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A literature survey on fatigue analysis approaches for rubber

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Abstract

Rubber components subjected to fluctuating loads often fail due to the nucleation and growth of defects or cracks. The prevention of such failures depends upon an understanding of the mechanics underlying the failure process. This paper reviews analysis approaches that are currently available for predicting fatigue life in rubber. Both crack nucleation and crack growth approaches are considered. A discussion of each approach's strengths and limitations, and examples of how these approaches have been applied in engineering analysis are presented. © 2002 Elsevier Science Ltd. All rights reserved.

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1. Introduction

Rubber's ability to withstand very large strains without permanent deformation or fracture makes it ideal for many applications. Applications include tires, vibration isolators, seals, hoses, belts, structural bearings, impact bumpers, medical devices, and footwear, to name a few. These applications impose large static and time-varying strains over a long time. Long-term durability is therefore a critical issue. While many factors contribute to long-term durability, mechanical fatigue, the nucleation and growth of cracks in the rubber, is often the primary consideration. To address the issue effectively and economically, engineers need to model and design for mechanical fatigue early in the product development process. This need has partially been addressed by the development of simulation software capable of predicting stress and strain histories [1–4]. The question remains, however, of how to use these histories to estimate component life.

The objective of this paper is to review analytical approaches that are currently available for predicting fatigue life in rubber. Typically, the fatigue failure process involves two distinct phases. The first phase is a period during which cracks nucleate in regions that were initially free of observable cracks. The second phase is a period during which nucleated cracks grow to the point of failure. It will be seen that nucleation, growth, and final failure may all be rationalized in terms of the fracture mechanical behavior of rubber. There are, however, issues unique to the crack nucleation phase, which deserve careful study.

Models for predicting fatigue life in rubber follow two overall approaches. One approach focuses on predicting crack nucleation life, given the history of quantities that are defined at a material point, in the sense of continuum mechanics. Stress and strain are examples of such quantities. The other approach, based on ideas from fracture mechanics, focuses on predicting the growth of a particular crack, given the initial geometry and energy release rate history of the crack. For each approach, existing theories are presented. A discussion of each approach's strengths and limitations, and examples of how these approaches have been applied in engineering analysis are also included.

Some of the information presented in this paper has been reviewed previously [5–13]. This literature survey updates these existing reviews to reflect recent and previously unnoticed developments. This survey also offers new interpretations of existing studies and theories, and identifies areas where additional research is needed. Another paper reviews factors that affect the fatigue life of rubber [14]. These include the effects of mechanical

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loading history, environmental effects, effects of rubber formulation, and effects due to dissipative aspects of the constitutive response of rubber.

2. Crack nucleation approaches

The crack nucleation approach considers that a material has an intrinsic life determined by the history of stresses or strains at a point. This approach is convenient because it is formulated in terms of stresses and strains, which are familiar to designers. The fatigue crack nucleation life may be defined as the number of cycles required to cause the appearance of a crack of a certain size. The earliest known study of this type was August Wöhler's work with railroad axles in the 1860s [15]. A similar analysis approach was applied to rubber as early as the 1940s [16,17], and remains in use today [18,19]. The approach is particularly appropriate in applications where the initial flaws that eventually determine component life are several orders of magnitude smaller than component features, and where it is desired to analyze the spatial distribution of fatigue life.

The two widely used fatigue life parameters for crack nucleation prediction in rubber are maximum principal strain (or stretch), and strain energy density. The octahedral shear strain has also been used, but less commonly. Strain is a natural choice because it can be directly determined from displacements, which can be readily measured in rubber. When strain energy density is applied to fatigue analysis in rubber, it is often estimated from a hyperelastic strain energy density function, which is defined entirely in terms of strains. Stress, apparently, has rarely been used as a fatigue life parameter in rubber [20]. This seems to be related to the fact that fatigue testing in rubber has traditionally been conducted in displacement control, and that accurate stress determination in rubber components can be formidably difficult.

2.1. Maximum principal strain

A common hypothesis [21], implicit in many studies, is that the alternating and mean values of the maximum principal strain uniquely predict nucleation life. Note that it is commonly observed in rubber that cracks initiate on a plane normal to the maximum tensile strain. The earliest fatigue studies in rubber focused on developing an empirical description of the number of cycles to failure as a function of alternating strain and minimum strain. In 1940, Cadwell and co-workers [16] studied unfilled, vulcanized natural rubber. They investigated minimum engineering strains in the range -40% to greater than $+500\%$, and strain amplitudes in the range 12.5% to 350%. They found that, for constant strain amplitude, the fatigue life of natural rubber improves with increasing minimum strain, up to a moderately high minimum strain level (200%), beyond which additional minimum strain decreased the life. Similar effects were observed in both axial and shear fatigue tests. Several years later, Fielding [17] applied the same approach (based simply on axial engineering strain) in studying the effect of minimum strain on two newly developed synthetic rubbers. In general, for rubbers that strain crystallize, increasing the minimum strain (i.e. increasing R-ratio) of the strain cycle can significantly lengthen the fatigue life. A detailed discussion of the minimum strain effects on fatigue crack nucleation as well as growth for both crystallized and non-crystallized rubber is presented in [14].

While both uniaxial and shear experiments had been performed in the study of Cadwell et al. [16], it was not attempted to quantitatively reconcile the results from different strain states, or to explicitly develop a theory of how to relate relatively simple lab tests to more complicated strain histories.

Roberts and Benzies [22] and Roach [23] investigated fatigue life under conditions of simple and equibiaxial tension. When plotted against the maximum principal stretch (or strain), fatigue life is longer in simple tension than in equibiaxial tension. The difference was pronounced for natural rubber (NR), and much less pronounced in styrene butadiene rubber (SBR). Ro [24] reanalyzed the data from these studies, using other strain-based parameters, including octahedral shear strain, and maximum shear strain. Ro concluded that none of these parameters were generally optimal for unifying simple and equibiaxial tension data.

2.2. Strain energy density

In the late 1950s and early 1960s, success with crack growth models [25–37] had a significant impact on the subsequent development of the nucleation life approach in rubber. Prior to this work, the independent variables in fatigue studies were usually taken as alternating and minimum tensile strain, or stretch. After the development of fracture mechanics for rubber, strain energy density came into use as a parameter to predict fatigue crack initiation [12].

Under certain conditions, the energy release rate is proportional to the product of strain energy density (far from the crack), and the crack size [36,38,39]. When this is the case, it may be considered that the strain energy density is a measure of the energy release rate of naturally occurring flaws. The conditions under which the strain energy density may be uniquely related to the energy release rate are limited. For the relationship to hold, it is assumed that crack growth is self-similar, that the far-field strain gradient across the crack is negligible, and that the stress state is one of simple tension. Several

researchers have investigated strain energy density as a fatigue life parameter in rubber [22–24,40–42].

Roberts and Benzies [22], and Roach [23], found that for NR, equibiaxial tension fatigue life was longer than simple tension fatigue life by a factor of approximately four, when compared based on equal strain energy density. For SBR, equibiaxial tension fatigue life was longer than simple tension fatigue life by a factor of approximately 16, when compared based on equal strain energy density. Note that this ranking is opposite of that found when compared on the basis of maximum principal strain. Roach proposed that these differences could be explained by considering only that portion of the strain energy density that was actually available for flaw growth. For simple tension, Roach proposed that all of the strain energy density is available for flaw growth, while for equibiaxial tension, only one half of the strain energy density is available for flaw growth. This hypothesis gave the best correlation between simple and equibiaxial tension fatigue data.

Ro [24] re-analyzed the data of Roach, and Roberts and Benzies and concluded that strain energy density is a better correlation parameter for high-cycle fatigue of rubber than other strain-based parameters. It should be noted, however, that Ro's analysis is entirely dependent upon assumptions of a contrived dependence of Poisson's ratio on strain, and of linear elastic stress–strain behavior. Curiously, Ro did not further investigate Roach's idea of an available energy density, despite the fact that it appeared to give the most consistent explanation of Roach's results. To this point, no distinction has been made between total strain energy density, and distortional strain energy density. Ro correctly pointed out that, because of rubber's near incompressibility, the distinction is usually unnecessary.

