2.3 – The Terminated, Lossless Transmission Line

Reading Assignment: *pp. 57-64*

We now know that a **lossless** transmission line is **completely** characterized by **real** constants Z_0 and β.

Likewise, the **2 waves** propagating on a transmission line are **completely** characterized by **complex** constants V_0^+ and V_0^-.

Q: *Z_0 and β are determined from L, C, and ω. How do we find V_0^+ and V_0^- ?*

A: Apply **Boundary Conditions**!

Every transmission line has **2** "boundaries"

1) At one end of the transmission line.
2) At the **other** end of the trans line!

Typically, there is a **source** at one end of the line, and a **load** at the other.

→ The purpose of the transmission line is to get power **from** the source, **to** the load!

Let's apply the **load** boundary condition!

HO: The Terminated, Lossless Transmission Line

Q: *So, the purpose of the transmission line is to transfer **E.M. energy** from the source to the load. Exactly how much **power** is flowing in the transmission line, and how much is **delivered** to the load?*

A: HO: Incident, Reflected, and Absorbed Power

Let's look at several "special" values of **load impedance**, as well as the interesting transmission line behavior they create.

HO: Special Values of Load Impedance

Q: *So the line impedance at the **end** of a line must be load impedance Z_L (i.e., $Z(z = z_L) = Z_L$); what is the line impedance at the **beginning** of the line (i.e., $Z(z = z_L - \ell) = ?$)?*

A: The **input impedance** !

HO: Transmission Line Input Impedance

Example: Input Impedance

Q: *For a given Z_L we can determine an equivalent Γ_L. Is there an equivalent Γ_{in} for each Z_{in}?*

A: HO: The Reflection Coefficient Transformation

Note that we can **specify** a load with its impedance Z_L or equivalently, its reflection coefficient Γ_L.

Q: *But these are both complex values. Isn't there a way of specifying a load with a real value?*

A: Yes (sort of)! The two most common methods are Return Loss and **VSWR**.

HO: RETURN LOSS AND VSWR

Q: *What happens if our transmission line is terminated by something* ***other*** *than a load? Is our transmission line theory* ***still*** *valid?*

A: As long as a transmission line is connected to linear devices our theory **is** valid. However, we must be careful to properly apply the **boundary conditions** associated with each linear device!

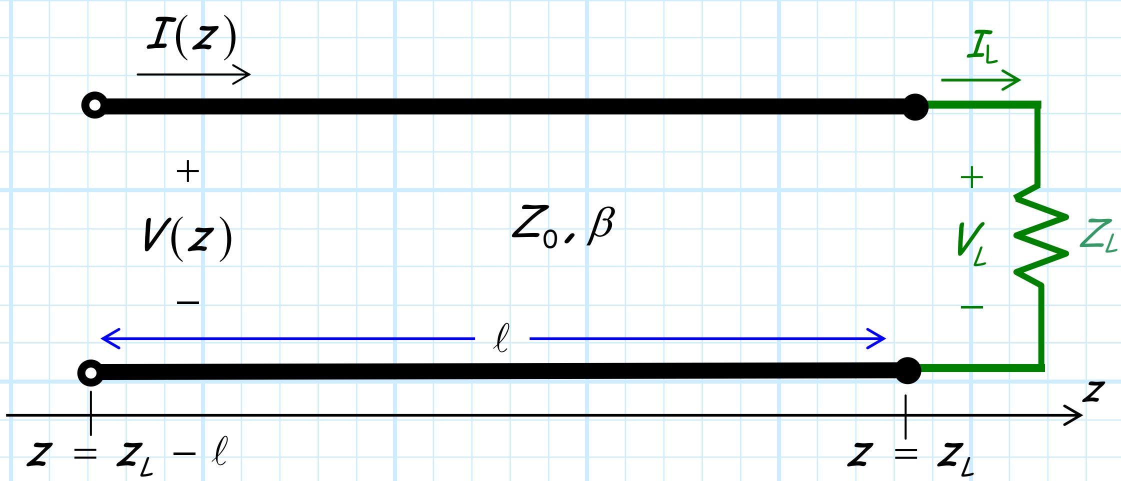
EXAMPLE: THE TRANSMISSION COEFFICIENT

EXAMPLE: APPLYING BOUNDARY CONDITIONS

EXAMPLE: ANOTHER BOUNDARY CONDITION PROBLEM

The Terminated, Lossless Transmission Line

Now let's **attach** something to our transmission line. Consider a **lossless** line, length ℓ, terminated with a **load** Z_L.



Q: *What is the **current** and **voltage** at each and **every** point on the transmission line (i.e., what is $I(z)$ and $V(z)$ for **all** points z where $z_L - \ell \leq z \leq z_L$?)?*

A: To find out, we must apply **boundary conditions**!

In other words, at the **end** of the transmission line ($z = z_L$)—where the load is **attached**—we have **many** requirements that **all** must be satisfied!

Requirement 1. To begin with, the voltage and current ($I(z=z_L)$ and $V(z=z_L)$) must be consistent with a valid **transmission line solution**:

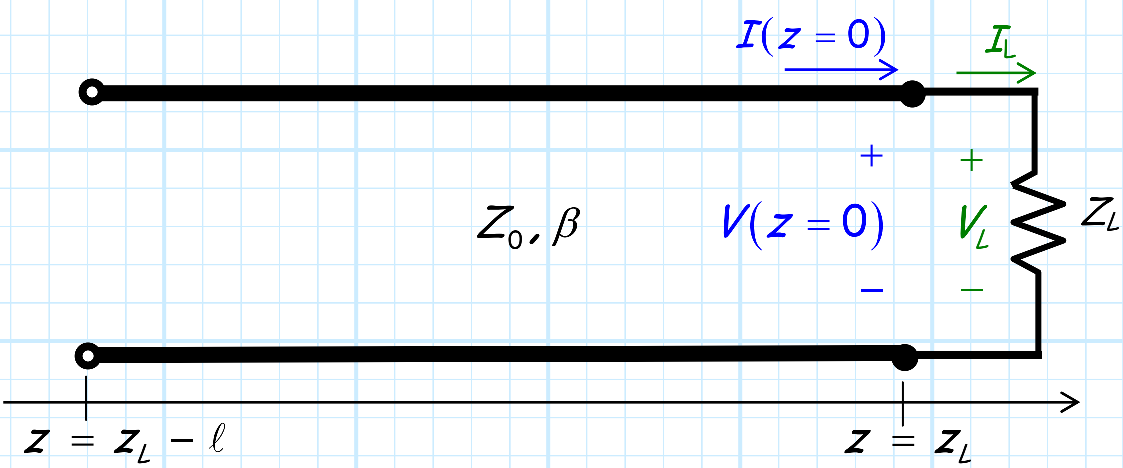
$$
\begin{aligned}
V(z=z_L) &= V^+(z=z_L) + V^-(z=z_L) \\
&= V_0^+\, e^{-j\beta z_L} + V_0^-\, e^{+j\beta z_L}
\end{aligned}
$$

$$
\begin{aligned}
I(z=z_L) &= \frac{V_0^+(z=z_L)}{Z_0} - \frac{V_0^-(z=z_L)}{Z_0} \\
&= \frac{V_0^+}{Z_0}\, e^{-j\beta z_L} - \frac{V_0^-}{Z_0}\, e^{+j\beta z_L}
\end{aligned}
$$

Requirement 2. Likewise, the load voltage and current must be related by **Ohm's law**:

$$V_L = Z_L\, I_L$$

Requirement 3. Most importantly, we recognize that the values $I(z=z_L)$, $V(z=z_L)$ and I_L, V_L are **not** independent, but in fact are strictly related by **Kirchoff's Laws**!

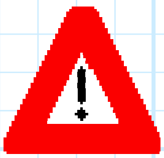


From KVL and KCL we find these requirements:

$$V(z = z_L) = V_L$$

$$I(z = z_L) = I_L$$

These are the **boundary conditions** for **this** particular problem.



→ **Careful**! Different transmission line problems lead to **different** boundary conditions—**you** must access each problem **individually** and **independently**!

Combining these equations and boundary conditions, we find that:

$$V_L = Z_L \, I_L$$

$$V(z = z_L) = Z_L \, I(z = z_L)$$

$$V^+(z = z_L) + V^-(z = z_L) = \frac{Z_L}{Z_0}\left(V^+(z = z_L) - V^-(z = z_L)\right)$$

Rearranging, we can conclude:

$$\frac{V^-(z = z_L)}{V^+(z = z_L)} = \frac{Z_L - Z_0}{Z_L + Z_0}$$

Q: *Hey wait as second! We earlier defined $V^-(z)/V^+(z)$ as* ***reflection coefficient*** *$\Gamma(z)$. How does this relate to the expression* ***above****?*

A: Recall that $\Gamma(z)$ is a **function** of transmission line position z. The value $V^-(z=z_L)/V^+(z=z_L)$ is simply the value of function $\Gamma(z)$ **evaluated** at $z=z_L$ (i.e., evaluated at the **end** of the line):

$$\frac{V^-(z=z_L)}{V^+(z=z_L)} = \Gamma(z=z_L) = \frac{Z_L - Z_0}{Z_L + Z_0}$$

This value is of **fundamental** importance for the terminated transmission line problem, so we provide it with its **own** special symbol (Γ_L)!

$$\boxed{\Gamma_L \doteq \Gamma(z=z_L) = \frac{Z_L - Z_0}{Z_L + Z_0}}$$

Q: *Wait! We* ***earlier*** *determined that:*

$$\Gamma(z) = \frac{Z(z) - Z_0}{Z(z) + Z_0}$$

so it would seem that:

$$\Gamma_L = \Gamma(z=z_L) = \frac{Z(z=z_L) - Z_0}{Z(z=z_L) + Z_0}$$

Which *expression is correct??*

A: They **both** are! It is evident that the two expressions:

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} \qquad \text{and} \qquad \Gamma_L = \frac{Z(z = z_L) - Z_0}{Z(z = z_L) + Z_0}$$

are **equal** if:

$$Z(z = z_L) = Z_L$$

And since we know that from **Ohm's Law**:

$$Z_L = \frac{V_L}{I_L}$$

and from **Kirchoff's Laws**:

$$\frac{V_L}{I_L} = \frac{V(z = z_L)}{I(z = z_L)}$$

and that **line impedance** is:

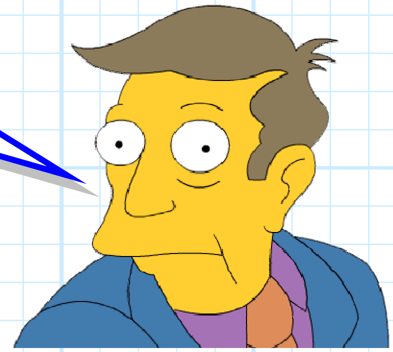
$$\frac{V(z = z_L)}{I(z = z_L)} = Z(z = z_L)$$

we find it apparent that the **line impedance** at the **end** of the transmission line is **equal** to the **load impedance**:

$$\boxed{Z(z = z_L) = Z_L}$$

The above expression is essentially **another** expression of the **boundary condition** applied at the **end** of the transmission line.

Q: *I'm confused! Just what are were we trying to accomplish in this handout?*



A: We are trying to find $V(z)$ and $I(z)$ when a lossless transmission line is terminated by a load Z_L!

We can now determine the value of V_0^- in terms of V_0^+. Since:

$$\Gamma_L = \frac{V^-(z = z_L)}{V^+(z = z_L)} = \frac{V_0^- e^{+j\beta z_L}}{V_0^+ e^{-j\beta z_L}}$$

We find:

$$V_0^- = e^{-2j\beta z_L}\,\Gamma_L\,V_0^+$$

And therefore we find:

$$V^-(z) = \left(e^{-2j\beta z_L}\,\Gamma_L\,V_0^+\right)e^{+j\beta z}$$

$$V(z) = V_0^+\left[e^{-j\beta z} + \left(e^{-2j\beta z_L}\,\Gamma_L\right)e^{+j\beta z}\right]$$

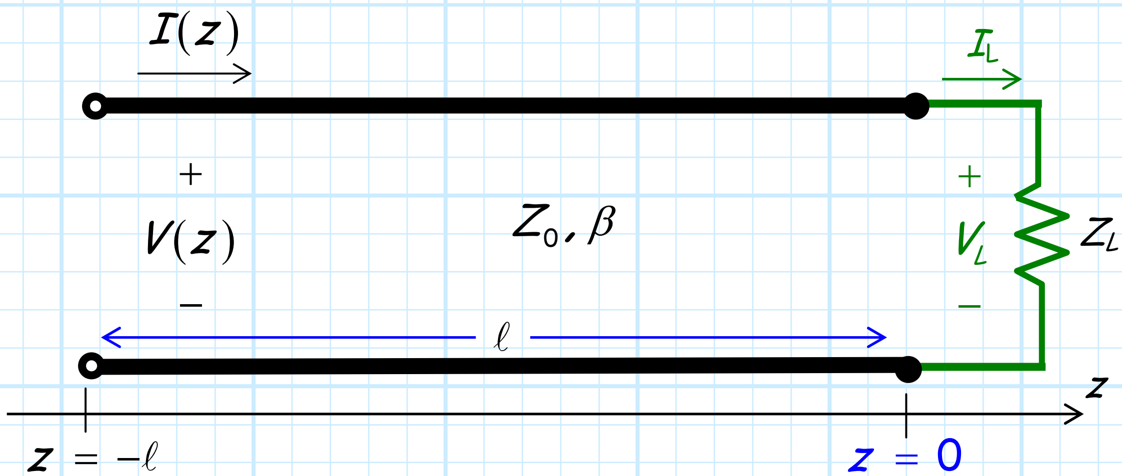
$$I(z) = \frac{V_0^+}{Z_0}\left[e^{-j\beta z} - \left(e^{-2j\beta z_L}\,\Gamma_L\right)e^{+j\beta z}\right]$$

where:

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}$$

$$z_L = 0$$

Now, we can further **simplify** our analysis by **arbitrarily** assigning the end point z_L a **zero** value (i.e., $z_L = 0$):



If the load is located at z=0 (i.e., **if** $z_L = 0$), we find that:

$$\begin{aligned} V(z=0) &= V^+(z=0) + V^-(z=0) \\ &= V_0^+ e^{-j\beta(0)} + V_0^- e^{+j\beta(0)} \\ &= V_0^+ + V_0^- \end{aligned}$$

$$\begin{aligned} I(z=0) &= \frac{V_0^+(z=0)}{Z_0} - \frac{V_0^-(z=0)}{Z_0} \\ &= \frac{V_0^+}{Z_0} e^{-j\beta(0)} - \frac{V_0^-}{Z_0} e^{+j\beta(0)} \\ &= \frac{V_0^+ - V_0^-}{Z_0} \end{aligned}$$

$$Z(z=0) = Z_0 \left(\frac{V_0^+ + V_0^-}{V_0^+ - V_0^-} \right)$$

Likewise, it is apparent that **if** $z_L = 0$, Γ_L and Γ_0 are the same:

$$\Gamma_L = \Gamma(z = z_L) = \frac{V^-(z=0)}{V^+(z=0)} = \frac{V_0^-}{V_0^+} = \Gamma_0$$

Therefore **if** $z_L = 0$:

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \Gamma_0$$

Thus, we can write the line current and voltage simply as:

$$V(z) = V_0^+\left[e^{-j\beta z} + \Gamma_0 \, e^{+j\beta z}\right]$$

$$\left[\text{for } z_L = 0\right]$$

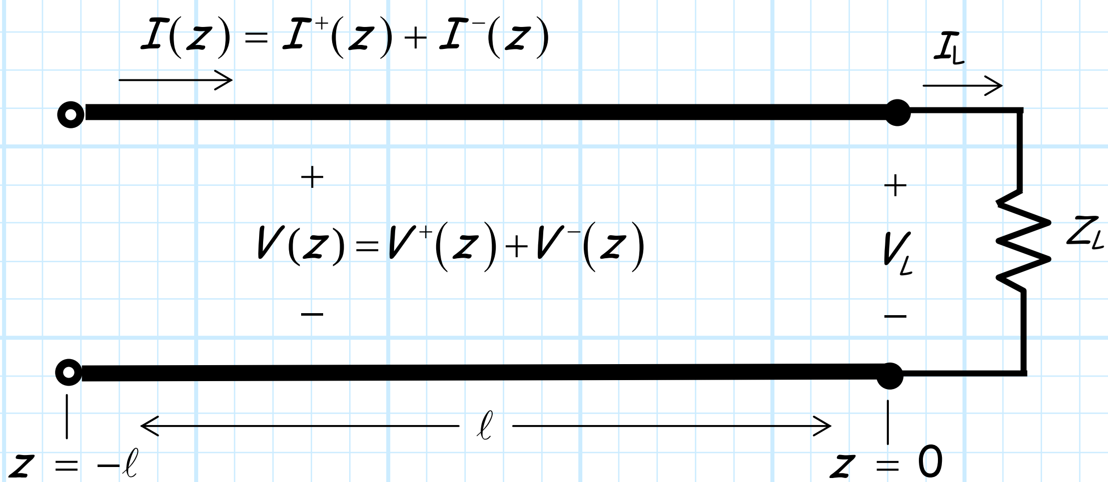
$$I(z) = \frac{V_0^+}{Z_0}\left[e^{-j\beta z} - \Gamma_0 \, e^{+j\beta z}\right]$$

Q: *But, how do we determine* V_0^+ *??*

A: We require a **second** boundary condition to determine V_0^+. The only boundary left is at the **other end** of the transmission line. Typically, a **source** of some sort is located there. This makes physical sense, as something must generate the **incident** wave !

Incident, Reflected, and Absorbed Power

We have discovered that **two waves propagate** along a transmission line, one in each direction ($V^+(z)$ and $V^-(z)$).



The result is that electromagnetic energy flows along the transmission line at a given rate (i.e., **power**).

Q: *How much power flows along a transmission line, and where does that power go?*

A: We can answer that question by determining the power **absorbed** by the **load**!

You of course recall that the **time-averaged** power (a **real** value!) absorbed by a **complex** impedance Z_L is:

$$P_{abs} = \frac{1}{2}\mathrm{Re}\left\{V_L I_L^*\right\}$$

Of course, the **load** voltage and current is simply the voltage an current at the **end** of the transmission line (at $z=0$). A **happy** result is that we can then use our **transmission line theory** to determine this absorbed power:

$$\begin{aligned}
P_{abs} &= \frac{1}{2}\mathrm{Re}\left\{V_L I_L^*\right\} \\
&= \frac{1}{2}\mathrm{Re}\left\{V(z=0)I(z=0)^*\right\} \\
&= \frac{1}{2Z_0}\mathrm{Re}\left\{\left(V_0^+\left[e^{-j\beta 0}+\Gamma_0 e^{+j\beta 0}\right]\right)\left(V_0^+\left[e^{-j\beta 0}-\Gamma_0 e^{+j\beta 0}\right]\right)^*\right\} \\
&= \frac{\left|V_0^+\right|^2}{2Z_0}\mathrm{Re}\left\{1-\left(\Gamma_0^*-\Gamma_0\right)-\left|\Gamma_0\right|^2\right\} \\
&= \frac{\left|V_0^+\right|^2}{2Z_0}\left(1-\left|\Gamma_0\right|^2\right)
\end{aligned}$$

The significance of this result can be seen by **rewriting** the expression as:

$$P_{abs} = \frac{\left|V_0^+\right|^2}{2Z_0}\left(1-\left|\Gamma_0\right|^2\right) = \frac{\left|V_0^+\right|^2}{2Z_0} - \frac{\left|V_0^+\Gamma_0\right|^2}{2Z_0} = \frac{\left|V_0^+\right|^2}{2Z_0} - \frac{\left|V_0^-\right|^2}{2Z_0}$$

The two terms in above expression have a very definite **physical meaning**. The **first** term is the time-averaged **power of the wave** propagating along the transmission line **toward the load**.

