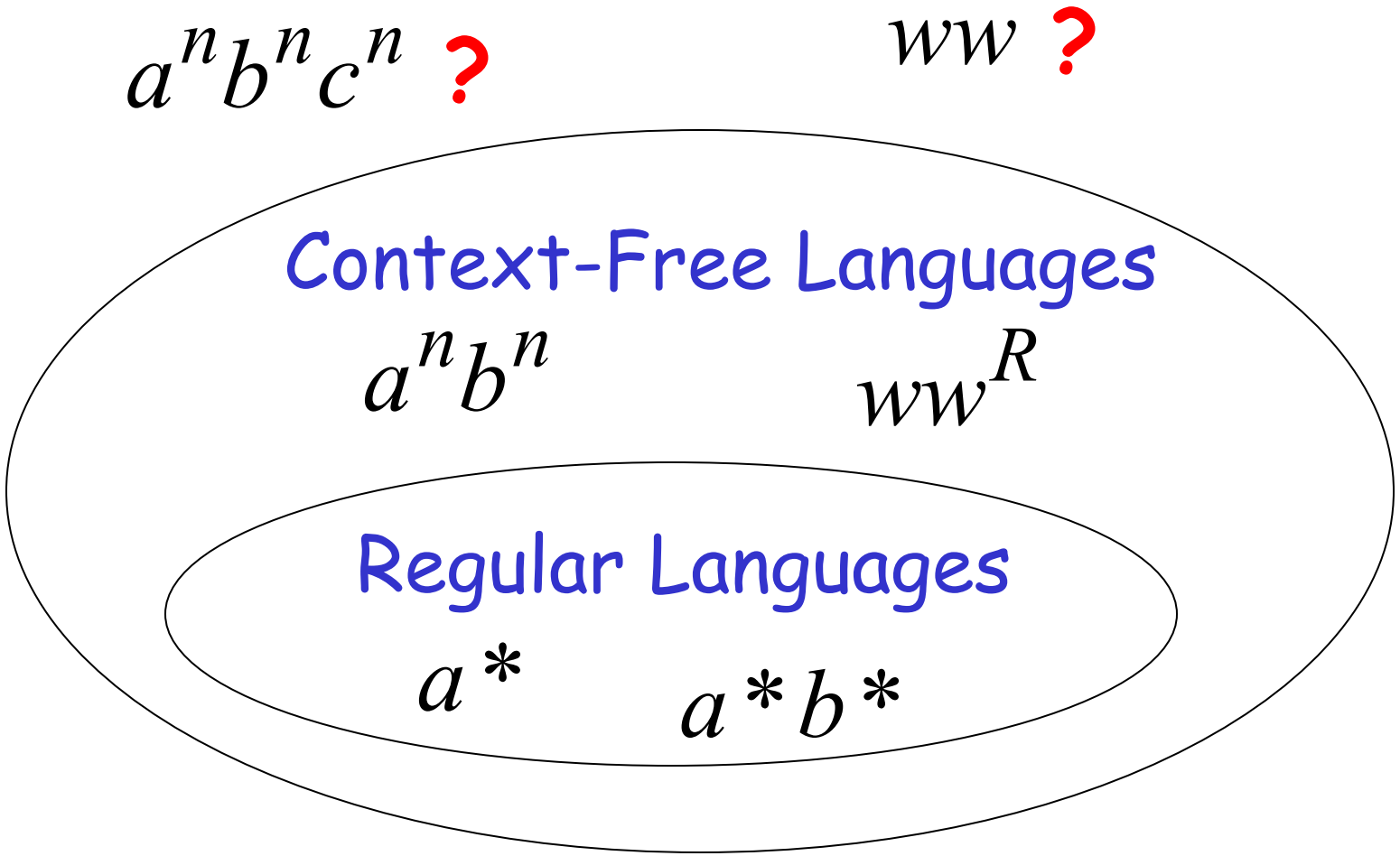
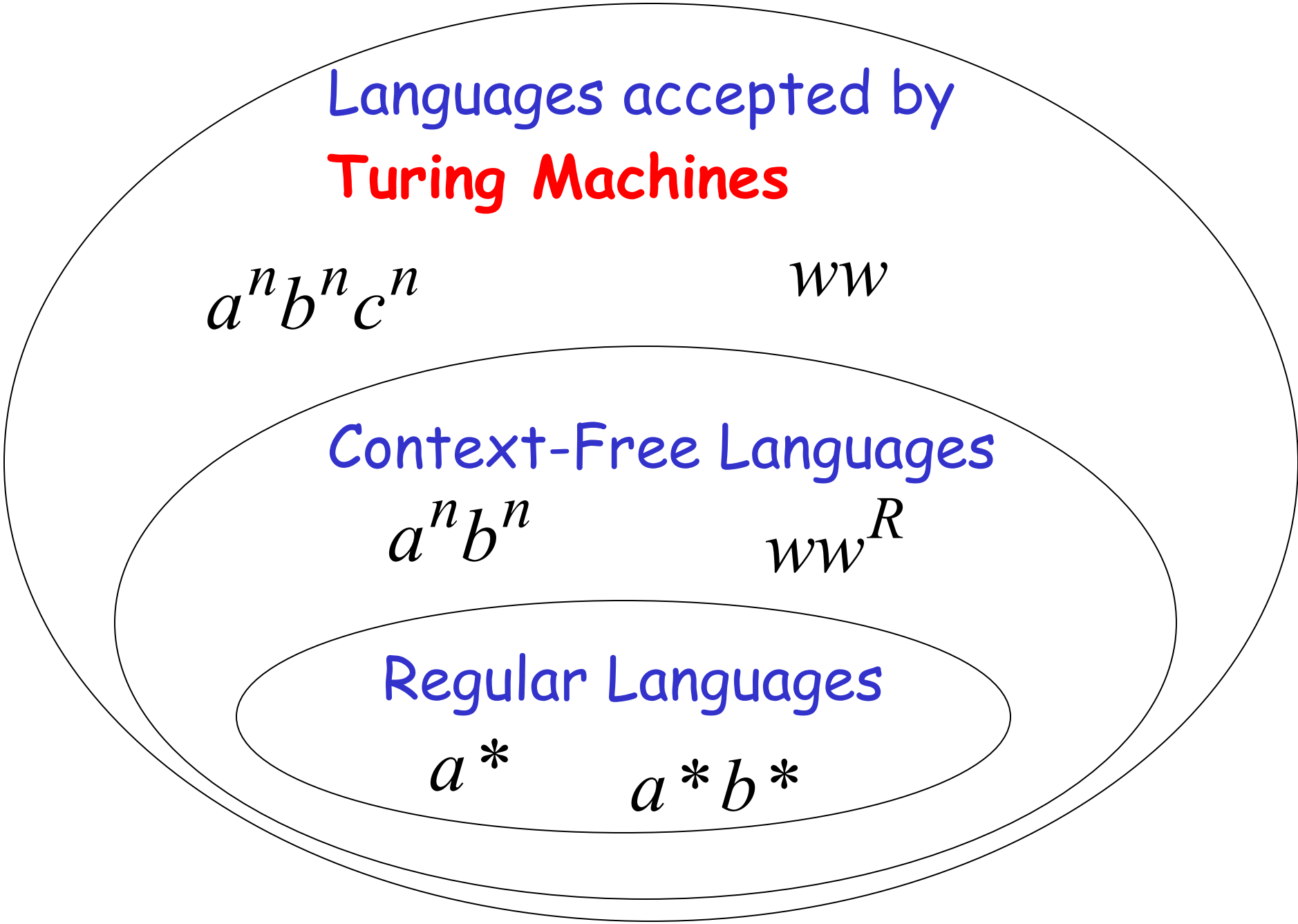
Turing Machines

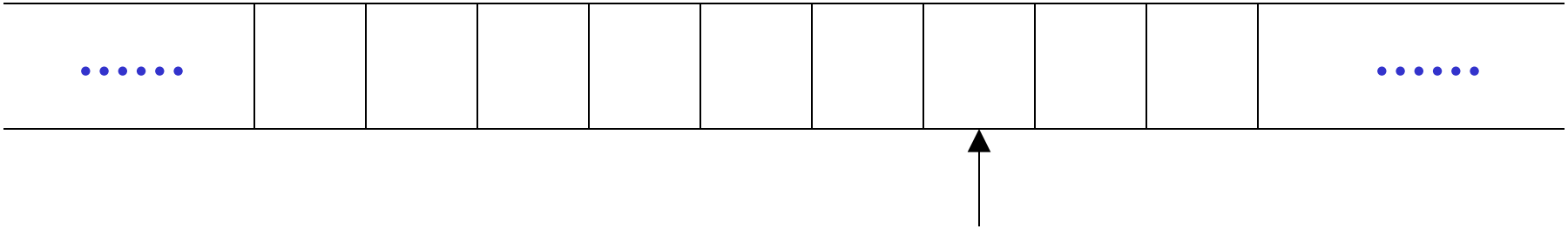
The Language Hierarchy




Languages accepted by
Turing Machines
$a^n b^n c^n$
ww
Context-Free Languages
$a^n b^n$
ww^R
Regular Languages
$a*$
$a*b*$

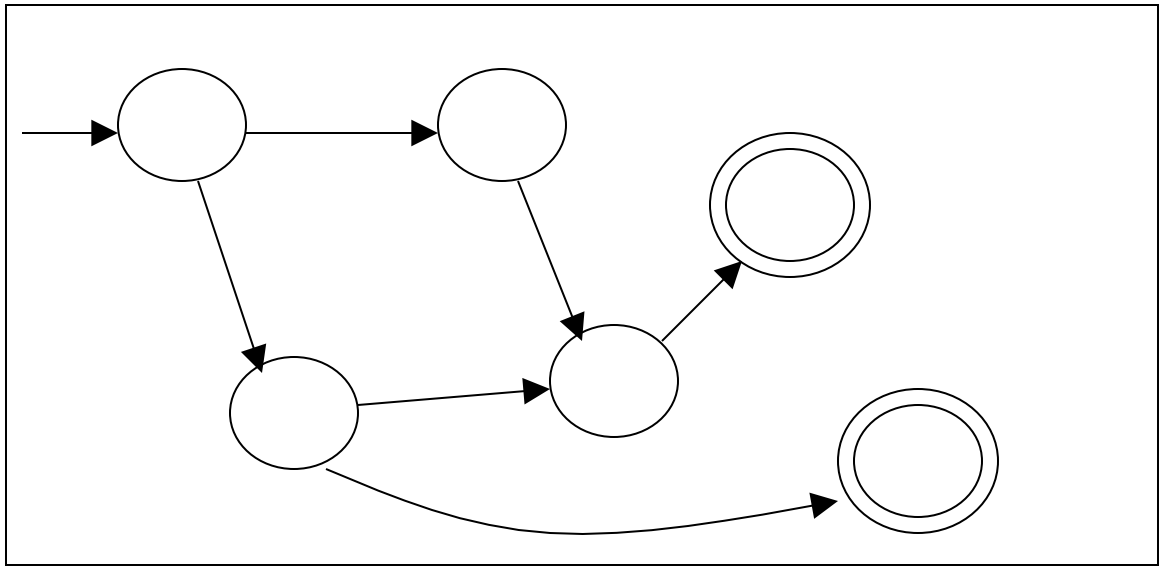
A Turing Machine

Tape



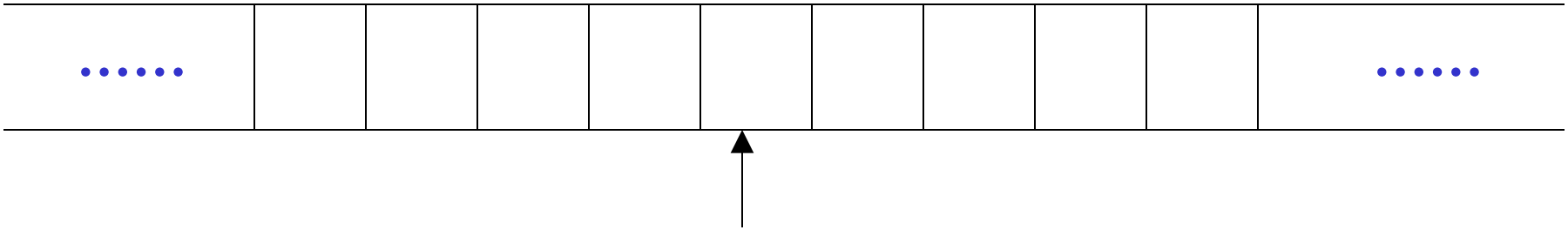
Read-Write head

Control Unit



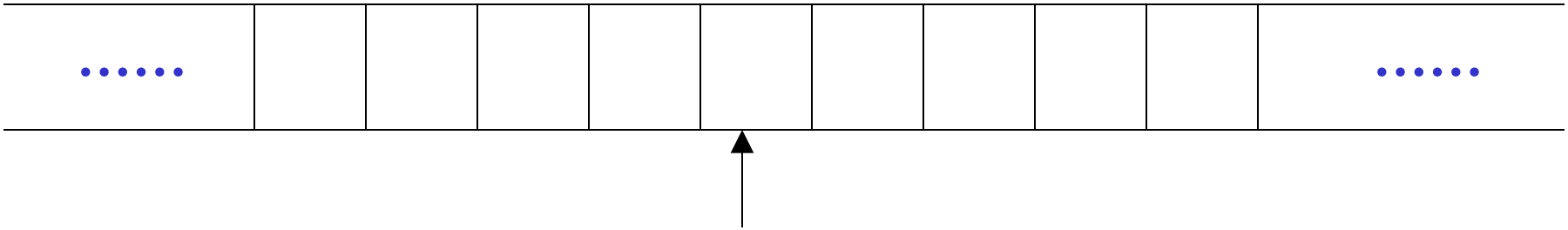
The Tape

No boundaries -- infinite length



Read-Write head

The head moves Left or Right



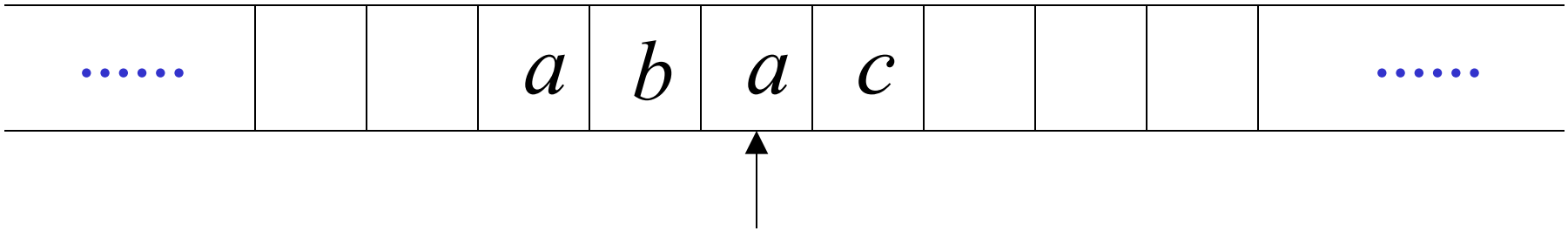
Read-Write head

The head at each transition (time step):

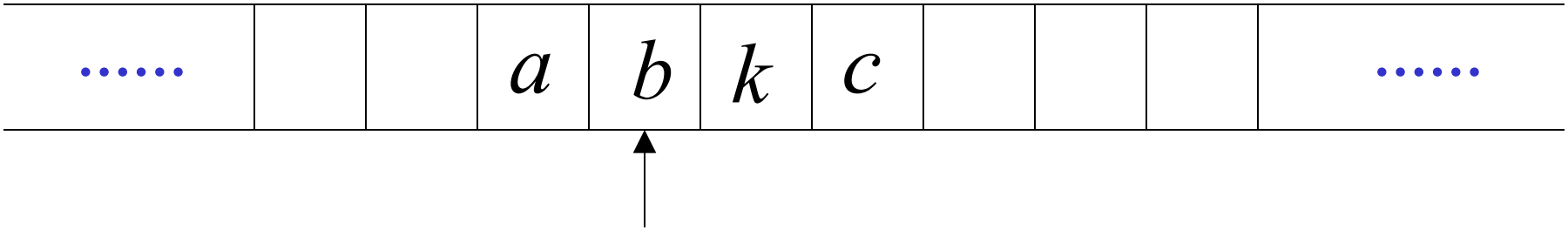
1. Reads a symbol
2. Writes a symbol
3. Moves Left or Right

Example:

Time 0

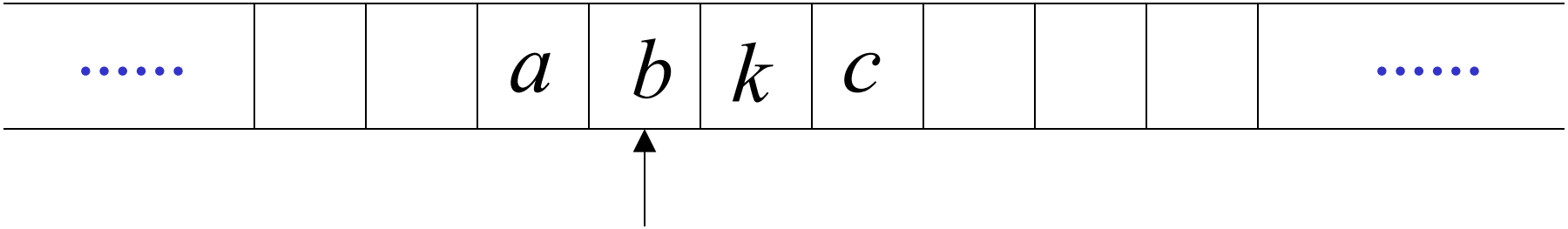


Time 1

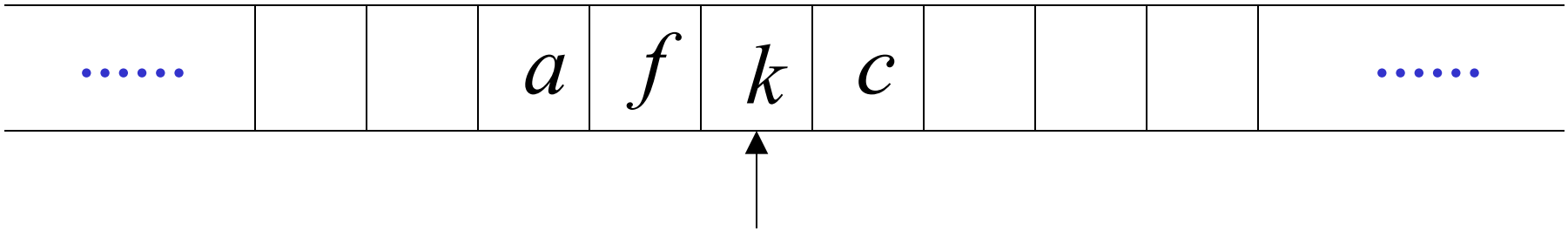


1. Reads a
2. Writes k
3. Moves Left

Time 1

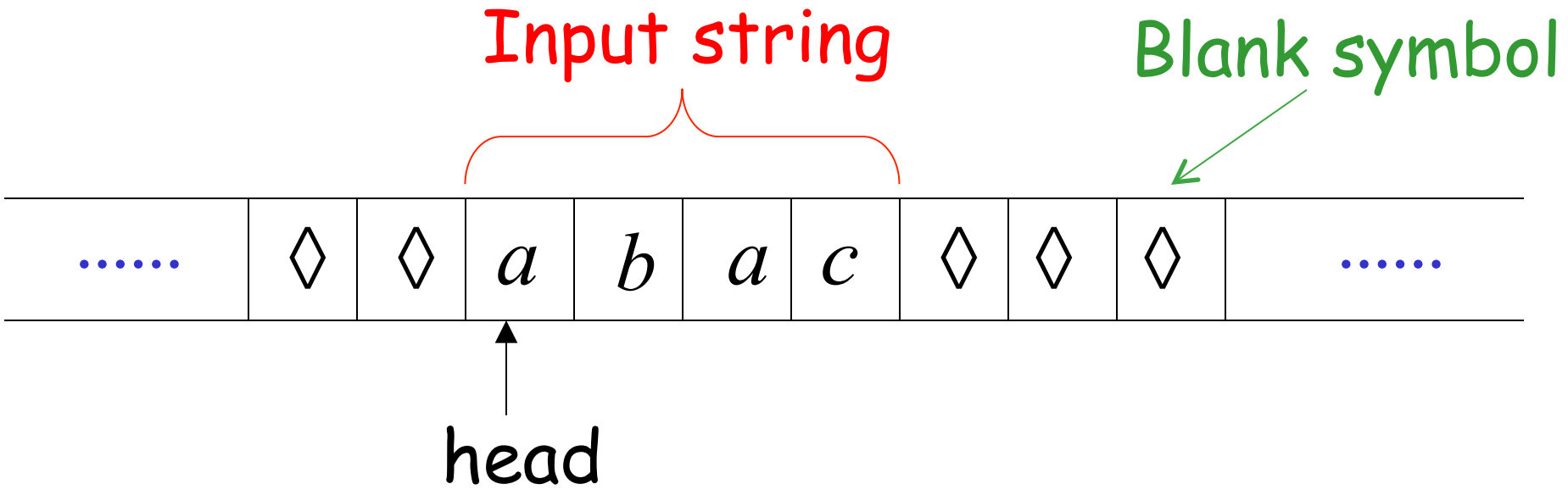


Time 2



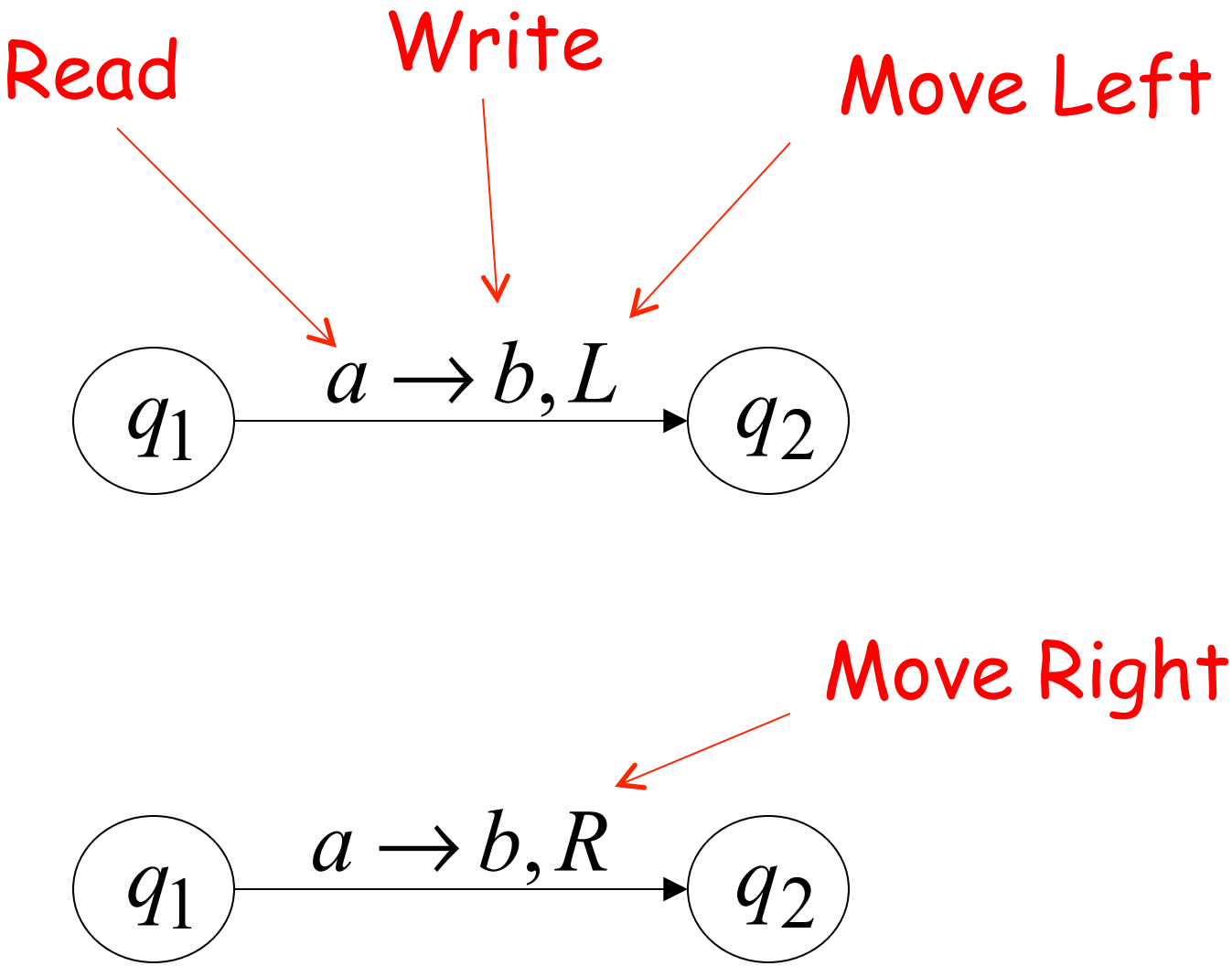
1. Reads b
2. Writes f
3. Moves Right

The Input String



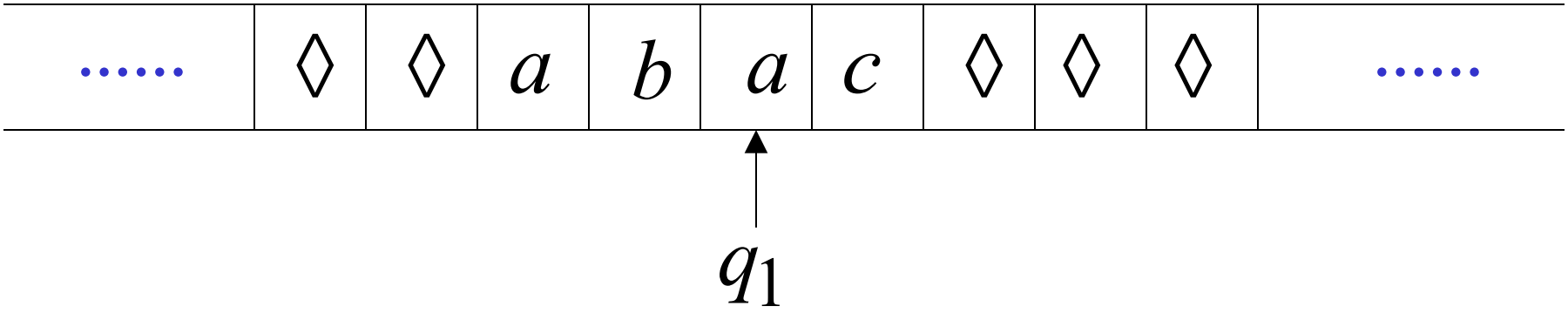
Head starts at the leftmost position of the input string

States & Transitions

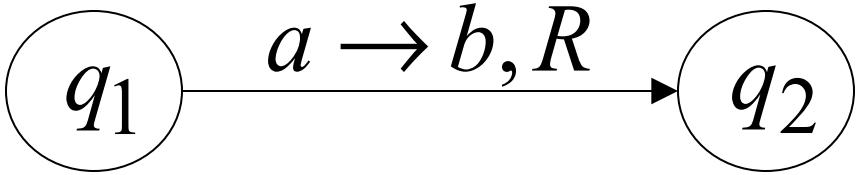


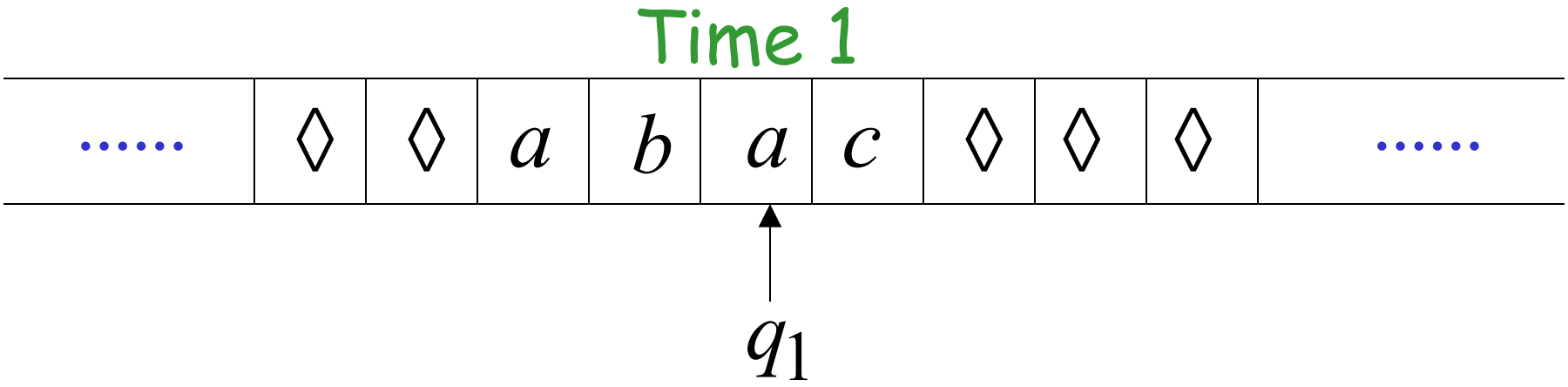
Example:

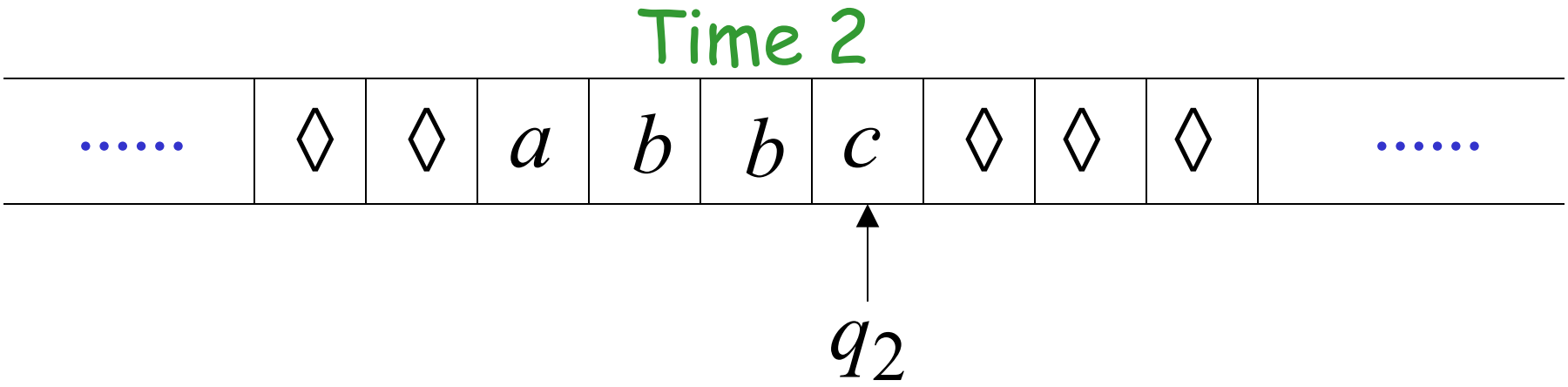
Time 1

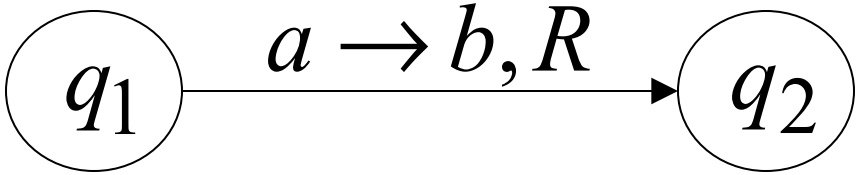


current state

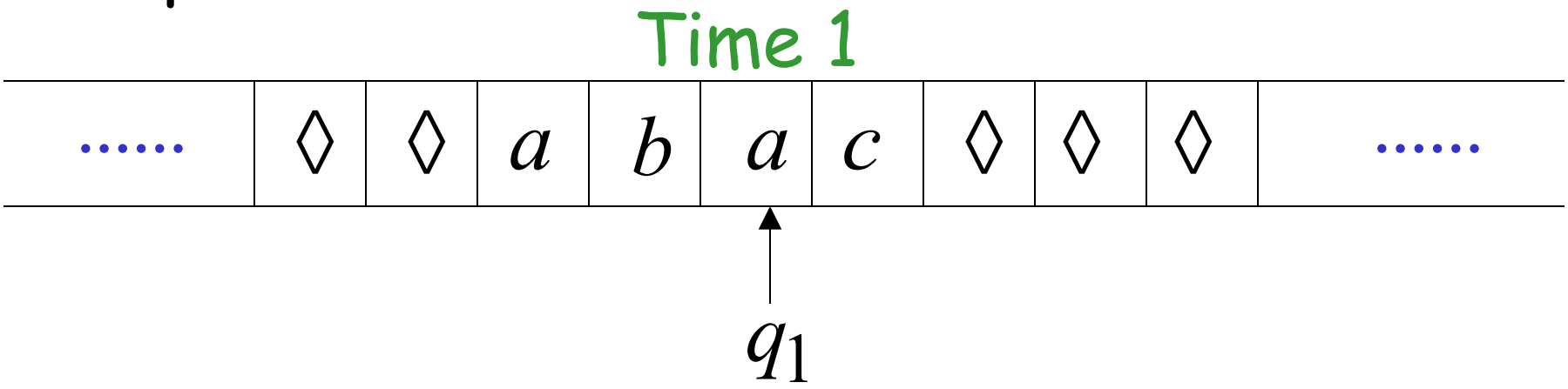


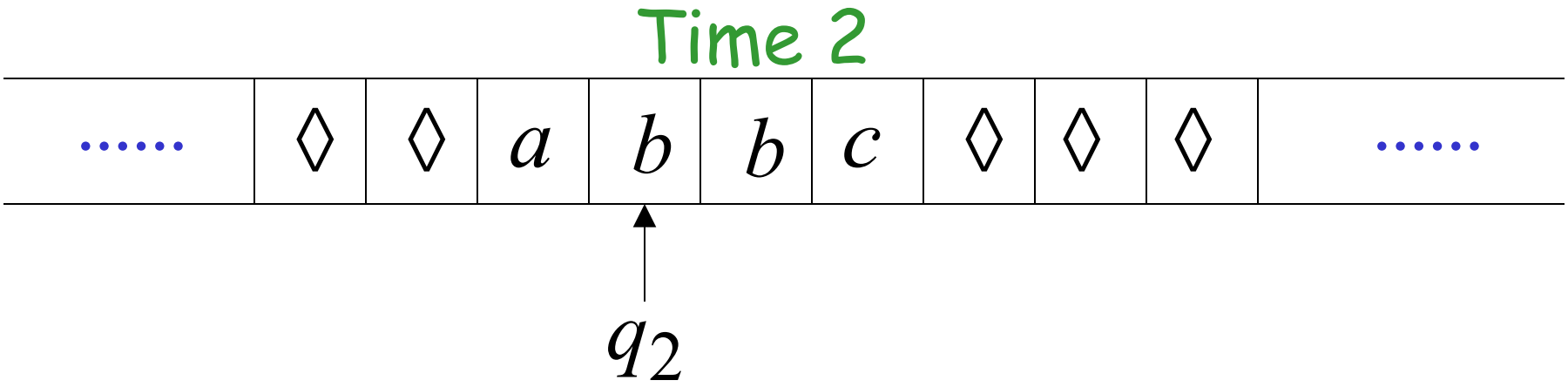

Time 1
.....
◊ ◊ a b a c ◊ ◊ ◊
.....
q_1

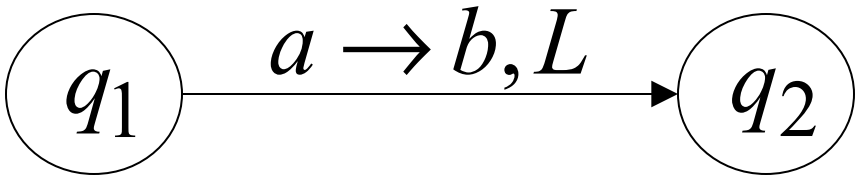

Time 2
.....
◊ ◊ a b b c ◊ ◊ ◊
.....
q_2


q_1
$a \to b, R$
q_2

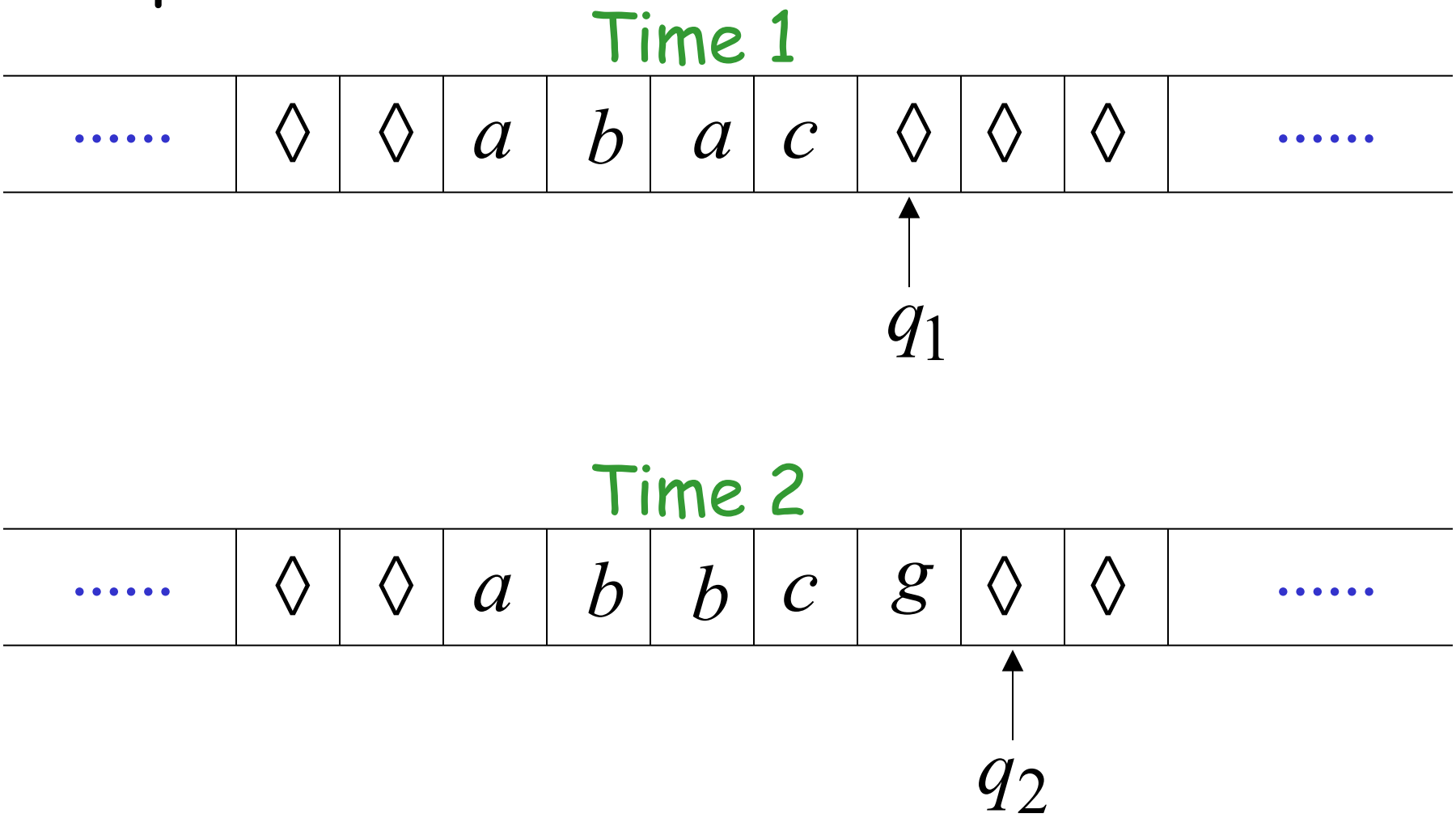
Example:

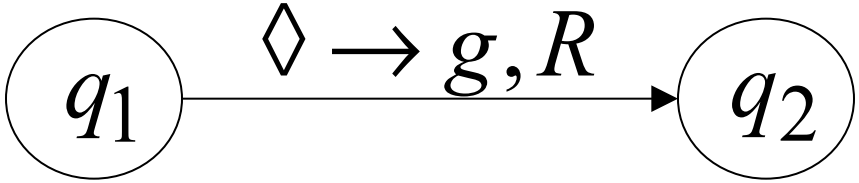






Example:

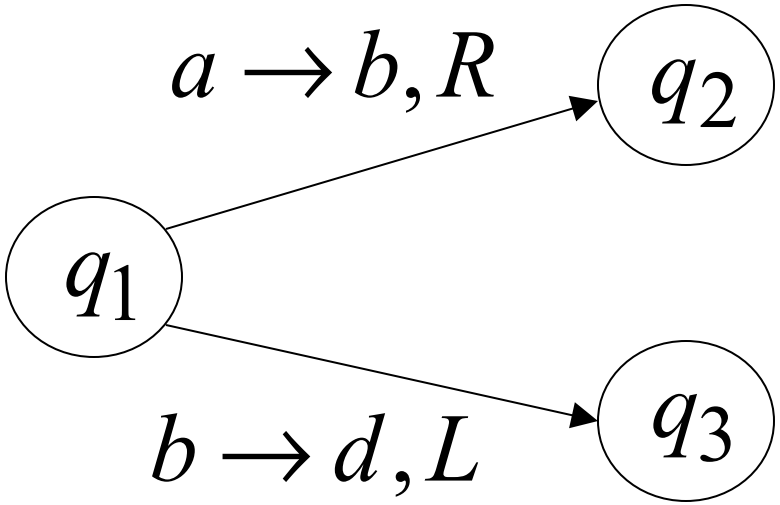




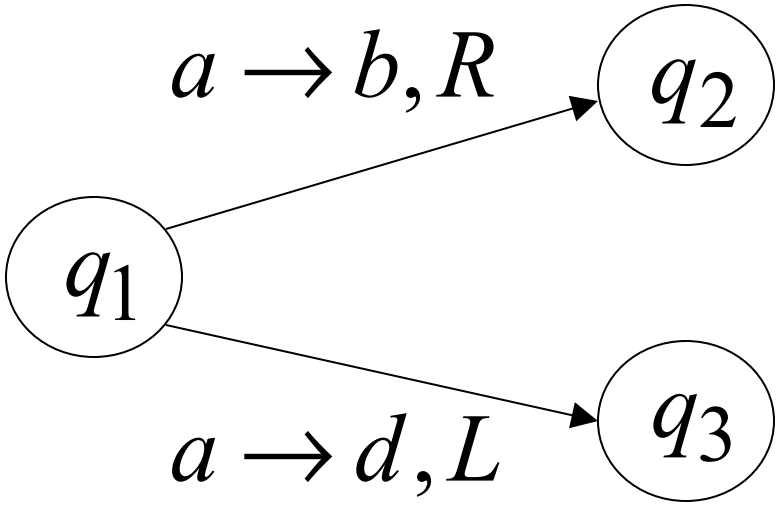
Determinism

Turing Machines are deterministic

Allowed



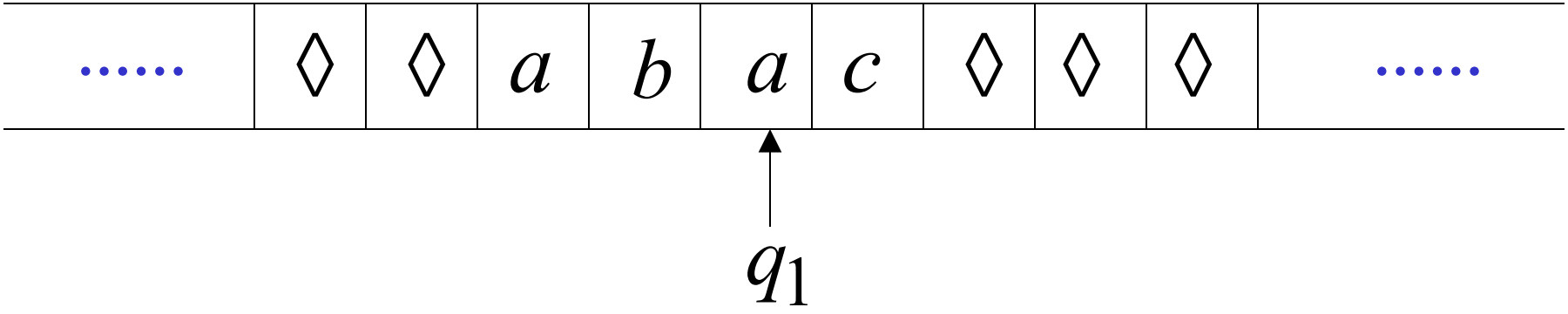
Not Allowed

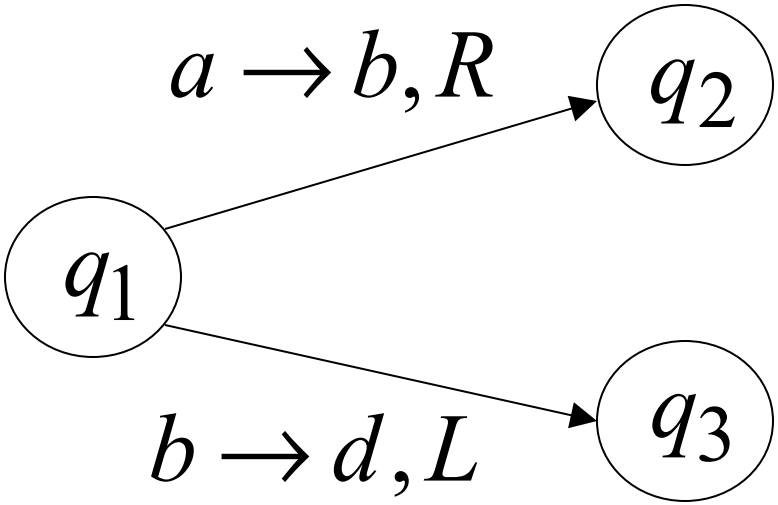


No ϵ-transitions allowed

Partial Transition Function

Example:





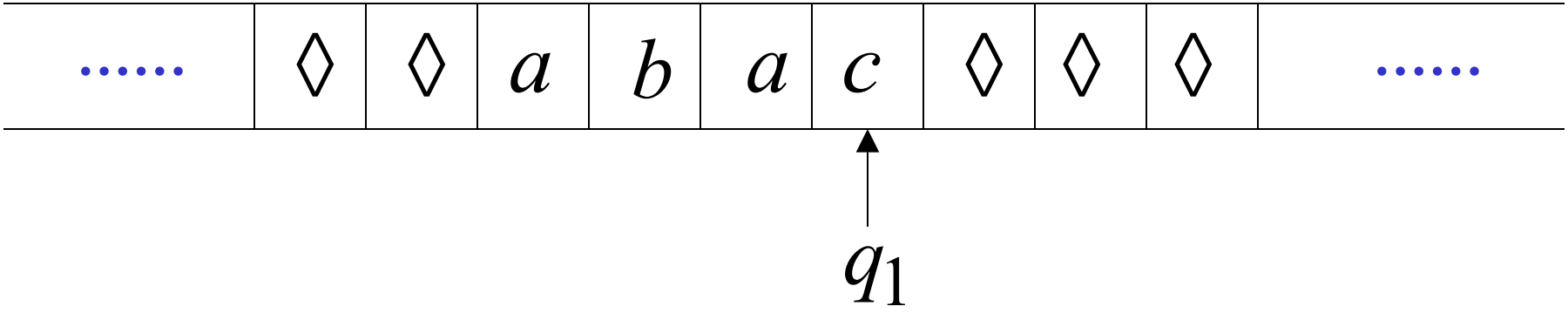
Allowed:

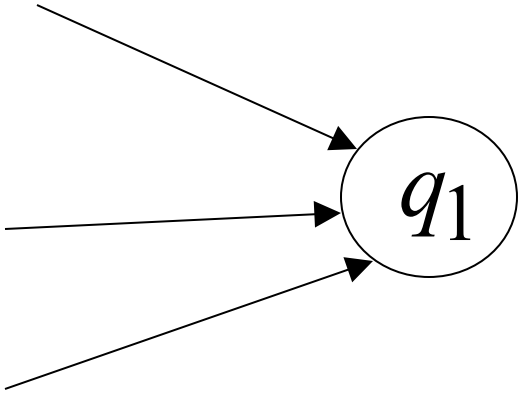
No transition for input symbol c

Halting

The machine ***halts*** in a state if there is no transition to follow

Halting Example 1:

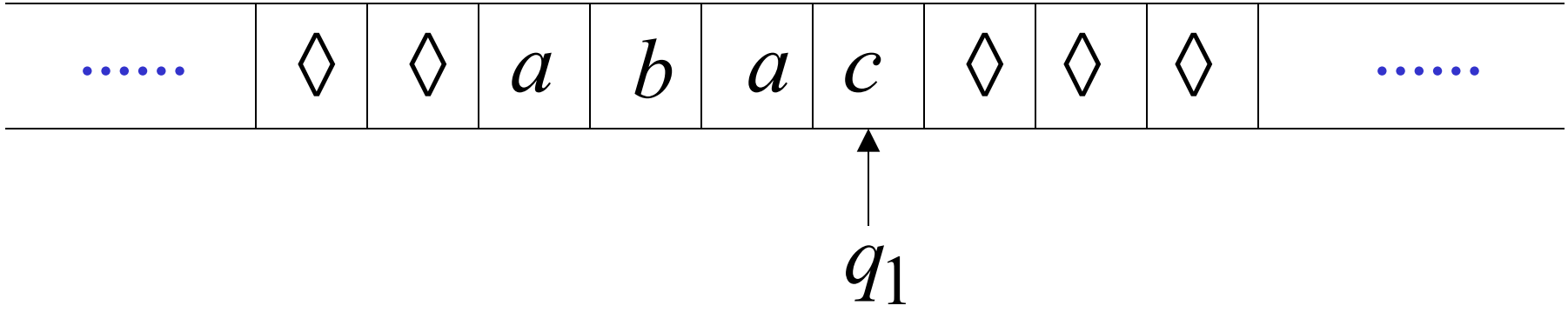


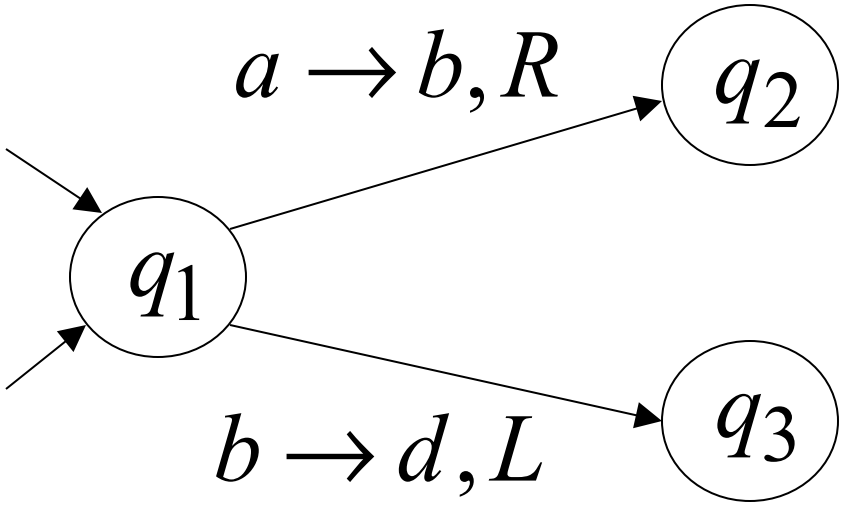


No transition from q_1

HALT!!!

Halting Example 2:

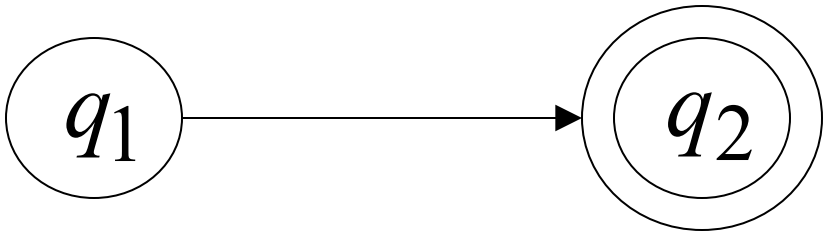




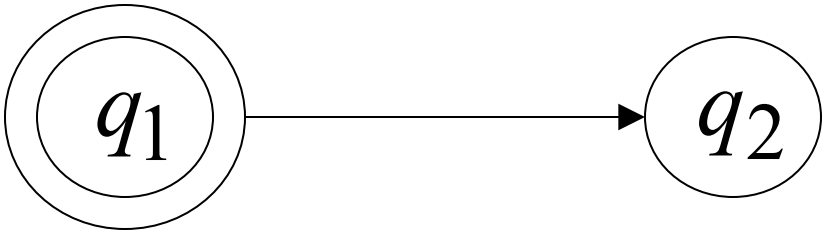
No possible transition from q_1 and symbol c

HALT!!!