Strain energy density was proposed and studied as a fatigue parameter in metals [43], but the correlation was not satisfactory, and theoretical objections have been raised [44,45]. Findley et al. [43] devised an experiment in which the stresses were cycled, but the strain energy density remained constant. Specimen failure was still observed to occur. Applied as a scalar criterion, strain energy density does not predict the fact that cracking appears in a specific orientation. In addition, the strain energy density cannot be a general measure of energy release rate, since the energy released depends on how the flaw is oriented with respect to the strains.

A number of other approaches have been proposed and evaluated for multiaxial fatigue nucleation life in metals [46–57]. It is particularly worthwhile mentioning the critical plane approaches, which have enjoyed a great deal of success. In critical plane approaches, the history of parameters associated with specific material planes are used to predict fatigue life. For rubber, however, multiaxial loading effects are not yet well understood.

Rubber components are quite commonly subjected to compressive loading, and this fact must be considered carefully in any analysis of fatigue life. Compressive loading along one direction is almost always associated with simultaneous shear and/or tensile loading in other directions. The only exception would be the case of pure hydrostatic compression. Although planes perpendicular to a compression axis experience closure, planes in other orientations experience shear and/or tension. Cracks will tend to nucleate and grow on these planes. Fatigue crack nucleation criteria (maximum principal strain, strain energy density) which do not consider crack closure may be particularly unreliable for cases involving compressive loading [58].

2.3. Applications of crack nucleation approaches

Many engineers and researchers have used strain energy density to correlate analysis results to experimental component life data. Often, such studies refer to the original work of Gent, Lindley, and Thomas [25,36], or the follow-up studies of Lake and Lindley [59,60], reflecting the argument presented in the previous section, that the strain energy density is a measure of the energy release rate of naturally occurring flaws.

Grosch [61] developed a simple analytical model to predict the endurance mileage of tires under various operating conditions. This model uses an analytical estimate of the strain energy density in a tire, along with a semi-empirical relationship between strain energy density and fatigue life. His model predicts relative differences in failure mileage, given differences in operating conditions. The model does not attempt to account for differences due to tire design.

Building on Ro's results [24], DeEskinazi et al. [62] used the Finite Element Method to compute the strain energy density in three tires with design differences. They correlated observed differences in fatigue life to computed strain energy density levels. Oh [63] and Yamashita [64] used strain energy density to predict the fatigue life of bushings and vibration damping devices, respectively.

The presumption of a unique relationship between strain energy density and crack nucleation life is implicit in these studies. While many in the rubber industry have used strain energy density as a predictive parameter, few have tried to ascertain its range of validity under the general conditions experienced by components in service.

Taken together, the studies cited here show that a nucleation life approach to design analysis with rubber is often needed, and that the approach must be general enough to handle situations involving multiaxial loading. Existing approaches have not been very successful in this regard, based on the limited data that have been reported for multiaxial conditions.

3. Crack growth approach

The crack growth approach explicitly considers pre-existing cracks or flaws. The idea of focusing attention on individual flaws was introduced by Inglis [65] in 1913, and Griffith [66] in 1920. Griffith proposed a fracture criterion based on an energy balance including both the mechanical energy of a cracked body, and the energy associated with the crack surfaces. Griffith's approach was further developed for rubber by Thomas, Greensmith, Lake, Lindley, Mullins, and Rivlin in the 1950s and 1960s [25–37]. Irwin [67–69], Rice [70], and others developed the approach in metals. While the original application of this approach to rubber was to predict static strength [25,71–78], in the late 1950s Thomas [29] extended the approach to analyze the growth of cracks under cyclic loads in natural rubber. He discovered a square-law relationship between peak energy release rate and crack growth rate for unfilled natural rubber. Paris et al. [79] independently found a similar power-law relationship in metals.

Two important developments in the fracture mechanics of rubber predated the analogous developments in metals. In his work on the J-integral, Rice [70] credits Thomas [26] for first showing the connection between the energy release rate and the strain concentration at the crack tip. Also, Thomas' proposal that the relationship between the cyclic energy release rate and the rate of fatigue crack growth follows a square-law [29], predated Paris' work with a power-law [79] by 3 years.

3.1. The energy release rate

Griffith's [66] hypothesis was that crack growth is due to the conversion of a structure's stored potential energy to surface energy associated with new crack surfaces. He was able to show that the surface energy associated with the crack faces of a broken glass filament was equal to the elastic energy released by the fracture. In rubber, the potential energy released from surrounding material is spent on both reversible and irreversible changes to create the new surfaces [5,25,80]. In any case, the energy release rate is simply the change in the stored mechanical energy $\mathrm{d}U$, per unit change in crack surface area $\mathrm{d}A$. In the rubber literature, this quantity is often called the tearing energy T, regardless of whether the applied loading results in fatigue crack growth or sudden fracture ("tearing").

$$T = -\frac{\mathrm{d}U}{\mathrm{d}A} \tag{1}$$

The energy release rate was first applied to the analysis of rubber specimens under static loading [25]. It was quickly realized, however, that the concept also applied to crack growth under cyclic loading. In this case, it was found that the maximum energy release rate achieved during a cycle determined the crack growth rate, for $R = 0$ cycles [29].

3.2. Fracture mechanics test specimens for rubber

Rivlin and Thomas [25] showed that static crack growth occurs above a critical value of the energy release rate, independent of the type of test specimen they used, suggesting that the critical energy release rate may be considered a true material property. Their initial study used three specimen types: a center-cracked sheet, an edge-cracked sheet, and a "trouser" test piece. A subsequent study by Thomas with three additional test specimen types confirmed the independence of the critical energy release rate from specimen geometry [30]. Other studies [29,36,37,81,82], using the same specimen types, showed that the fatigue crack growth rate is also uniquely determined by the energy release rate.

The aforementioned studies did not investigate the effect of specimen thickness on the fatigue or fracture properties of the material. Thickness effects have been reported by Kadir and Thomas [83], and by Mazich et al. [84]. Kadir and Thomas showed that thickness dependence is related to the development of crack tip roughness. They hypothesized that under hydrostatic tension, rubber can cavitate, and that this may be the cause of the roughness developed around the tip. When the fracture surface remained smooth during growth, little dependence of the growth rate on specimen thickness was observed. Thicknesses ranging from 0.1 mm to 10 mm were investigated. When so called rough, or stick–slip crack growth occurs, however, a thickness effect can arise. On a plot of crack growth rate, at constant energy release rate, as a function of specimen thickness, two plateaus were observed. Below 0.5 mm, and above 5 mm, thickness had little effect. Between these thicknesses, however, the crack growth rate changes by more than an order of magnitude. The crack growth rate for thin specimens was larger than for thick specimens, by an order of magnitude, at equal energy release rate. Their explanation is that the transition in crack growth rate with thickness is related to the scale of the observed crack tip roughness, which is of the same order of magnitude as the transition thickness.

Mazich et al. [84] looked at the dependence of the critical energy release rate for fracture on specimen thickness. They found that, in the range from 1 mm to 3 mm, the critical energy release rate for crack growth increased by a factor of two. Note that these results are broadly consistent with those of Kadir and Thomas, since higher crack growth rates are associated with lower critical energy release rates. Both of these studies were conducted with gum SBR.

The two most commonly used specimens in rubber fatigue crack growth studies are the single edge cut planar tension specimen and single edge cut simple ten-

sion specimen. The planar tension specimen geometry (also known as the pure shear specimen in the rubber literature) has proven to be especially useful in fracture mechanics tests for rubber [81,85]. This specimen is short and wide such that lateral contraction is prevented by the grips, while an axial strain is introduced in the direction of the short dimension. The pure shear designation arises because, for small, volume-conserving strains, this strain state is one of pure shear, in a coordinate system suitably rotated with respect to the specimen.