We say that this wave is **incident** on the load:

$$\boxed{P_{inc} = P_{+} = \frac{|V_0^+|^2}{2Z_0}}$$

Likewise, the second term of the P_{abs} equation describes the **power of the wave** moving in the other direction (**away from the load**). We refer to this as the wave **reflected** from the load:

$$\boxed{P_{ref} = P_{-} = \frac{|V_0^-|^2}{2Z_0} = \frac{|\Gamma_L V_0^+|^2}{2Z_0} = |\Gamma_L|^2 \frac{|V_0^+|^2}{2Z_0} = |\Gamma_L|^2 P_{inc}}$$

Thus, the power **absorbed** by the load (i.e., the power **delivered to** the load) is simply:

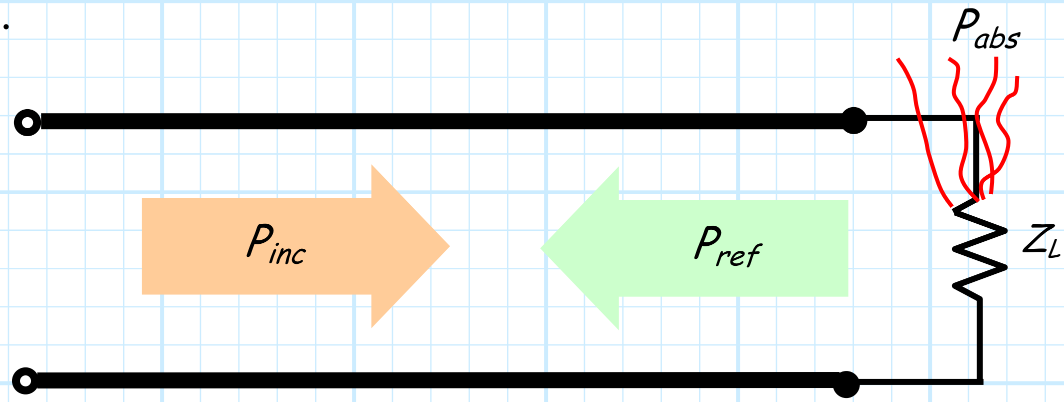
$$P_{abs} = \frac{|V_0^+|^2}{2Z_0} - \frac{|V_0^-|^2}{2Z_0} = P_{inc} - P_{ref}$$

or, rearranging, we find:

$$\boxed{P_{inc} = P_{abs} + P_{ref}}$$

This equation is simply an expression of the **conservation of energy** !

It says that power flowing **toward** the load (P_{inc}) is either **absorbed** by the load (P_{abs}) or **reflected** back from the load (P_{ref}).



Now let's consider some special cases:

1. $\left|\Gamma_L\right|^2 = 1$

For this case, we find that the load absorbs **no power**!

$$P_{abs} = P_{inc}\left(1-\left|\Gamma_0\right|^2\right) = P_{inc}\left(1-1\right) = 0$$

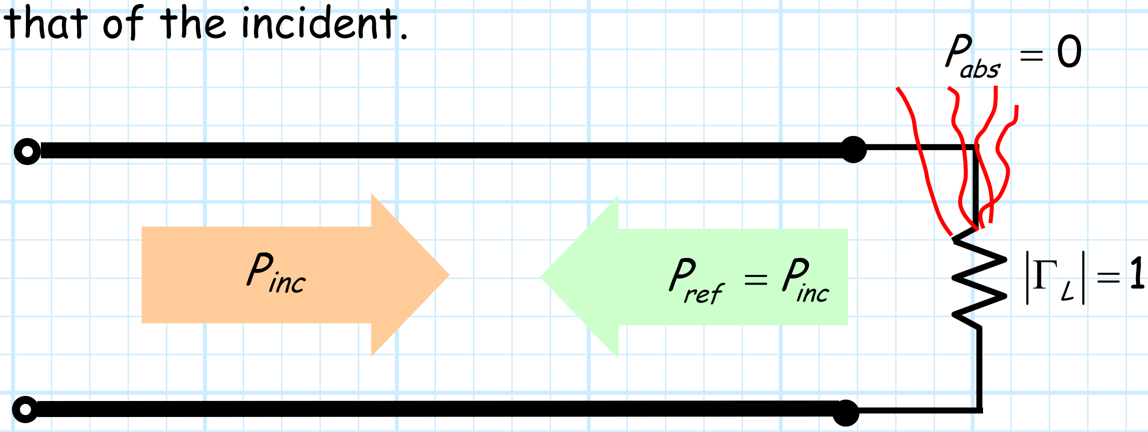
Likewise, we find that the reflected power is **equal** to the incident:

$$P_{ref} = \left|\Gamma_L\right|^2 P_{inc} = \left(1\right)P_{inc} = P_{inc}$$

Note these two results are completely **consistent**—by conservation of energy, if one is true the other **must** also be:

$$P_{inc} = P_{abs} + P_{ref} = 0 + P_{ref} = P_{ref}$$

In this case, **no power** is absorbed by the load. **All** of the incident power is **reflected**, so that the reflected power is **equal** to that of the incident.



2. $|\Gamma_L| = 0$

For this case, we find that there is **no reflected power**!

$$P_{ref} = |\Gamma_L|^2 P_{inc} = (0) P_{inc} = 0$$

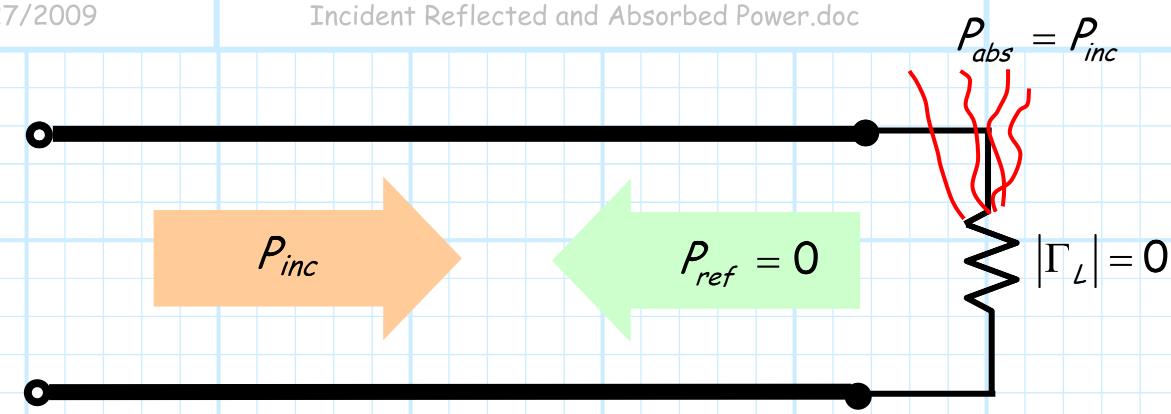
Likewise, we find that the absorbed power is **equal** to the incident:

$$P_{abs} = P_{inc}\left(1 - |\Gamma_0|^2\right) = P_{inc}\left(1 - 0\right) = P_{inc}$$

Note these two results are completely **consistent**—by conservation of energy, if one is true the other **must** also be:

$$P_{inc} = P_{abs} + P_{ref} = P_{abs} + 0 = P_{abs}$$

In this case, **all** the incident power is absorbed by the load. **None** of the incident power is **reflected**, so that the absorbed power is **equal** to that of the incident.



3. $0 < |\Gamma_L| < 1$

For this case, we find that the **reflected** power is greater than zero, but **less** than the incident power.

$$0 < P_{ref} = |\Gamma_L|^2 P_{inc} < P_{inc}$$

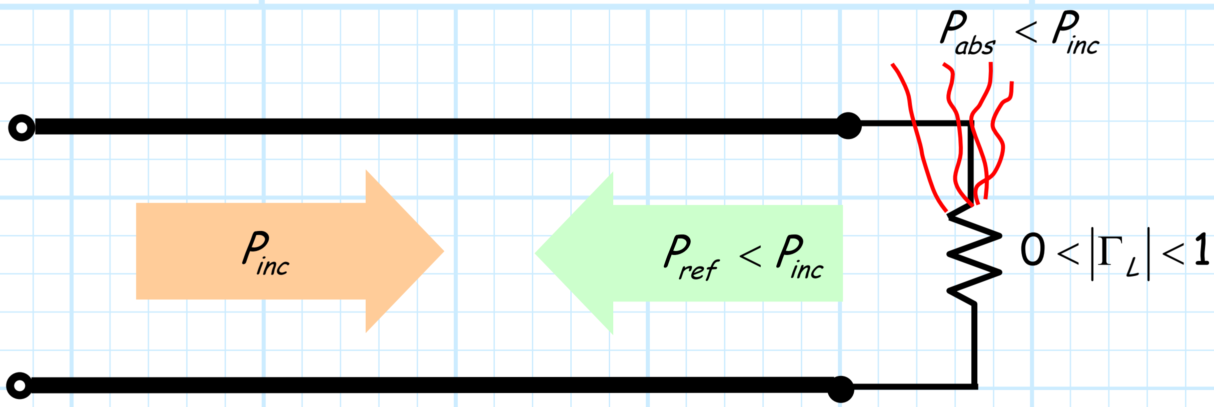
Likewise, we find that the **absorbed** power is **also** greater than zero, but **less** than the incident power.

$$0 < P_{abs} = P_{inc}\left(1 - |\Gamma_0|^2\right) < P_{inc}$$

Note these two results are completely **consistent**—by conservation of energy, if one is true the other **must** also be:

$$0 < P_{ref} = P_{inc} - P_{abs} < P_{inc} \qquad \text{and} \qquad 0 < P_{abs} = P_{inc} - P_{ref} < P_{inc}$$

In this case, the incident power is divided. Some of the incident power is absorbed by the load, while the remainder is reflected from the load.



4. $|\Gamma_L| > 1$

For this case, we find that the **reflected** power is **greater** than the **incident** power.

$$0 < P_{ref} = |\Gamma_L|^2 P_{inc} < P_{inc}$$

Q: *Yikes! What's up with that? This result does **not** seem at all consistent with your conservation of energy argument. How can the reflected power be **larger** than the incident?*

A: Quite insightful! It is indeed a result quite **askew** with our conservation of energy analysis. To see why, let's determine the **absorbed** power for this case.

$$P_{abs} = P_{inc}\left(1 - |\Gamma_L|^2\right) < 0$$

The power absorbed by the load is **negative**!

This result actually has a **physical interpretation**. A negative absorbed power indicates that the load is not absorbing power at all—it is instead **producing** power!

This makes sense if **you think** about it. The power flowing away from the load (the reflected power) can be larger than the power flowing toward the load (the incident power) **only** if the load itself is creating this extra power. The load is not a power **sink**, it is a power **source**.

Q: *But how could a **passive** load be a power source?*

A: It **can't**. A passive device cannot produce power. Thus, we have come to an important conclusion. The reflection coefficient of any and all passive loads **must** have a **magnitude** that is **less than one**.

$$\boxed{|\Gamma_L| \leq 1 \quad \text{for all passive loads}}$$

Q: *Can $|\Gamma_L|$ every be **greater** than one?*

A: Sure, if the "load" is an **active** device. In other words, the load must have some **external power** source connected to it.

Q: *What about the case where $|\Gamma_L| < 0$, shouldn't we examine **that** situation as well?*

A: That would be just plain **silly**; do **you** see why?

Special Values of Load Impedance

It's interesting to note that the load Z_L enforces a boundary condition that explicitly determines neither $V(z)$ nor $I(z)$—but **completely** specifies **line impedance** $Z(z)$!

$$Z(z) = Z_0 \frac{e^{-j\beta z} + \Gamma_L e^{+j\beta z}}{e^{-j\beta z} - \Gamma_L e^{+j\beta z}} = Z_0 \frac{Z_L \cos\beta z - jZ_0 \sin\beta z}{Z_0 \cos\beta z - jZ_L \sin\beta z}$$

$$\Gamma(z) = \Gamma_L e^{+j2\beta z} = \frac{Z_L - Z_0}{Z_L + Z_0} e^{+j2\beta z}$$

Likewise, the load boundary condition leaves $V^+(z)$ and $V^-(z)$ undetermined, but **completely** determines **reflection coefficient function** $\Gamma(z)$!

Let's look at some **specific** values of load impedance $Z_L = R_L + jX_L$ and see what functions $Z(z)$ and $\Gamma(z)$ result!

1. $Z_L = Z_0$

In this case, the **load** impedance is **numerically** equal to the **characteristic** impedance of the transmission line. Assuming the line is **lossless**, then Z_0 is **real**, and thus:

$$R_L = Z_0 \qquad \text{and} \qquad X_L = 0$$

It is evident that the resulting load reflection coefficient is **zero**:

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{Z_0 - Z_0}{Z_0 + Z_0} = 0$$

This result is very interesting, as it means that there is **no reflected wave** $V^-(z)$!

$$\begin{aligned} V^-(z) &= \left(e^{-2j\beta z_L}\,\Gamma_L\,V_0^+\right)e^{+j\beta z} \\ &= \left(e^{-2j\beta z_L}\,(0)\,V_0^+\right)e^{+j\beta z} \\ &= 0 \end{aligned}$$

Thus, the **total** voltage and current along the transmission line is simply voltage and current of the **incident** wave:

$$V(z) = V^+(z) = V_0^+ e^{-j\beta z}$$

$$I(z) = I^+(z) = \frac{V_0^+}{Z_0} e^{-j\beta z}$$

Meaning that the **line** impedance is likewise **numerically** equal to the **characteristic** impedance of the transmission line for **all** line position z:

$$Z(z) = \frac{V(z)}{I(z)} = Z_0 \frac{V_0^+ e^{-j\beta z}}{V_0^+ e^{-j\beta z}} = Z_0$$

And likewise, the reflection coefficient is **zero** at **all** points along the line:

$$\Gamma(z) = \frac{V^-(z)}{V^+(z)} = \frac{0}{V^+(z)} = 0$$

We call this condition (when $Z_L = Z_0$) the **matched** condition, and the load $Z_L = Z_0$ a **matched load**.

2. $Z_L = 0$

A device with **no** impedance is called a **short** circuit! I.E.:

$$R_L = 0 \quad \text{and} \quad X_L = 0$$

In this case, the **voltage** across the load—and thus the voltage at the end of the transmission line—is **zero**:

$$V_L = Z_L \, I_L = 0 \qquad \text{and} \qquad V(z = z_L) = 0$$

Note that this does **not** mean that the **current** is zero!

$$I_L = I(z = z_L) \neq 0$$

For a **short**, the resulting load reflection coefficient is therefore:

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{0 - Z_0}{0 + Z_0} = -1$$

Meaning (assuming $z_L = 0$):

$$V_0^- = -V_0^+$$

As a result, the total **voltage** and **current** along the transmission line is simply:

$$V(z) = V_0^+ \left(e^{-j\beta z} - e^{+j\beta z}\right) = -j2V_0^+ \sin(\beta z)$$

$$I(z) = \frac{V_0^+}{Z_0}\left(e^{-j\beta z} + e^{+j\beta z}\right) = \frac{2V_0^+}{Z_0} \cos(\beta z)$$

Meaning that the **line impedance** can likewise be written in terms of a **trigonometric** function:

$$Z(z) = \frac{V(z)}{I(z)} = -jZ_0 \tan(\beta z)$$

Note that this impedance is **purely reactive**. This means that the current and voltage on the transmission line will be everywhere 90° **out of phase**.

Hopefully, this was likewise apparent to **you** when you **observed** the expressions for $V(z)$ and $I(z)$!

Note at the **end** of the line (i.e., $z = z_L = 0$), we find that:

$$V(z=0) = -j2V_0^+ \sin(0) = 0$$

$$I(z=0) = \frac{2V_0^+}{Z_0} \cos(0) = \frac{2V_0^+}{Z_0}$$

As expected, the voltage is **zero** at the end of the transmission line (i.e. the voltage across the **short**). Likewise, the **current** at the end of the line (i.e., the current through the short) is at a **maximum**!

Finally, we note that the **line** impedance at the **end** of the transmission line is:

$$Z(z=0) = -jZ_0 \tan(0) = 0$$

Just as we expected—a **short** circuit!

Finally, the reflection coefficient **function** is (assuming $z_L = 0$):

$$\Gamma(z) = \frac{V^-(z)}{V^+(z)} = \frac{-V_0^+ e^{+j\beta z}}{V_0^+ e^{-j\beta z}} = -e^{j\beta z}$$

Note that for **this** case $|\Gamma(z)| = 1$, meaning that:

$$|V^-(z)| = |V^+(z)|$$

In other words, the **magnitude** of each **wave** on the transmission line is the **same**—the reflected wave is **just** as big as the incident wave!

$\boxed{\textbf{3.}\ \ Z_L = \infty}$

A device with **infinite** impedance is called an **open** circuit! I.E.:

$$R_L = \infty \quad \text{and/or} \quad X_L = \pm\infty$$

In this case, the **current** through the load—and thus the current at the end of the transmission line—is **zero**:

$$I_L = \frac{V_L}{Z_L} = 0 \qquad \text{and} \qquad I(z = z_L) = 0$$

Note that this does **not** mean that the **voltage** is zero!

$$V_L = V(z = z_L) \neq 0$$

For an **open**, the resulting load reflection coefficient is:

$$\Gamma_L = \lim_{Z_L \to \infty} \frac{Z_L - Z_0}{Z_L + Z_0} = \lim_{Z_L \to \infty} \frac{Z_L}{Z_L} = 1$$

Meaning (assuming $z_L = 0$):

$$V_0^- = V_0^+$$

As a result, the **total** voltage and current along the transmission line is simply (assuming $z_L = 0$):

$$V(z) = V_0^+ \left(e^{-j\beta z} + e^{+j\beta z}\right) = 2V_0^+ \cos(\beta z)$$

$$I(z) = \frac{V_0^+}{Z_0}\left(e^{-j\beta z} - e^{+j\beta z}\right) = -j\,\frac{2V_0^+}{Z_0} \sin(\beta z)$$

Meaning that the **line impedance** can likewise be written in terms of trigonometric function:

$$Z(z) = \frac{V(z)}{I(z)} = jZ_0 \cot(\beta z)$$

Again note that this impedance is **purely reactive**— *V(z)* and *I(z)* are again 90° **out of phase**!

Note at the **end** of the line (i.e., $z = z_L = 0$), we find that

$$V(z=0) = 2V_0^+ \cos(0) = \frac{2V_0^+}{Z_0}$$

$$I(z=0) = -j\,\frac{2V_0^+}{Z_0}\sin(0) = 0$$

As expected, the **current is zero** at the end of the transmission line (i.e. the current through the **open**). Likewise, the **voltage** at the end of the line (i.e., the voltage across the open) is at a **maximum**!

Finally, we note that the **line impedance** at the **end** of the transmission line is:

$$Z(z=0) = jZ_0\, cot(0) = \infty$$

Just as we expected—an **open** circuit!

Finally, the reflection coefficient is (assuming $z_L = 0$):

$$\Gamma(z) = \frac{V^-(z)}{V^+(z)} = \frac{V_0^+ e^{+j\beta z}}{V_0^+ e^{-j\beta z}} = e^{+j2\beta z}$$

Note that likewise for **this** case $|\Gamma(z)| = 1$, meaning **again** that:

$$|V^-(z)| = |V^+(z)|$$

In other words, the **magnitude** of each **wave** on the transmission line is the **same**—the reflected wave is **just** as big as the incident wave!

4. $Z_L = j X_L$

For this case, the load impedance is **purely reactive** (e.g. a capacitor of inductor), and thus the resistive portion is zero:

$$R_L = 0$$

Thus, **both** the current through the load, and voltage across the load, are **non-zero**:

$$I_L = I(z = z_L) \neq 0 \qquad\qquad V_L = V(z = z_L) \neq 0$$

The resulting load reflection coefficient is:

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{jX_L - Z_0}{jX_L + Z_0}$$

Given that Z_0 is real (i.e., the line is **lossless**), we find that this load reflection coefficient is generally some **complex** number.

We can rewrite this value explicitly in terms of its **real** and **imaginary** part as:

$$\Gamma_L = \frac{jX_L - Z_0}{jX_L + Z_0} = \left(\frac{X_L^2 - Z_0^2}{X_L^2 + Z_0^2}\right) + j\left(\frac{2\,Z_0\,X_L}{X_L^2 + Z_0^2}\right)$$

Yuck! This ***isn't*** *much help!*

Let's instead write this complex value Γ_L in terms of its **magnitude** and **phase**. For **magnitude** we find a much more straightforward result!

$$|\Gamma_L|^2 = \frac{|jX_L - Z_0|^2}{|jX_L + Z_0|^2} = \frac{X_L^2 + Z_0^2}{X_L^2 + Z_0^2} = 1$$

Its magnitude is **one**! Thus, we find that for reactive loads, the reflection coefficient can be simply expressed as:

$$\Gamma_L = e^{j\theta_\Gamma}$$

where

$$\theta_\Gamma = tan^{-1}\left[\frac{2\,Z_0\,X_L}{X_L^2 - Z_0^2}\right]$$

We can therefore conclude that for a **reactive load**:

$$V_0^- = e^{j\theta_\Gamma} V_0^+$$

As a result, the **total** voltage and current along the transmission line is simply (assuming $z_L = 0$):

$$\begin{aligned} V(z) &= V_0^+ \left(e^{-j\beta z} + e^{+j\theta_L} e^{+j\beta z} \right) \\ &= V_0^+ \; e^{+j\theta_\Gamma/2} \left(e^{-j(\beta z + \theta_\Gamma/2)} + e^{+j(\beta z + \theta_\Gamma/2)} \right) \\ &= 2V_0^+ \; e^{+j\theta_\Gamma/2} \cos\left(\beta z + \theta_\Gamma/2\right) \end{aligned}$$

$$\begin{aligned} I(z) &= \frac{V_0^+}{Z_0}\left(e^{-j\beta z} - e^{+j\beta z} \right) \\ &= \frac{V_0^+}{Z_0} e^{+j\theta_L/2} \left(e^{-j(\beta z + \theta_L/2)} - e^{+j(\beta z + \theta_L/2)} \right) \\ &= -j \, \frac{2V_0^+}{Z_0} e^{+j\theta_L/2} \sin\left(\beta z + \theta_L/2\right) \end{aligned}$$

Meaning that the **line impedance** can again be written in terms of trigonometric function:

$$Z(z) = \frac{V(z)}{I(z)} = jZ_0 \cot\left(\beta z + \theta_\Gamma/2\right)$$

Again note that this impedance is **purely reactive**— *V(z)* and *I(z)* are once again 90° out of phase!