Accepting States



Allowed

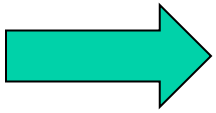


Not Allowed

- Accepting states have no outgoing transitions
- The machine halts and accepts

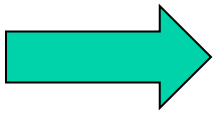
Acceptance

Accept Input string



If machine halts in an accept state

Reject Input string



If machine halts in a non-accept state

or

If machine enters an *infinite loop*

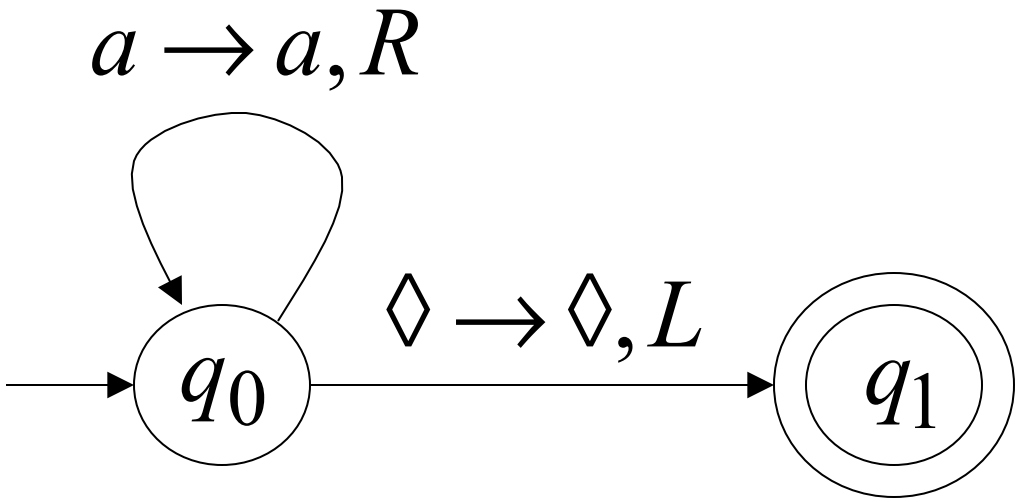
Observation:

In order to accept an input string, it is not necessary to scan all the symbols of the input string