In the single edge cut, planar tension specimen, the energy release rate T has an especially simple form, provided that the cut is sufficiently deep and that growth of the cut results only in translation of the crack tip fields. The expression depends only on the strain energy density W remote from the crack and from specimen edges, and the specimen height h. Note that, in displacement control, the energy release rate for this specimen is independent of the crack size [25].

$$T = Wh \tag{2}$$

In the single edge cut, simple tension specimen, the energy release rate T depends on the gauge section strain energy density W, the size of the crack a, and a strain-dependent parameter k.

$$T = 2kWa \tag{3}$$

The dependence of the parameter k on strain was studied by Greensmith [34], and Lindley [72], for the case in which the crack is much smaller than the specimen width. Nait-Abdelaziz et al. [82] have extended the parameter for cracks of finite size, relative to specimen width. A plot of Greensmith's data is shown in Fig. 1. A practical approximation of the data is given in terms of the engineering strain ε by [72]:

$$k = \frac{2.95 - 0.08\varepsilon}{(1+\varepsilon)^{1/2}} \tag{4}$$

The trouser specimen, shown in Fig. 2, was used in early studies of fatigue crack growth in rubber [29] to demonstrate geometry independence of the relationship between the energy release rate and the fatigue crack growth rate. The energy release rate T depends on the applied force F, the extension ratio λ and strain energy density W in the "legs" of the specimen, the specimen thickness t, and the "leg" width b.

$$T = \frac{2F\lambda}{t} - bW \tag{5}$$

Energy release rate calculations have been developed for other specimen types that have occasionally been used in fracture mechanics studies of rubber. These include the so-called angled, and split specimens. Since these have been reviewed previously [5–11,25,30,59,81,85,86], and since their application has been limited primarily to static strength measurements, the corresponding energy release rate calculations are not given here.

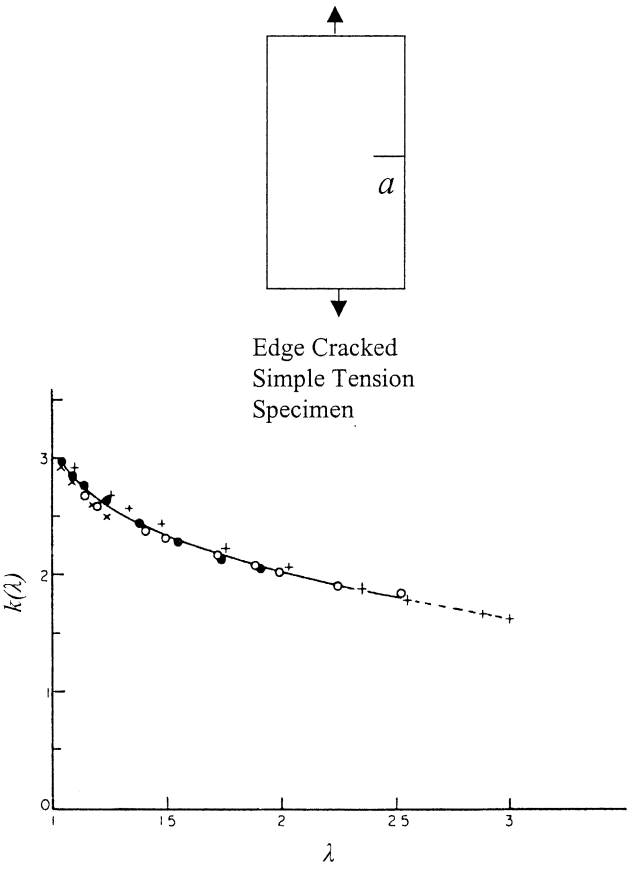

Fig. 1. Greensmith's [34] data for variation of k with maximum principal stretch λ, in the energy release rate of a crack in the simple tension specimen, $T = 2kWa$. Different data point types are for different rubber compounds.

3.3. Regimes of fatigue crack growth

Lake and Lindley [59] identified four distinct regimes of fatigue crack growth behavior, based on the maximum energy release rate per cycle, T, for $R = 0$ cycles in rubber. The full range of behavior is shown in Fig. 3, for unfilled NR and SBR.

So long as the peak energy release rate T remains below a threshold T_o, crack growth proceeds at a constant rate r, due solely to environmental attack. The crack growth rate da/dN below T_o is independent of the mechanical loading, and is denoted Regime 1.

$$\frac{da}{dN} = r \quad T \leq T_o \tag{6}$$

There is then a range of T, between T_o and T_t, over which there is a transition. The transition is described

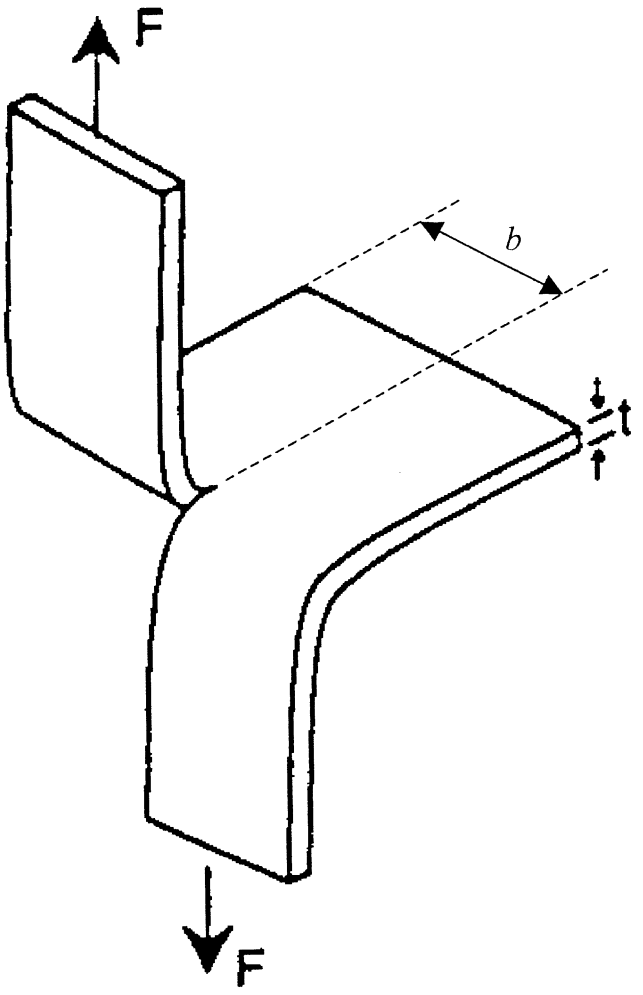

Fig. 2. Trouser test specimen used in early fatigue crack growth studies.

by the following relationship, in which A is a material property. This is denoted Regime 2.

$$\frac{da}{dN} = A(T - T_o) + r \quad T_o \le T \le T_t \tag{7}$$

After the transition, there is a range between T_t and T_c, over which the relationship between the fatigue crack growth rate and the energy release rate obeys a power-law. The associated material properties are B and F. This is denoted Regime 3.

$$\frac{da}{dN} = BT^F \quad T_t \le T \le T_c \tag{8}$$

Finally, beyond T_c, unstable crack growth ensues. In this regime, the crack growth rate is essentially infinite. This is denoted Regime 4.

$$\frac{da}{dN} = \infty \quad T = T_c \tag{9}$$

Aglan and Moet [87] developed a single relationship that predicts Regimes 2, 3, and 4, for $R = 0$ loading.

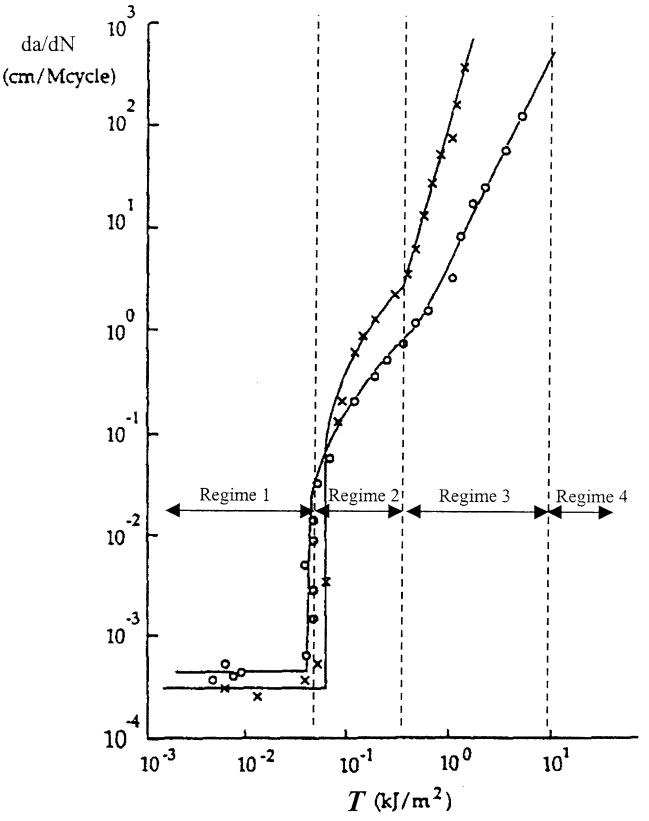

Fig. 3. Regimes of fatigue crack growth behavior in unfilled rubber under $R = 0$ loading (×, SBR; ○, NR) [59].