Note at the **end** of the line (i.e., $z = z_L = 0$), we find that

$$V(z=0) = 2V_0^+ \cos\left(\theta_\Gamma/2\right)$$

$$I(z=0) = -j\,\frac{2V_0^+}{Z_0}\sin\left(\theta_\Gamma/2\right)$$

As expected, **neither** the current **nor** voltage at the end of the line are zero.

We also note that the line impedance at the **end** of the transmission line is:

$$Z(z=0) = jZ_0\,cot\left(\theta_\Gamma/2\right)$$

With a little trigonometry, we can show (trust me!) that:

$$cot\left(\theta_\Gamma/2\right) = \frac{X_L}{Z_0}$$

and therefore:

$$Z(z=0) = jZ_0\,cot\left(\theta_\Gamma/2\right) = j\,X_L = Z_L$$

Just as we **expected**!

Finally, the reflection coefficient **function** is (assuming $z_L = 0$):

$$\Gamma(z) = \frac{V^-(z)}{V^+(z)} = \frac{V_0^+ e^{+j\theta_\Gamma} e^{+j\beta z}}{V_0^+ e^{-j\beta z}} = e^{+j2\left(\beta z+\theta_\Gamma/2\right)}$$

Note that likewise for **this** case $|\Gamma(z)| = 1$, meaning **once again**:

$$|V^-(z)| = |V^+(z)|$$

In other words, the **magnitude** of each **wave** on the transmission line is the **same**—the reflected wave is **just** as big as the incident wave!

Q: *Gee, a **reactive** load leads to results very **similar** to that of an **open** or **short** circuit. Is this just **coincidence**?*

A: Hardly! An open and short **are** in fact reactive loads—they **cannot absorb power** (think about this!).

Specifically, for an **open**, we find $\theta_\Gamma = 0$, so that:

$$\Gamma_L = e^{j\theta_\Gamma} = 1$$

Likewise, for a **short**, we find that $\theta_\Gamma = \pi$, so that:

$$\Gamma_L = e^{j\theta_\Gamma} = -1$$

5. $Z_L = R_L$

For this case, the load impedance is **purely real** (e.g. a **resistor**), meaning its reactive portion is zero:

$$X_L = 0$$

Thus, **both** the current through the load, and voltage across the load, are **non-zero**:

$$I_L = I(z = z_L) \neq 0 \qquad\qquad V_L = V(z = z_L) \neq 0$$

The resulting **load reflection coefficient** is:

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{R - Z_0}{R + Z_0}$$

Given that Z_0 is real (i.e., the line is **lossless**), we find that this load reflection coefficient must be a purely **real** value! In other words:

$$Re\{\Gamma_L\} = \frac{R - Z_0}{R + Z_0} \qquad\qquad \mathrm{Im}\{\Gamma_L\} = 0$$

The magnitude is thus:

$$|\Gamma_L| = \left|\frac{R - Z_0}{R + Z_0}\right|$$

whereas the phase θ_Γ can take on one of two values:

$$\theta_\Gamma = \begin{cases} 0 & if \quad \mathrm{Re}\{\Gamma_L\} > 0 \quad (\text{i.e., if } R_L > Z_0) \\ \\ \pi & if \quad \mathrm{Re}\{\Gamma_L\} < 0 \quad (\text{i.e., if } R_L < Z_0) \end{cases}$$

For this case, the impedance at the **end** of the line must be **real** ($Z(z = z_L) = R_L$). Thus, the current and the voltage at this point are precisely **in phase**.

However, even though the **load** impedance is real, the **line** impedance at all other points on the line is *generally* **complex**!

Moreover, the **general** current and voltage expressions, as well as reflection coefficient function, **cannot** be further simplified for the case where $Z_L = R_L$.

Q: *Why is that? When the load was purely **imaginary** (reactive), we where able to **simply** our general expressions, and likewise deduce all sorts of interesting results!*

A: True! And here's **why**. Remember, a **lossless** transmission line has series inductance and shunt capacitance **only**. In other words, a length of lossless transmission line is a **purely reactive** device (it absorbs **no** energy!).

* If we attach a **purely reactive** load at the end of the transmission line, we still have a **completely** reactive system (load and transmission line). Because this system has **no** resistive (i.e., real) component, the general expressions for line impedance, line voltage, etc. can be significantly **simplified**.

* However, if we attach a **purely real** load to our reactive transmission line, we now have a **complex** system, with **both** real and imaginary (i.e., resistive and reactive) components.

This **complex** case is exactly what our general expressions **already** describes—**no** further simplification is possible!

5. $Z_L = R_L + jX_L$

Now, let's look at the **general** case, where the **load** has both a **real** (resitive) and **imaginary** (reactive) component.

Q: *Haven't we **already** determined all the **general** expressions (e.g., $\Gamma_L, V(z), I(z), Z(z), \Gamma(z)$) for this general case? Is there **anything** else left to be determined?*

A: There is **one** last thing we need to discuss. It seems trivial, but its ramifications are **very** important!

For you see, the "general" case is **not**, in reality, quite so general. Although the reactive component of the load can be **either** positive or negative ($-\infty < X_L < \infty$), the resistive component of a passive load **must** be positive ($R_L > 0$)—there's **no** such thing as a (passive) **negative** resistor!

This leads to one **very** important and useful result. Consider the **load reflection coefficient**:

$$\begin{aligned}\Gamma_L &= \frac{Z_L - Z_0}{Z_L + Z_0} \\ &= \frac{(R_L + jX_L) - Z_0}{(R_L + jX_L) + Z_0} \\ &= \frac{(R_L - Z_0) + jX_L}{(R_L + Z_0) + jX_L}\end{aligned}$$

Now let's look at the **magnitude** of this value:

$$\begin{aligned}|\Gamma_L|^2 &= \left|\frac{(R_L - Z_0) + jX_L}{(R_L + Z_0) + jX_L}\right|^2 \\ &= \frac{(R_L - Z_0)^2 + X_L^2}{(R_L + Z_0)^2 + X_L^2} \\ &= \frac{(R_L^2 - 2R_L Z_0 + Z_0^2) + X_L^2}{(R_L^2 + 2R_L Z_0 + Z_0^2) + X_L^2} \\ &= \frac{(R_L^2 + Z_0^2 + X_L^2) - 2R_L Z_0}{(R_L^2 + Z_0^2 + X_L^2) + 2R_L Z_0}\end{aligned}$$

It is apparent that since both R_L and Z_0 are **positive**, the **numerator** of the above expression must be **less** than (or equal to) the **denominator** of the above expression.

→ In other words, the magnitude of the load reflection coefficient is always **less** than or equal to one!

$$\boxed{\left|\Gamma_L\right| \leq 1 \quad (\text{for } R_L \geq 0)}$$

Moreover, we find that this means the reflection coefficient **function** likewise always has a magnitude **less** than or equal to one, for **all** values of position *z*.

$$\left|\Gamma(z)\right| \leq 1 \quad (\text{for all } z)$$

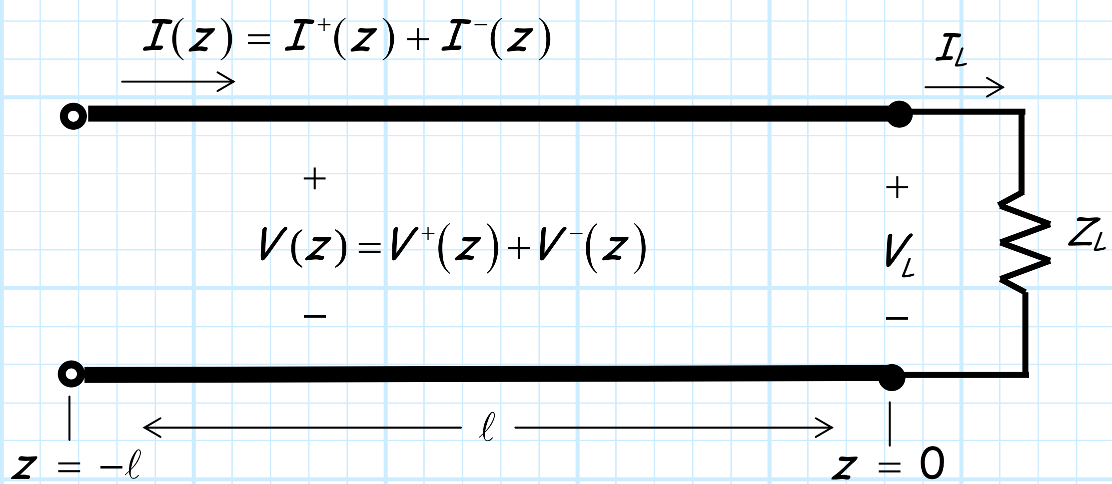
Which means, of course, that the **reflected** wave will always have a magnitude **less** than that of the **incident** wave magnitude:

$$\boxed{\left|V^-(z)\right| \leq \left|V^+(z)\right| \qquad (\text{for all } z)}$$

We will find out later that this result is consistent with **conservation of energy**—the reflected wave from a passive load **cannot** be larger than the wave incident on it.

Transmission Line Input Impedance

Consider a **lossless** line, length ℓ, terminated with a **load** Z_L.



Let's determine the **input impedance** of this line!

Q: *Just what do you mean by **input** impedance?*

A: The input impedance is simply the line impedance seen at the **beginning** ($z = -\ell$) of the transmission line, i.e.:

$$Z_{in} = Z(z = -\ell) = \frac{V(z = -\ell)}{I(z = -\ell)}$$

Note Z_{in} equal to **neither** the load impedance Z_L **nor** the characteristic impedance Z_0!

$$Z_{in} \neq Z_L \quad \text{and} \quad Z_{in} \neq Z_0$$

To determine exactly what Z_{in} is, we first must determine the voltage and current at the **beginning** of the transmission line ($z=-\ell$).

$$V(z=-\ell) = V_0^+\left[e^{+j\beta\ell} + \Gamma_0\, e^{-j\beta\ell}\right]$$

$$I(z=-\ell) = \frac{V_0^+}{Z_0}\left[e^{+j\beta\ell} - \Gamma_0\, e^{-j\beta\ell}\right]$$

Therefore:

$$Z_{in} = \frac{V(z=-\ell)}{I(z=-\ell)} = Z_0\left(\frac{e^{+j\beta\ell} + \Gamma_0\, e^{-j\beta\ell}}{e^{+j\beta\ell} - \Gamma_0\, e^{-j\beta\ell}}\right)$$

We can explicitly write Z_{in} in terms of load Z_L using the previously determined relationship:

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \Gamma_0$$

Combining these two expressions, we get:

$$\begin{aligned} Z_{in} &= Z_0\,\frac{(Z_L+Z_0)\,e^{+j\beta\ell} + (Z_L-Z_0)\,e^{-j\beta\ell}}{(Z_L+Z_0)\,e^{+j\beta\ell} - (Z_L-Z_0)\,e^{-j\beta\ell}} \\ &= Z_0\left(\frac{Z_L\left(e^{+j\beta\ell}+e^{-j\beta\ell}\right) + Z_0\left(e^{+j\beta\ell}-e^{-j\beta\ell}\right)}{Z_L\left(e^{+j\beta\ell}+e^{-j\beta\ell}\right) - Z_0\left(e^{+j\beta\ell}-e^{-j\beta\ell}\right)}\right) \end{aligned}$$

Now, recall **Euler's equations**:

$$e^{+j\beta\ell} = \cos\beta\ell + j\,\sin\beta\ell$$
$$e^{-j\beta\ell} = \cos\beta\ell - j\,\sin\beta\ell$$

Using Euler's relationships, we can likewise write the input impedance without the **complex** exponentials:

$$Z_{in} = Z_0 \left(\frac{Z_L \cos\beta\ell + j\, Z_0 \sin\beta\ell}{Z_0 \cos\beta\ell + j\, Z_L \sin\beta\ell} \right)$$

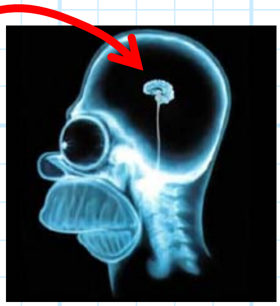
$$= Z_0 \left(\frac{Z_L + j\, Z_0 \tan\beta\ell}{Z_0 + j\, Z_L \tan\beta\ell} \right)$$

Note that depending on the values of β, Z_0 and ℓ, the input impedance can be **radically** different from the load impedance Z_L!

Special Cases

Now let's look at the Z_{in} for some important **load** impedances and **line lengths**.

→ You should commit these results to **memory**!



1. $\ell = \lambda/2$

If the length of the transmission line is exactly **one-half** wavelength ($\ell = \lambda/2$), we find that:

$$\beta\ell = \frac{2\pi}{\lambda}\frac{\lambda}{2} = \pi$$

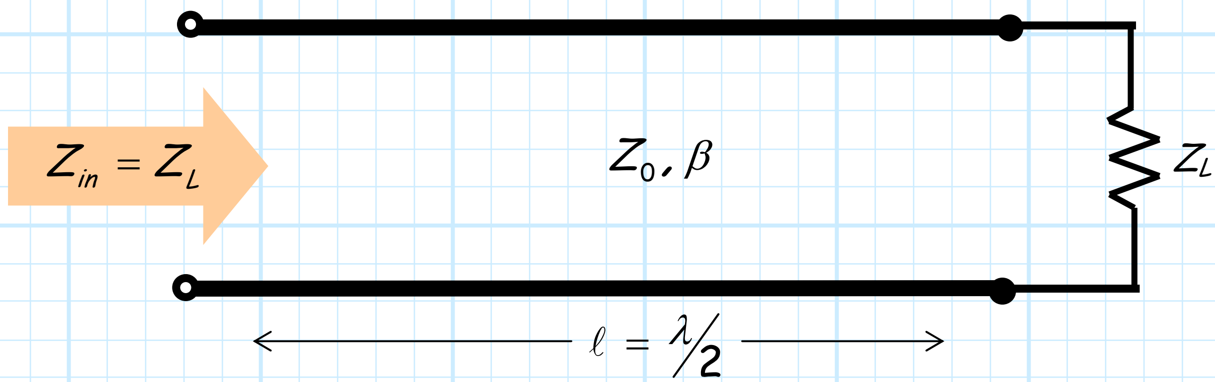
meaning that:

$$\cos\beta\ell = \cos\pi = -1 \qquad \text{and} \qquad \sin\beta\ell = \sin\pi = 0$$

and therefore:

$$
\begin{aligned}
Z_{in} &= Z_0 \left(\frac{Z_L \cos\beta\ell + j\, Z_0 \sin\beta\ell}{Z_0 \cos\beta\ell + j\, Z_L \sin\beta\ell} \right) \\
&= Z_0 \left(\frac{Z_L\,(-1) + j\, Z_L\,(0)}{Z_0\,(-1) + j\, Z_L\,(0)} \right) \\
&= Z_L
\end{aligned}
$$

In other words, if the transmission line is precisely **one-half wavelength** long, the **input** impedance is equal to the **load** impedance, **regardless** of Z_0 or β.



2. $\ell = \lambda/4$

If the length of the transmission line is exactly **one-quarter** wavelength ($\ell = \lambda/4$), we find that:

$$\beta\ell = \frac{2\pi}{\lambda}\frac{\lambda}{4} = \frac{\pi}{2}$$

meaning that:

$$\cos\beta\ell = \cos\ \pi/2 = 0 \qquad \text{and} \qquad \sin\beta\ell = \sin\ \pi/2 = 1$$

and therefore:

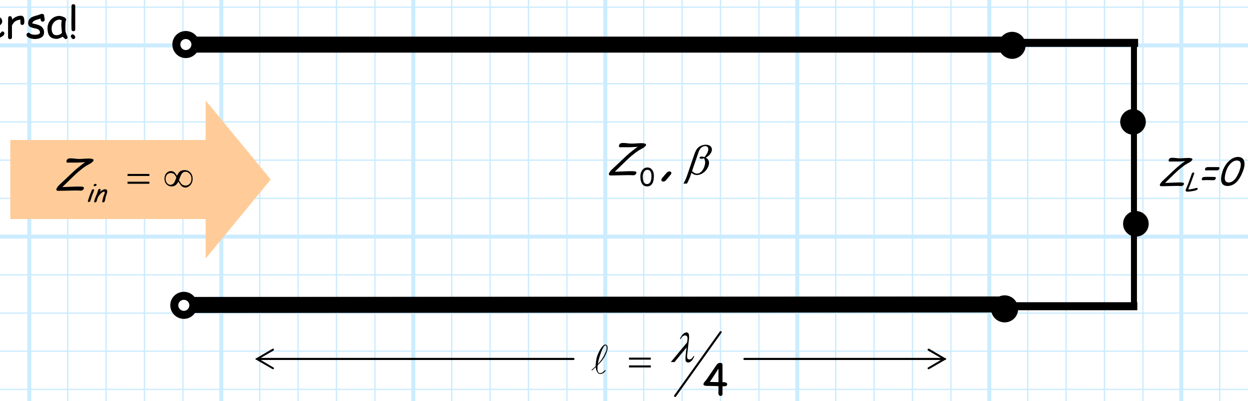
$$
\begin{aligned}
Z_{in} &= Z_0\left(\frac{Z_L \cos\beta\ell + j\,Z_0 \sin\beta\ell}{Z_0 \cos\beta\ell + j\,Z_L \sin\beta\ell}\right)\\
&= Z_0\left(\frac{Z_L\,(0) + j\,Z_0\,(1)}{Z_0\,(0) + j\,Z_L\,(1)}\right)\\
&= \frac{(Z_0)^2}{Z_L}
\end{aligned}
$$

In other words, if the transmission line is precisely **one-quarter wavelength** long, the **input** impedance is **inversely** proportional to the **load** impedance.

Think about what this means! Say the load impedance is a **short** circuit, such that $Z_L = 0$. The **input impedance** at beginning of the $\lambda/4$ transmission line is therefore:

$$Z_{in} = \frac{(Z_0)^2}{Z_L} = \frac{(Z_0)^2}{0} = \infty$$

$Z_{in} = \infty$! This is an **open** circuit! The quarter-wave transmission line **transforms** a short-circuit into an open-circuit—and vice versa!

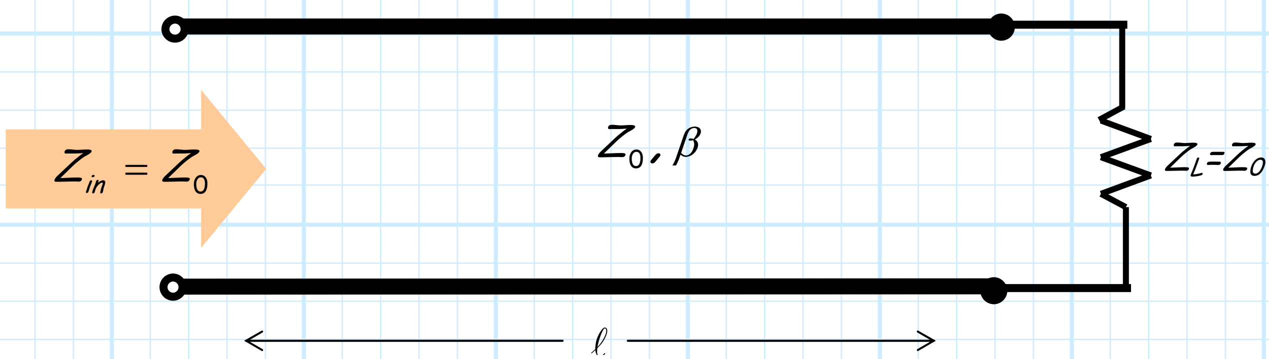


3. $Z_L = Z_0$

If the load is **numerically equal** to the characteristic impedance of the transmission line (a real value), we find that the input impedance becomes:

$$\begin{aligned} Z_{in} &= Z_0\left(\frac{Z_L \cos\beta\ell + j\,Z_0 \sin\beta\ell}{Z_0 \cos\beta\ell + j\,Z_L \sin\beta\ell}\right) \\ &= Z_0\left(\frac{Z_0 \cos\beta\ell + j\,Z_0 \sin\beta\ell}{Z_0 \cos\beta\ell + j\,Z_0 \sin\beta\ell}\right) \\ &= Z_0 \end{aligned}$$

In other words, if the **load** impedance is equal to the transmission line **characteristic** impedance, the **input** impedance will be likewise be equal to Z_0 regardless of the transmission line length ℓ.

