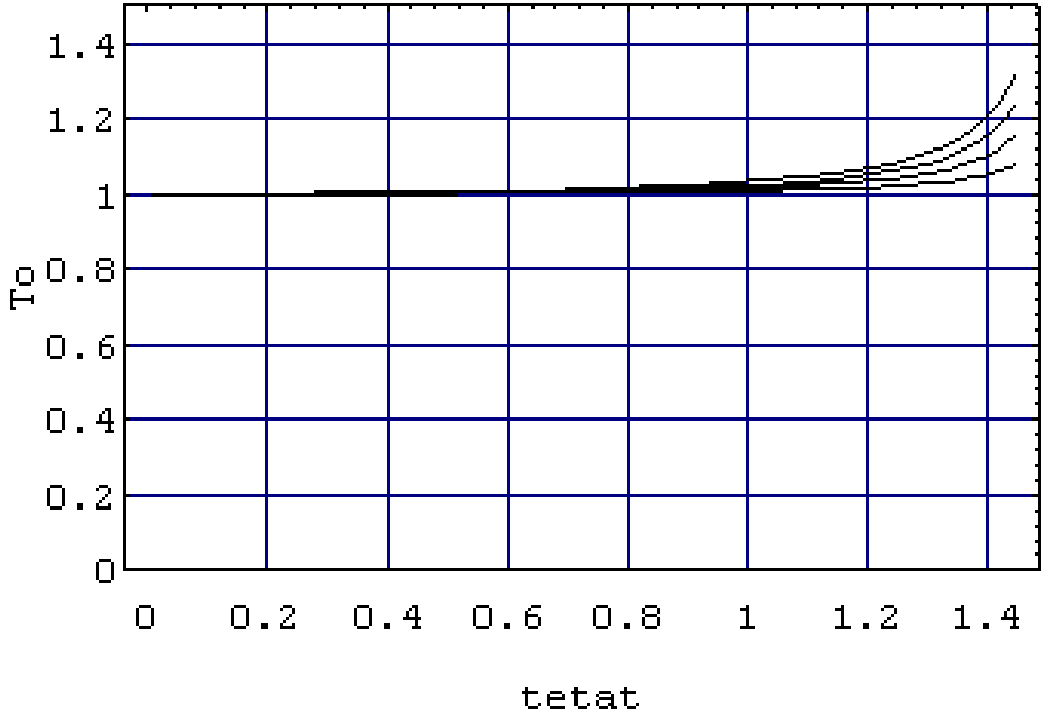
$$d_X(\theta) = sen\theta$$

$$\eta = C_T / C_D$$

Approximate solution under constant current profile

Normalized tension at TDP, referred to the catenary solution, vs catenary angle at upper end parametrized wrt the current intensity force

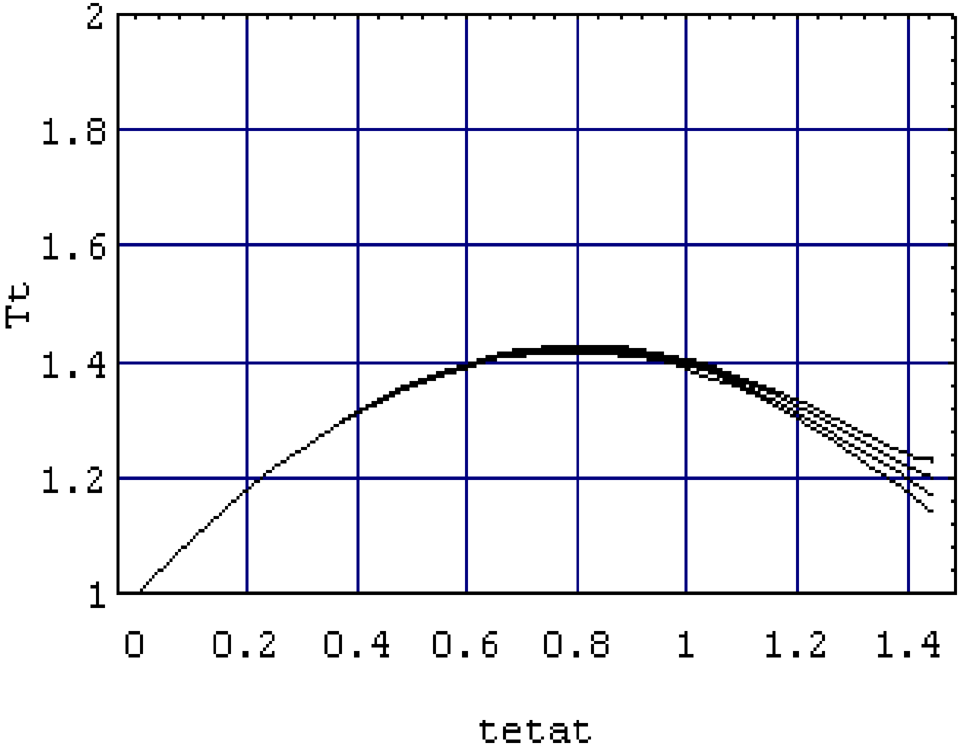
$$C_D \frac{\alpha}{q} = \frac{1}{2} C_D \frac{\rho_a D U_0^{\,2}}{q} = 2{,}5;\ \ 5{,}0;\ \ 7{,}5\ \ \text{e}\ \ 10\%$$