Their model, the Crack–Layer theory for rubber, is based on the irreversible thermodynamics of an "active zone", preceding the crack tip [87–89]. The model takes the thermodynamic flux to be the crack growth rate, and the conjugate driving force to be the energy release rate range ΔT. The model assumes that the energy dissipation associated with crack growth is proportional to the square of the energy-release rate range, in analogy to the Dugdale strip-yield model [90–92]. In addition to the critical energy release rate T_c, the model uses material parameters β and μ.

$$\frac{da}{dN} = \frac{\beta \Delta T^2}{\mu T_c - \Delta T} \tag{10}$$

Chow and Lu also developed a multi-regime model, based on thermodynamic arguments [93,94]. Their model is claimed to be valid for a wide range of materials. In addition to the critical energy release rate T_c, the model uses material parameters β and m.

$$\frac{da}{dN} = \frac{\beta \Delta T^m}{T_c - T_{max}} \tag{11}$$

These multi-regime models resemble models proposed for fatigue crack growth in metals by Foreman [95], and by Weertman [96]. Foreman's model depends on the R

ratio, the range of the stress intensity factor ΔK, the fracture toughness K_c, and material parameters β and n.

$$\frac{da}{dN} = \frac{\beta \Delta K^n}{(1-R)K_c - \Delta K} \tag{12}$$

Weertman's model depends also on the peak stress intensity factor, K_{max}.

$$\frac{da}{dN} = \frac{\beta \Delta K^4}{K_c^2 - K_{max}^2} \tag{13}$$

In Foreman's model, the resemblance can be seen by taking $n = 4$, and comparing to Aglan and Moet mode with $\mu = 1$. Note that while the Foreman equation is in terms of the stress intensity factor, the Aglan–Moet and Chow–Lu equations are in terms of energy release rate T. The energy release rate is proportional to the square of the stress intensity factor. All models also contain a proportionality constant, herein denoted by the common symbol β to emphasize algebraic similarity.

In some elastomers, the occurrence of compressive loading during a fatigue cycle can also have a dramatic effect on crack growth rate through the mechanism of strain crystallization at the crack tip. For cases in which the minimum tensile stress of the crack tip during the load cycle is sufficient to induce crystallization, the material remains crystalline at all times, retarding crack growth. For cases where the loading conditions permit the crack tip crystalline region to "melt" at some point during the cycle, the fatigue crack growth rate increases [97]. The effect of R-ratio on strain-crystallizing rubbers can influence crack growth rate and fatigue life by several orders of magnitude.

A major shortcoming of the aforementioned models, when applied to strain-crystallizing rubbers, is that they predict increased crack growth rates for $R > 0$ conditions. In contrast, for strain-crystallizing rubbers, the crack growth rate is significantly retarded by $R > 0$ conditions, as shown by Lindley [98]. Environmental effects can interact significantly with R ratio effects [97]. It appears that no current multi-regime fatigue crack growth model predicts the observed $R > 0$ effects for an important class of rubbers.

3.4. Relationship of energy release rate to crack tip conditions

The success of the energy release rate as a parameter for predicting fatigue crack growth and fracture has been attributed to its unique relationship to the local conditions at the crack tip. Thomas was the first to demonstrate this relationship experimentally [26]. He studied the strain distribution around the tip of a model crack in a sheet of rubber, in several specimen geometries. He found that the average strain energy density of material surrounding an idealized crack tip was uniquely related to the energy release rate, independent of the specimen type.

The relationship between the energy release rate and local crack tip conditions in rubber has also been confirmed in independent studies by Andrews [99,100], Knauss [101], Lee and Donovan [102], and Morman et al. [103]. Andrews used a microscopic, photoelastic technique to quantify the strain field around the crack tip. He showed that a combination of hysteresis and large displacements result in blunting of the crack tip in highly deformable materials. Knauss used a printed-grid technique. Lee and Donovan used Thomas' original approach. Morman et al. developed an analytical expression relating the energy release rate to the crack tip radius. The studies of Andrews and Knauss produced a more detailed map of the crack tip strain distribution than Thomas's; and all studies confirmed Thomas' conclusion.

Rice's development of the J-integral [70] provided a mathematical argument to explain the relationship between the energy release rate and the local crack tip conditions. The J-integral expresses an energy balance on a volume surrounding a crack tip. Rice proved that the value of the integral is independent of the choice of integration path. Since the integral is independent of the integration path, the path can be chosen close to the crack tip. The integral is therefore a measure of local crack tip conditions. The integration path may also be chosen to follow the boundaries of the specimen. In this case, the integral turns out to be equivalent to the energy release rate [84,102]. The energy release rate is therefore a measure of the intensity of local crack tip fields, for a given material and crack tip geometry. Rice's original formulation was valid for nonlinear elastic materials and infinitesimal strains. Chang generalized the J-integral for nonlinear elastic materials at finite strains [104].

A practical consequence of the J-integral is that the details of the processes occurring at the crack tip often do not need to be quantified in any way other than the J-integral in order to model crack growth. Instead, the details are accounted for by treating them as intrinsic to the material/crack-tip system. In this manner, nonlinearities due to finite network extensibility [105], strain crystallization [16,17,106–108], frictional losses due to filler interactions [109–111], and the Mullins effect [112–114] may be rolled into the fracture properties of the material [5,6]. A general theory to account for the effects of general nonlinear, dissipative constitutive behavior on crack tip fields has been proposed by Andrews [115]. Of course, this approach remains subject to the assumption that the crack propagates through an isotropic, homogeneous continuum. It seems that the energy release rate approach avoids the necessity of modeling crack tip dissipative processes by focusing on where the energy for driving the crack comes from (from strain energy stored beyond the J-integral boundary), instead

of where that energy is spent (on dissipative processes near the crack tip).

In situations where the stress–strain behavior is strongly time-dependent, the J-integral does not uniquely characterize local crack tip conditions. Lindley presented an approach for addressing time-dependent crack growth in SBR [116]. A time-dependent path-integral approach has also been developed [117–121], but it does not appear that this approach has been applied to elastomers.

Compressive loading must be considered carefully in any analysis of crack growth [122]. One consideration is that crack growth can only occur due to loads which result in tensile stresses at the crack tip. Purely compressive loading of a crack results in closure of crack faces until contact is achieved. Further compressive loading is then transmitted across the crack without causing crack growth. The strain energy release rate in this case is zero, despite the fact that the compressive loading may be large. When compressive and shear loads are both present, the crack tip can experience tensile loading. Determination of the strain energy release rate in this case can be complex, depending not only on the crack geometry and loading conditions, but also on the frictional properties of the crack faces.

Of course, the details of crack-tip processes are of interest when a deeper understanding of the failure process is sought. Theories for the relationship between the failure properties of rubber and rubber's molecular characteristics have been proposed and studied by several researchers [123–125].

3.5. Applications of the crack growth approach

An early design application of the fracture mechanics approach in rubber came from Lake and Clapson [126]. They developed an estimate of the energy release rate cycle at the base of tread pattern grooves in tires, based on the crack mouth opening displacement. They successfully predicted the rate of growth of cracks in tires on a road test from fatigue crack growth data generated in the lab.

Southern and Thomas [127] applied a fracture mechanics approach to develop a model for abrasive wear of rubber. Their model is based on fatigue crack growth at the base of an incipient wear particle. The model relates observed wear rates to measured fatigue crack growth properties, for the particular wear mechanism studied.

Huang and Yeoh [128] developed a fatigue crack growth model to rationalize the nucleation phase of the belt edge separation process in tires. The model was based on an estimate of the energy release rate of an array of penny-shaped cracks, each located at the end of a cord. Their model was validated via fatigue testing of a model cord–rubber composite. Choi, Roland, and Bissonette [129] analyzed failure in a large, elastomeric torpedo launcher. Medri and Strozzi [130] analyzed crack growth in elastomeric seals. Lindley [131] analyzed the life of metal–rubber bonds using a fracture mechanics approach. Stevenson et al. [132], and Gunderson et al. [133] have used FEA to evaluate energy release rate cycles experienced by cracks in rubber supports on off-shore structures. Stevenson and Malek [134] developed a model to predict the puncture behavior of thick rubber components penetrated by a sharp-cornered cylindrical indenter.