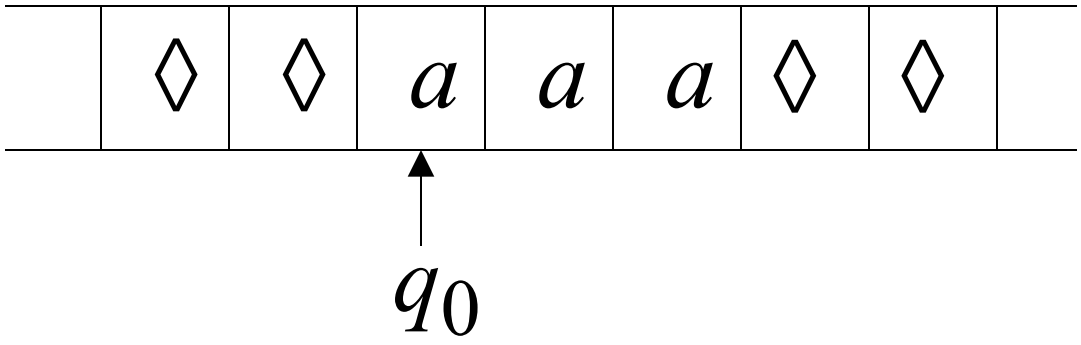
Turing Machine Example

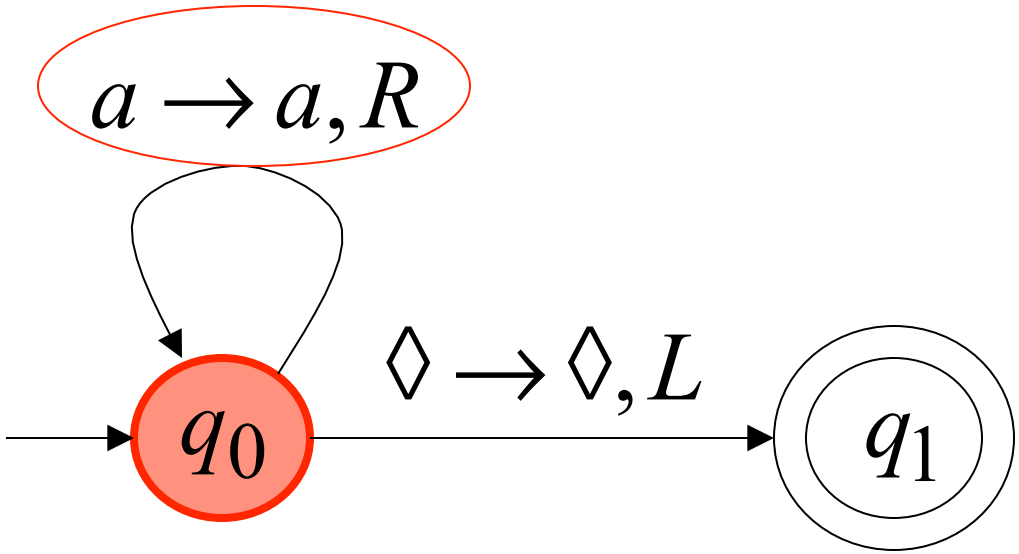
Input alphabet $\Sigma = \{a, b\}$

Accepts the language: a^*

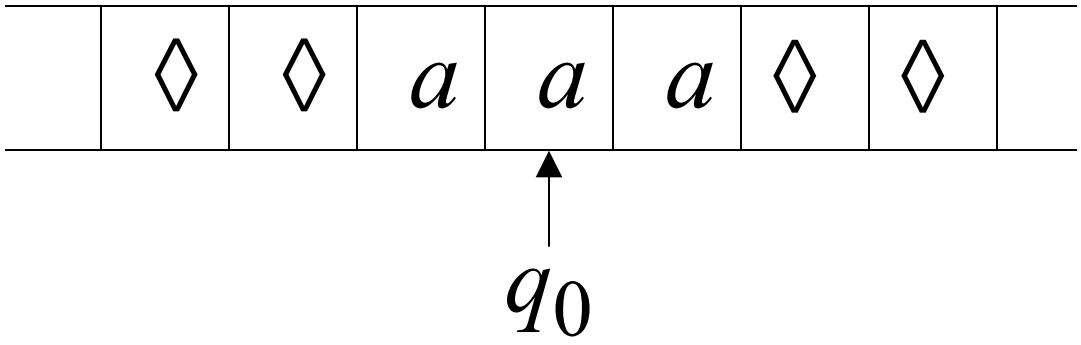


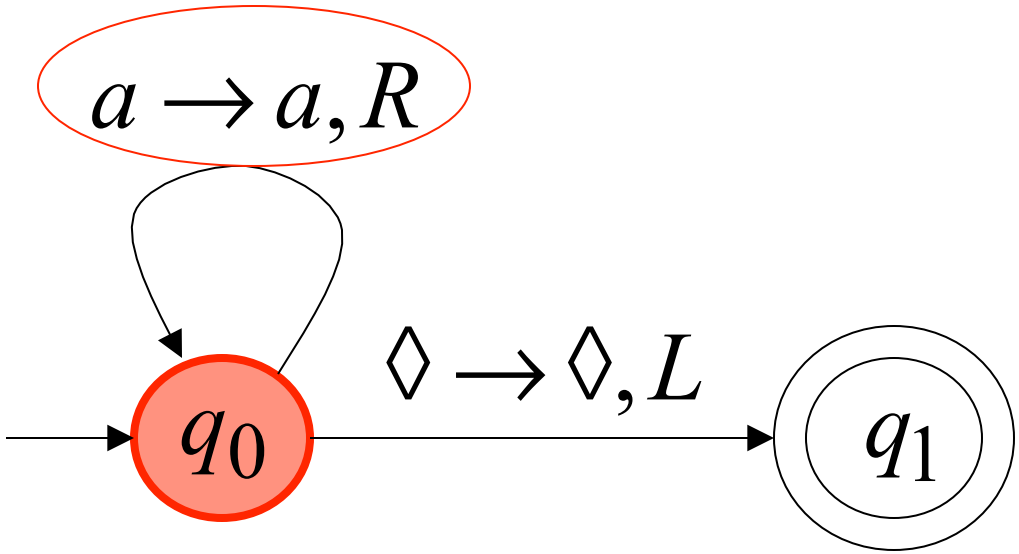
Time 0



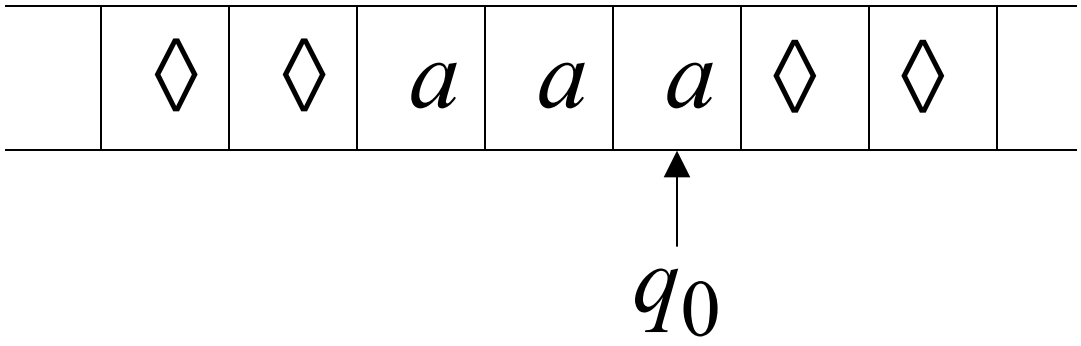


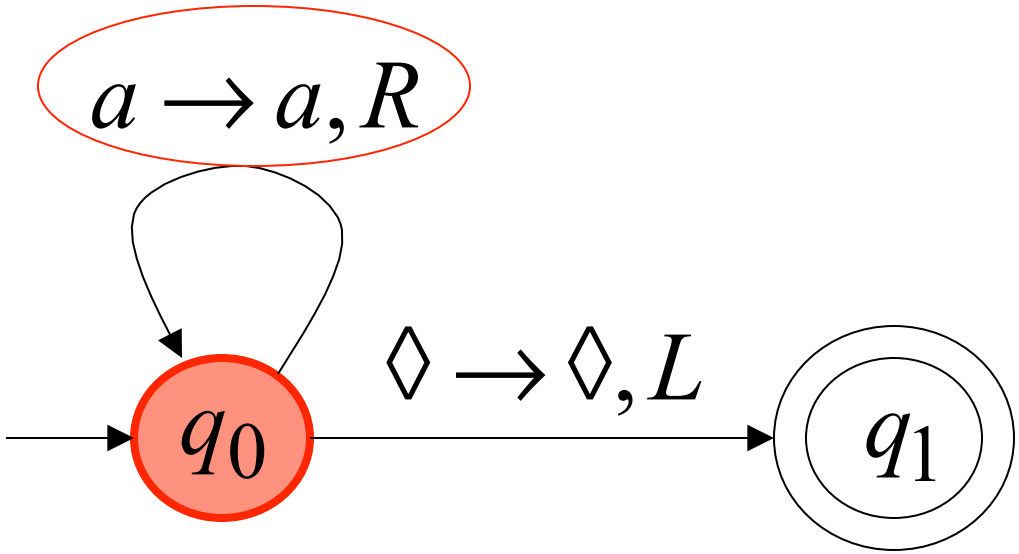
Time 1



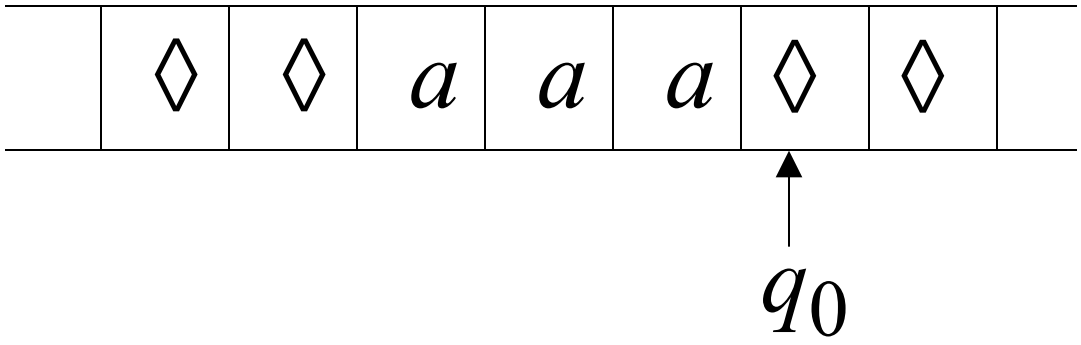


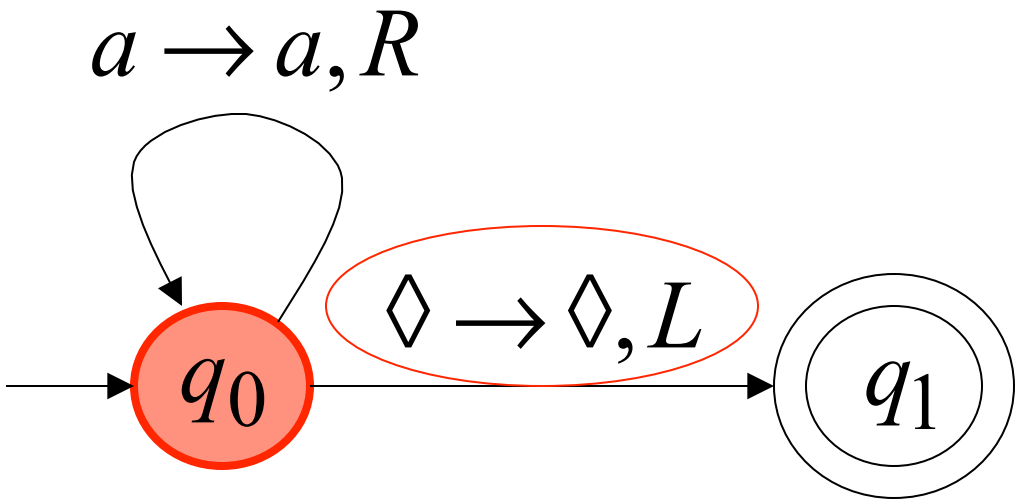
Time 2



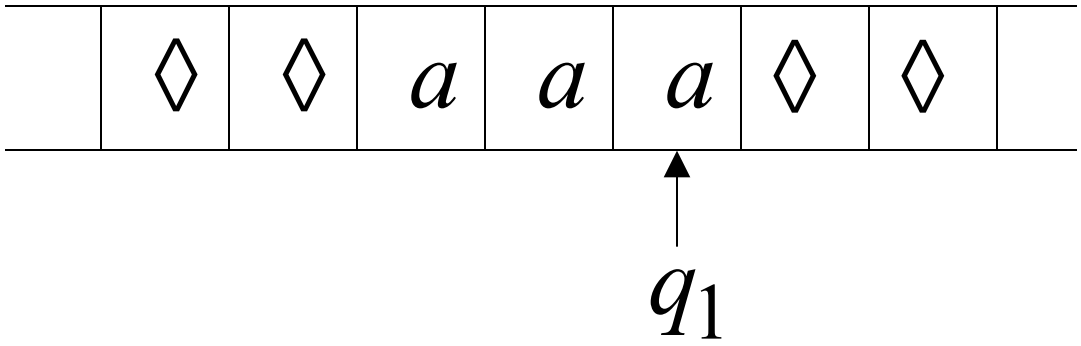


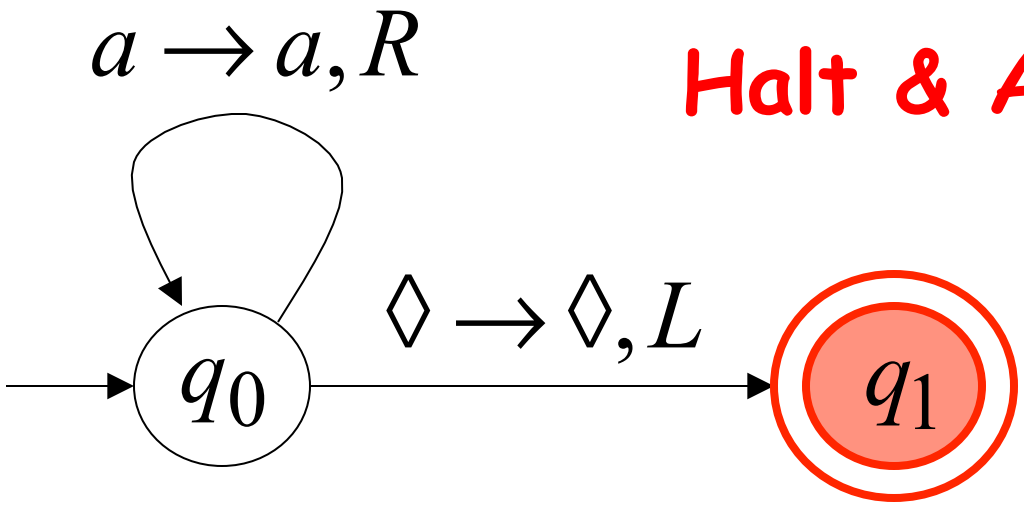
Time 3





Time 4

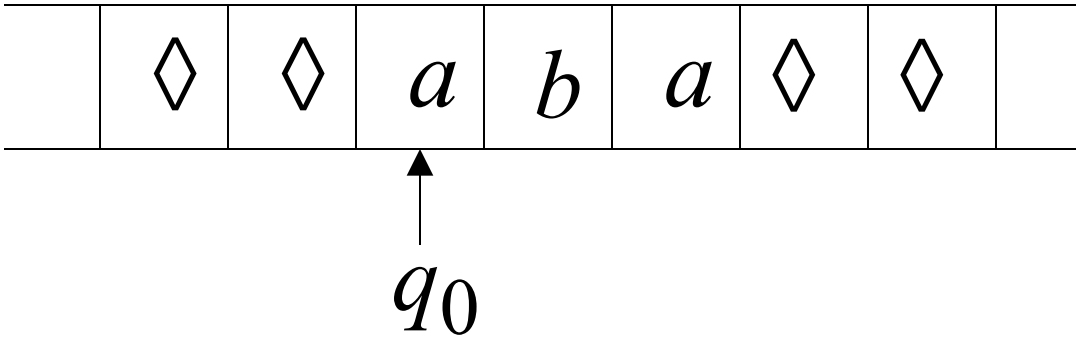


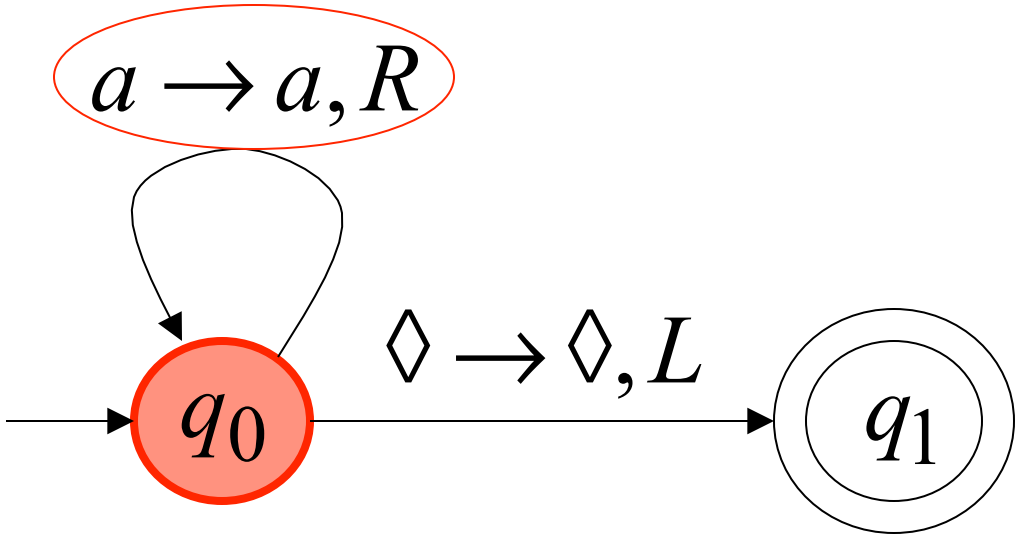


Halt & Accept

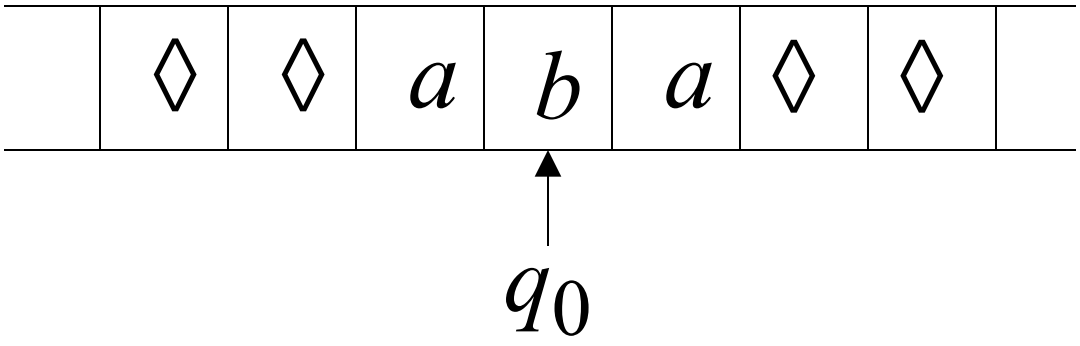
Rejection Example

Time 0



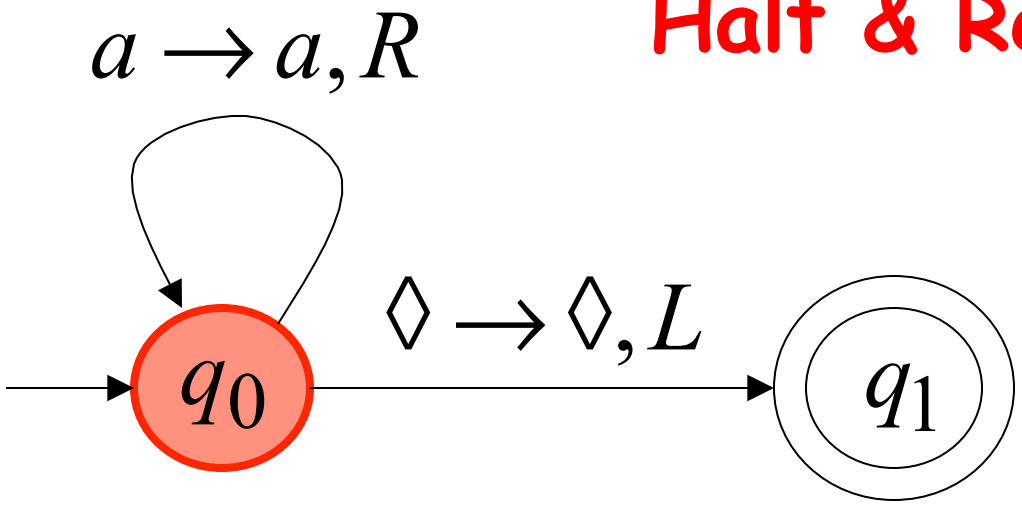


Time 1



No possible Transition

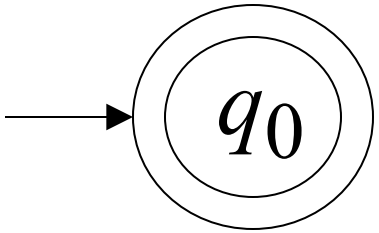
Halt & Reject



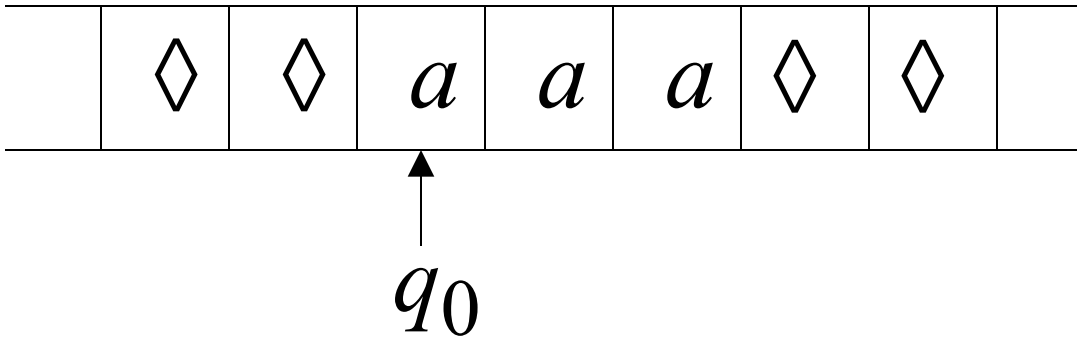
A simpler machine for same language

but for input alphabet $\Sigma = \{a\}$

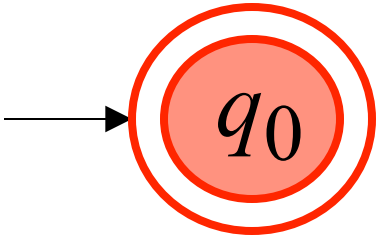
Accepts the language: a^*



Time 0



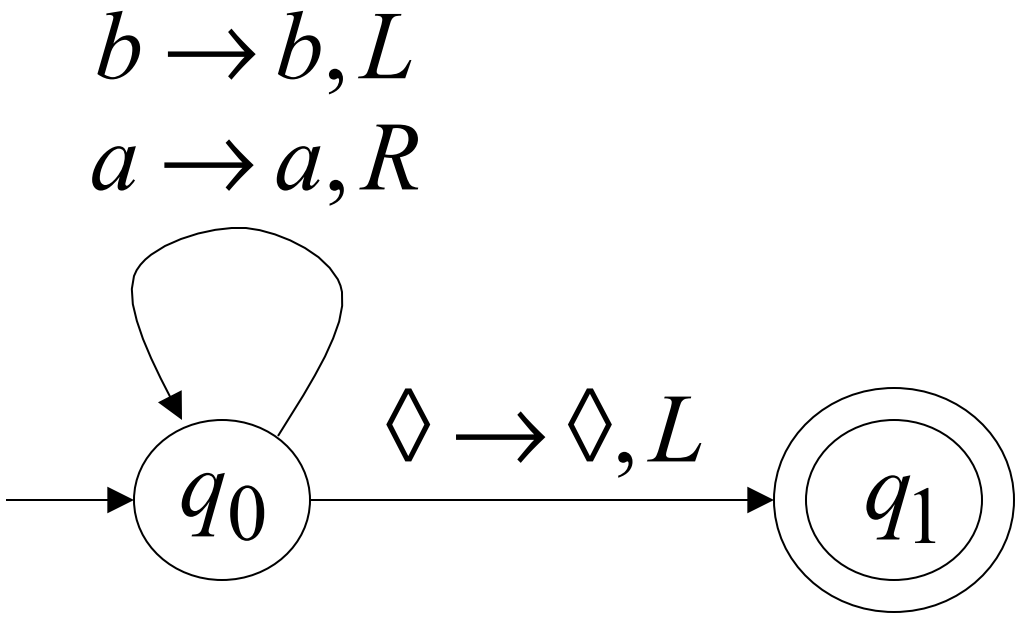
Halt & Accept



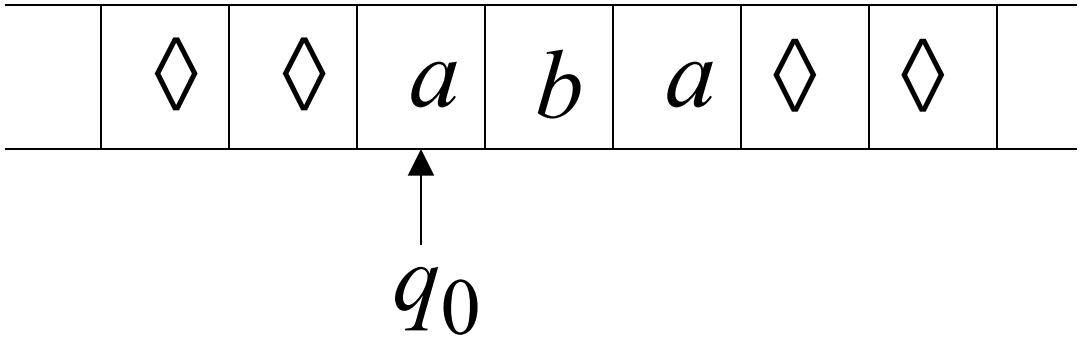
Not necessary to scan input

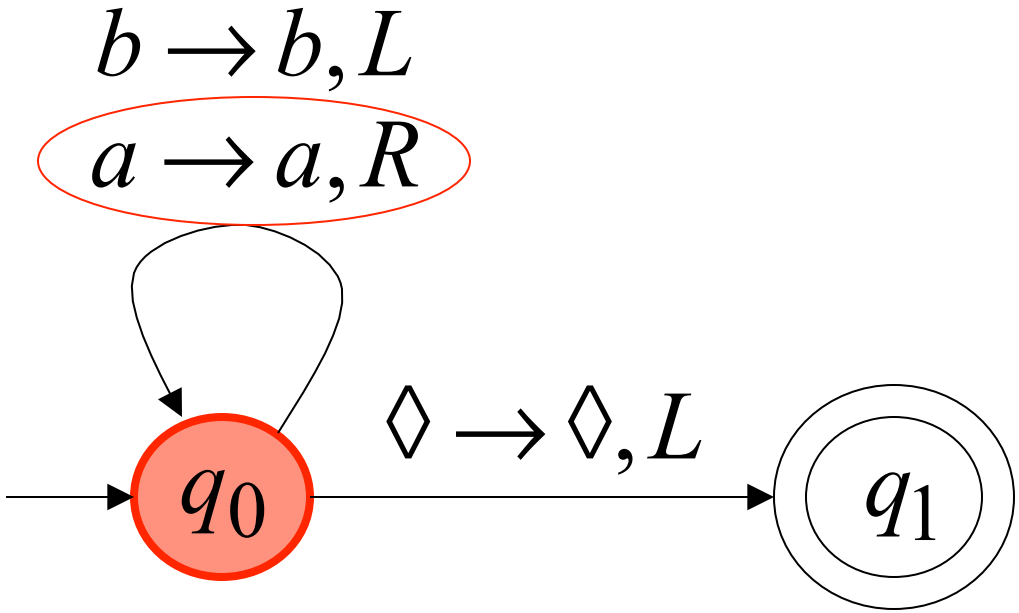
Infinite Loop Example

A Turing machine for language $a^*+b(a+b)^*$

