DEPARTAMENTO DE ENGENHARIA NAVAL E OCEÂNICA – EPUSP

PNV-2322 Dinâmica de Sistemas II

Lista de Exercícios Nº 1

5.1.1 (Ellipses and energy conservation for the harmonic oscillator) Consider the harmonic oscillator $\dot{x} = v$, $\dot{v} = -\omega^2 x$.

a) Show that the orbits are given by ellipses $\omega^2 x^2 + v^2 = C$, where C is any non-negative constant. (Hint: Divide the $\dot{x}$ equation by the $\dot{v}$ equation, separate the v's from the x's, and integrate the resulting separable equation.)

b) Show that this condition is equivalent to conservation of energy.

Write the following systems in matrix form.

5.1.3 $\dot{x} = -y, \ \dot{y} = -x$

5.1.4 $\dot{x} = 3x - 2y, \ \dot{y} = 2y - x$

5.1.5 $\dot{x} = 0, \ \dot{y} = x + y$

5.1.6 $\dot{x} = x, \ \dot{y} = 5x + y$

For each of the following systems, decide whether the origin is attracting, Liapunov stable, asymptotically stable, or none of the above.

a) $\dot{x} = y$, $\dot{y} = -4x$.

b) $\dot{x} = 2y$, $\dot{y} = x$

c) $\dot{x} = 0$, $\dot{y} = x$

d) $\dot{x} = 0$, $\dot{y} = -y$

e) $\dot{x} = -x$, $\dot{y} = -5y$

f) $\dot{x} = x$, $\dot{y} = y$

5.2.1 Consider the system $\dot{x} = 4x - y$, $\dot{y} = 2x + y$.

a) Write the system as $\dot{\mathbf{x}} = A\mathbf{x}$. Show that the characteristic polynomial is $\lambda^2 - 5\lambda + 6$, and find the eigenvalues and eigenvectors of A.

b) Find the general solution of the system.

c) Classify the fixed point at the origin.

d) Solve the system subject to the initial condition $(x_0, y_0) = (3, 4)$.

5.2.2 (Complex eigenvalues) This exercise leads you through the solution of a linear system where the eigenvalues are complex. The system is $\dot{x} = x - y$, $\dot{y} = x + y$.

a) Find A and show that it has eigenvalues $\lambda_1 = 1 + i$, $\lambda_2 = 1 - i$, with eigenvectors $\mathbf{v}_1 = (i, 1)$, $\mathbf{v}_2 = (-i, 1)$. (Note that the eigenvalues are complex conjugates, and so are the eigenvectors—this is always the case for real A with complex eigenvalues.)
b) The general solution is $\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2$. So in one sense we're done! But this way of writing $\mathbf{x}(t)$ involves complex coefficients and looks unfamiliar. Express $\mathbf{x}(t)$ purely in terms of real-valued functions. (Hint: Use $e^{i\omega t} = \cos \omega t + i \sin \omega t$ to rewrite $\mathbf{x}(t)$ in terms of sines and cosines, and then separate the terms that have a prefactor of i from those that don't.)

5.2.12 (*LRC* circuit) Consider the circuit equation $L\ddot{I} + R\dot{I} + I/C = 0$, where $L, C > 0$ and $R \geq 0$.

a) Rewrite the equation as a two-dimensional linear system.
b) Show that the origin is asymptotically stable if $R > 0$ and neutrally stable if $R = 0$.
c) Classify the fixed point at the origin, depending on whether $R^2 C - 4L$ is positive, negative, or zero, and sketch the phase portrait in all three cases.

5.2.13 (Damped harmonic oscillator) The motion of a damped harmonic oscillator is described by $m\ddot{x} + b\dot{x} + kx = 0$, where $b > 0$ is the damping constant.

a) Rewrite the equation as a two-dimensional linear system.
b) Classify the fixed point at the origin and sketch the phase portrait. Be sure to show all the different cases that can occur, depending on the relative sizes of the parameters.
c) How do your results relate to the standard notions of overdamped, critically damped, and underdamped vibrations?

EXAMPLE 6.5.2:

Consider a particle of mass $m=1$ moving in a double-well potential $V(x)=-\frac{1}{2}x^2+\frac{1}{4}x^4$. Find and classify all the equilibrium points for the system. Then plot the phase portrait and interpret the results physically.