Ebbott [135] and Wei et al. [136] have used FEA to evaluate the energy release rate cycle experienced by a crack at the edge of cord–rubber tire components. Ebbott used a global–local analysis procedure in which a coarse mesh of the whole tire was first analyzed, followed by a refined-mesh analysis involving only the region of interest. Wei et al. performed the analysis with a single mesh. In both cases, the assumed crack geometry was built into the FE model. Both studies reported reasonable estimates of the crack growth rate for the tires analyzed. A limitation of this approach is that each potential failure mode requires its own mesh and analysis. In addition, each mesh applies only to a given crack size. A full life analysis with this approach requires large effort and expense in creating and analyzing multiple models. Automated mesh adaptation is required to make these approaches suitable for general use.

A common difficulty of using the crack growth approach in rubber is that it requires up-front knowledge of the initial location and state of the crack that causes the final failure. Often, this information is not available, or it is the very information the designer needs to predict. In addition, the changing geometry of the problem must be considered. Numerical implementations of a direct fracture mechanics approach remain labor intensive and computationally expensive. There is a great need for robust, general-purpose algorithms for crack growth analyses in rubber.

4. A flaw growth model for crack nucleation

A crack growth approach has been successfully used to predict uniaxial fatigue crack nucleation life from fatigue crack growth measurements [36,59]. This unified analysis is based on integration of the fatigue crack growth rate. The energy release rate of an assumed, pre-existing flaw is estimated from the flaw size and the strain energy density. The analysis only applies to small cracks, when the energy release rate may be factored into a product proportional to both strain energy density and crack size. Nevertheless, the small crack assumption often covers the most important portion of a component's life, since the presence of a large crack usually means the component has already failed.

4.1. Integrated power-law model

Fatigue life is ultimately determined by a pre-existing flaw that is the first to grow to a critical size. The life is obtained by integrating the growth rate of the fastest growing flaw, from its initial size to its critical or final size. For the purposes of integration, it is assumed that flaw growth is planar and self-similar.

Piece-wise integration over the four regimes of fatigue crack growth behavior is possible, and yields the most accurate fatigue life predictions [9,137]. A practical shortcut assumes power-law behavior over the entire life of the flaw [5,85,138]. This results in a closed-form relationship between the fatigue life and the fatigue crack growth properties. Combining Eqs. (8) and (3),

$$\frac{da}{dN} = f[T(a,W)] = BT^{F} = B(2kWa)^{F} \tag{14}$$

Then integrating,

$$N_f = \int_0^{N_f} dN = \int_{a_o}^{a_f} \frac{1}{f[T(a,W)]} da = \int_{a_o}^{a_f} \frac{1}{B(2kW)^{F}} a^{-F} da \tag{15}$$

$$N_f = \frac{1}{F-1} \frac{1}{B(2kW)^{F}} \left[\frac{1}{a_0^{F-1}} - \frac{1}{a_f^{F-1}} \right] \tag{16}$$

From Eq. (16), we see that if the initial flaw size a_o is much smaller than the critical flaw size, a_f, then the life becomes independent of the critical flaw size.

$$N_f = \frac{1}{F-1} \frac{1}{B(2kW)^{F}} \frac{1}{a_0^{F-1}} \tag{17}$$

If the initial flaw size is regarded as a property intrinsic to a given virgin material, and if variation of k with strain is neglected, the constants of the equation may be combined into a single material constant D.

$$N_f = DW^{-F} \tag{18}$$

This derivation applies only to uniaxial loading, where the energy release rate can be factored into the strain energy density and the crack size. For multiaxial situations, not all of the elastically stored energy is available to be released [23]. A general-purpose calculation of the available strain energy density has not been published.

4.2. Intrinsic flaws in rubber

The preceding theory suggests a way to estimate the size of naturally occurring flaws in rubber, by using the initial flaw size as a curve fit parameter to obtain agreement between crack nucleation and crack growth experiments [34,59,139]. The resulting flaw size is actually an effective flaw size, reflecting both the size and shape of the flaws [36]. Effective flaw sizes in the range of 20×10^{-6} m to 50×10^{-6} m were observed in a study by Lake and Lindley [59], which covered eight different polymer types, and various fillers, curatives, and other compounding variables. It has been confirmed that the measured flaw size is independent of temperature [140]. Flaw size has some dependence on crosslink density [59,141], carbon black type [59,142], and degree of dispersion of compound ingredients [138]. The flaw size can also be deduced from static strength measurements [10,140], and optical microscopy techniques [142], independent of fatigue measurements. Agreement of these methods to within a factor of 2 has been reported [139,142]. Damage characterization of elastomeric composites using X-ray attenuation has been reported by Bathias et al. [143].

A basic assumption of fracture mechanics is continuity and homogeneity of the material. In materials where the initial flaws are small enough that this assumption is not true, additional considerations are necessary, as is the case in metals [144–146]. In rubber, the intrinsic flaws are larger than features of the molecular network structure by a factor of more than 10,000, and larger than individual filler particles by a factor of more than 100. Agglomerations of carbon black particles can exhibit dimensions of the same scale of intrinsic flaws [142]. The actual scale of such features seems likely to depend on manufacturing processes such as mixing. Table 1 summarizes the size scales of importance in filled rubber.

The independent agreement of nucleation and growth approaches, using initial flaw size as the sole fitting parameter suggests that it is appropriate to assume that the initial flaws are embedded in continuous, homogeneous material. The precise nature of such flaws remains obscure because it appears that there are multiple sources for flaws of the observed effective size. These sources may include naturally occurring contaminants or voids in the base polymer, imperfectly dispersed compounding

Table 1
Geometric features of filled rubber

Feature	Size
Large carbon black agglomerate [142]	200×10^{-6} m
Smallest flaw visible to naked eye	100×10^{-6} m
Typical size of intrinsic defects [139]	40×10^{-6} m
Small carbon black agglomerate [142]	20×10^{-6} m
Coarse carbon black particle [147]	500×10^{-9} m
Fine carbon black particle [147]	10×10^{-9} m
Distance along polymer chain between crosslinks (assuming 300% macro-stretch = 100% chain extension, crosslink density of $5.8 \times 10^{19}/\mathrm{cm}^3$) [148]	1×10^{-9} m
Length of a single monomer unit (isoprene) [123,149]	500×10^{-12} m
Spatial distance between crosslinks (based on crosslink density of $5.8 \times 10^{19}/cm^3$) [148]	300×10^{-12} m
Length of a main-chain, polysulfidic bond [150]	100×10^{-12} m

ingredients, filler agglomerates, mold lubricants, and imperfections in mold surfaces.

5. Summary

Two approaches have developed for analyzing fatigue life in rubber components, the crack nucleation approach, and the crack growth approach. In rubber, the crack growth approach has been studied and used extensively. The nucleation approach has received less attention in the literature, although many engineers still use this approach for its simplicity and familiarity.

The nucleation approach is advantageous for analyzing the spatial distribution of fatigue life, since it is based on quantities that are defined at a material point, in the sense of continuum mechanics. In rubber, uniaxial fatigue life results are most commonly correlated based on maximum principal strain (or stretch), and strain energy density. Neither of these parameters has been robustly successful in correlating results from different strain states, particularly simple tension and equibiaxial tension.

Some success has been achieved in developing and applying the crack growth approach in rubber. A major practical challenge is computation of the energy release rate associated with the crack of interest, and predicting the location and path of the fastest growing crack, especially when the geometry and loading are complicated. Robust numerical procedures are inevitably required, but are not widely available. When the crack of interest is small, another problem is determining the initial size and shape of the crack. Small flaws are often of particular importance, since most of a component's life may be spent on the growth of small flaws.

For uniaxial situations in which failure initiates from a small flaw, the strain energy density can be used to estimate the energy release rate of the flaw, from which fatigue life can be computed, given the fatigue crack growth curve. For multiaxial situations, the strain energy density is not generally appropriate because not all of the energy is available to be released by the growth of a flaw. An adequate multiaxial nucleation life approach is needed to accurately predict fatigue life in rubber components.