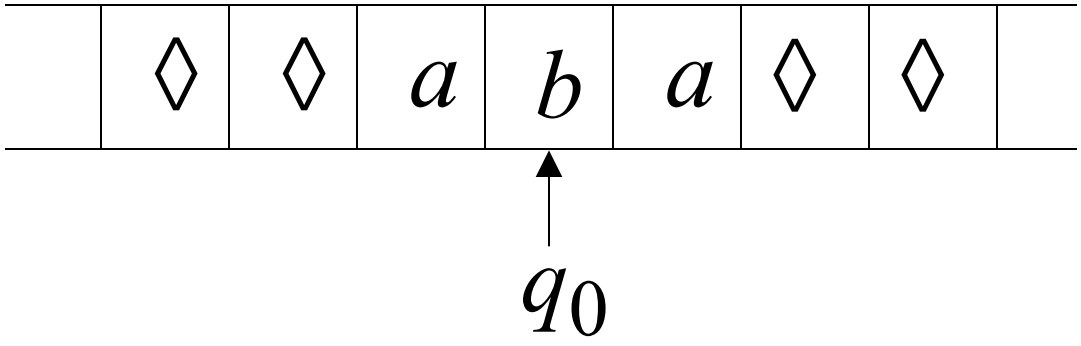


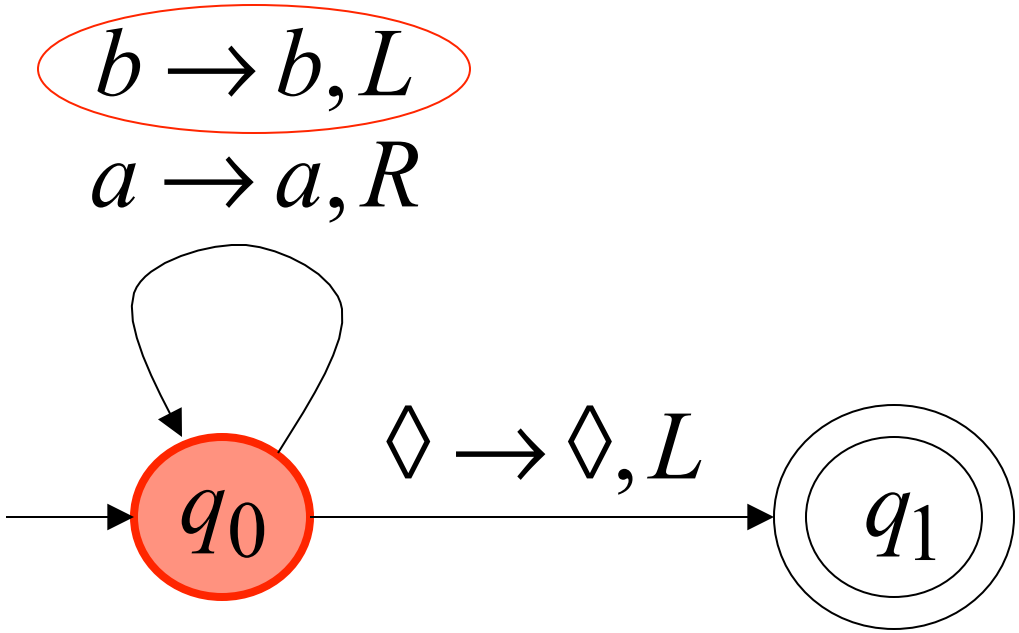
Time 0



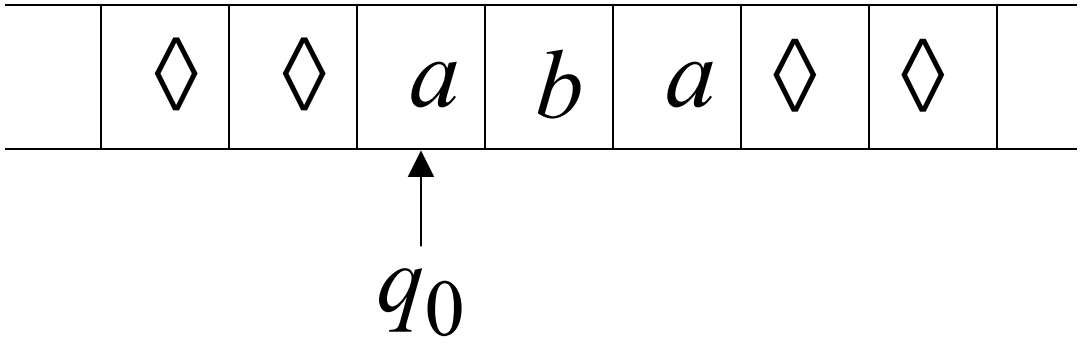


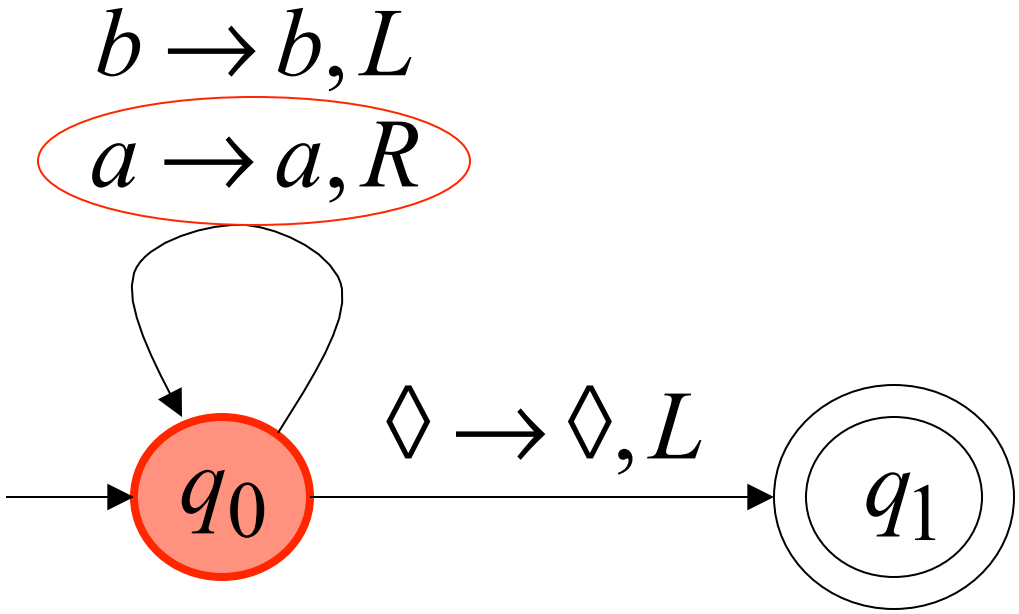
Time 1



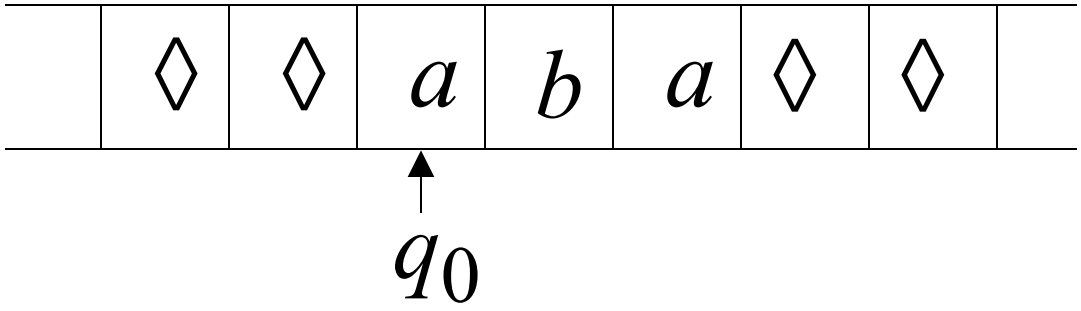


Time 2

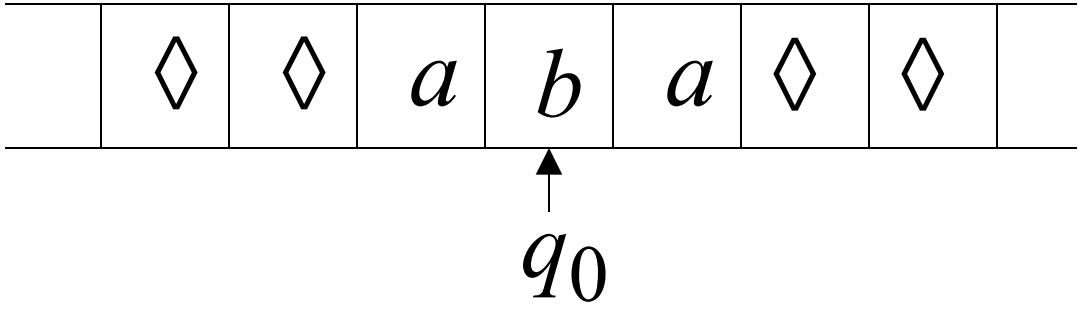




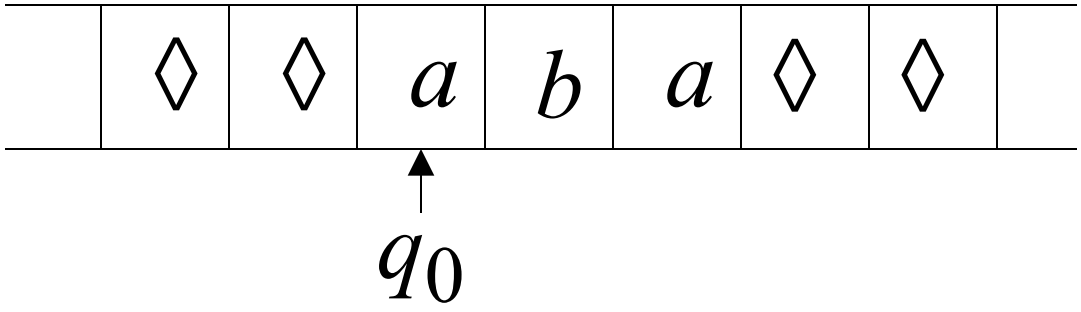
Time 2



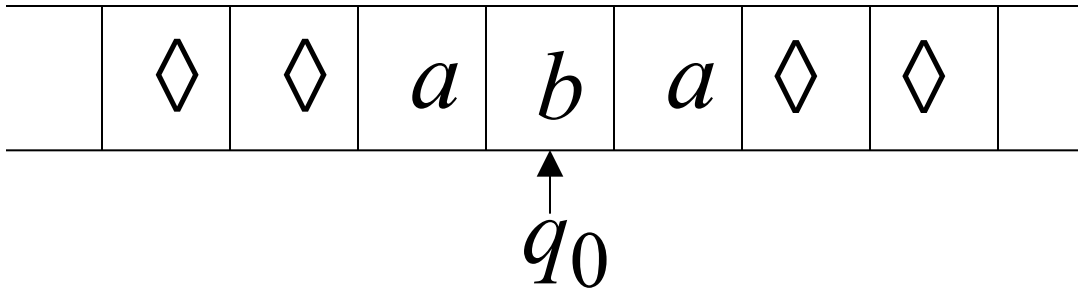
Time 3



Time 4



Time 5



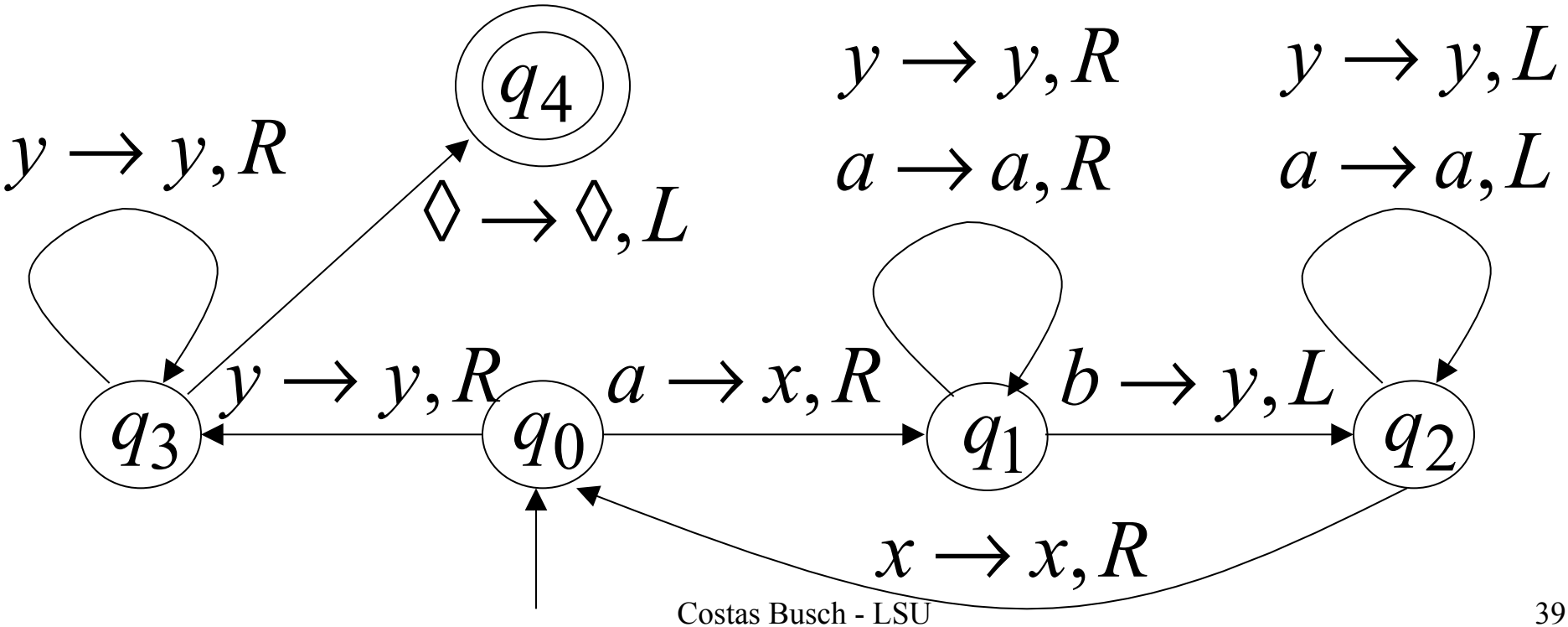
Infinite loop

Because of the infinite loop:

- The accepting state cannot be reached
- The machine never halts
- The input string is rejected

Another Turing Machine Example

Turing machine for the language $\{a^n b^n\}$, $n \geq 1$



Basic Idea:

Match a's with b's:

Repeat:

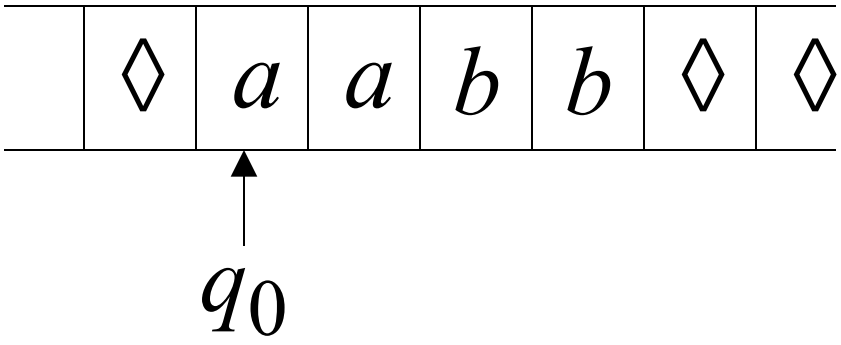
replace leftmost a with x

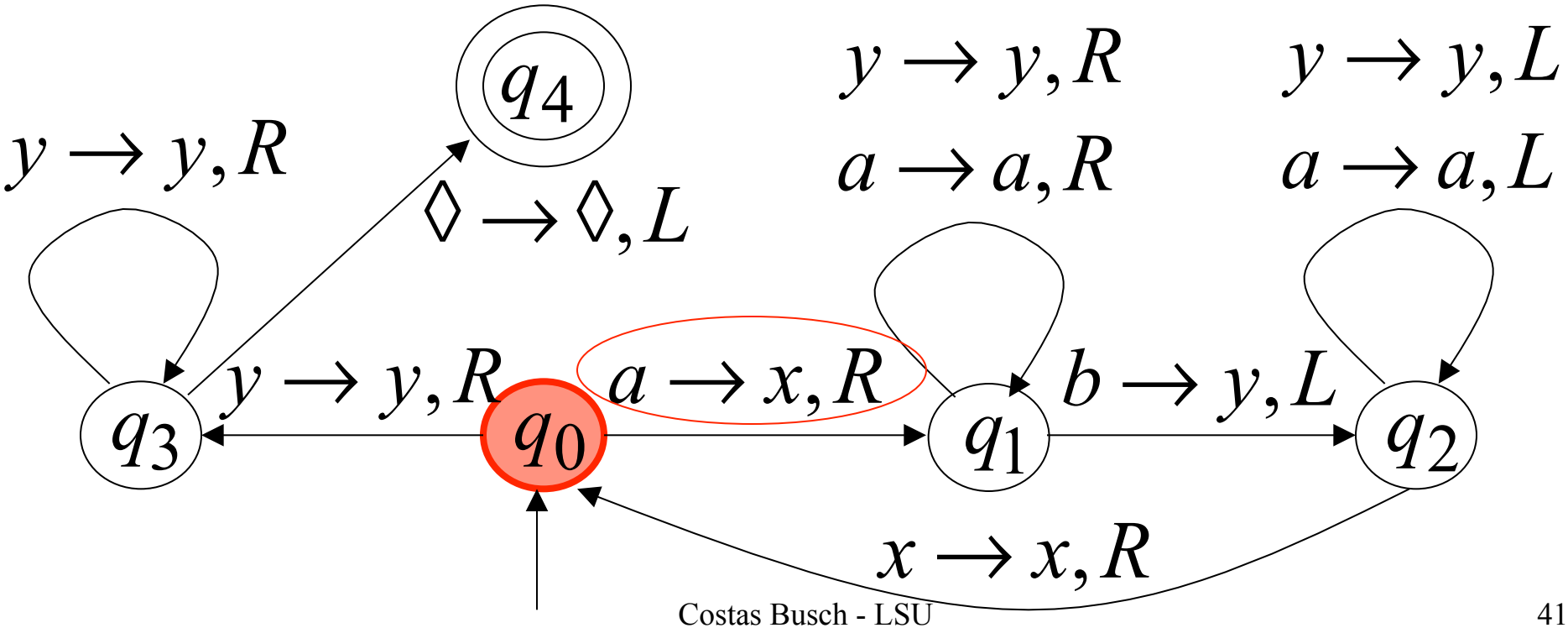
find leftmost b and replace it with y

Until there are no more a's or b's

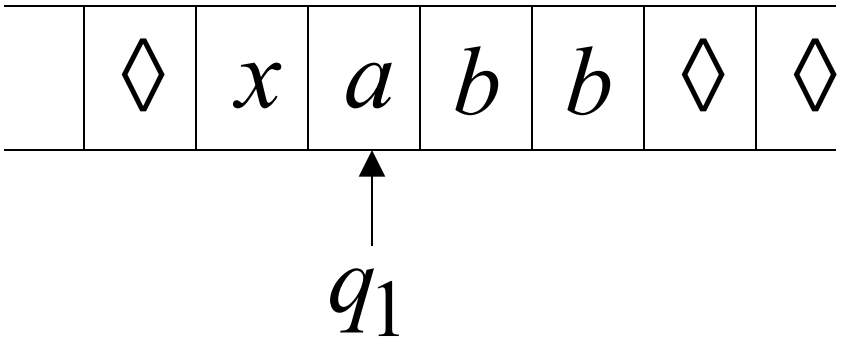
If there is a remaining a or b reject

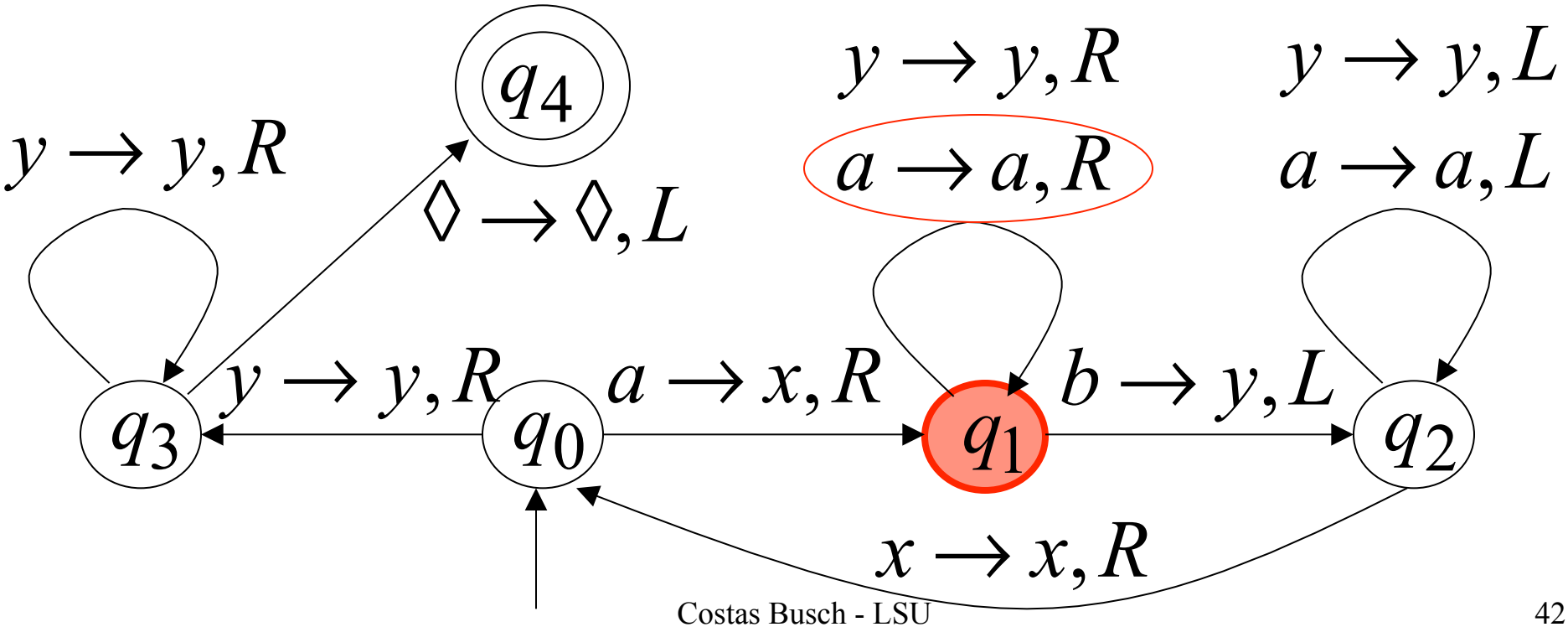
Time 0


◊ a a b b ◊ ◊
q_0

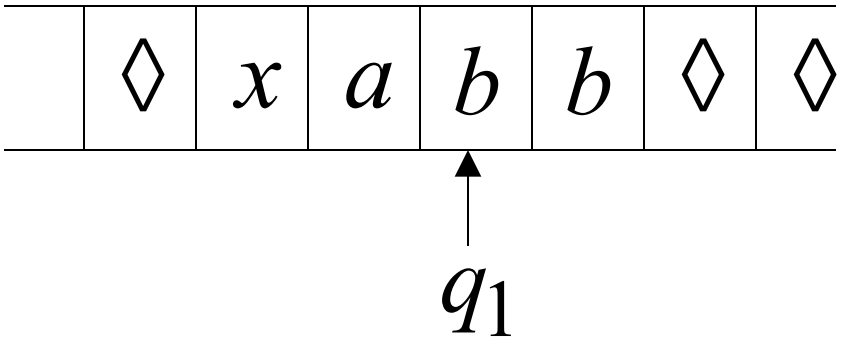

q_4
$y \to y, R$
$a \to a, R$
$y \to y, L$
$a \to a, L$
$y \to y, R$
$\Diamond \to \Diamond, L$
$y \to y, R$
$a \to x, R$
$b \to y, L$
q_3
q_0
q_1
q_2
$x \to x, R$