Solution: The force is $-dV/dx = x - x^3$, so the equation of motion is

$$\ddot{x} = x - x^3.$$

This can be rewritten as the vector field

$$\dot{x} = y$$
$$\dot{y} = x - x^3$$

where y represents the particle's velocity. Equilibrium points occur where $(\dot{x}, \dot{y}) = (0,0)$. Hence the equilibria are $(x^*, y^*) = (0,0)$ and $(\pm 1, 0)$. To classify these fixed points we compute the Jacobian:

$$A = \begin{pmatrix} 0 & 1 \\ 1 - 3x^2 & 0 \end{pmatrix}.$$

At (0,0), we have $\Delta = -1$, so the origin is a saddle point. But when $(x^*, y^*) = (\pm 1, 0)$, we find $\tau = 0$, $\Delta = 2$; hence these equilibria are predicted to be centers.

At this point you should be hearing warning bells—in Section 6.3 we saw that small nonlinear terms can easily destroy a center predicted by the linear approximation. But that's not the case here, because of energy conservation. The trajectories are closed curves defined by the ***contours*** of constant energy, i.e.,

$$E = \tfrac{1}{2} y^2 - \tfrac{1}{2} x^2 + \tfrac{1}{4} x^4 = \text{constant}.$$

Figure 6.5.1 shows the trajectories corresponding to different values of E. To decide which way the arrows point along the trajectories, we simply compute the vector $(\dot{x}, \dot{y})$ at a few convenient locations. For example, $\dot{x} > 0$ and $\dot{y} = 0$ on the positive y-axis, so the motion is to the right. The orientation of neighboring trajectories follows by continuity.

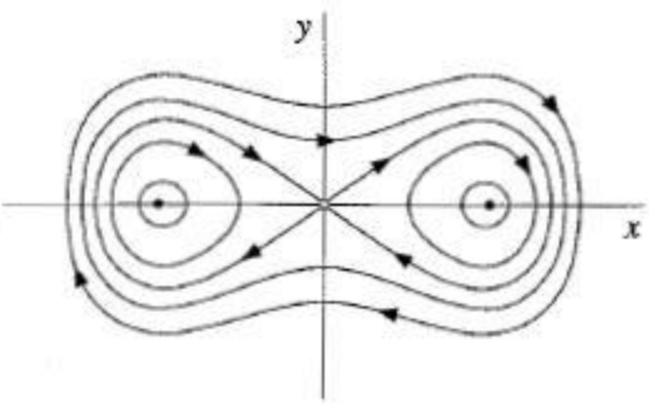

Figure 6.5.1

As expected, the system has a saddle point at (0,0) and centers at (1,0) and (−1,0). Each of the neutrally stable centers is surrounded by a family of small closed orbits. There are also large closed orbits that encircle all three fixed points.

Thus solutions of the system are typically *periodic*, except for the equilibrium solutions and two very special trajectories: these are the trajectories that appear to start and end at the origin. More precisely, these trajectories approach the origin as $t \to \pm\infty$. Trajectories that start and end at the same fixed point are called ***homoclinic orbits***. They are common in conservative systems, but are rare otherwise. Notice that a homoclinic orbit does *not* correspond to a periodic solution, because the trajectory takes forever trying to reach the fixed point.

Finally, let's connect the phase portrait to the motion of an undamped particle in a double-well potential (Figure 6.5.2).

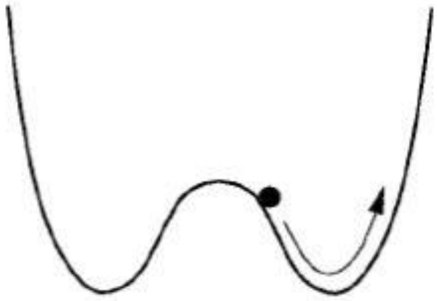

Figure 6.5.2

The neutrally stable equilibria correspond to the particle at rest at the bottom of one of the wells, and the small closed orbits represent small oscillations about these equilibria. The large orbits represent more energetic oscillations that repeatedly take the particle back and forth over the hump. Do you see what the saddle point and the homoclinic orbits mean physically? ■

For each of the following systems, find the fixed points, classify them, sketch the neighboring trajectories, and try to fill in the rest of the phase portrait.

6.3.1 $\dot{x} = x - y$, $\dot{y} = x^2 - 4$

6.3.2 $\dot{x} = \sin y$, $\dot{y} = x - x^3$

6.3.3 $\dot{x} = 1 + y - e^{-x}$, $\dot{y} = x^3 - y$

6.3.4 $\dot{x} = y + x - x^3$, $\dot{y} = -y$

6.3.5 $\dot{x} = \sin y$, $\dot{y} = \cos x$

6.3.6 $\dot{x} = xy - 1$, $\dot{y} = x - y^3$

Fonte: Strogatz, S. H. "Nonlinear Dynamic and Chaos", Perseus Books, 1994.