References

[1] White L. Extreme oilfield conditions push elastomers to the limit. European Rubber Journal 1999;181(1):24–7.
[2] Nicholson DW, Nelson NW. Finite-element analysis in design with rubber. Rubber Chemistry and Technology 1990;63:368–406.
[3] Nonlinear finite element analysis of elastomers. Los Angeles (CA): MSC Software Corporation, 2000 www.mscsoftware.com.
[4] Morman KN, Pan TY. Application of finite-element analysis in the design of automotive elastomeric components. Rubber Chemistry and Technology 1988;61:503–33.
[5] Lake GJ. Fatigue and fracture of elastomers. Rubber Chemistry and Technology 1995;68:435–60.
[6] Thomas AG. The development of fracture mechanics for elastomers. Rubber Chemistry and Technology 1994;67:G50–G60.
[7] Ellul MD. Mechanical fatigue. In: Gent A, editor. Engineering with rubber, How to design rubber components. Munich: Carl Hanser Verlag; 1992. [chapter 6].
[8] Lake GJ, Thomas AG. Strength. In: Gent A, editor. Engineering with rubber, How to design rubber components. Munich: Carl Hanser Verlag; 1992. [chapter 5].
[9] Lake GJ. Aspects of fatigue and fracture of rubber. Progress of Rubber Technology 1983;45:89–143.
[10] Gent AN. Strength of elastomers. In: Eirich FR, editor. Science and technology of rubber. New York: Academic Press; 1978. p. 419–54.
[11] Lake GJ. Mechanical fatigue of rubber. Rubber Chemistry and Technology 1972;45:309–28.
[12] Beatty JR. Fatigue of rubber. Rubber Chemistry and Technology 1964;37:1341–64.
[13] Legorju-Jago K, Bathias C. Fatigue initiation and propagation in natural and synthetic rubbers. 2nd International Conference on Fatigue of Composites, Jun 4–7, 2000; Williamsburg, VA.
[14] Mars WV, Fatemi A. Factors that affect the fatigue life of rubber: a literature survey. Rubber Chemistry and Technology; in press.
[15] Wöhler A. Wöhler's experiments on the strength of metals. Engineering 1867;2:160.
[16] Cadwell SM, Merrill RA, Sloman CM, Yost FL. Dynamic fatigue life of rubber. Industrial and Engineering Chemistry, Analytical Edition 1940;12:19–23 [reprinted in Rubber Chemistry and Technology 1940;13:304–315].
[17] Fielding JH. Flex life and crystallization of synthetic rubber. Industrial and Engineering Chemistry 1943;35:1259–61.
[18] Standard test method for rubber property — extension cycling fatigue, ASTM D 4482-85; 1994.
[19] Klenke D, Beste A. Ensurance of the fatigue-life of metal–rubber components. Kautschuk und Gummi Kunstoffe 1987;40:1067–71.
[20] Andre N, Cailletaud G, Piques R. Haigh diagram for fatigue crack initiation prediction of natural rubber components. Kautschuk Und Gummi Kunstoffe 1999;52:120–3.
[21] Handbook of molded and extruded rubber, 3rd ed. Goodyear Tire and Rubber Company, 1969.
[22] Roberts BJ, Benzies JB. The relationship between uniaxial and equibiaxial fatigue in gum and carbon black filled vulcanizates. In: Proceedings of Rubbercon '77, vol. 2.1. 1977. pp. 2.1–2.13.
[23] Roach JF. Crack growth in elastomers under biaxial stresses. Ph.D. Dissertation. USA: University of Akron; May 1982.
[24] Ro HS. Modeling and interpretation of fatigue failure initiation in rubber related to pneumatic tires. Ph.D. Dissertation. USA: Purdue University; 1989.
[25] Rivlin RS, Thomas AG. Rupture of rubber. I. Characteristic energy for tearing. Journal of Polymer Science 1953;10:291–318.
[26] Thomas AG. Rupture of rubber. II. The strain concentration at an inclusion. Journal of Polymer Science 1955;18:177–88.
[27] Greensmith HW, Thomas AG. Rupture of rubber. III. Determination of tear properties. Journal of Polymer Science 1955;18:189–200.
[28] Greensmith HW. Rupture of rubber. IV. Tear properties of vulcanizates containing carbon black. Journal of Polymer Science 1956;21:175–87.
[29] Thomas AG. Rupture of rubber. V. Cut growth in natural rubber vulcanizates. Journal of Polymer Science 1958;31:467–80.
[30] Thomas AG. Rupture of rubber. VI. Further experiments on the tear criterion. Journal of Polymer Science 1960;3(8):168–74.
[31] Greensmith HW. Rupture of rubber. VII. Effect of rate of exten-

