DEPARTAMENTO DE ENGENHARIA NAVAL E OCEÂNICA – EPUSP

PNV-3414 Dinâmica de Sistemas II

2ª Lista de Exercícios

Os exercícios abaixo são extraídos de Strogatz, S. “Nonlinear Dynamics and Chaos”.
Alguns exercícios envolvem o uso de computador; esses exercícios são opcionais, servindo para complementar o estudo. Para os alunos interessados, sugere-se o emprego do programa Phaser (demonstrado em sala de aula), disponível gratuitamente na internet para teste por 30 dias.

7.1.5 (From polar to Cartesian coordinates) Show that the system $\dot{r} = r(1-r^2)$, $\dot{\theta} = 1$ is equivalent to

$$\dot{x} = x - y - x(x^2 + y^2), \qquad \dot{y} = x + y - y(x^2 + y^2),$$

where $x = r\cos\theta$, $y = r\sin\theta$. (Hint: $\dot{x} = \frac{d}{dt}(r\cos\theta) = \dot{r}\cos\theta - r\dot{\theta}\sin\theta$.)

7.1.8 (A circular limit cycle) Consider $\ddot{x} + a\dot{x}(x^2 + \dot{x}^2 - 1) + x = 0$, where $a > 0$.

a) Find and classify all the fixed points.
b) Show that the system has a circular limit cycle, and find its amplitude and period.
c) Determine the stability of the limit cycle.

d) Give an argument which shows that the limit cycle is unique, i.e., there are no other periodic trajectories.

8.1 Saddle-Node, Transcritical, and Pitchfork Bifurcations

8.1.1 For the following prototypical examples, plot the phase portraits as μ varies:

a) $\dot{x} = \mu x - x^2$, $\dot{y} = -y$ (transcritical bifurcation)
b) $\dot{x} = \mu x + x^3$, $\dot{y} = -y$ (subcritical pitchfork bifurcation)

For each of the following systems, find the eigenvalues at the stable fixed point as a function of μ, and show that one of the eigenvalues tends to zero as $\mu \to 0$.

8.1.2 $\dot{x} = \mu - x^2$, $\dot{y} = -y$

8.1.3 $\dot{x} = \mu x - x^2$, $\dot{y} = -y$

8.1.4 $\dot{x} = \mu x + x^3$, $\dot{y} = -y$

8.1.7 Find and classify all bifurcations for the system $\dot{x} = y - ax$, $\dot{y} = -by + x/(1+x)$.

8.1.11 In a study of isothermal autocatalytic reactions, Gray and Scott (1985) considered a hypothetical reaction whose kinetics are given in dimensionless form by

$$\dot{u} = a(1-u) - uv^2, \qquad \dot{v} = uv^2 - (a+k)v,$$

where $a, k > 0$ are parameters. Show that saddle-node bifurcations occur at $k = -a \pm \frac{1}{2}\sqrt{a}$.

8.2.1 Consider the biased van der Pol oscillator $\ddot{x} + \mu(x^2 - 1)\dot{x} + x = a$. Find the curves in (μ, a) space at which Hopf bifurcations occur.

For each of the following systems, a Hopf bifurcation occurs at the origin when $\mu = 0$. Using a computer, plot the phase portrait and determine whether the bifurcation is subcritical or supercritical.

8.2.5 $\dot{x} = y + \mu x, \quad \dot{y} = -x + \mu y - x^2 y$

8.2.6 $\dot{x} = \mu x + y - x^3, \quad \dot{y} = -x + \mu y + 2y^3$

8.2.7 $\dot{x} = \mu x + y - x^2, \quad \dot{y} = -x + \mu y + 2x^2$

8.2.11 (Degenerate bifurcation, not Hopf) Consider the damped Duffing oscillator $\ddot{x} + \mu\dot{x} + x - x^3 = 0$.

a) Show that the origin changes from a stable to an unstable spiral as μ decreases though zero.

b) Plot the phase portraits for $\mu > 0$, $\mu = 0$, and $\mu < 0$, and show that the bifurcation at $\mu = 0$ is a degenerate version of the Hopf bifurcation.

8.2.12 (Analytical criterion to decide if a Hopf bifurcation is subcritical or supercritical) Any system at a Hopf bifurcation can be put into the following form by suitable changes of variables:

$$\dot{x} = -\omega y + f(x,y), \qquad \dot{y} = \omega x + g(x,y),$$

where f and g contain only higher-order nonlinear terms that vanish at the origin. As shown by Guckenheimer and Holmes (1983, pp. 152–156), one can decide whether the bifurcation is subcritical or supercritical by calculating the sign of the following quantity:

$$\begin{aligned} 16a &= f_{xxx} + f_{xyy} + g_{xxy} + g_{yyy} \\ &\quad + \frac{1}{\omega}\left[f_{xy}(f_{xx} + f_{yy}) - g_{xy}(g_{xx} + g_{yy}) - f_{xx}g_{xx} + f_{yy}g_{yy}\right] \end{aligned}$$

where the subscripts denote partial derivatives evaluated at $(0,0)$. The criterion is: If $a < 0$, the bifurcation is supercritical; if $a > 0$, the bifurcation is subcritical.

a) Calculate a for the system $\dot{x} = -y + xy^2$, $\dot{y} = x - x^2$.

b) Use part (a) to decide which type of Hopf bifurcation occurs for $\dot{x} = -y + \mu x + xy^2$, $\dot{y} = x + \mu y - x^2$ at $\mu = 0$. (Compare the results of Exercises 8.2.2–8.2.4.)

(You might be wondering what a measures. Roughly speaking, a is the coefficient of the cubic term in the equation $\dot{r} = ar^3$ governing the radial dynamics at the bifurcation. Here r is a slightly transformed version of the usual polar coordinate. For details, see Guckenheimer and Holmes (1983) or Grimshaw (1990).)

For each of the following systems, a Hopf bifurcation occurs at the origin when $\mu = 0$. Use the analytical criterion of Exercise 8.2.12 to decide if the bifurcation is sub- or supercritical. Confirm your conclusions on the computer.

8.2.13 $\dot{x} = y + \mu x$, $\quad \dot{y} = -x + \mu y - x^2 y$

8.2.14 $\dot{x} = \mu x + y - x^3$, $\quad \dot{y} = -x + \mu y + 2y^3$

8.2.15 $\dot{x} = \mu x + y - x^2$, $\quad \dot{y} = -x + \mu y + 2x^2$

8.2.16 In Example 8.2.1, we argued that the system $\dot{x} = \mu x - y + xy^2$, $\dot{y} = x + \mu y + y^3$ undergoes a subcritical Hopf bifurcation at $\mu = 0$. Use the analytical criterion to confirm that the bifurcation is subcritical.