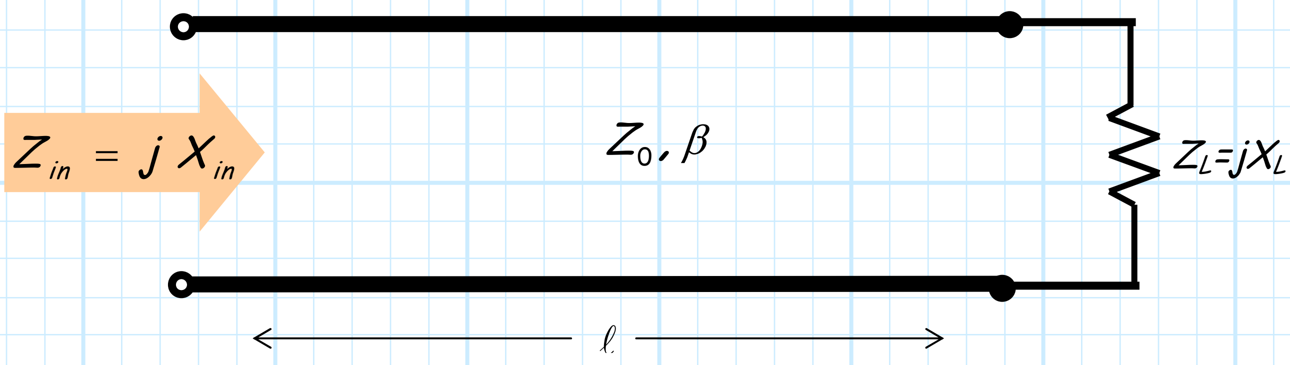


4. $Z_L = j\,X_L$

If the load is **purely reactive** (i.e., the resistive component is zero), the input impedance is:

$$
\begin{aligned}
Z_{in} &= Z_0 \left(\frac{Z_L \cos\beta\ell + j\, Z_0 \sin\beta\ell}{Z_0 \cos\beta\ell + j\, Z_L \sin\beta\ell} \right) \\
&= Z_0 \left(\frac{j\, X_L \cos\beta\ell + j\, Z_0 \sin\beta\ell}{Z_0 \cos\beta\ell + j^2\, X_L \sin\beta\ell} \right) \\
&= j\, Z_0 \left(\frac{X_L \cos\beta\ell + Z_0 \sin\beta\ell}{Z_0 \cos\beta\ell - X_L \sin\beta\ell} \right)
\end{aligned}
$$

In other words, if the load is purely reactive, then the input impedance will **likewise** be purely reactive, **regardless** of the line length ℓ.



Note that the **opposite** is **not** true: even if the load is **purely resistive** ($Z_L = R$), the input impedance will be **complex** (both resistive and reactive components).

Q: *Why is this?*

A:

5. $\ell \ll \lambda$

If the transmission line is **electrically small**—its length ℓ is small with respect to signal wavelength λ--we find that:

$$\beta\ell = \frac{2\pi}{\lambda}\ell = 2\pi\frac{\ell}{\lambda} \approx 0$$

and thus:

$$\cos\beta\ell = \cos 0 = 1 \qquad \text{and} \qquad \sin\beta\ell = \sin 0 = 0$$

so that the input impedance is:

$$\begin{aligned} Z_{in} &= Z_0\left(\frac{Z_L\cos\beta\ell + j\,Z_0\sin\beta\ell}{Z_0\cos\beta\ell + j\,Z_L\sin\beta\ell}\right) \\ &= Z_0\left(\frac{Z_L\,(1) + j\,Z_L\,(0)}{Z_0\,(1) + j\,Z_L\,(0)}\right) \\ &= Z_L \end{aligned}$$

In other words, if the transmission line length is much smaller than a wavelength, the **input** impedance Z_{in} will **always** be equal to the **load** impedance Z_L.

This is the assumption we used in all previous circuits courses (e.g., EECS 211, 212, 312, 412)! In those courses, we assumed that the signal frequency ω is relatively **low**, such that the signal wavelength λ is **very large** ($\lambda \gg \ell$).

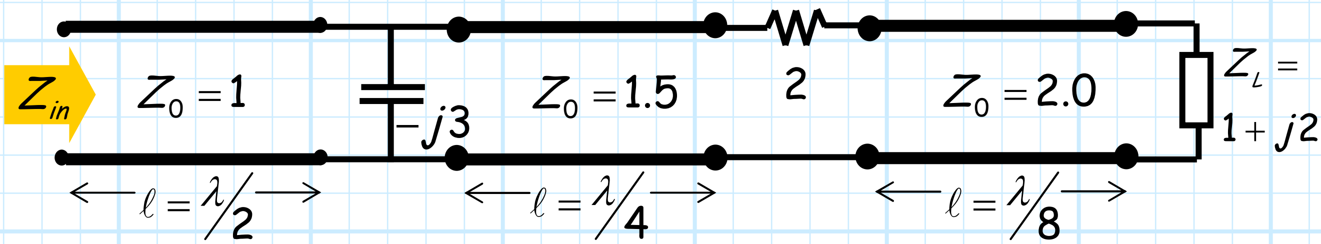
Note also for this case (the electrically short transmission line), the voltage and current at each end of the transmission line are approximately the **same**!

$$V(z=-\ell) \approx V(z=0) \quad \text{and} \quad I(z=-\ell) \approx I(z=0) \quad \text{if} \quad \ell \ll \lambda$$

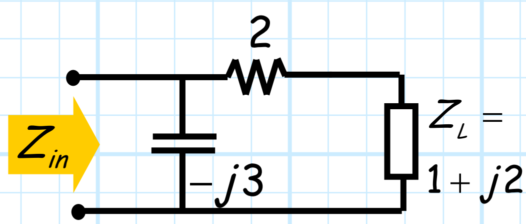
If $\ell \ll \lambda$, our "wire" behaves **exactly** as it did in EECS 211 !

Example: Input Impedance

Consider the following circuit:



If we **ignored** our new μ-wave knowledge, we might **erroneously** conclude that the input impedance of this circuit is:

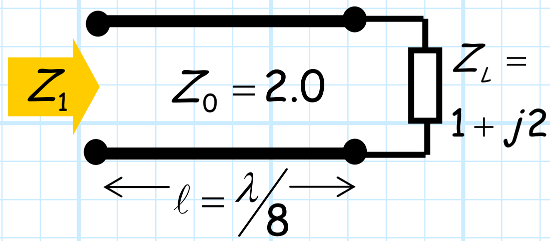


Therefore:

$$Z_{in} = \frac{-j3(2+1+j2)}{-j3+2+1+j2} = \frac{6-j9}{3-j} = 2.7 - j2.1$$

Of course, this is **not** the correct answer!

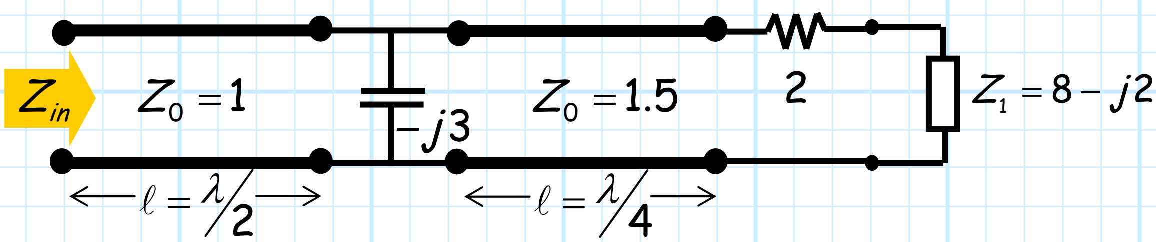
We must use our **transmission line theory** to determine an accurate value. Define Z_1 as the input impedance of the last section:



we find that Z_1 is :

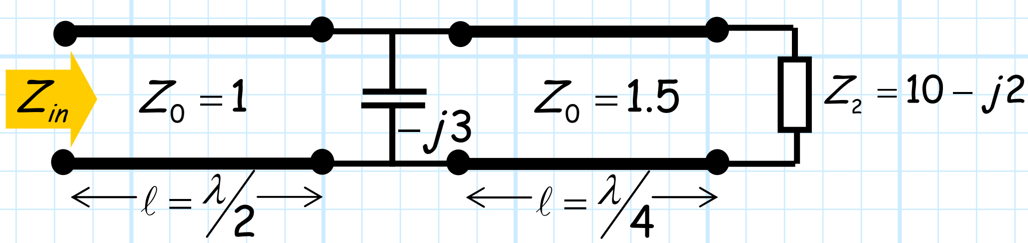
$$
\begin{aligned}
Z_1 &= Z_0\left(\frac{Z_L\cos\beta\ell + j\,Z_0\sin\beta\ell}{Z_0\cos\beta\ell + j\,Z_L\sin\beta\ell}\right)\\
&= 2\left(\frac{(1+j2)\cos\left(\pi/4\right) + j\,2\sin\left(\pi/4\right)}{2\cos\left(\pi/4\right) + j(1+j2)\;\sin\left(\pi/4\right)}\right)\\
&= 2\left(\frac{1+j4}{j}\right)\\
&= 8-j2
\end{aligned}
$$

Therefore, our circuit now becomes:

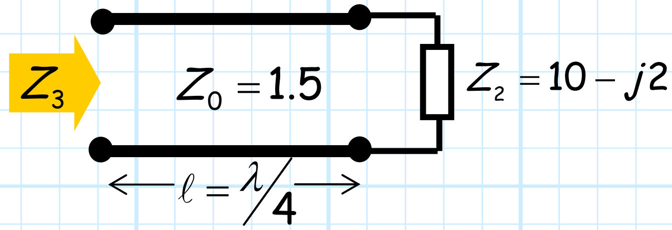


Note the resistor is in **series** with impedance Z_1. We can **combine** these two into one impedance defined as Z_2:

$$Z_2 = 2 + Z_1 = 2 + (8-j2) = 10-j2$$



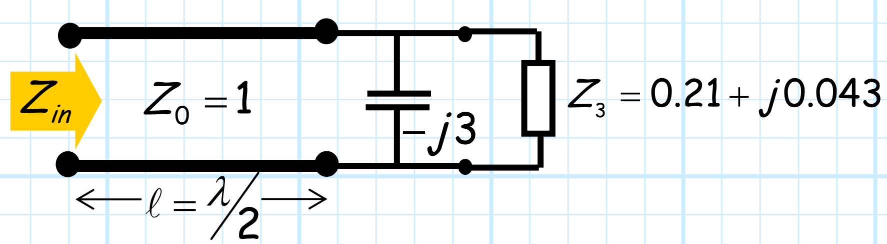
Now let's define the input impedance of the **middle** transmission line section as Z_3:



Note that this transmission line is a **quarter wavelength** ($\ell = \lambda/4$). This is one of the **special** cases we considered earlier! The input impedance Z_3 is:

$$\begin{aligned} Z_3 &= \frac{Z_0^{\,2}}{Z_L} \\ &= \frac{Z_0^{\,2}}{Z_2} \\ &= \frac{1.5^2}{10 - j2} \\ &= 0.21 + j0.043 \end{aligned}$$

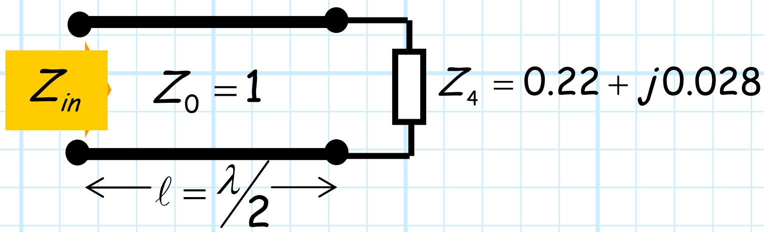
Thus, we can further **simplify** the original circuit as:



Now we find that the impedance Z_3 is **parallel** to the capacitor. We can **combine** the two impedances and define the result as impedance Z_4:

$$
\begin{aligned}
Z_4 &= -j3 \| (0.21 + j0.043) \\
&= \frac{-j3(0.21 + j0.043)}{-j3 + 0.21 + j0.043} \\
&= 0.22 + j0.028
\end{aligned}
$$

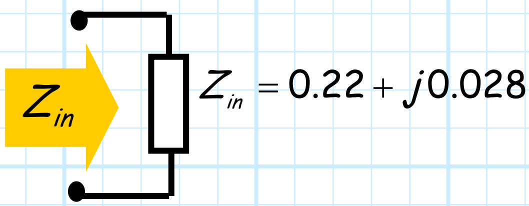
Now we are left with **this** equivalent circuit:



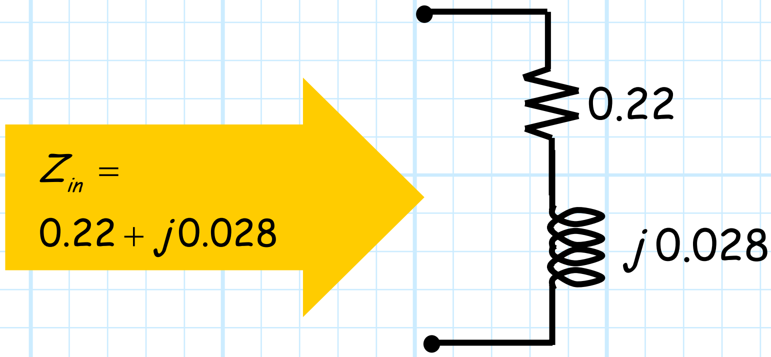
Note that the remaining transmission line section is a **half wavelength**! This is one of the **special** situations we discussed in a previous handout. Recall that the **input** impedance in this case is simply equal to the **load** impedance:

$$
Z_{in} = Z_L = Z_4 = 0.22 + j0.028
$$

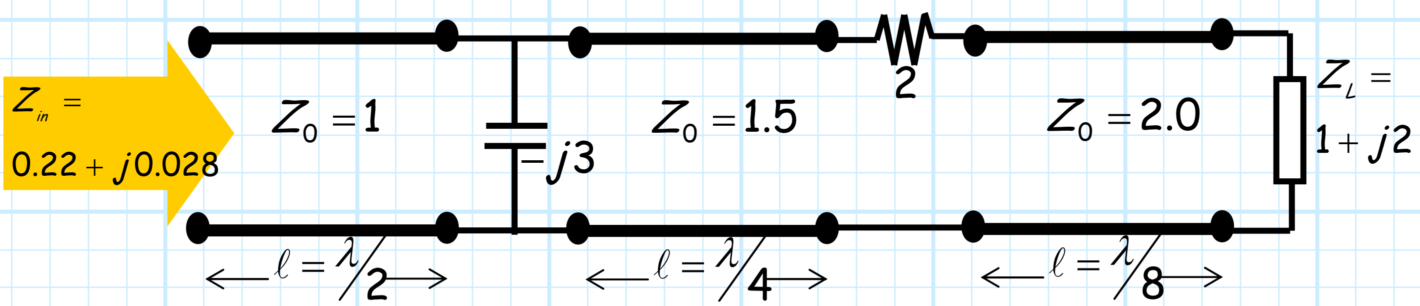
Whew! We are **finally** done. The **input impedance** of the original circuit is:



Note this means that **this** circuit:



and **this** circuit:



are precisely the **same**! They have **exactly** the same impedance, and thus they "behave" precisely the **same** way in any circuit (but **only** at frequency ω_0!).

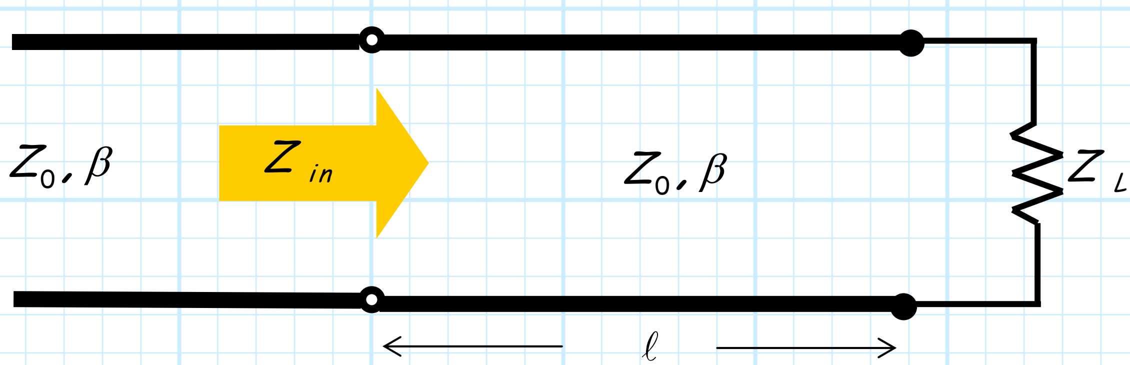
The Reflection Coefficient Transformation

The **load** at the end of some length of a transmission line (with characteristic impedance Z_0) can be specified in terms of its impedance Z_L **or** its reflection coefficient Γ_L.

Note **both** values are complex, and **either one** completely specifies the load—if you know **one**, you know the **other**!

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} \qquad \text{and} \qquad Z_L = Z_0\left(\frac{1+\Gamma_L}{1-\Gamma_L}\right)$$

Recall that we determined how a length of transmission line **transformed** the load **impedance** into an input **impedance** of a (generally) different value:

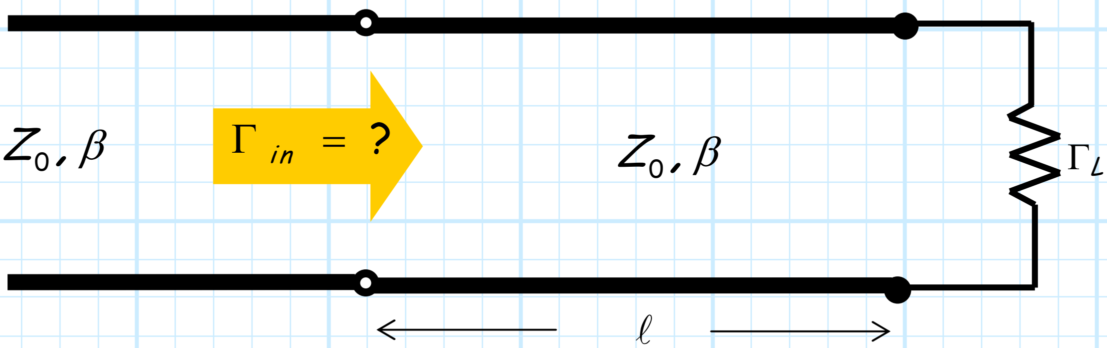


where:

$$Z_{in} = Z_0 \left(\frac{Z_L \cos \beta\ell + j\, Z_0 \sin \beta\ell}{Z_0 \cos \beta\ell + j\, Z_L \sin \beta\ell} \right)$$

$$= Z_0 \left(\frac{Z_L + j\, Z_0 \tan \beta\ell}{Z_0 + j\, Z_L \tan \beta\ell} \right)$$

Q: *Say we know the load in terms of its **reflection coefficient**. How can we express the **input** impedance in terms its **reflection coefficient** (call this Γ_{in})?*



A: Well, we **could** execute these **three** steps:

1. Convert Γ_L to Z_L:

$$Z_L = Z_0 \left(\frac{1+\Gamma_L}{1-\Gamma_L} \right)$$

2. Transform Z_L down the line to Z_{in}:

$$Z_{in} = Z_0 \left(\frac{Z_L \cos \beta\ell + j\, Z_0 \sin \beta\ell}{Z_0 \cos \beta\ell + j\, Z_L \sin \beta\ell} \right)$$

3. Convert Z_{in} to Γ_{in}:

$$\Gamma_{in} = \frac{Z_{in} - Z_0}{Z_{in} + Z_0}$$

Q: *Yikes! This is a **ton** of complex arithmetic—isn't there an **easier** way?*

A: Actually, there **is**!

Recall in an **earlier handout** that the input impedance of a transmission line length ℓ, terminated with a load Γ_L, is:

$$Z_{in} = \frac{V(z=-\ell)}{I(z=-\ell)} = Z_0 \left(\frac{e^{+j\beta\ell} + \Gamma_L e^{-j\beta\ell}}{e^{+j\beta\ell} - \Gamma_L e^{-j\beta\ell}} \right)$$

Note this **directly** relates Γ_L to Z_{in} (steps 1 and 2 combined!).