sion in tensile tests. Journal of Polymer Science 1960;3(8):175–82.
[32] Greensmith HW. Rupture of rubber. VIII. Comparisons of tear and tensile rupture measurements. Journal of Polymer Science 1960;3(8):183–93.
[33] Mullins L. Rupture of rubber. IX. Role of hysteresis in the tearing of rubber. Transactions of the Institution of the Rubber Industry 1959;35:213–22.
[34] Greensmith HW. Rupture of rubber. X. The change in stored energy on making a small cut in a test piece held in simple extension. Journal of Polymer Science 1963;7:993–1002.
[35] Greensmith HW. Rupture of rubber. XI. Tensile rupture and crack growth in a noncrystallizing rubber. Journal of Polymer Science 1964;8:1113–28.
[36] Gent AN, Lindley PB, Thomas AG. Cut growth and fatigue of rubbers. I. The relationship between cut growth and fatigue. Journal of Applied Polymer Science 1964;8:455–66 [reprinted in Rubber Chemistry and Technology 1965;38:292–300].
[37] Lake GJ, Lindley PB. Cut growth and fatigue of rubbers. II. Experiments on a noncrystallizing rubber. Journal of Applied Polymer Science 1964;8:455–66 [reprinted in Rubber Chemistry and Technology 1965;38:301–313].
[38] Lee MP, Moet A. Analysis of fatigue crack propagation in NR/BR rubber blend. Rubber Chemistry and Technology 1993;66:304–16.
[39] Hocine NA, Abdelaziz MN, Mesmacque G. Experimental and numerical investigation on single specimen methods of determination of J in rubber materials. International Journal of Fracture 1998;94:321–38.
[40] James AG. Fatigue in elastomers. Kautschuk und Gummi Kunststoffe 1973;26(3):87–91.
[41] Hirakawa H, Urano F. Analysis of the fatigue process. International Polymer Science and Techology 1976;3(10):T18–T26.
[42] Hirakawa H, Urano F, Kida M. Analysis of fatigue process of rubber vulcanizates. Rubber Chemistry and Technology 1978;51(2):201–14.
[43] Findley WN, Mathur PN, Szczepanski E, Temel AO. Energy versus stress theories for combined stress — fatigue experiment using a rotating disc. Journal of Basic Engineering, Trans. ASME 1961;83(D):10–4.
[44] Findley WN. Combined stress fatigue strength of 76S–T61 aluminum alloy with superimposed mean stresses and corrections for yielding. Tech Notes NACA, Wash 2924; 1953.
[45] Brown MW, Miller KJ. A theory for fatigue failure under multiaxial stress–strain conditions. Proceedings of the Institution of Mechanical Engineers, Part C, Journal of Mechanical Engineering Science 1973;187(65):745–55.
[46] Findley WN. Theories relating to fatigue of materials under combinations of stress. In: Colloquium on Fatigue, IUTAM, Stockholm. 1955. p. 35–42.
[47] Garud YS. A new approach to evaluation of fatigue under multiaxial loadings. Engineering Material Technology, Trans. ASME 1981;103:118–25.
[48] Bannantine JA, Socie DF. Multiaxial fatigue life estimation techniques. In: Mitchell MR, Landgraf RW, editors. Advances in fatigue lifetime predictive techniques. ASTM STP 1122. Philadelphia (PA): American Society for Testing and Materials; 1992. p. 249–58.
[49] Socie DF. Critical plane approaches for multiaxial fatigue damage assessment. In: McDowell DL, Ellis R, editors. Advances in multiaxial fatigue. ASTM STP 1191. Philadelphia (PA): American Society for Testing and Materials; 1993. p. 7–36.
[50] Fatemi A, Socie DF. A critical plane approach to multiaxial fatigue damage including out-of-phase loading. Fatigue and Fracture of Engineering Materials and Structures 1988;11(3):149–65.
[51] Fatemi A, Kurath P. Multiaxial fatigue life predictions under the influence of mean-stress. Engineering Material Technology, Trans ASME 1988;110:380–8.
[52] McDiarmid DL. A general criterion of high cycle multiaxial fatigue failure. Fatigue and Fracture of Engineering Materials and Structures 1991;14:429–53.
[53] Glinka G, Wang G, Plumtree A. Mean stress effects in multiaxial fatigue. Fatigue and Fracture of Engineering Materials and Structures 1995;18:755–64.
[54] Glinka G, Shen G, Plumtree A. A multiaxial fatigue strain energy density parameter related to the critical fracture plane. Fatigue and Fracture of Engineering Materials and Structures 1995;18:37–46.
[55] Fatemi A, Yang L. Cumulative fatigue damage and life prediction theories: a survey of the state of the art for homogeneous materials. International Journal of Fatigue 1998;20:9–34.
[56] Pan WF, Hung CY, Chen LL. Fatigue life estimation under multiaxial loadings. International Journal of Fatigue 1999;21:3–10.
[57] Socie DF. Multiaxial fatigue damage models. Journal of Engineering Materials and Technology, Trans. ASME 1987;109:293–8.
[58] Mars WV, Fatemi A. Criteria for fatigue crack nucleation in rubber under multiaxial loading. In: Besdo, Schuster, Ihlemann, editors. Constitutive models for rubber II: Proceedings of the European Conference on Constitutive Models for Rubber II. Hannover (Germany): Swets and Zeitlinger; 2001. p. 213–22.
[59] Lake GJ, Lindley PB. The mechanical fatigue limit for rubber. Journal of Applied Polymer Science 1965;9:1233–51 [reprinted in Rubber Chemistry and Technology 1966;39:348–364].
[60] Lake GJ, Lindley PB. Fatigue of rubber at low strains. Journal of Applied Polymer Science 1966;10:343–51.
[61] Grosch K. Rolling resistance and fatigue life of tires. Rubber Chemistry and Technology 1988;61:42–63.
[62] DeEskinazi J, Ishihara K, Volk H, Warholic TC. Towards predicting relative belt edge endurance with the finite element method. Tire Science and Technology 1990;18:216–35.
[63] Oh HL. A fatigue-life model of a rubber bushing. Rubber Chemistry and Technology 1980;53:1226–38.
[64] Yamashita S. Selecting damping materials (service environment, strain and endurance). International Polymer Science and Technology 1992;19(4):T/41–T/56.
[65] Inglis CE. Stresses in a plate due to the presence of cracks and sharp corners. Transactions of the Institute of Naval Architects 1913;55:219–41.
[66] Griffith AA. The phenomena of rupture and flow in solids. Philosophical Transactions of the Royal Society of London, Series A 1920;221:163–98.
[67] Irwin GR. Fracture dynamics. In: Fracturing of metals. American Society for Metals; 1948. p. 147–66.
[68] Irwin GR. Onset of fast crack propagation in high strength steel and aluminum alloys. Sagamore Research Conference Proceedings 1956;2:289–305.
[69] Irwin GR. Analysis of stresses and strains near the end of a crack traversing a plate. Journal of Applied Mechanics, Trans ASME 1957;24:361–4.
[70] Rice JR. A path-independent integral and the approximate analysis of strain concentration by notches and cracks. Journal of Applied Mechanics, Trans ASME 1968;35:379–86.
[71] Harwood JAC, Payne AR. Hysteresis and strength of rubbers. Journal of Applied Polymer Science 1968;12:889–901.
[72] Lindley PB. Energy for crack growth in model rubber components. Journal of Strain Analysis 1972;7:132–40.
[73] Ahagon A, Gent AN. Threshold fracture energies for elastomers. Journal of Polymer Science, Polymer Physics Edition 1975;13:1903–11.
[74] Gent AN, Tobias RH. Threshold tear strength of elastomers. Journal of Polymer Science, Polymer Physics Edition 1982;20:2051–8.
[75] Lake GJ, Yeoh OH. Effect of crack tip sharpness on the strength

of vulcanized rubbers. Journal of Polymer Science, Part B, Polymer Physics 1987;25:1157–90.
[76] Chow CL, Wang J, Tse PN. Rubber fracture characterization using J-integral. Tire Science and Technology 1988;16:44–60.
[77] Hamed GR. Energy dissipation and the fracture of rubber vulcanizates. Rubber Chemistry and Technology 1991;64:493–500.
[78] Stommel M, Haberstroh E, Reiter R. Finite element analysis — an efficient tool for failure prediction in technical rubber parts. ANTEC 1997;3:3351–5.
[79] Paris PC, Gomez MP, Anderson WP. A rational analytic theory of fatigue. The Trend in Engineering 1961;13:9–14.
[80] Neumeister L, Koenig JL. Crack analysis of unfilled natural rubber using IR microscopy. Rubber Chemistry and Technology 1997;70:271–82.
[81] Young DG. Application of fatigue methods based on fracture mechanics for tire compound development. Rubber Chemistry and Technology 1990;63:567–81.
[82] Nait-Abdelaziz M, Ghfiri H, Mesmacque G, Neviere RG. The J Integral as a fracture criterion of rubber-like materials: a comparative study between a compliance method and an energy separation method. In: Erdogan F editor. Fracture Mechanics, vol. 25, ASTM STP 1220. Philadelphia (PA): American Society for Testing and Materials; 1995. p. 380–96.
[83] Kadir A, Thomas AG. Tear behavior of rubbers over a wide range of rates. Rubber Chemistry and Technology 1981;54:15–23.
[84] Mazich KA, Morman KN, Oblinger FG, Pan TY, Killgoar PC. The effect of specimen thickness on the tearing energy of a gum vulcanizate. Rubber Chemistry and Technology 1989;62:850–62.
[85] Young DG. Dynamic property and fatigue propagation research on tire sidewall and model compounds. Rubber Chemistry and Technology 1985;58:785–805.
[86] Stephens RI, Fatemi A, Stephens RR, Fuchs HO. Metal fatigue in engineering, 2nd ed. New York: Wiley-Interscience, 2000.
[87] Aglan H, Moet A. The resistance of rubber compounds to brittle crack propagation. Rubber Chemistry and Technology 1989;62:98–106.
[88] Chudnovsky A, Moet A. Thermodynamics of translational crack layer propagation. Journal of Materials Science 1985;20:630–5.
[89] Chudnovsky A, Moet A. A theory for crack layer propagation in polymers. Journal of Elastomers and Plastics 1986;18:50–5.
[90] Anderson TL. Fracture mechanics — fundamentals and applications. CRC Press, 1995.
[91] Dugdale DS. Yielding of steel sheets containing slits. Journal of Mechanics and Physics of Solids 1960;8:100–4.
[92] Barenblatt GI. The mathematical theory of equilibrium cracks in brittle fracture. Advances in Applied Mechanics 1962;7:55–129.
[93] Chow CL, Lu TJ. Fatigue crack propagation in metals and polymers with a unified formulation. Tire Science and Technology 1992;20:106–29.
[94] Chow CL, Lu TJ. On the cyclic J-integral applied to fatigue cracking. International Journal of Fracture 1989;40:53–9.
[95] Kearny VE, Engle RM, Forman RG. Numerical analysis of crack propagation in cyclic loaded structures. Journal of Basic Engineering 1967;89:459–64.
[96] Weertman J. Rate of growth of fatigue cracks calculated from the theory of infinitesimal dislocations distributed on a plane. International Journal of Fracture Mechanics 1966;2:460–7.
[97] Bathias C, Le Gorju K, Lu C, Menabeuf L. Fatigue crack growth damage in elastomeric materials. In: Piascik RS, Newman JC, Dowling NE editors. Fatigue and fracture, vol. 27, ASTM STP 1296. Philadelphia (PA): American Society for Testing and Materials; 1997. p. 505–13.
[98] Lindley PB. Relation between hysteresis and the dynamic crack growth resistance of natural rubber. International Journal of Fracture 1973;9:449–61.
[99] Andrews EH. Stresses at a crack in an elastomer. Proceedings of the Physical Society 1961;77:483–98.
[100] Andrews EH. Rupture propagation in hysteresial materials: stress at a notch. Journal of Mechanics and Physics of Solids 1963;11:231–42.
[101] Knauss WG. Stresses near a crack in a rubber sheet. Experimental Mechanics 1968;8:177–81.
[102] Lee RF, Donovan JA. J-integral and crack opening displacement as crack initiation criteria in rubber. Rubber Chemistry and Technology 1986;59:787–99.
[103] Morman KN, Mazich KA, Oblinger FG, Zhang F, Killgoar PC. Critical tearing energy in a circumferentially-cracked circular cylinder of rubber under finite deformation. International Journal of Fracture 1992;53:129–57.
[104] Chang SJ. Path-independent integral for rupture of perfectly elastic materials. Journal of Applied Mathematics and Physics (ZAMP) 1972;23:149–52.
[105] Treloar LRG. Physics of rubber elasticity. London: Oxford University Press, 1958.
[106] Roland CM. Network recovery from uniaxial extension. I. Elastic equilibrium. Rubber Chemistry and Technology 1989;62:863–79.
[107] Elyashevich GK. Thermodynamics and kinetics of orientational crystallization of flexible-chain polymers. Advances in Polymer Science 1982;43:205–45.
[108] Choi IS, Roland CM. Strain-crystallization of guayle and hevea rubbers. Rubber Chemistry and Technology 1997;70:202–9.
[109] Luchini JR. A mathematical model for hysteresis in filled rubber. Ph.D. Dissertation. USA: University of Michigan; 1977.
[110] Becker A, Dorsch V, Kaliske M, Rothert H. A material model for simulating the hysteretic behavior of filled rubber for rolling tires. Tire Science and Technology 1998;26:132–48.
[111] Coveney VA, Jamil S, Johnson DE, Keavey MA, Menger C, Williams HT. Implementation in finite element analysis of a triboelastic law for dynamic behaviour of filled elastomers. In: Boast D, Coveny VA, editors. Finite element analysis of elastomers. United Kingdom: IMechE; 1999.
[112] Mullins L. Softening of rubber by deformation. Rubber Chemistry and Technology 1969;42:339–62.
[113] Roland CM. Network recovery from uniaxial extension. II. The origin of the Mullins effect. Rubber Chemistry and Technology 1989;62:880–95.
[114] Gent AN, Hindi M. Effect of oxygen on the tear strength of elastomers. Rubber Chemistry and Technology 1990;63:123–34.
[115] Andrews EH. A generalized theory of fracture mechanics. Journal of Materials Science 1974;9:887–94.
[116] Lindley PB. Non-relaxing crack growth and fatigue in a non-crystallizing rubber. Rubber Chemistry and Technology 1974;47:1253–64.
[117] Schapery RA. A theory of crack initiation and growth in viscoelastic media. I: Theoretical development. International Journal of Fracture 1975;11:141–59.
[118] Schapery RA. A theory of crack initiation and growth in viscoelastic media. II. Approximate methods of analysis. International Journal of Fracture 1975;11:369–88.
[119] Schapery RA. A theory of crack initiation and growth in viscoelastic media. III. Analysis of continuous growth. International Journal of Fracture 1975;11:549–62.
[120] Schapery RA. Correspondence principles and a generalized J integral for large deformation and fracture analysis of viscoelastic media. International Journal of Fracture 1984;25:195–223.
[121] Schapery RA. On some path independent integrals and their use in fracture of nonlinear viscoelastic media. International Journal of Fracture 1990;42:189–207.
[122] Lindley PB, Stevenson A. Fatigue resistance of natural rubber in compression. Rubber Chemistry and Technology 1982;55:337–51.
[123] Lake GJ, Thomas AG. The strength of highly elastic materials.