Approximate solution under constant current profile

Normalized tension at TOP, referred to the catenary solution, vs catenary angle at upper end parametrized wrt the current intensity force

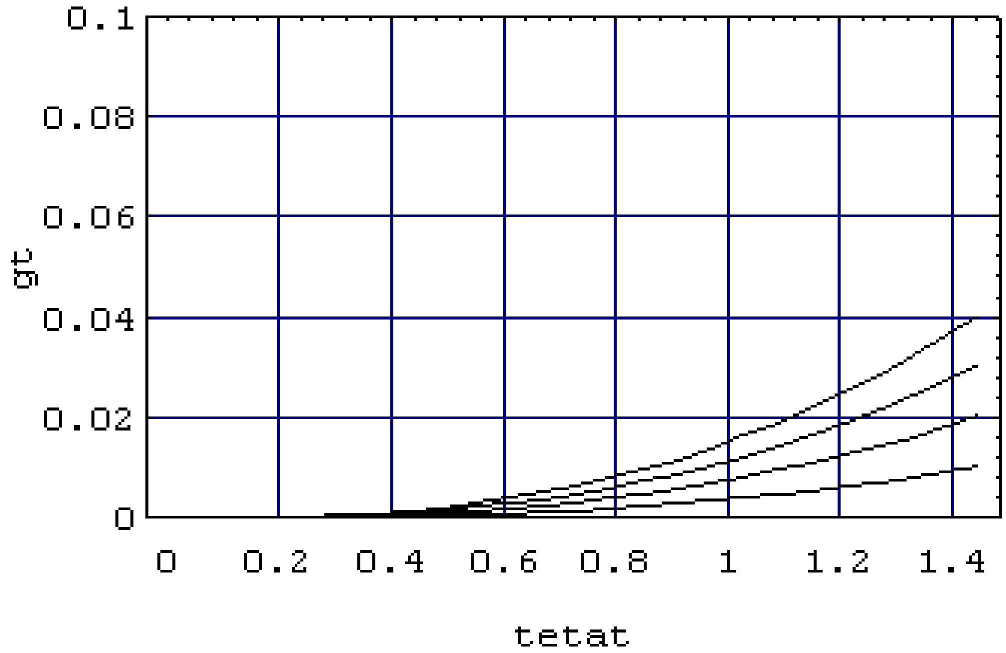
$$C_D \frac{\alpha}{q} = \frac{1}{2} C_D \frac{\rho_a D U_0^{\ 2}}{q} = 2{,}5;\ \ 5{,}0;\ \ 7{,}5 \ \text{ e } \ 10\%$$



Approximate solution under constant current profile

Angle variation (rad) at TOP vs catenary angle at upper end parametrized wrt the current intensity force

