Transformada de Laplace de $f(t) = \cos(at)$.

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

Sendo $f(t) = \cos(at)$, $a \in \mathbb{R}$

$$\mathcal{L}\{\cos(at)\} = \int_0^\infty e^{-st} \cos(at)\, dt$$

Teremos que integrar por partes duas vezes.

1ª vez

$$\boxed{\begin{array}{ll} u(t) = \cos(at) & du = -a\,\mathrm{sen}(at)\, dt \\ dv = e^{-st}\, dt & v = -\frac{1}{s} e^{-st} \\ \multicolumn{2}{c}{\int_0^\infty u\, dv = uv\Big|_0^\infty - \int_0^\infty v\, du} \end{array}}$$

$$\mathcal{L}\{\cos(at)\} = -\frac{1}{s} e^{-st} \cos(at)\Big|_0^\infty - \frac{a}{s} \int_0^\infty e^{-st}\, \mathrm{sen}(at)\, dt$$

$$" \quad = -\frac{1}{s}\left[\lim_{t\to\infty}\left(e^{-st}\cos(at)\right) - \lim_{t\to 0}\left(e^{-st}\cos(at)\right)\right] - \frac{a}{s}\int_0^\infty e^{-st}\, \mathrm{sen}(at)\, dt$$

- se $s > 0$ então $\lim_{t\to\infty}\left[e^{-st}\cos(at)\right] = 0$

$\cancel{\mathcal{L}\{\cos(at)\} =}$

- $\lim_{t\to 0}\left[e^{-st}\cos(at)\right] = e^0 \cdot \cos(0) = 1 \cdot 1 = 1$.

$$\boxed{\mathcal{L}\{\cos(at)\} = \frac{1}{s} - \frac{a}{s}\int_0^\infty e^{-st}\, \mathrm{sen}(at)\, dt} \quad (I)$$

②

Para calcular $\int_0^\infty e^{-st} sen(at)\,dt$ vamos ~~faze~~ calcula integral por partes pela segunda vez

<u>2^{da} vez</u>

$$\boxed{\begin{aligned} u &= sen(at) \rightarrow du = a\cos(at)\,dt \\ dv &= e^{-st}dt \rightarrow v = -\tfrac{1}{s}e^{-st} \\ \int_0^\infty u\,dv &= uv\Big|_0^\infty - \int_0^\infty v\,du \end{aligned}}$$

$$\int_0^\infty e^{-st} sen(at)\,dt = -\frac{1}{s}e^{-st} sen(at)\Big|_0^\infty + \frac{a}{s}\int_0^\infty e^{-st}\cos(at)\,dt$$

$$\lim_{t\to\infty}\Big[\underbrace{e^{-st}}_{\substack{\text{tende}\\ \text{a zero}\\ \text{quando } t\to\infty}}\underbrace{sen(at)}_{\text{limitada}}\Big] = 0 \quad \text{quando } s>0$$

$$\lim_{t\to 0}\left[e^{-st} sen(at)\right] = e^0 \cdot sen(0) = 1\cdot 0 = 0$$

$$\int_0^\infty e^{-st} sen(at)\,dt = \frac{a}{s}\underbrace{\int_0^\infty e^{-st}\cos(at)\,dt}$$

$$\boxed{\int_0^\infty e^{-st} sen(at)\,dt = \frac{a}{s}\mathcal{L}\{\cos(at)\}} \quad (II)$$

Voltando com (II) em (I) temos

$$\mathcal{L}\{\cos(at)\} = \frac{1}{s} - \frac{a}{s}\,\frac{a}{s}\mathcal{L}\{\cos(at)\} = \frac{1}{s} - \frac{a^2}{s^2}\mathcal{L}\{\cos(at)\}$$

$$\mathcal{L}\{\cos(at)\} + \frac{a^2}{s^2}\mathcal{L}\{\cos(at)\} = \frac{1}{s}$$

$$\mathcal{L}\{\cos(at)\}\left(1+\frac{a^2}{s^2}\right) = \frac{1}{s}.$$

$$\mathcal{L}\{\cos(at)\}\left(\frac{s^2+a^2}{s^2}\right) = \frac{1}{s}$$

$$\boxed{\mathcal{L}\{\cos(at)\} = \frac{s}{s^2+a^2}, \quad s>0}$$