Proceedings of the Royal Society of London, Series A. Mathematical and Physical Sciences 1967;300:108–19.
[124] Mazich KA, Samus MA, Smith CA, Rossi G. Threshold fracture of lightly cross-linked networks. Macromolecules 1991;24:2766–9.
[125] Hamed G. Molecular aspects of the fatigue and fracture of rubber. Rubber Chemistry and Technology 1994;67:529–36.
[126] Lake GJ, Clapson BE. Truck tire groove cracking, theory and practice. Rubber Journal 1970;152(12):36–52 [reprinted in Rubber Chemistry and Technology 1971;44:1186–1202].
[127] Southern E, Thomas AG. Studies of rubber abrasion. Plastic and Rubber Materials and Applications 1978;3:133–8.
[128] Huang YS, Yeoh OH. Crack initiation and propagation in model cord–rubber composites. Rubber Chemistry and Technology 1989;62:709–31.
[129] Choi IS, Roland CM, Bissonette LC. An elastomeric ejection system. Rubber Chemistry and Technology 1994;67:892–903.
[130] Medri G, Strozzi A. Stress–strain fields in compressed elastomeric seals and their extension to fracture mechanics. Rubber Chemistry and Technology 1986;59:709–21.
[131] Lindley PB. Ozone attack at a rubber–metal bond. Journal of the Institute of the Rubber Industry 1971;5:243–8.
[132] Stevenson A, Harris JA, Hawkes J, Becker E, Miller T, McMullen R. Fatige life calculations for elastomeric engineering components. Paper 44E4, Rubbercon ’95, Gothenburg, Sweden; 1995.
[133] Gunderson RH, Stevenson A, Harris JA, Gahagan P, Chilton TS. Fatigue life of TLP flexelements. In: Offshore Technology Conference, May 1992; Houston, TX. 1992. p. 225–35 [Paper No. OTC 6898].
[134] Stevenson A, Malek KA. On the puncture mechanics of rubber. Rubber Chemistry and Technology 1994;67:743–60.
[135] Ebbott TG. Application of finite element-based fracture mechanics analysis to cord–rubber structures. Tire Science and Technology 1996;24:220–35.
[136] Wei YT, Tian ZH, Du XW. A finite element model for the rolling loss prediction and fracture analysis of radial tires. Tire Science and Technology 1999;27:250–76.
[137] Fukahori Y. Estimation of fatigue life in elastomers. Theoretical analysis and applications of S–N curves with fracture mechanics. International Polymer Science and Technology 1985;12(10):T/37–T/45.
[138] Fielding-Russell GS, Rongone RL. Fatiguing of rubber–rubber interfaces. Rubber Chemistry and Technology 1983;56:838–44.
[139] Choi IS, Roland CM. Intrinsic defects and the failure properties of cis-1,4-polyisoprenes. Rubber Chemistry and Technology 1996;69:591–9.
[140] Braden M, Gent AN. The attack of ozone on stretched rubber vulcanizates. II. Conditions for cut growth. Journal of Applied Polymer Science 1960;3(7):100–6.
[141] Hamed GR. Effect of crosslink density on the critical flaw size of a simple elastomer. Rubber Chemistry and Technology 1983;56:244–51.
[142] Dizon ES, Hicks AE, Chirico VE. The effect of carbon black parameters on the fatigue life of filled rubber compounds. Rubber Chemistry and Technology 1974;47:231–49.
[143] Bathias C, Le Gorju K, Houel P, Berete YN. Damage characterization of elastomeric composites using X-ray attenuation. In: Reifsnider KL, Dillard DA, Cardon AH, editors. Progress in durability analysis of composite systems: Third International Conference, 1997. Balkema; 1998. p. 103–10.
[144] Tanaka K. Mechanics and micromechanics of fatigue crack propagation. In: Wei RP, Gangloff RP, editors. Fracture mechanics: perspectives and directions, 20th Symposium. ASTM STP 1020. Philadelphia (PA): American Society for Testing and Materials; 1989. p. 151–83.
[145] Kandil FA, Brown MW, Miller KJ. Biaxial low cycle fatigue of 316 stainless steel at elevated temperature. Metals Society, London 1982;280:203–9.
[146] Miller KJ. The two thresholds of fatigue behavior. Fatigue and Fracture of Engineering Materials and Structures 1993;16:931–9.
[147] Kraus G. Reinforcement of elastomers by particulate fillers. In: Science and Technology of Rubber. Academic Press; 1978. p. 339–65.
[148] Roland CM, Smith CR. Defect accumulation in rubber. Rubber Chemistry and Technology 1985;58:806–14.
[149] Morton M. Polymerization. In: Eirich FR, editor. Science and technology of rubber. Academic Press; 1978. p. 23–74.
[150] Chun H, Gent AN. Strength of sulfur-linked elastomers. Rubber Chemistry and Technology 1996;69:577–90.