If we directly **insert** this equation into:

$$\Gamma_{in} = \frac{Z_{in} - Z_0}{Z_{in} + Z_0}$$

we get an equation **directly** relating Γ_L to Γ_{in} :

$$\begin{aligned}\Gamma_{in} &= \frac{Z_0\left(e^{+j\beta\ell}+\Gamma_L e^{-j\beta\ell}\right)-\left(e^{+j\beta\ell}-\Gamma_L e^{-j\beta\ell}\right)}{Z_0\left(e^{+j\beta\ell}+\Gamma_L e^{-j\beta\ell}\right)+\left(e^{+j\beta\ell}-\Gamma_L e^{-j\beta\ell}\right)}\\ &= \frac{2\Gamma_L e^{-j\beta\ell}}{2e^{+j\beta\ell}}\\ &= \Gamma_L e^{-j\beta\ell} e^{-j\beta\ell}\\ &= \Gamma_L e^{-j2\beta\ell}\end{aligned}$$

Q: *Hey! This result looks **familiar**. Haven't we seen something like this **before**?*

A: Absolutely! Recall that we found that the reflection coefficient **function** $\Gamma(z)$ can be expressed as:

$$\Gamma(z) = \Gamma_0 e^{j2\beta z}$$

Evaluating this function at the **beginning** of the line (i.e., at $z = z_L - \ell$):

$$\begin{aligned}\Gamma(z = z_L - \ell) &= \Gamma_0 e^{j2\beta(z_L-\ell)}\\ &= \Gamma_0 e^{j2\beta z_L} e^{-j2\beta\ell}\end{aligned}$$

But, we recognize that:

$$\Gamma_0 e^{j2\beta z_L} = \Gamma(z = z_L) = \Gamma_L$$

And so:

$$\begin{aligned}\Gamma(z = z_L - \ell) &= \Gamma_0 e^{j2\beta z_L} e^{-j2\beta\ell}\\ &= \Gamma_L e^{-j2\beta\ell}\end{aligned}$$

Thus, we find that Γ_{in} is simply the value of function $\Gamma(z)$ **evaluated** at the line **input** of $z = z_L - \ell$!

$$\boxed{\Gamma_{in} = \Gamma\left(z = z_L - \ell\right) = \Gamma_L \, e^{-j2\beta\ell}}$$

Makes sense! After all, the input impedance is **likewise** simply the line impedance evaluated at the line input of $z = z_L - \ell$:

$$Z_{in} = Z\left(z = z_L - \ell\right)$$

It is apparent that from the above expression that the reflection coefficient at the input is simply related to Γ_L by a **phase shift** of $2\beta\ell$.

In other words, the **magnitude** of Γ_{in} is the **same** as the magnitude of Γ_L!

$$\begin{aligned} \left|\Gamma_{in}\right| &= \left|\Gamma_L\right| \left|e^{j(\theta_\Gamma - 2\beta\ell)}\right| \\ &= \left|\Gamma_L\right| (1) \\ &= \left|\Gamma_L\right| \end{aligned}$$

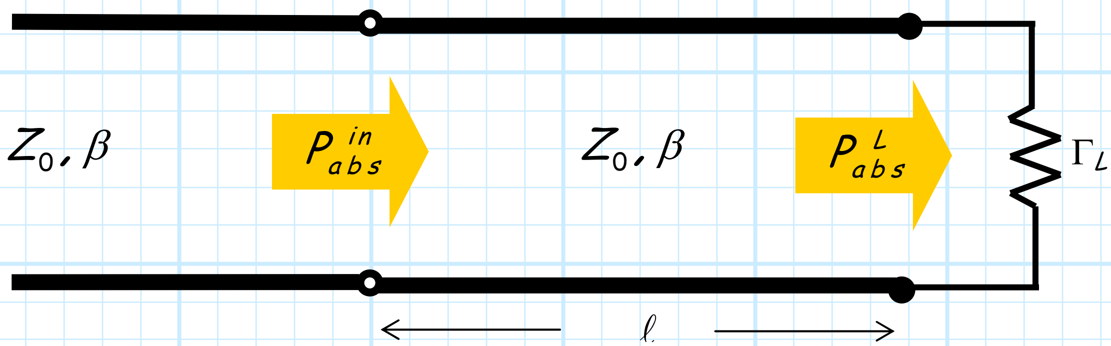
If we **think** about this, it makes **perfect sense**!

Recall that the power **absorbed** by the load Γ_{in} would be:

$$P_{abs}^{in} = \frac{\left|V_0^+\right|^2}{2\,Z_0}\left(1-\left|\Gamma_{in}\right|^2\right)$$

while that absorbed by the **load** Γ_L is:

$$P_{abs}^{L} = \frac{\left|V_0^+\right|^2}{2\,Z_0}\left(1-\left|\Gamma_{L}\right|^2\right)$$



Recall, however, that a lossless transmission line can absorb **no** power! By adding a length of transmission line to load Γ_L , we have added only **reactance**. Therefore, the power absorbed by load Γ_{in} is **equal** to the power absorbed by Γ_L:

$$P_{abs}^{in} = P_{abs}^{L}$$

$$\frac{\left|V_0^+\right|^2}{2\,Z_0}\left(1-\left|\Gamma_{in}\right|^2\right) = \frac{\left|V_0^+\right|^2}{2\,Z_0}\left(1-\left|\Gamma_{L}\right|^2\right)$$

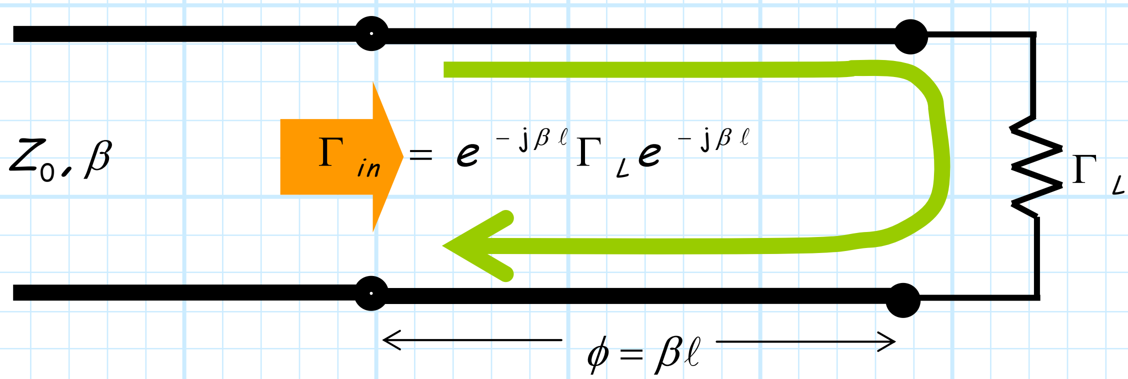
$$1-\left|\Gamma_{in}\right|^2 = 1-\left|\Gamma_{L}\right|^2$$

Thus, we can conclude from **conservation of energy** that:

$$|\Gamma_{in}| = |\Gamma_L|$$

Which of course is **exactly** the result we just found!

Finally, the **phase shift** associated with transforming the load Γ_L down a transmission line can be attributed to the phase shift associated with the wave propagating a length ℓ down the line, reflecting from load Γ_L, and then propagating a length ℓ back up the line:



To **emphasize** this wave interpretation, we recall that by definition, we can write Γ_{in} as:

$$\Gamma_{in} = \Gamma(z = z_L - \ell) = \frac{V^-(z = z_L - \ell)}{V^+(z = z_L - \ell)}$$

Therefore:

$$\begin{aligned} V^-(z = z_L - \ell) &= \Gamma_{in}\, V^+(z = z_L - \ell) \\ &= e^{-j\beta\ell}\,\Gamma_L\, e^{-j\beta\ell}\, V^+(z = z_L - \ell) \end{aligned}$$

Return Loss and VSWR

The **ratio** of the reflected power from a load, to the incident power on that load, is known as **return loss**. Typically, return loss is expressed in **dB**:

$$R.L. = -10\ log_{10}\left[\frac{P_{ref}}{P_{inc}}\right] = -10\ log_{10}\left|\Gamma_L\right|^2$$

The return loss thus tells us the percentage of the incident power reflected by load (expressed in decibels!).

For **example**, if the return loss is **10dB**, then **10%** of the incident power is **reflected** at the load, with the remaining **90%** being **absorbed** by the load—we "lose" 10% of the incident power

Likewise, if the return loss is **30dB**, then **0.1 %** of the incident power is **reflected** at the load, with the remaining **99.9%** being **absorbed** by the load—we "lose" 0.1% of the incident power.

Thus, a **larger** numeric value for return loss **actually** indicates **less** lost power! An **ideal** return loss would be ∞ dB, whereas a return loss of 0 dB indicates that $\left|\Gamma_L\right| = 1$--the load is **reactive**!

Return loss is helpful, as it provides a **real-valued** measure of load match (as opposed to the complex values Z_L and Γ_L).

Another traditional real-valued measure of load match is **Voltage Standing Wave Ratio** (VSWR). Consider again the **voltage** along a terminated transmission line, as a function of **position** z:

$$V(z) = V_0^+ \left[e^{-j\beta z} + \Gamma_L e^{+j\beta z} \right]$$

Recall this is a **complex** function, the magnitude of which expresses the **magnitude** of the **sinusoidal signal** at position z, while the phase of the complex value represents the relative **phase** of the sinusoidal signal.

Let's look at the **magnitude** only:

$$\begin{aligned} |V(z)| &= |V_0^+| \; |e^{-j\beta z} + \Gamma_L e^{+j\beta z}| \\ &= |V_0^+| |e^{-j\beta z}| |1 + \Gamma_L e^{+j2\beta z}| \\ &= |V_0^+| |1 + \Gamma_L e^{+j2\beta z}| \end{aligned}$$

It can be shown that the **largest** value of $|V(z)|$ occurs at the location z where:

$$\Gamma_L e^{+j2\beta z} = |\Gamma_L| + j0$$

while the **smallest** value of $|V(z)|$ occurs at the location z where:

$$\Gamma_L e^{+j2\beta z} = -|\Gamma_L| + j0$$

As a result we can conclude that:

$$\left|V(z)\right|_{max} = \left|V_0^+\right|\left(1+\left|\Gamma_L\right|\right)$$

$$\left|V(z)\right|_{min} = \left|V_0^+\right|\left(1-\left|\Gamma_L\right|\right)$$

The ratio of $\left|V(z)\right|_{max}$ to $\left|V(z)\right|_{min}$ is known as the **Voltage Standing Wave Ratio** (VSWR):

$$\boxed{\text{VSWR} \doteq \frac{\left|V(z)\right|_{max}}{\left|V(z)\right|_{min}} = \frac{1+\left|\Gamma_L\right|}{1-\left|\Gamma_L\right|} \qquad \therefore \qquad 1 \le VSWR \le \infty}$$

Note if $\left|\Gamma_L\right| = 0$ (i.e., $Z_L = Z_0$), then VSWR = 1. We find for this case:

$$\left|V(z)\right|_{\max} = \left|V(z)\right|_{\min} = \left|V_0^+\right|$$

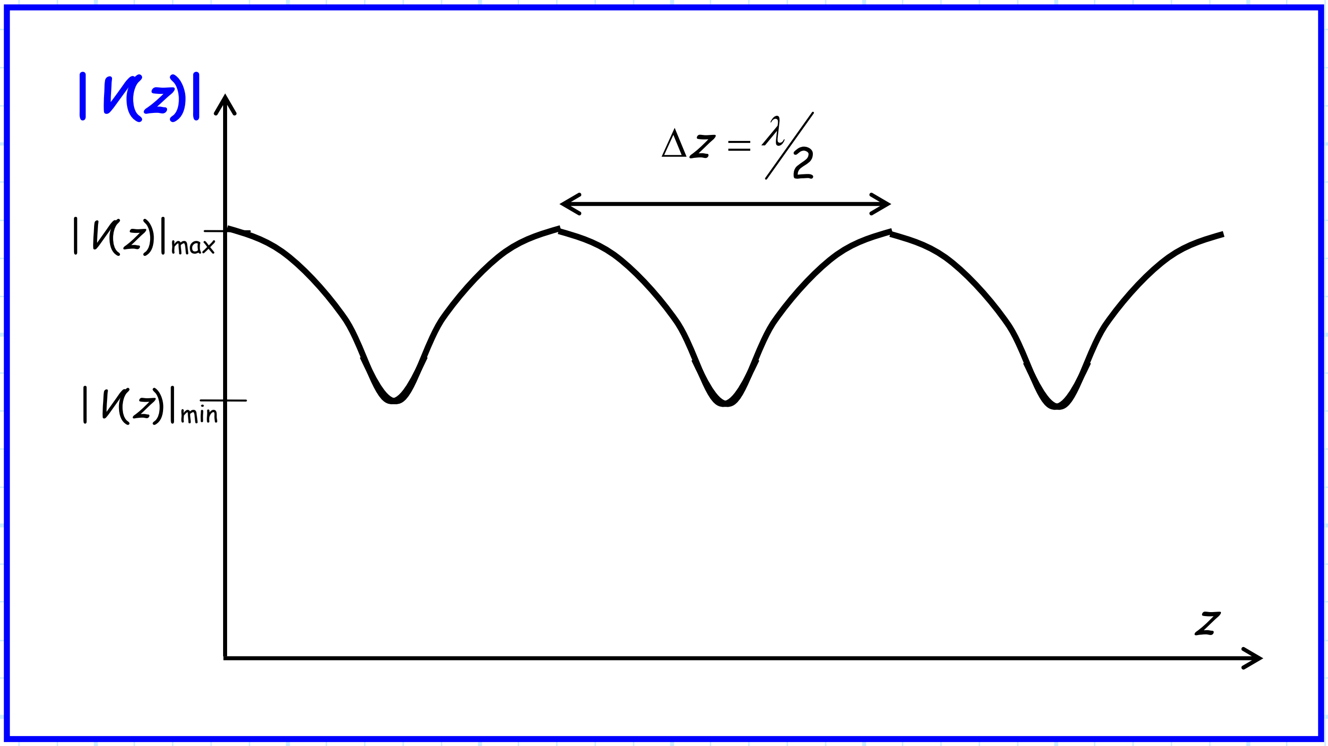
In other words, the voltage magnitude is a **constant** with respect to position *z*.

Conversely, if $\left|\Gamma_L\right| = 1$ (i.e., $Z_L = jX$), then VSWR = ∞. We find for **this** case:

$$\left|V(z)\right|_{\min} = 0 \qquad \text{and} \qquad \left|V(z)\right|_{\max} = 2\left|V_0^+\right|$$

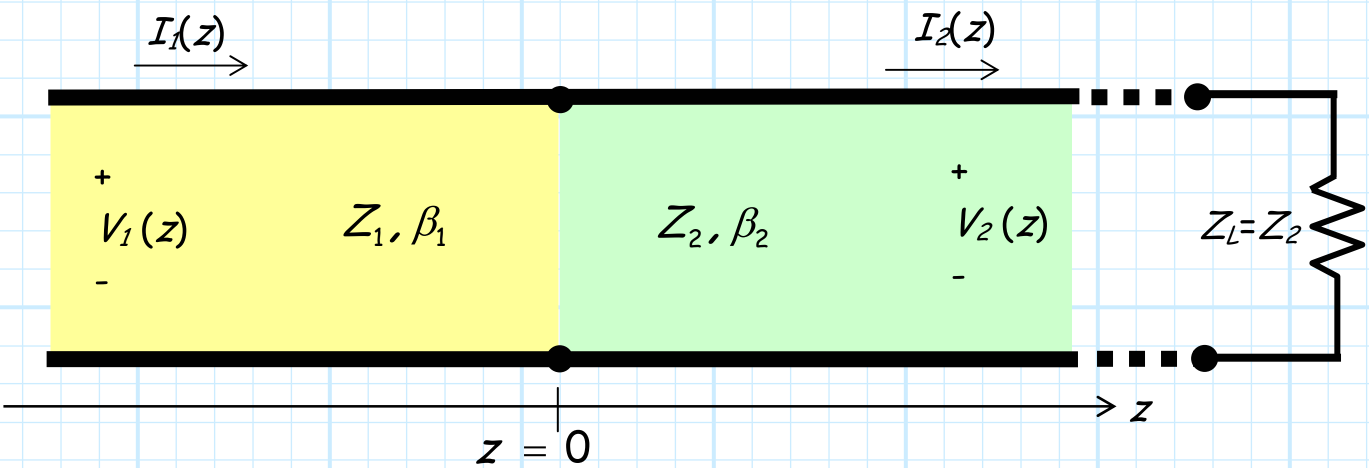
In other words, the voltage magnitude varies **greatly** with respect to position *z*.

As with **return loss**, VSWR is dependent on the **magnitude** of Γ_L (i.e, $|\Gamma_L|$) **only** !



Example: The Transmission Coefficient T

Consider this circuit:



I.E., a transmission line with characteristic impedance Z_1 **transitions** to a **different** transmission line at location z=0. This second transmission line has different **characteristic impedance** Z_2 $(Z_1 \neq Z_2)$. This second line is **terminated** with a load Z_L = Z_2 (i.e., the second line is **matched**).

Q: *What is the voltage and current along each of these two transmission lines? More specifically, what are* V_{01}^+, V_{01}^-, V_{02}^+ and V_{02}^- *??*

A: Since a source has not been specified, we can only determine V_{01}^-, V_{02}^+ and V_{02}^- in terms of complex constant V_{01}^+. To accomplish this, we must apply a **boundary condition** at z=0!

z < 0

We know that the voltage along the **first** transmission line is:

$$V_1(z) = V_{01}^{+}\, e^{-j\beta_1 z} + V_{01}^{-}\, e^{+j\beta_1 z} \qquad \left[\text{for } z<0\right]$$

while the **current** along that same line is described as:

$$I_1(z) = \frac{V_{01}^{+}}{Z_1} e^{-j\beta_1 z} - \frac{V_{01}^{-}}{Z_1} e^{+j\beta_1 z} \qquad \left[\text{for } z<0\right]$$

z > 0

We likewise know that the voltage along the **second** transmission line is:

$$V_2(z) = V_{02}^{+}\, e^{-j\beta_2 z} + V_{02}^{-}\, e^{+j\beta_2 z} \qquad \left[\text{for } z>0\right]$$

while the **current** along that same line is described as:

$$I_2(z) = \frac{V_{02}^{+}}{Z_2} e^{-j\beta_2 z} - \frac{V_{02}^{-}}{Z_2} e^{+j\beta_2 z} \qquad \left[\text{for } z>0\right]$$

Moreover, since the second line is terminated in a **matched load**, we know that the **reflected** wave from this load must be zero:

$$V_2^{-}(z) = V_{02}^{-}\, e^{-j\beta_2 z} = 0$$

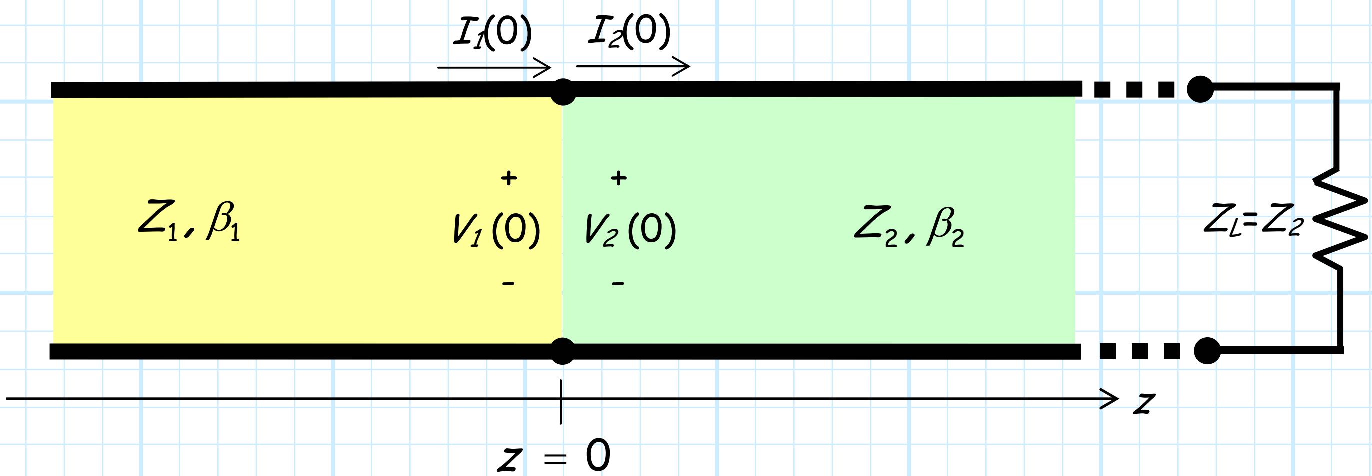
The voltage and current along the **second** transmission line is thus simply:

$$V_2(z) = V_2^+(z) = V_{02}^+ \, e^{-j\beta_2 z} \qquad [\text{for } z > 0]$$

$$I_2(z) = I_2^+(z) = \frac{V_{02}^+}{Z_2} e^{-j\beta_2 z} \qquad [\text{for } z > 0]$$

z=0

At the location where these two transmission lines meet, the current and voltage expressions each **must** satisfy some specific **boundary conditions**:



The **first** boundary condition comes from **KVL**, and states that:

$$V_1(z=0) = V_2(z=0)$$

$$V_{01}^+ \, e^{-j\beta_1(0)} + V_{01}^- \, e^{+j\beta_1(0)} = V_{02}^+ \, e^{-j\beta_2(0)}$$

$$V_{01}^+ + V_{01}^- = V_{02}^+$$

while the **second** boundary condition comes from **KCL**, and states that:

$$I_1(z=0) = I_2(z=0)$$

$$\frac{V_{01}^+}{Z_1}e^{-j\beta_1(0)} - \frac{V_{01}^-}{Z_1}e^{+j\beta_1(0)} = \frac{V_{02}^+}{Z_2}e^{-j\beta_2(0)}$$

$$\frac{V_{01}^+}{Z_1} - \frac{V_{01}^-}{Z_1} = \frac{V_{02}^+}{Z_2}$$

We now have **two** equations and **two** unknowns (V_{01}^- and V_{02}^+)! We can **solve** for each in terms of V_{01}^+ (i.e., the **incident** wave).