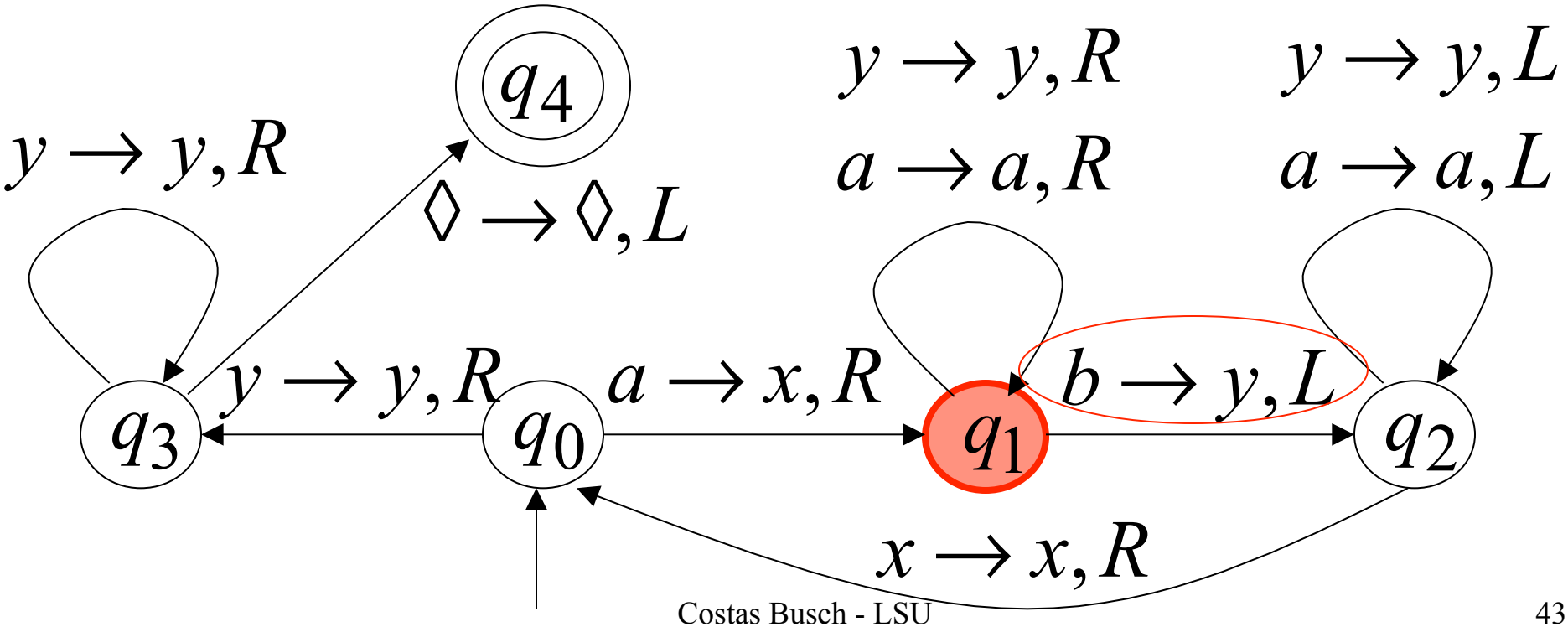
Time 1


◊ x a b b ◊ ◊
q_1

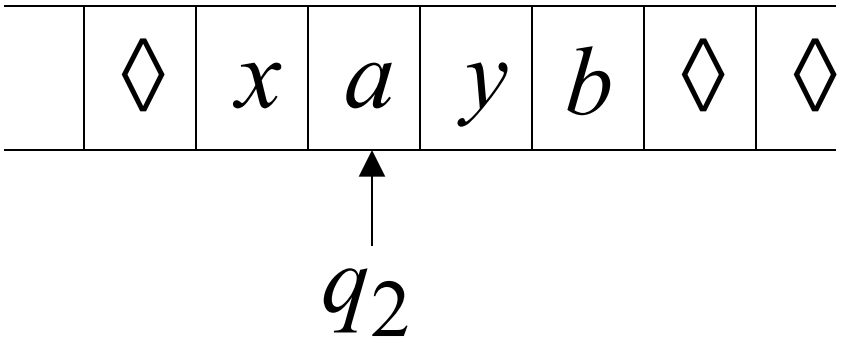

q_4
$y \to y, R$
$\Diamond \to \Diamond, L$
$y \to y, R$
$a \to a, R$
$y \to y, L$
$a \to a, L$
q_3
$y \to y, R$
q_0
$a \to x, R$
q_1
$b \to y, L$
q_2
$x \to x, R$

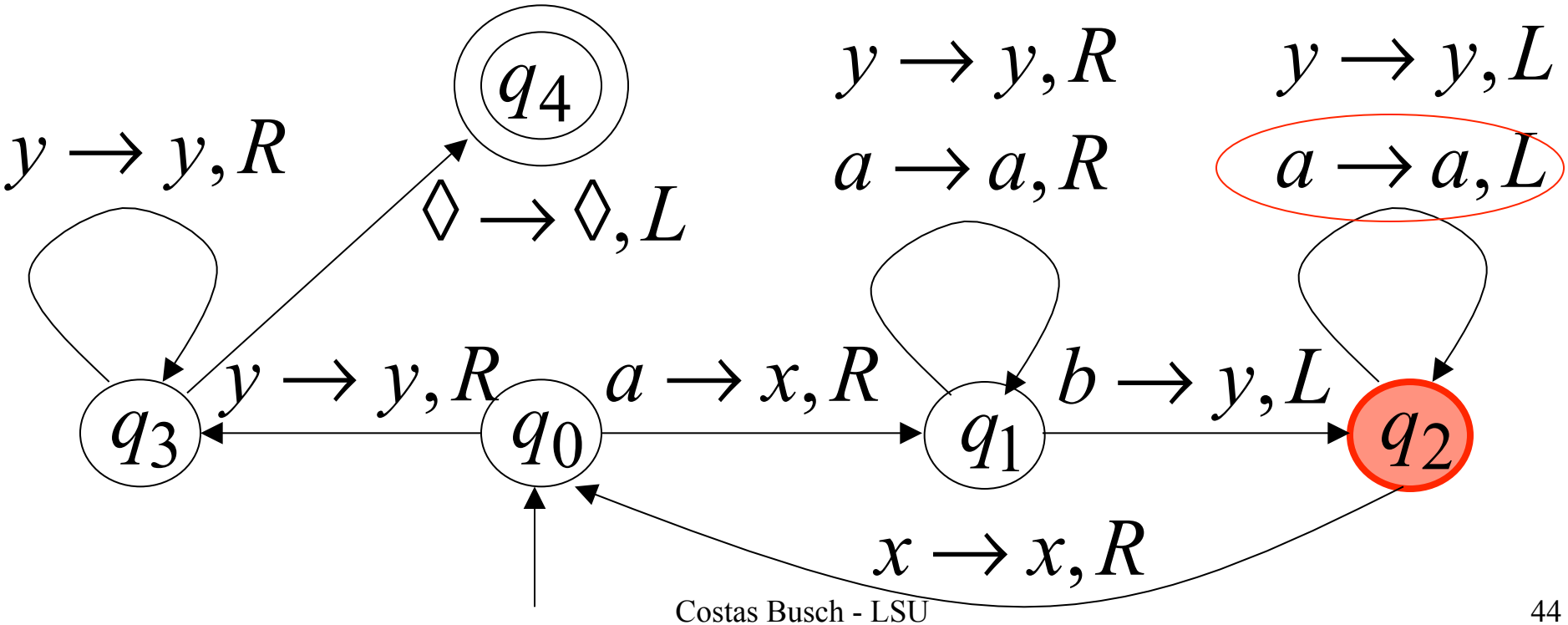
Time 2


◊ x a b b ◊ ◊
q_1

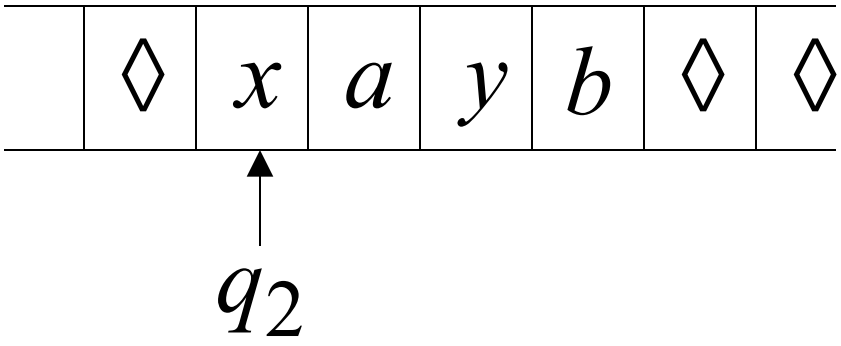

q_4
$y \to y, R$
$\Diamond \to \Diamond, L$
$y \to y, R$
$a \to a, R$
$y \to y, L$
$a \to a, L$
q_3
$y \to y, R$
q_0
$a \to x, R$
q_1
$b \to y, L$
q_2
$x \to x, R$

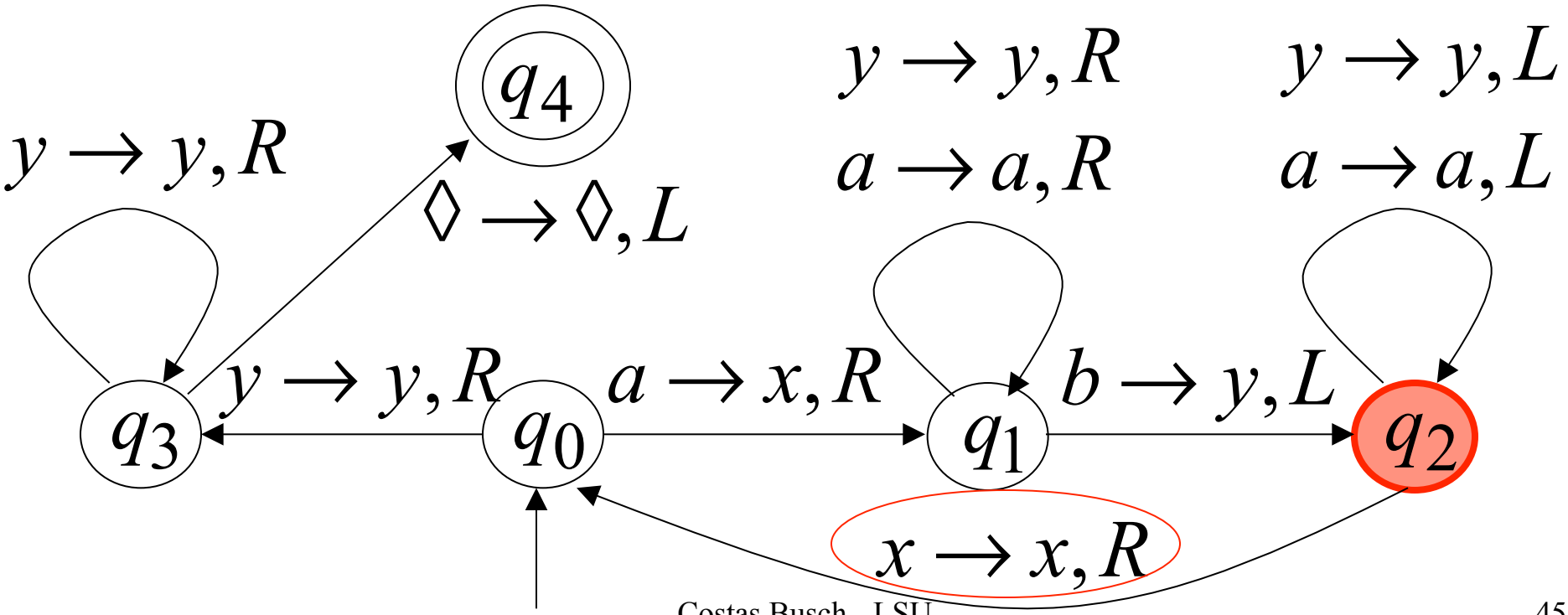
Time 3


◊ x a y b ◊ ◊
q_2

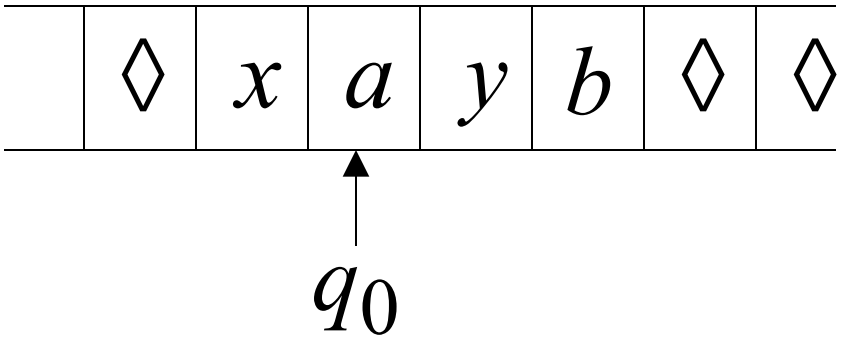

q_4
$y \to y, R$
$\Diamond \to \Diamond, L$
$y \to y, R$
$a \to a, R$
$y \to y, L$
$a \to a, L$
q_3
$y \to y, R$
q_0
$a \to x, R$
q_1
$b \to y, L$
q_2
$x \to x, R$

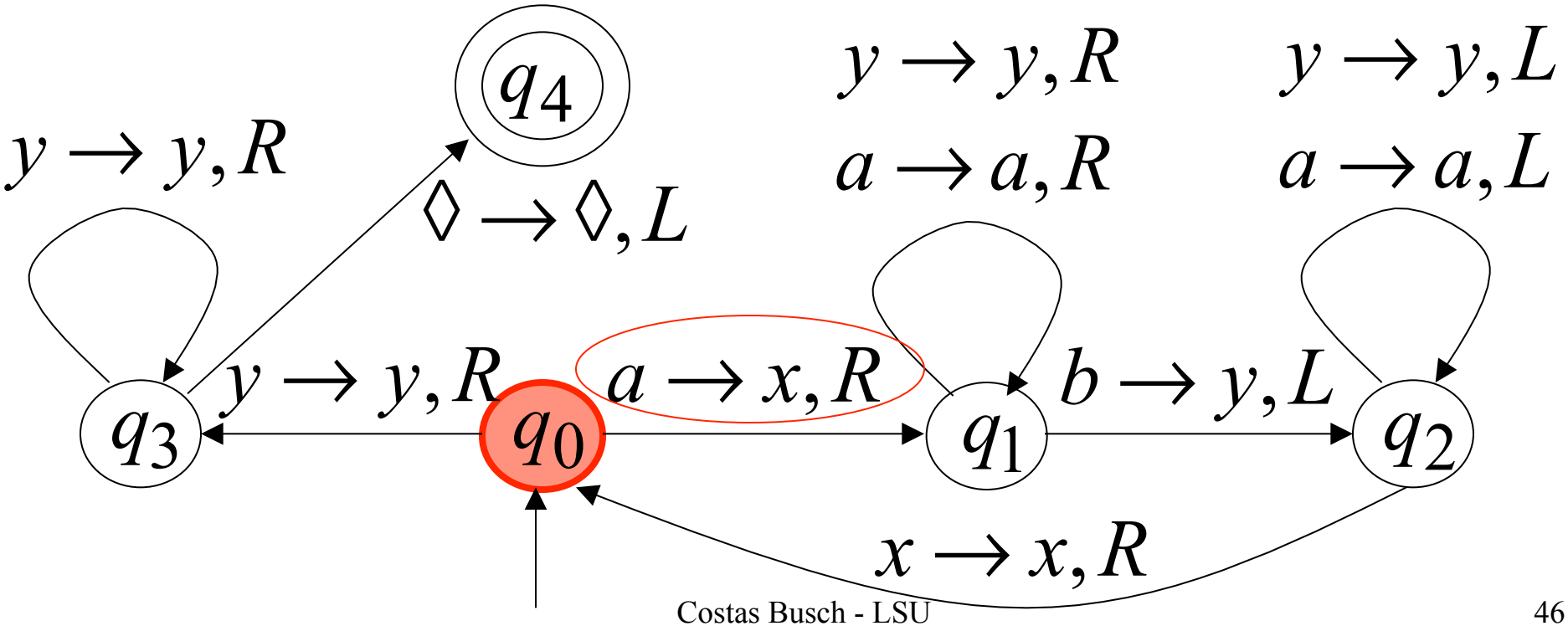
Time 4


◊ x a y b ◊ ◊
q_2

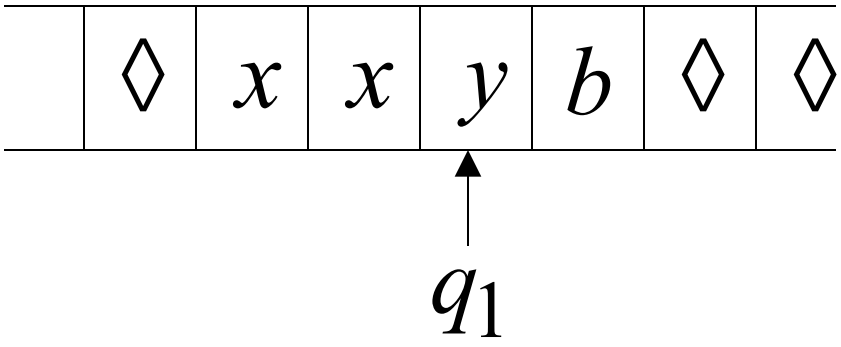

q_4
$y \to y, R$
$\Diamond \to \Diamond, L$
$y \to y, R$
$a \to a, R$
$y \to y, L$
$a \to a, L$
q_3
$y \to y, R$
q_0
$a \to x, R$
q_1
$b \to y, L$
q_2
$x \to x, R$