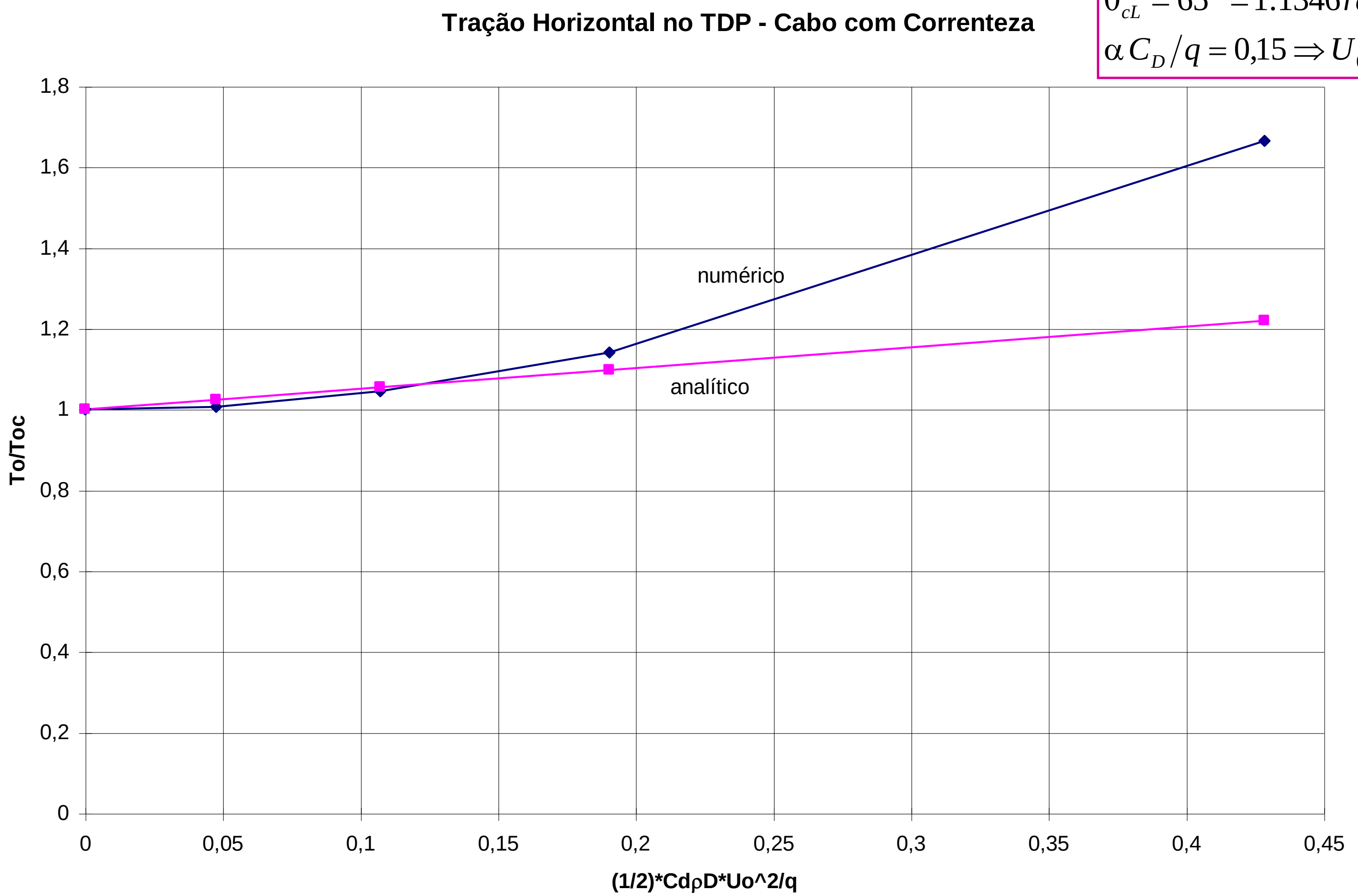
$$C_D \frac{\alpha}{q} = \frac{1}{2} C_D \frac{\rho_a D U_0^{\,2}}{q} = 2{,}5;\ \ 5{,}0;\ \ 7{,}5\ \ \text{e}\ \ 10\%$$



Approximate vs 'exact' numerical solution

SCR:10"3/4 in 910m deep waters

$$\theta_{cL} = 65^o = 1.1346\, rad$$
$$\alpha C_D / q = 0{,}15 \Rightarrow U_0 = 0{,}9\, \mathrm{m}/s$$



Summary

- In the planar problem, curvature at TDP depends only on tension and immersed weight. Tension brings implicitly all information from the equilibrium configuration.

- A first-order approximation gives the static configuration under a constant current profile.

- Such an approximation is fair enough in mild current conditions, up to 1.0m/s for a standard 10 inches SCR.

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