From the **first** boundary condition we can state:

$$V_{01}^- = V_{02}^+ - V_{01}^+$$

Inserting this into the **second** boundary condition, we find an expression involving **only** V_{02}^+ and V_{01}^+:

$$\frac{V_{01}^+}{Z_1} - \frac{V_{01}^-}{Z_1} = \frac{V_{02}^+}{Z_2}$$

$$\frac{V_{01}^+}{Z_1} - \frac{V_{02}^+ - V_{01}^+}{Z_1} = \frac{V_{02}^+}{Z_2}$$

$$\frac{2V_{01}^+}{Z_1} = \frac{V_{02}^+}{Z_2} + \frac{V_{02}^+}{Z_1}$$

Solving this expression, we find:

$$V_{02}^+ = \left(\frac{2Z_2}{Z_1 + Z_2}\right)V_{01}^+$$

We can therefore define a **transmission coefficient**, which relates V_{02}^{+} to V_{01}^{+}:

$$\boxed{T_0 \doteq \frac{V_{02}^{+}}{V_{01}^{+}} = \frac{2Z_2}{Z_1 + Z_2}}$$

Meaning that $V_{02}^{+} = T\,V_{01}^{+}$, and thus:

$$V_2(z) = V_2^{+}(z) = T\,V_{01}^{+}\,e^{-j\beta_2 z} \qquad [\text{for } z > 0]$$

We can **likewise** determine the constant V_{01}^{-} in terms of V_{01}^{+}. We again start with the **first** boundary condition, from which we concluded:

$$V_{02}^{+} = V_{01}^{+} + V_{01}^{-}$$

We can insert this into the **second** boundary condition, and determine an expression involving V_{01}^{-} and V_{01}^{+} **only**:

$$\frac{V_{01}^{+}}{Z_1} - \frac{V_{01}^{-}}{Z_1} = \frac{V_{02}^{+}}{Z_2}$$

$$\frac{V_{01}^{+}}{Z_1} - \frac{V_{01}^{-}}{Z_1} = \frac{V_{01}^{+} + V_{01}^{-}}{Z_2}$$

$$\left(\frac{1}{Z_1} - \frac{1}{Z_2}\right)V_{01}^{+} = \left(\frac{1}{Z_1} + \frac{1}{Z_2}\right)V_{01}^{-}$$

Solving this expression, we find:

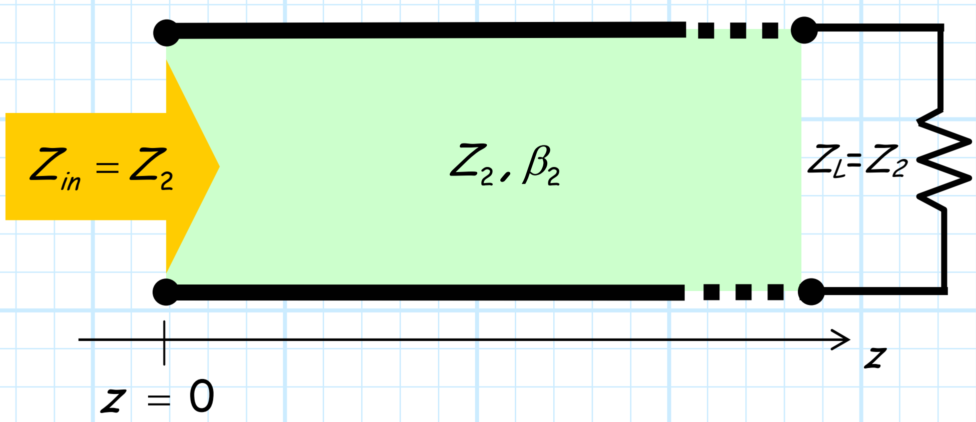
$$V_{01}^{-} = \left(\frac{Z_2 - Z_1}{Z_2 + Z_1}\right) V_{01}^{+}$$

We can therefore define a **reflection coefficient**, which relates V_{01}^{-} to V_{01}^{+}:

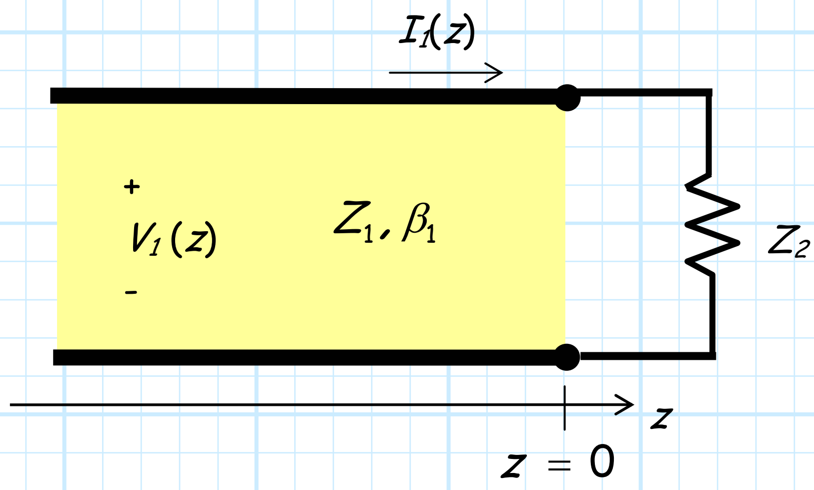
$$\boxed{\Gamma_0 \doteq \frac{V_{01}^{-}}{V_{01}^{+}} = \frac{Z_2 - Z_1}{Z_2 + Z_1}}$$

This result should **not** surprise us!

Note that because the **second** transmission line is **matched**, its **input** impedance is equal to Z_1:



and thus we can **equivalently** write the entire circuit as:



We have already analyzed **this** circuit! We know that:

$$\begin{aligned} V_{01}^- &= \Gamma_L V_{01}^+ \\ &= \left(\frac{Z_2 - Z_1}{Z_2 + Z_1}\right) V_{01}^+ \end{aligned}$$

Which is **exactly** the same result as we determined earlier!

The values of the reflection coefficient Γ_0 and the transmission coefficient T_0 are **not** independent, but in fact are directly **related**. Recall the **first** boundary expressed was:

$$V_{01}^+ + V_{01}^- = V_{02}^+$$

Dividing this by V_{01}^+:

$$1 + \frac{V_{01}^-}{V_{01}^+} = \frac{V_{02}^+}{V_{01}^+}$$

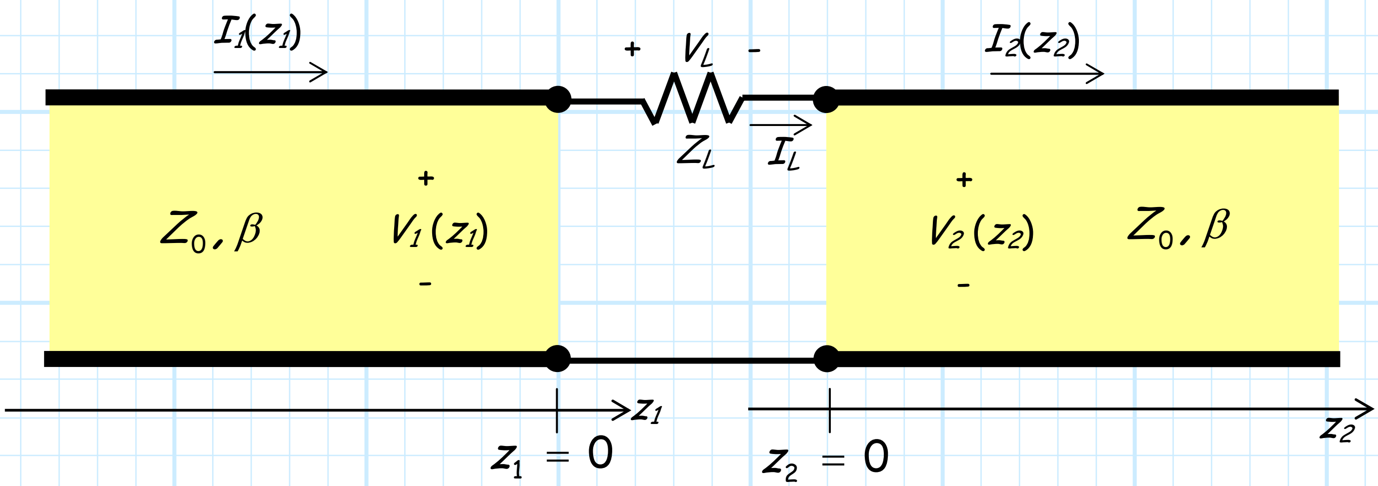
Since $\Gamma_0 = V_{01}^{-}/V_{01}^{+}$ and $T_0 = V_{02}^{+}/V_{01}^{+}$:

$$\boxed{1 + \Gamma_0 = T_0}$$

Note the result $T_0 = 1 + \Gamma_0$ is true for **this** particular circuit, and therefore is **not** a universally valid expression for two-port networks!

Example: Applying Boundary Conditions

Consider **this** circuit:



I.E., Two transmissions of **identical** characteristic impedance are connect by a **series** impedance Z_L. This second line is eventually **terminated** with a load $Z_L = Z_0$ (i.e., the second line is **matched**).

Q: *What is the voltage and current along each of these two transmission lines? More specifically, what are* V_{01}^+, V_{01}^-, V_{02}^+ and V_{02}^- *??*

A: Since a source has not been specified, we can only determine V_{01}^-, V_{02}^+ and V_{02}^- in terms of complex constant V_{01}^+. To accomplish this, we must apply a **boundary conditions** at the end of each line!

$z_1 < 0$

We know that the voltage along the **first** transmission line is:

$$V_1(z_1) = V_{01}^{+}\, e^{-j\beta z_1} + V_{01}^{-}\, e^{+j\beta z_1} \qquad [\text{for } z_1 < 0]$$

while the **current** along that same line is described as:

$$I_1(z_1) = \frac{V_{01}^{+}}{Z_0}\, e^{-j\beta z_1} - \frac{V_{01}^{-}}{Z_0}\, e^{+j\beta z_1} \qquad [\text{for } z_1 < 0]$$

$z_2 > 0$

We likewise know that the voltage along the **second** transmission line is:

$$V_2(z_2) = V_{02}^{+}\, e^{-j\beta z_2} + V_{02}^{-}\, e^{+j\beta z_2} \qquad [\text{for } z_2 > 0]$$

while the **current** along that same line is described as:

$$I_2(z_2) = \frac{V_{02}^{+}}{Z_0}\, e^{-j\beta z_2} - \frac{V_{02}^{-}}{Z_0}\, e^{+j\beta z_2} \qquad [\text{for } z_2 > 0]$$

Moreover, since the second line is terminated in a **matched load**, we know that the **reflected** wave from this load must be zero:

$$V_2^{-}(z_2) = V_{02}^{-}\, e^{-j\beta z_2} = 0$$

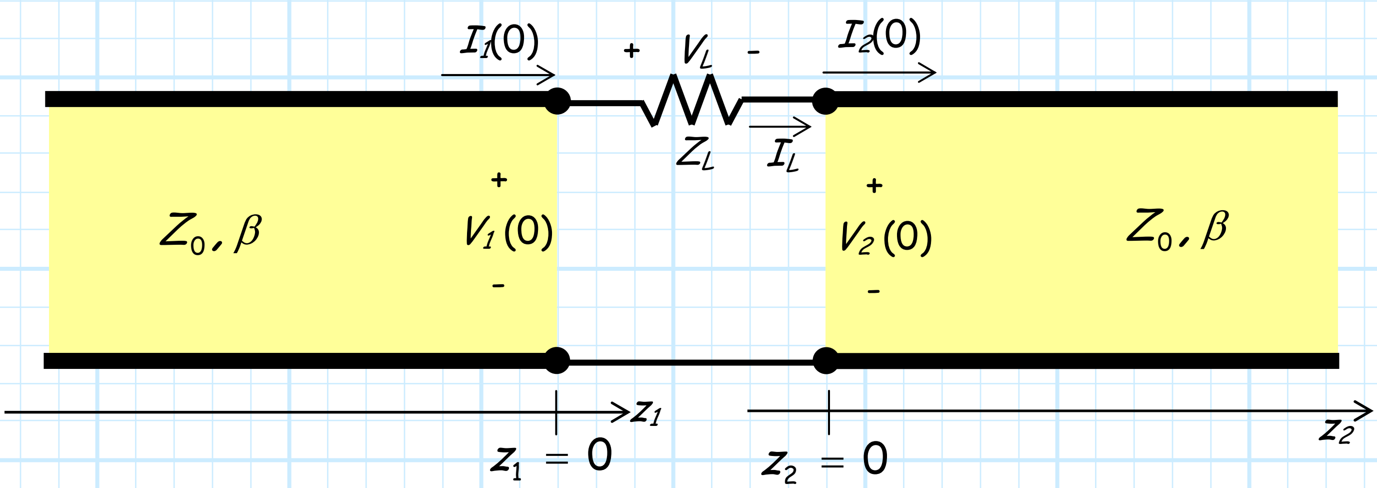
The voltage and current along the **second** transmission line is thus simply:

$$V_2(z_2) = V_2^+(z_2) = V_{02}^+ \, e^{-j\beta z_2} \qquad \left[\text{for } z_2 > 0\right]$$

$$I_2(z_2) = I_2^+(z_2) = \frac{V_{02}^+}{Z_2} e^{-j\beta z_2} \qquad \left[\text{for } z_2 > 0\right]$$

$z=0$

At the location where these two transmission lines meet, the current and voltage expressions each **must** satisfy some specific **boundary conditions**:



The **first** boundary condition comes from **KVL**, and states that:

$$V_1(z=0) - I_L Z_L = V_2(z=0)$$

$$V_{01}^+ \, e^{-j\beta(0)} + V_{01}^- \, e^{+j\beta(0)} - I_L Z_L = V_{02}^+ \, e^{-j\beta(0)}$$

$$V_{01}^+ + V_{01}^- - I_L Z_L = V_{02}^+$$

the **second** boundary condition comes from **KCL**, and states that:

$$I_1(z=0) = I_L$$

$$\frac{V_{01}^{+}}{Z_0} e^{-j\beta(0)} - \frac{V_{01}^{-}}{Z_0} e^{+j\beta(0)} = I_L$$

$$V_{01}^{+} - V_{01}^{-} = Z_0 I_L$$

while the **third** boundary condition likewise comes from **KCL**, and states that:

$$I_L = I_2(z=0)$$

$$I_L = \frac{V_{02}^{+}}{Z_0} e^{-j\beta(0)}$$

$$Z_0 I_L = V_{02}^{+}$$

Finally, we have Ohm's Law:

$$V_L = Z_L I_L$$

Note that we now have **four** equations and **four** unknowns ($V_{01}^{-}, V_{02}^{+}, V_L, I_L$)! We can **solve** for each in terms of V_{01}^{+} (i.e., the **incident** wave).

For **example**, let's determine V_{02}^{+} (in terms of V_{01}^{+}). We combine the **first** and **second** boundary conditions to determine:

$$V_{01}^{+} + V_{01}^{-} - I_L Z_L = V_{02}^{+}$$

$$V_{01}^{+} + \left(V_{01}^{+} - Z_0 I_L\right) - I_L Z_L = V_{02}^{+}$$

$$2V_{01}^{+} - I_L\left(Z_0 + Z_L\right) = V_{02}^{+}$$

And then adding in the **third** boundary condition:

$$2V_{01}^{+} - I_L\left(Z_0 + Z_L\right) = V_{02}^{+}$$

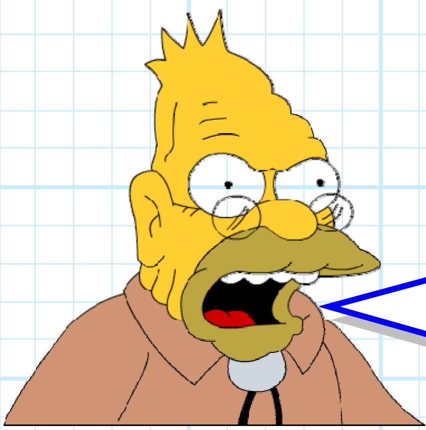
$$2V_{01}^{+} - \frac{V_{02}^{+}}{Z_0}\left(Z_0 + Z_L\right) = V_{02}^{+}$$

$$2V_{01}^{+} = V_{02}^{+}\left(\frac{2Z_0 + Z_L}{Z_0}\right)$$

Thus, we find that $V_{02}^{+} = T_0\, V_{01}^{+}$:

$$T_0 \doteq \frac{V_{02}^{+}}{V_{01}^{+}} = \frac{2Z_0}{2Z_0 + Z_L}$$

Now let's determine V_{01}^{-} (in terms of V_{01}^{+}).



Q: *Why are you wasting our time? Don't we **already** know that* $V_{01}^{-} = \Gamma_0\, V_{01}^{+}$*, where:*

$$\Gamma_0 = \frac{Z_L - Z_0}{Z_L + Z_0}$$

A: Perhaps. Humor me while I **continue** with our **boundary condition** analysis.

We combine the **first** and **third** boundary conditions to determine:

$$V_{01}^{+} + V_{01}^{-} - I_L Z_L = V_{02}^{+}$$
$$V_{01}^{+} + V_{01}^{-} - I_L Z_L = Z_0 I_L$$
$$V_{01}^{+} + V_{01}^{-} = I_L \left(Z_0 + Z_L \right)$$

And then adding the **second** boundary condition:

$$V_{01}^{+} + V_{01}^{-} = I_L \left(Z_0 + Z_L \right)$$
$$V_{01}^{+} + V_{01}^{-} = \frac{\left(V_{01}^{+} - V_{01}^{-}\right)}{Z_0} \left(Z_0 + Z_L \right)$$
$$V_{01}^{+} \left(\frac{Z_L}{Z_0} \right) = V_{01}^{-} \left(\frac{2Z_0 + Z_L}{Z_0} \right)$$

Thus, we find that $V_{01}^{-} = \Gamma_0 \, V_{01}^{+}$, where:

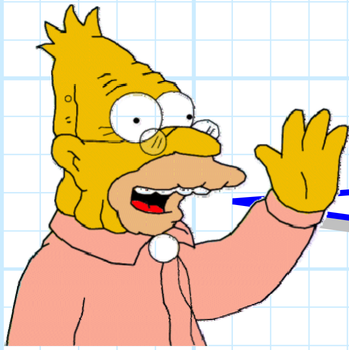
$$\Gamma_0 \doteq \frac{V_{01}^{-}}{V_{01}^{+}} = \frac{Z_L}{Z_L + 2Z_0}$$

Note this is **not** the expression:

$$\Gamma_0 \neq \frac{Z_L - Z_0}{Z_L + Z_0}$$

This is a completely **different** problem than the transmission line simply terminated by load Z_L. Thus, the **results** are likewise different. This shows that you must always **carefully** consider the problem you are attempting to solve, and guard against using "shortcuts" with previously derived expressions that may be **inapplicable**.

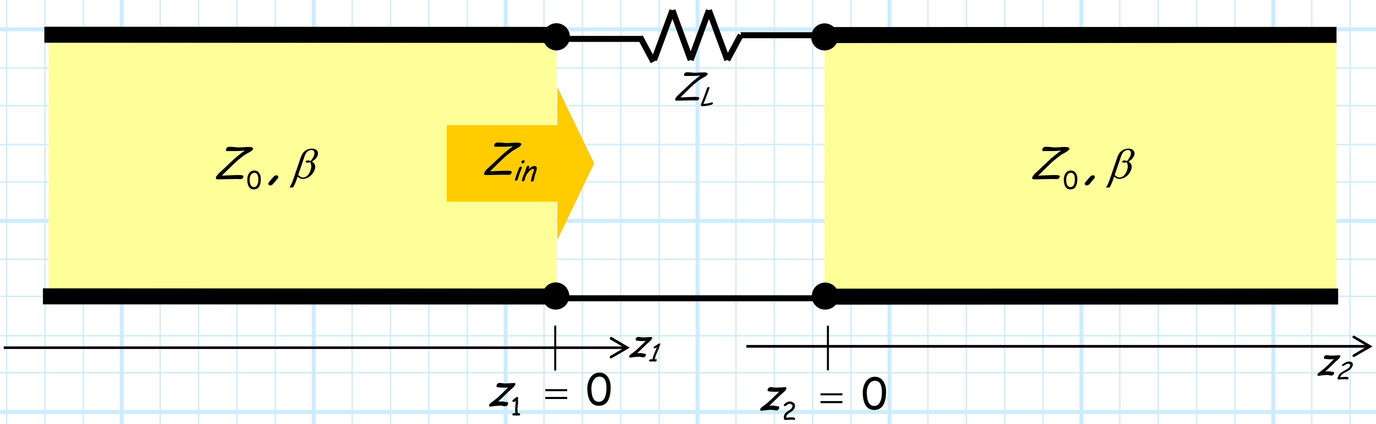
→ This is why you must know **why** a correct answer is correct!



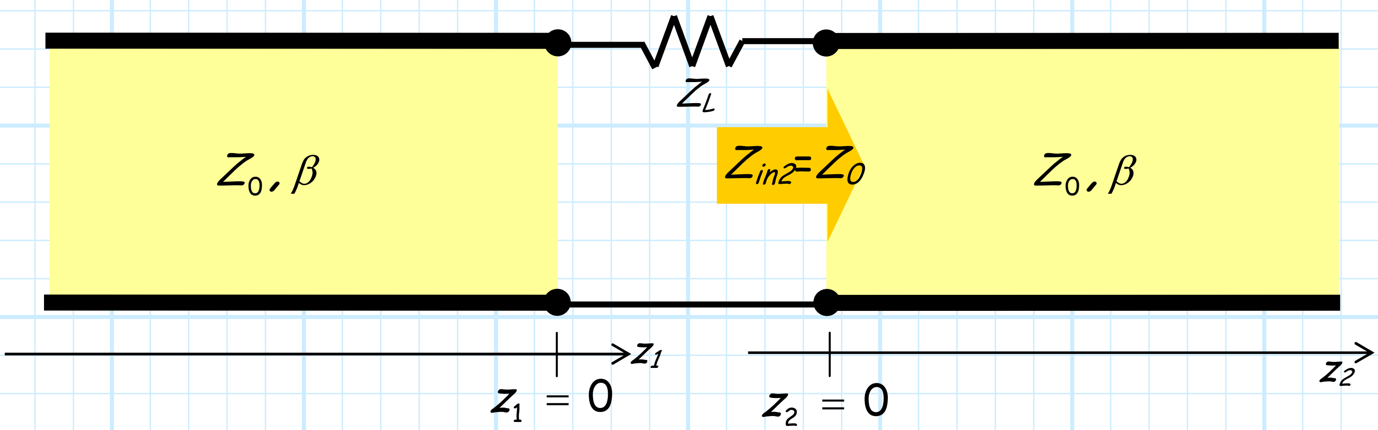
Q: *But, isn't there **some** way to solve this using our previous work?*

A: Actually, there is!

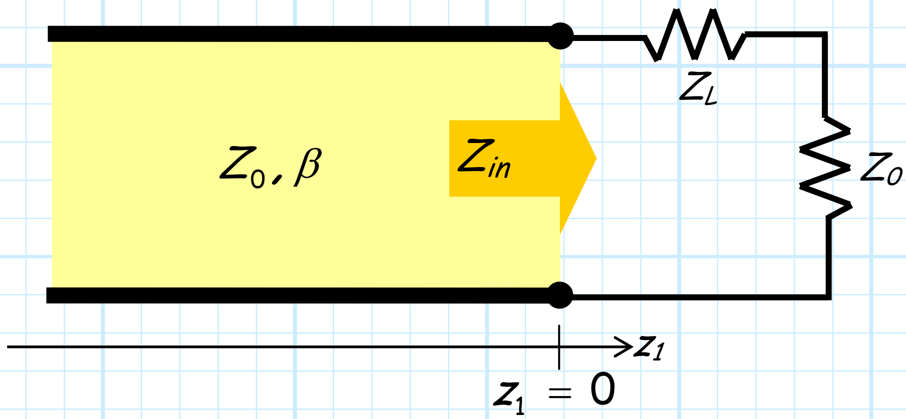
An **alternative** way for finding $\Gamma_0 = V_{01}^- / V_{01}^+$ is to determine the **input** impedance at the **end** of the **first** transmission line:



Note that since the second line is (eventually) terminated in a matched load, the input impedance at the **beginning** of the **second** line is simply equal to Z_0.



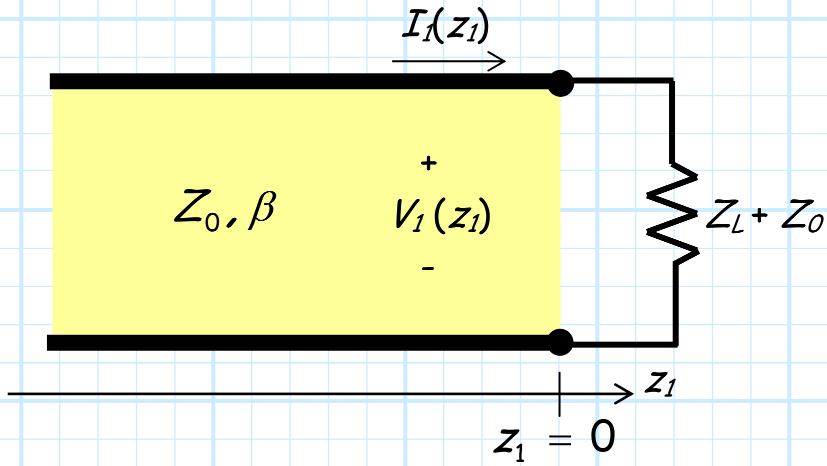
Thus, the **equivalent** circuit becomes:



And it is apparent that:

$$Z_{in} = Z_L + Z_0$$

As far as the first section of transmission line is concerned, it is **terminated** in a load with impedance $Z_L + Z_0$. The current and voltage along this first transmission line is **precisely** the same as if it **actually** were!



Thus, we find that $\Gamma_0 = V_{01}^{-} / V_{01}^{+}$, where:

$$\begin{aligned}\Gamma_0 &= \frac{Z(z_1=0)-Z_0}{Z(z_1=0)+Z_0} \\ &= \frac{(Z_L+Z_0)-Z_0}{(Z_L+Z_0)+Z_0} \\ &= \frac{Z_L}{Z_L+2Z_0}\end{aligned}$$

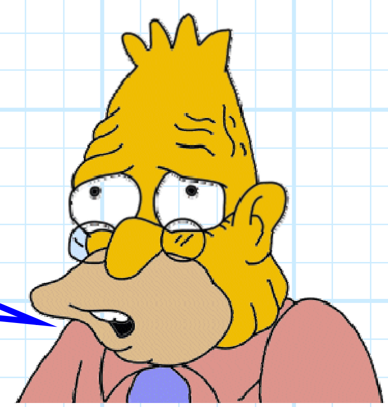
Precisely the same result as before!

Now, one **more** point. Recall we found in an **earlier** handout that $T_0 = 1+\Gamma_0$. But for this example we find that this statement is **not valid**:

$$1+\Gamma_0 = \frac{2(Z_L+Z_0)}{Z_L+2Z_0} \neq T_0$$

Again, be **careful** when analyzing microwave circuits!

Q: *But this seems so* ***difficult****. How will I* ***know*** *if I have made a mistake?*



A: An important engineering tool that **you** must master is commonly referred to as the "**sanity check**".

Simply put, a sanity check is simply **thinking** about your result, and determining whether or not it **makes sense**. A great **strategy** is to set one of the variables to a value so that the **physical** problem becomes **trivial**—so trivial that the correct answer is **obvious** to you. Then make sure your results **likewise** provide this obvious answer!

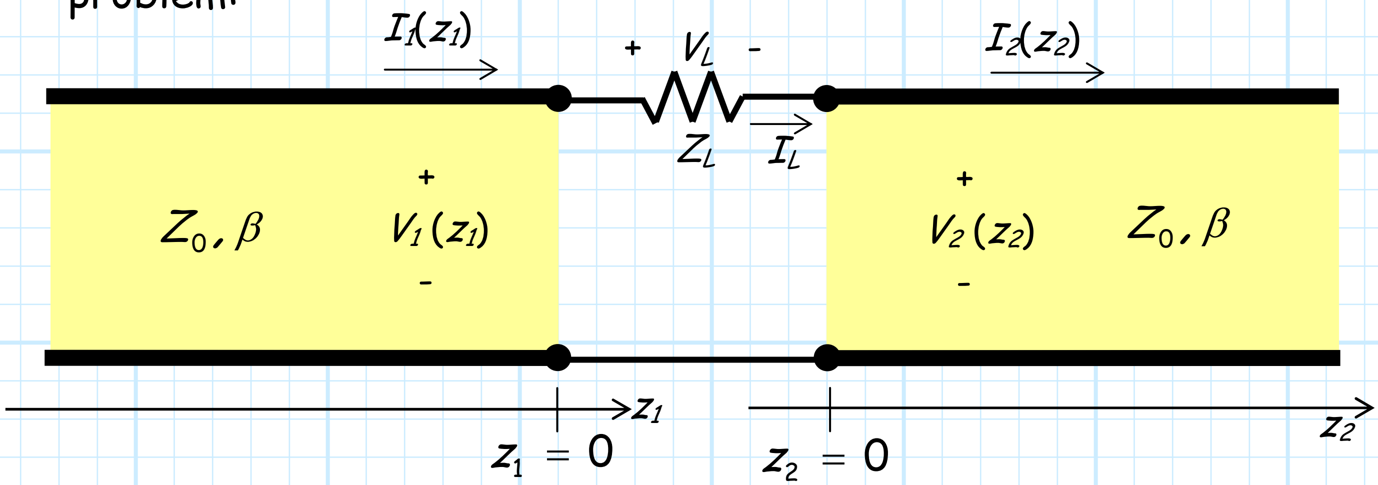
For example, consider the problem we just finished analyzing. Say that the impedance Z_L is actually a **short** circuit (Z_L=0). We find that:

$$\Gamma_0 = \left.\frac{Z_L}{Z_L + 2Z_0}\right|_{Z_L=0} = 0 \qquad\qquad T_0 = \left.\frac{2Z_0}{2Z_0 + Z_L}\right|_{Z_L=0} = 1$$

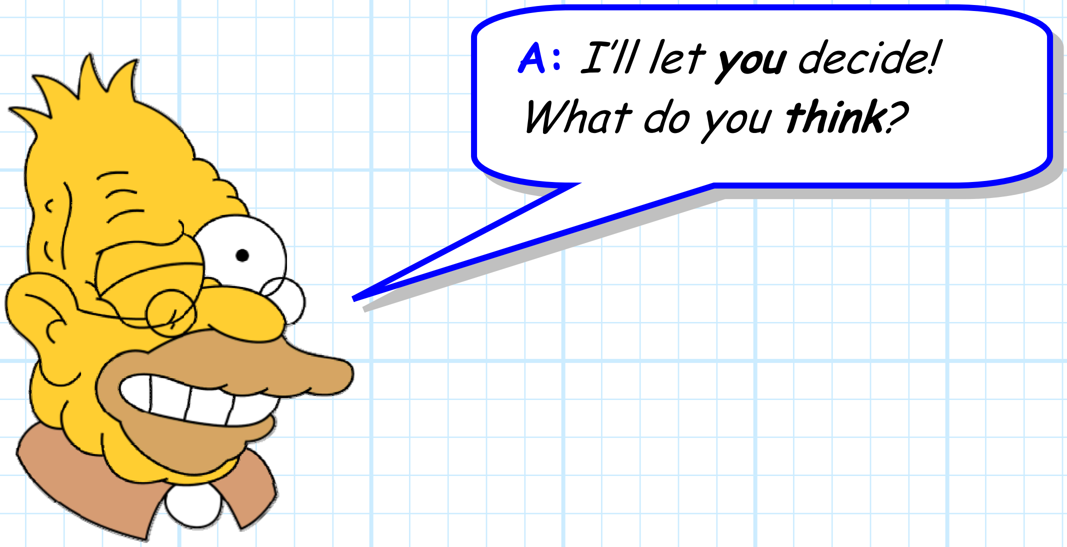
Likewise, consider the case where Z_L is actually an **open** circuit (Z_L=∞). We find that:

$$\Gamma_0 = \left.\frac{Z_L}{Z_L + 2Z_0}\right|_{Z_L=\infty} = 1 \qquad\qquad T_0 = \left.\frac{2Z_0}{2Z_0 + Z_L}\right|_{Z_L=\infty} = 0$$

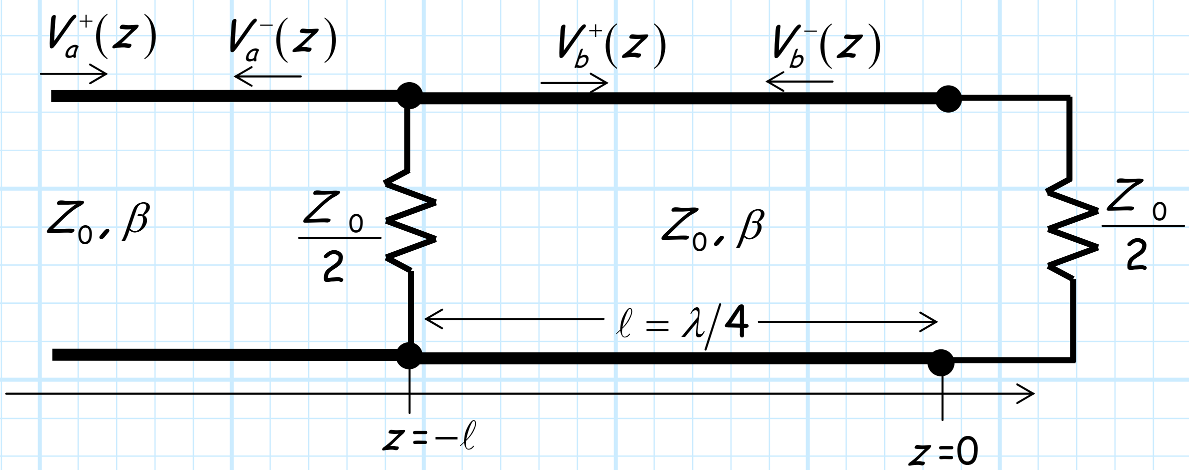
Think about what these results mean in terms of the **physical** problem:



Q: Do these results **make sense**? Have we **passed** the sanity check?



Example: Another Boundary Condition Problem



The **total** voltage along the transmission line shown above is expressed as:

$$V(z) = \begin{cases} V_a^+ e^{-j\beta z} + V_a^- e^{+j\beta z} & z < -\ell \\ \\ V_b^+ e^{-j\beta z} + V_b^- e^{+j\beta z} & -\ell < z < 0 \end{cases}$$

Carefully determine and apply boundary conditions at both $z = 0$ and $z = -\ell$ to find the three values:

$$\frac{V_a^-}{V_a^+}, \quad \frac{V_b^+}{V_a^+}, \quad \frac{V_b^-}{V_a^+}$$

Solution

From the telegrapher's equation, we likewise know that the current along the transmission lines is:

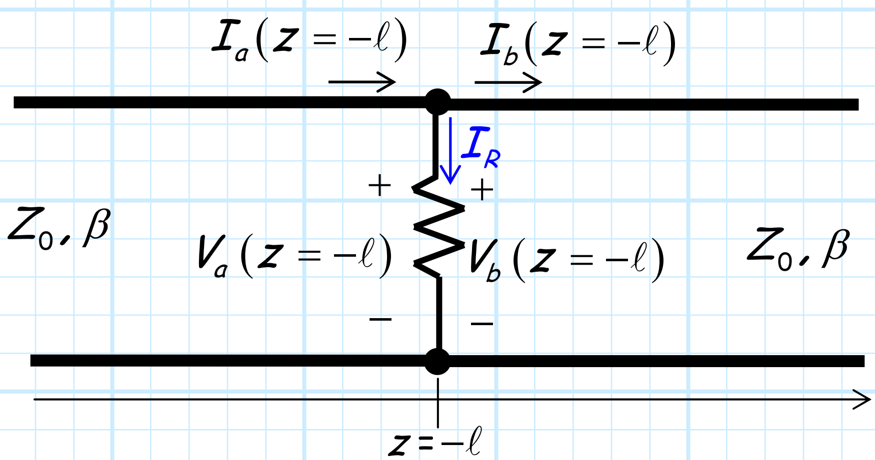
$$I(z) = \begin{cases} \dfrac{V_a^+}{Z_0} e^{-j\beta z} - \dfrac{V_a^-}{Z_0} e^{+j\beta z} & z < -\ell \\[2ex] \dfrac{V_b^+}{Z_0} e^{-j\beta z} - \dfrac{V_b^-}{Z_0} e^{+j\beta z} & -\ell < z < 0 \end{cases}$$

To find the values:

$$\frac{V_a^-}{V_a^+}, \quad \frac{V_b^+}{V_a^+}, \quad \frac{V_b^-}{V_a^+}$$

We need only to evaluate boundary conditions!

Boundary Conditions at $z = -\ell$



From KVL, we conclude:

$$V_a(z=-\ell) = V_b(z=-\ell)$$

From KCL:

$$I_a(z=-\ell) = I_b(z=-\ell) + I_R$$

And from Ohm's Law:

$$I_R = \frac{V_a(z=-\ell)}{Z_0/2} = \frac{2\,V_a(z=-\ell)}{Z_0} = \frac{2\,V_b(z=-\ell)}{Z_0}$$

We likewise know from the telegrapher's equation that:

$$\begin{aligned} V_a(z=-\ell) &= V_a^+\, e^{-j\beta(-\ell)} + V_a^-\, e^{+j\beta(-\ell)} \\ &= V_a^+\, e^{+j\beta\ell} + V_a^-\, e^{-j\beta\ell} \end{aligned}$$

And since $\ell = \lambda/4$, we find:

$$\beta\ell = \left(\frac{2\pi}{\lambda}\right)\left(\frac{\lambda}{4}\right) = \frac{\pi}{2}$$

And so:

$$\begin{aligned} V_a(z=-\ell) &= V_a^+\, e^{+j\beta\ell} + V_a^-\, e^{-j\beta\ell} \\ &= V_a^+\, e^{+j(\pi/2)} + V_a^-\, e^{-j(\pi/2)} \\ &= V_a^+\,(j) + V_a^-\,(-j) \\ &= j\left(V_a^+ - V_a^-\right) \end{aligned}$$

We similarly find that:

$$V_b(z=-\ell) = j\left(V_b^+ - V_b^-\right)$$

and for currents:

$$I_a(z=-\ell) = j\,\frac{V_a^+ + V_a^-}{Z_0}$$

$$I_b(z=-\ell) = j\,\frac{V_b^+ + V_b^-}{Z_0}$$

Inserting these results into our KVL boundary condition statement:

$$\begin{aligned} V_a(z=-\ell) &= V_b(z=-\ell) \\ j\left(V_a^+ - V_a^-\right) &= j\left(V_b^+ - V_b^-\right) \\ V_a^+ - V_a^- &= V_b^+ - V_b^- \end{aligned}$$

Normalizing to (i.e., dividing by) V_a^+, we conclude:

$$1 - \frac{V_a^-}{V_a^+} = \frac{V_b^+}{V_a^+} - \frac{V_b^-}{V_a^+}$$

From Ohm's Law:

$$I_R = \frac{2\,V_a(z=-\ell)}{Z_0} = \frac{2\,j\left(V_a^+ - V_a^-\right)}{Z_0}$$

$$I_R = \frac{2\,V_b(z=-\ell)}{Z_0} = \frac{2\,j\left(V_b^+ - V_b^-\right)}{Z_0}$$

And finally from our KCL boundary condition:

$$\begin{aligned} I_a(z=-\ell) &= I_b(z=-\ell) + I_R \\ j\,\frac{V_a^+ + V_a^-}{Z_0} &= j\,\frac{V_b^+ + V_b^-}{Z_0} + I_R \end{aligned}$$

After an **enjoyable** little bit of algebra, we can thus conclude:

$$V_a^+ + V_a^- = V_b^+ + V_b^- - jI_R Z_0$$

And inserting the result from Ohm's Law:

$$\begin{aligned} V_a^+ + V_a^- &= V_b^+ + V_b^- - jI_R Z_0 \\ &= V_b^+ + V_b^- - j\left(\frac{2\,j\left(V_b^+ - V_b^-\right)}{Z_0}\right) Z_0 \\ &= V_b^+ + V_b^- - 2\,j^2\left(V_b^+ - V_b^-\right)\left(\frac{Z_0}{Z_0}\right) \\ &= V_b^+ + V_b^- - 2\,(-1)\left(V_b^+ - V_b^-\right) \\ &= V_b^+ + V_b^- + 2V_b^+ - 2V_b^- \\ &= 3V_b^+ - V_b^- \end{aligned}$$

Again normalizing to V_a^+, we get a second relationship:

$$1 + \frac{V_a^-}{V_a^+} = 3\frac{V_b^+}{V_a^+} - \frac{V_b^-}{V_a^+}$$

Q: *But wait! We now have* ***two*** *equations:*

$$1 - \frac{V_a^-}{V_a^+} = \frac{V_b^+}{V_a^+} - \frac{V_b^-}{V_a^+} \qquad\qquad 1 + \frac{V_a^-}{V_a^+} = 3\frac{V_b^+}{V_a^+} - \frac{V_b^-}{V_a^+}$$

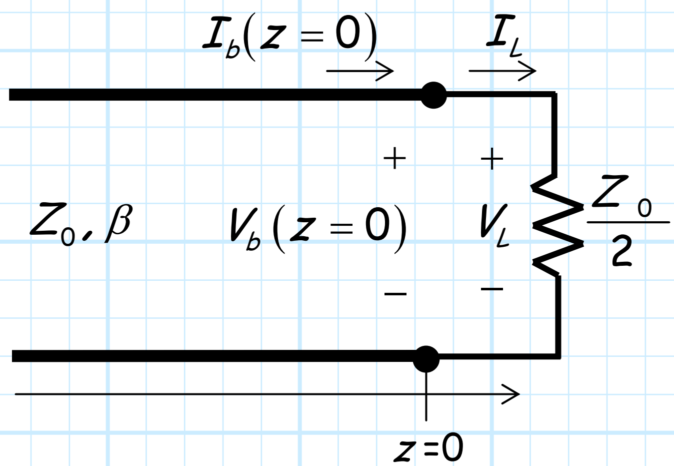
but ***three*** *unknowns:*

$$\frac{V_a^-}{V_a^+},\ \frac{V_b^+}{V_a^+},\ \frac{V_b^-}{V_a^+}$$

Did we make a ***mistake*** *somewhere?*

A: Nope! We just have more work to do. After all, there is a yet **another boundary** to be analyzed!

Boundary Conditions at $z = 0$



From KVL, we conclude:

$$V_b(z=0) = V_L$$

From KCL:

$$I_b(z=0) = I_L$$

And from Ohm's Law:

$$I_L = \frac{V_L}{Z_0/2} = \frac{2V_L}{Z_0}$$

We likewise know from the telegrapher's equation that:

$$\begin{aligned} V_b(z=0) &= V_b^+ e^{-j\beta(0)} + V_b^- e^{+j\beta(0)} \\ &= V_b^+ (1) + V_b^- (1) \\ &= V_b^+ + V_b^- \end{aligned}$$

We similarly find that:

$$I_b(z=0) = \frac{V_b^+ - V_b^-}{Z_0}$$

Combing this with the above results:

$$I_L = \frac{2\,V_L}{Z_0}$$

$$I_b(z=0) = \frac{2\,V_b(z=0)}{Z_0}$$

$$\frac{V_b^+ - V_b^-}{Z_0} = \frac{2\left(V_b^+ + V_b^-\right)}{Z_0}$$

From which we conclude:

$$V_b^+ - V_b^- = 2\left(V_b^+ + V_b^-\right) \quad \Rightarrow \quad -3V_b^- = V_b^+$$

And so:

$$V_b^- = -\frac{1}{3}V_b^+$$

Note that we could have also determined this using the load reflection coefficient:

$$\frac{V_b^-(z=0)}{V_b^+(z=0)} = \Gamma(z=0) = \Gamma_0$$

Where:

$$V_b^-(z=0) = V_b^-\ e^{+j\beta(0)} = V_b^-$$

$$V_b^+(z=0) = V_b^+\ e^{-j\beta(0)} = V_b^+$$

And we use the boundary condition:

$$\Gamma_0 = \Gamma_{Lb} = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{0.5Z_0 - Z_0}{0.5Z_0 + Z_0} = \frac{-0.5}{1.5} = -\frac{1}{3}$$

Therefore, we arrive at the **same result** as before:

$$\frac{V_b^-(z=0)}{V_b^+(z=0)} = \Gamma_0$$

$$\frac{V_b^-}{V_b^+} = -\frac{1}{3}$$

Either way, we can use this result to simplify our first set of boundary conditions:

$$\begin{aligned} 1 - \frac{V_a^-}{V_a^+} &= \frac{V_b^+}{V_a^+} - \frac{V_b^-}{V_a^+} \\ &= \frac{V_b^+}{V_a^+} - \frac{-V_b^+/3}{V_a^+} \\ &= \frac{V_b^+}{V_a^+} + \frac{1}{3}\frac{V_b^+}{V_a^+} \\ &= \frac{4}{3}\frac{V_b^+}{V_a^+} \end{aligned}$$

And:

$$\begin{aligned} 1+\frac{V_a^-}{V_a^+} &= 3\frac{V_b^+}{V_a^+}-\frac{V_b^-}{V_a^+} \\ &= 3\frac{V_b^+}{V_a^+}-\frac{-V_b^+/3}{V_a^+} \\ &= 3\frac{V_b^+}{V_a^+}+\frac{1}{3}\frac{V_b^+}{V_a^+} \\ &= \frac{10}{3}\frac{V_b^+}{V_a^+} \end{aligned}$$

NOW we have **two** equations and **two** unknowns:

$$1-\frac{V_a^-}{V_a^+}=\frac{4}{3}\frac{V_b^+}{V_a^+} \qquad\qquad 1+\frac{V_a^-}{V_a^+}=\frac{10}{3}\frac{V_b^+}{V_a^+}$$

Adding the two equations, we find:

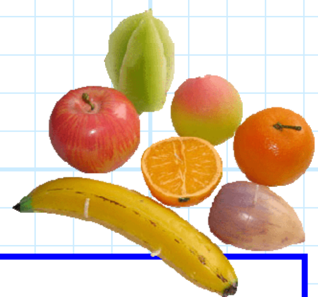
$$\begin{aligned} \left(1-\frac{V_a^-}{V_a^+}\right)+\left(1+\frac{V_a^-}{V_a^+}\right) &= \left(\frac{4}{3}\frac{V_b^+}{V_a^+}\right)+\left(\frac{10}{3}\frac{V_b^+}{V_a^+}\right) \\ 2 &= \frac{14}{3}\frac{V_b^+}{V_a^+} \\ \frac{3}{7} &= \frac{V_b^+}{V_a^+} \end{aligned}$$

And so using the second equation above:

$$\begin{aligned} \frac{V_a^-}{V_a^+} &= \frac{10}{3}\frac{V_b^+}{V_a^+}-1 \\ &= \frac{10}{3}\frac{3}{7}-1 \\ &= \frac{3}{7} \end{aligned}$$

And finally, from one of our original boundary conditions:

$$
\begin{aligned}
\frac{V_b^-}{V_a^+} &= \frac{V_b^+}{V_a^+} - 1 + \frac{V_a^-}{V_a^+} \\
&= \frac{3}{7} - 1 + \frac{3}{7} \\
&= -\frac{1}{7}
\end{aligned}
$$



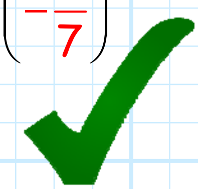
And so now we **summarize** the fruit of our labor:

$$
\boxed{\frac{V_a^-}{V_a^+} = \frac{3}{7} \qquad \frac{V_b^+}{V_a^+} = \frac{3}{7} \qquad \frac{V_b^-}{V_a^+} = -\frac{1}{7}}
$$

Yes it is! It's time for a **sanity check**!!!

The first of our boundary condition equations:

$$
\begin{aligned}
1 - \frac{V_a^-}{V_a^+} &= \frac{V_b^+}{V_a^+} - \frac{V_b^-}{V_a^+} \\
1 - \frac{3}{7} &= \frac{3}{7} - \left(-\frac{1}{7}\right) \\
\frac{4}{7} &= \frac{4}{7}
\end{aligned}
$$



And from the second:

$$1+\frac{V_a^-}{V_a^+}=3\frac{V_b^+}{V_a^+}-\frac{V_b^-}{V_a^+}$$

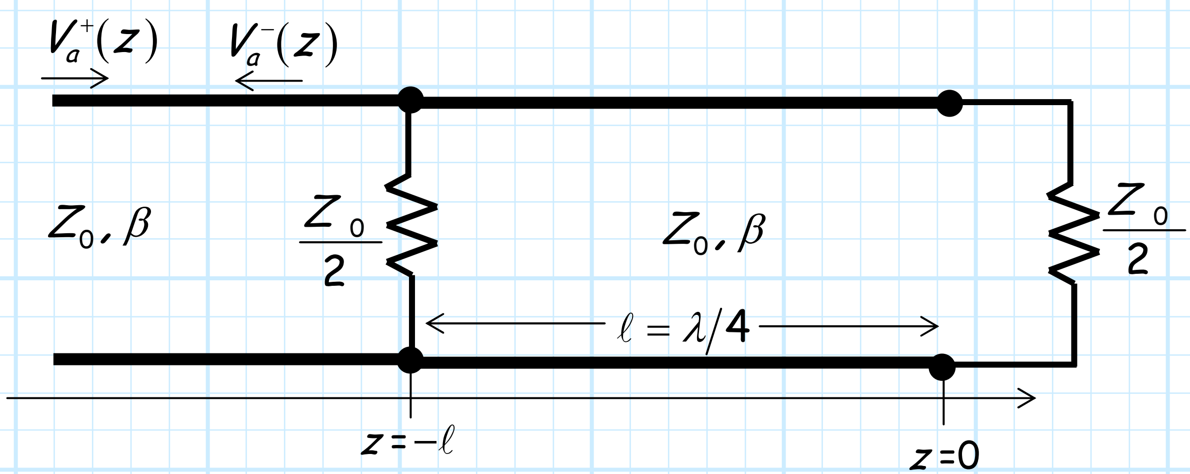
$$1+\frac{3}{7}=3\frac{3}{7}-\left(-\frac{1}{7}\right)$$

$$\frac{10}{7}=\frac{10}{7}$$

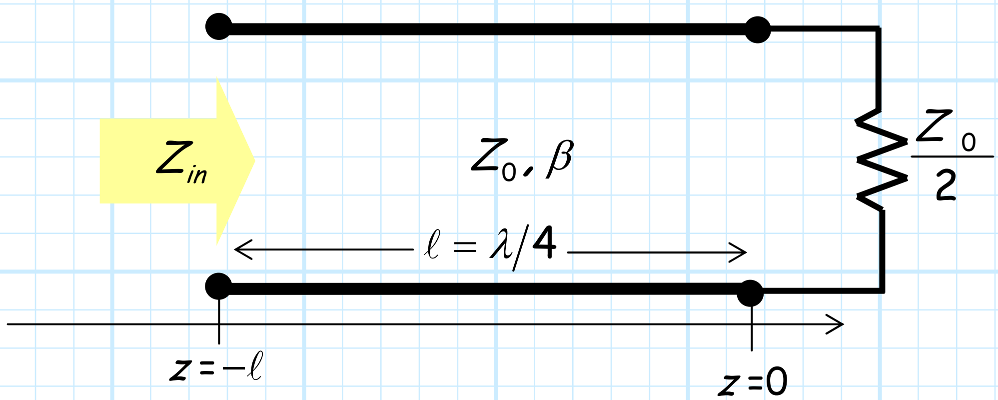
Notice that we can **also verify** the result:

$$\frac{V_a^-}{V_a^+}=\frac{3}{7}$$

By using the **equivalent circuit** of:



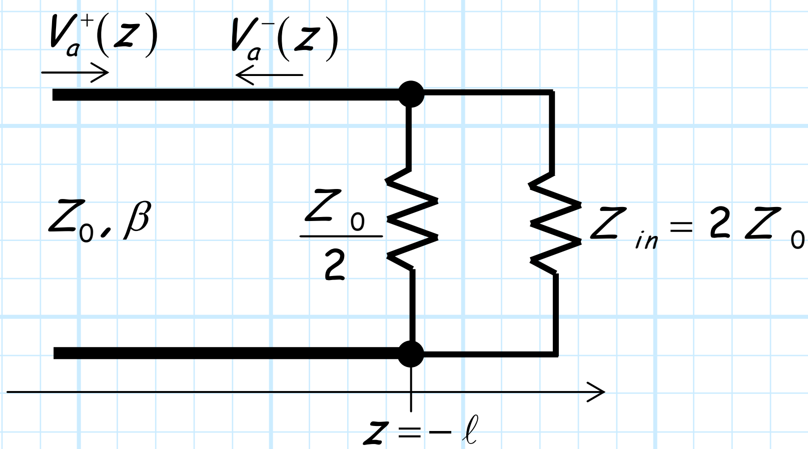
Specifically, we can determine the input impedance of this circuit:



Since the transmission line is the special case of one quarter wavelength, we know that:

$$Z_{in} = \frac{Z_0^2}{0.5Z_0} = 2.0Z_0$$

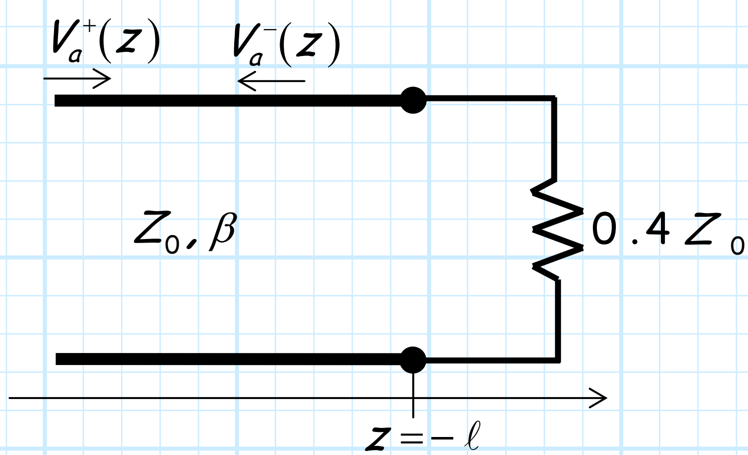
And so the equivalent circuit is



Where the two parallel impedances combine as:

$$0.5Z_0 \| 2Z_0 = \frac{Z_0}{2.5} = 0.4Z_0$$

And so the equivalent load at $z = -\ell$ is $0.4Z_0$:



Now, the reflection coefficient of this load is:

$$\Gamma_{La} = \frac{0.4Z_0 - Z_0}{0.4Z_0 + Z_0} = \frac{-0.6}{1.4} = -\frac{3}{7}$$

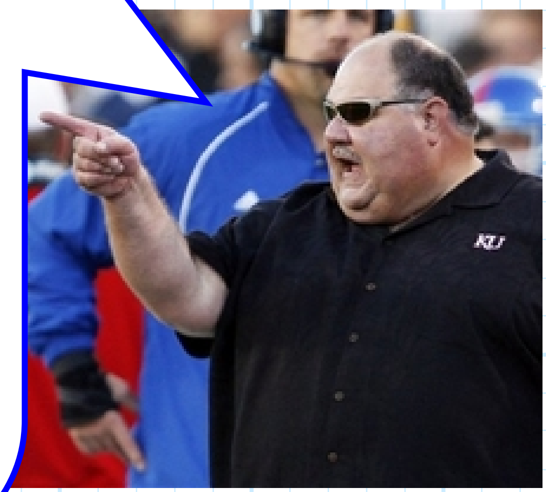
Q: *Wait a second! Using your fancy "boundary conditions" to solve the problem, you **earlier** arrived at the conclusion:*

$$\frac{V_a^-}{V_a^+} = \frac{3}{7}$$

*But **now** we find that instead:*

$$\frac{V_a^-}{V_a^+} = \Gamma_{La} = -\frac{3}{7}$$

*Apparently your annoyingly pretentious boundary condition analysis introduced some sort of **sign error** !*



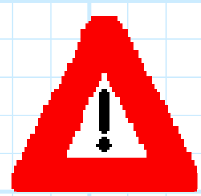
A: Absolutely not! The boundary condition analysis is perfectly correct, and:

$$\frac{V_a^-}{V_a^+} = \frac{3}{7}$$

is the right answer.

The statement:

$$\cancel{\frac{V_a^-}{V_a^+} = \Gamma_{La} = -\frac{3}{7}}$$



is **erroneous**!

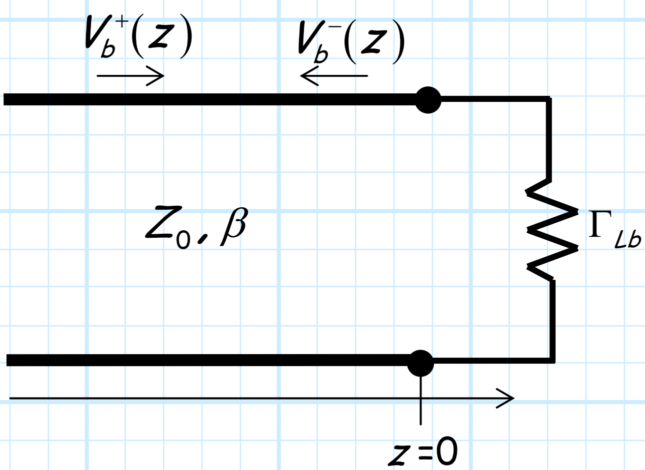
Q: *But how could that possibly be? **You** earlier determined that:*

$$\frac{V_b^-}{V_b^+} = \Gamma_{Lb} = -\frac{1}{3}$$

So why then is:

$$\frac{V_a^-}{V_a^+} \neq \Gamma_{La} \quad ????$$

A: In the first case, load Γ_{Lb} is located at position $z=0$, so that:

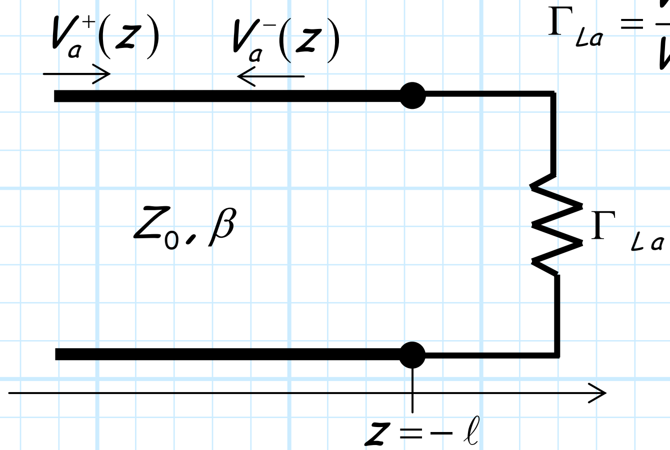


$$\Gamma_{Lb} = \frac{V_b^-(z=0)}{V_b^+(z=0)} = \frac{V_b^-}{V_b^+}$$

Note this result can be more compactly stated as a boundary condition requirement:

$$\Gamma_{Lb} = \Gamma(z=0) = \frac{V_b^-}{V_b^+} e^{+j\beta(0)} = \frac{V_b^-}{V_b^+}$$

For the second case, the load Γ_{Lb} is located at position $z = 0$, so that:



$$\Gamma_{La} = \frac{V_a^-(z=-\ell)}{V_a^+(z=-\ell)} = \frac{V_b^- e^{-j\beta\ell}}{V_b^+ e^{+j\beta\ell}} = \frac{V_b^-}{V_b^+} e^{-j2\beta\ell}$$

Note this result can be more compactly stated as a boundary condition requirement:

$$\Gamma_{La} = \Gamma(z=-\ell) = \frac{V_b^-}{V_b^+} e^{-j2\beta\ell}$$

From the equation above we find:

$$\frac{V_b^-}{V_b^+} = \Gamma_{La}\, e^{+j2\beta\ell} = -\frac{3}{7} e^{+j\pi} = +\frac{3}{7}$$

*That's **precisely** the same result as we determined earlier using our **boundary conditions**! Our answers are **good**!*