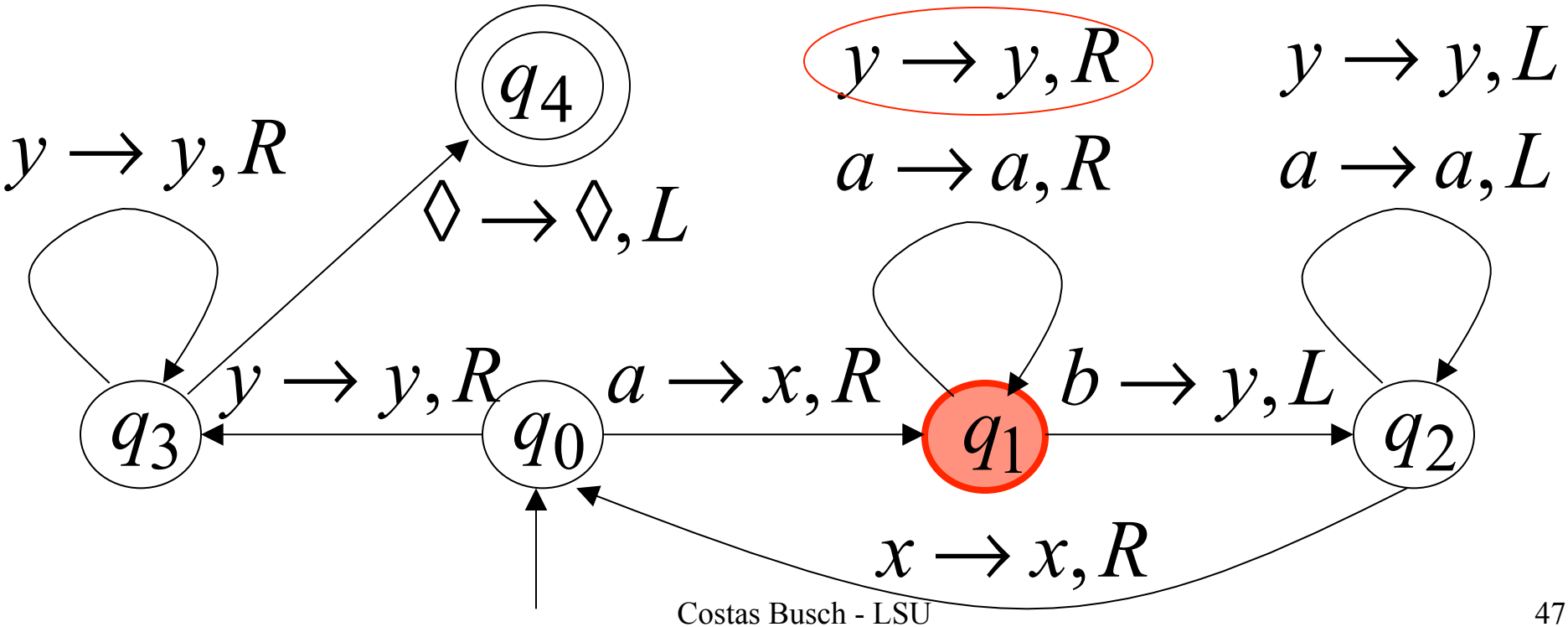
Time 5


◊ x a y b ◊ ◊
q_0

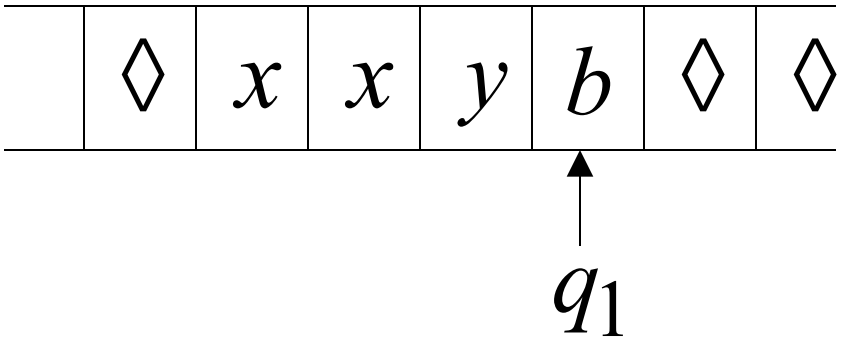

q_4
$y \to y, R$
$\Diamond \to \Diamond, L$
$y \to y, R$
$a \to a, R$
$y \to y, L$
$a \to a, L$
q_3
$y \to y, R$
q_0
$a \to x, R$
q_1
$b \to y, L$
q_2
$x \to x, R$

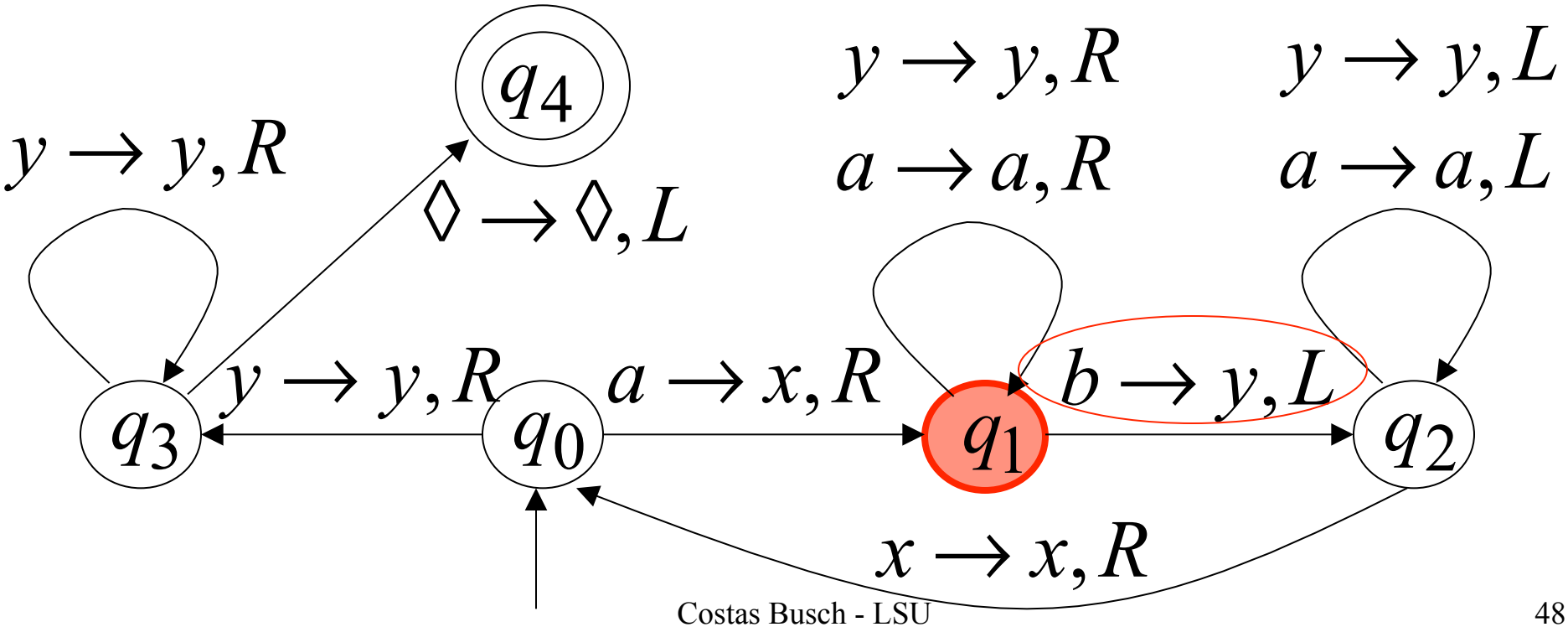
Time 6


◊ x x y b ◊ ◊
q_1

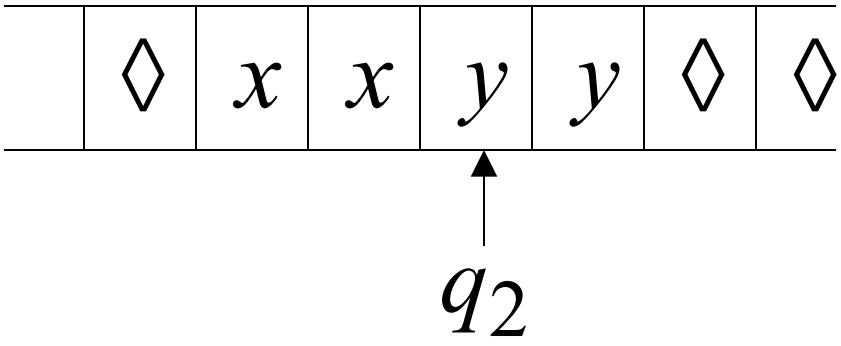

q_4
$y \to y, R$
$\lozenge \to \lozenge, L$
$y \to y, R$
$a \to a, R$
$y \to y, L$
$a \to a, L$
q_3
$y \to y, R$
q_0
$a \to x, R$
q_1
$b \to y, L$
q_2
$x \to x, R$

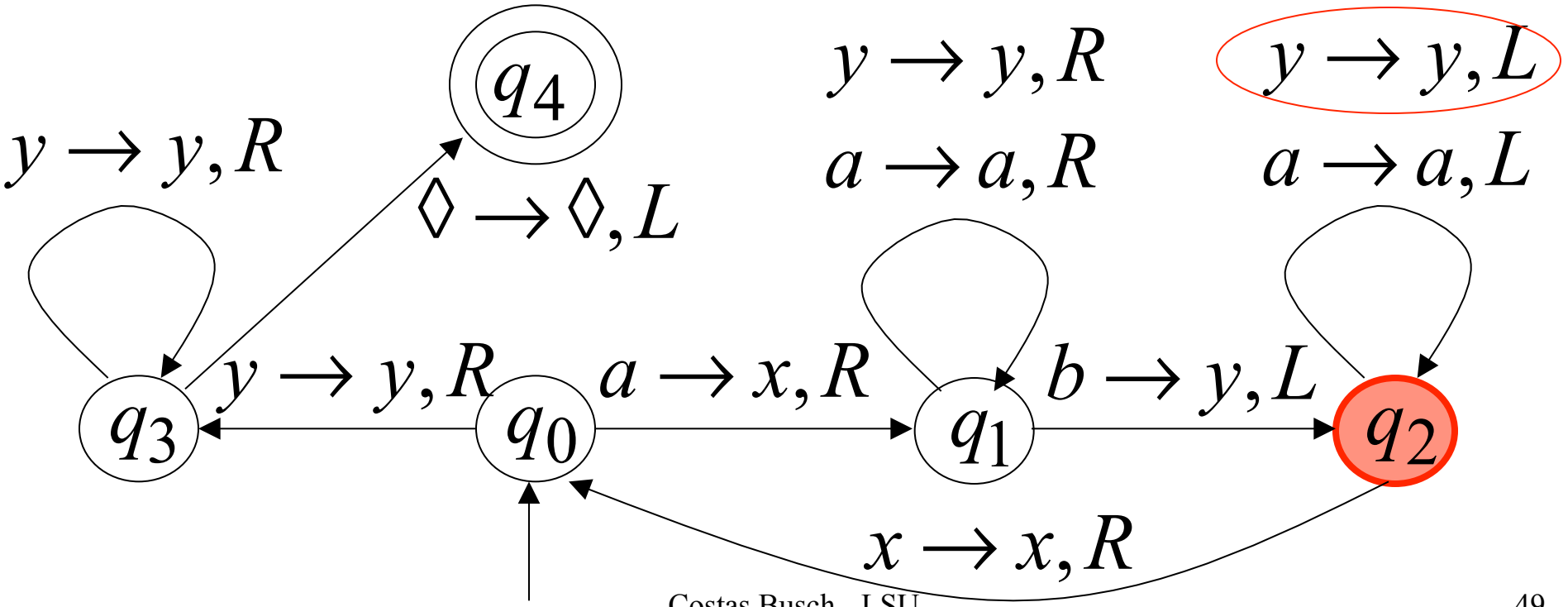
Time 7


◊ x x y b ◊ ◊
q_1

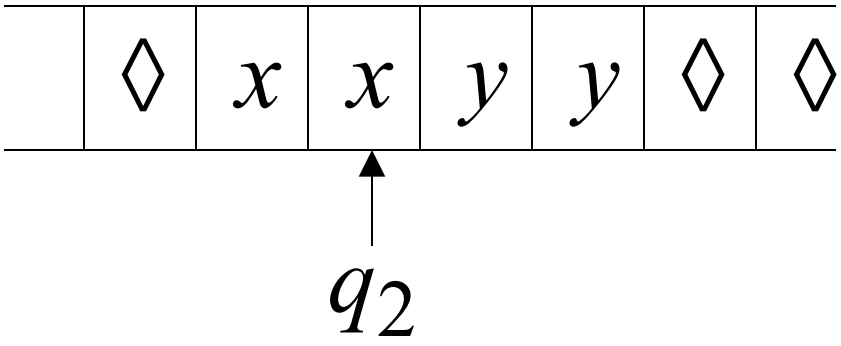

q_4
$y \to y, R$
$\Diamond \to \Diamond, L$
$y \to y, R$
$a \to a, R$
$y \to y, L$
$a \to a, L$
q_3
$y \to y, R$
q_0
$a \to x, R$
q_1
$b \to y, L$
q_2
$x \to x, R$

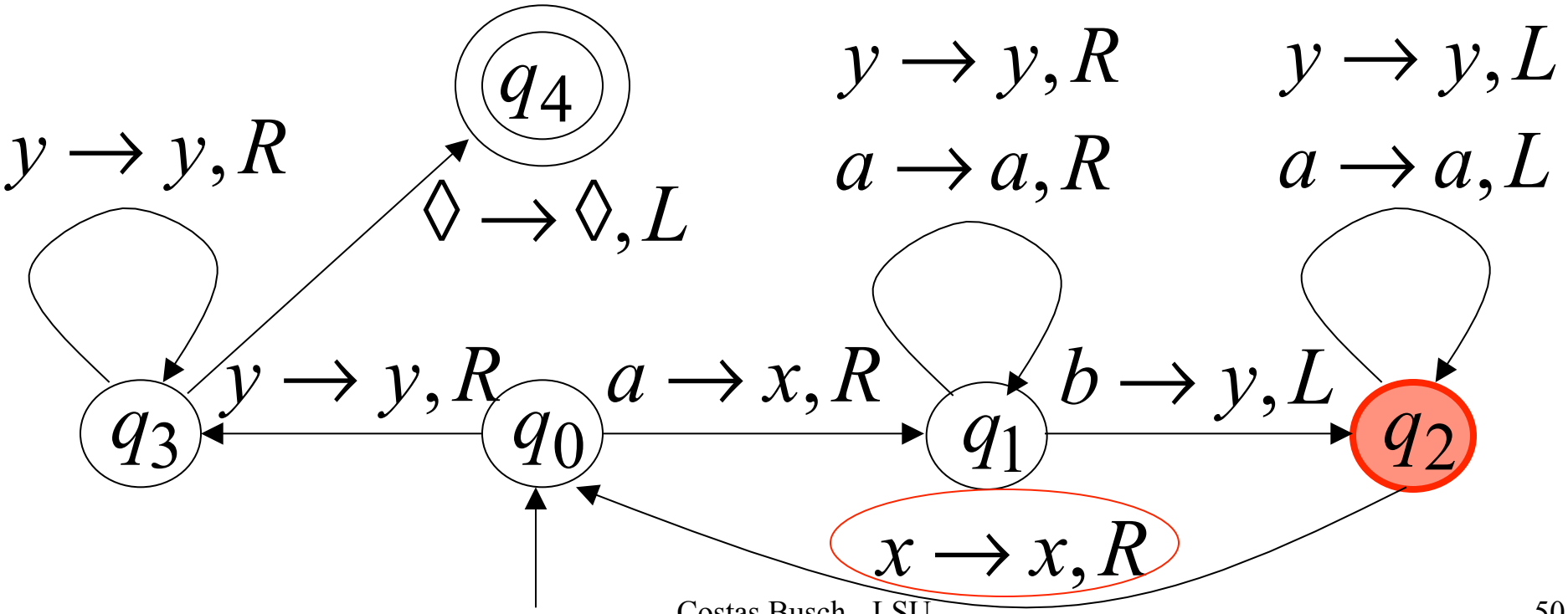
Time 8


◊ x x y y ◊ ◊
q_2

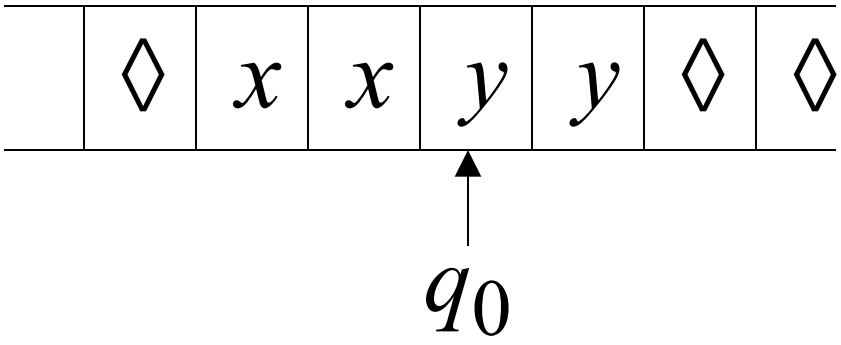

q_4
$y \to y, R$
$\Diamond \to \Diamond, L$
$y \to y, R$
$a \to a, R$
$y \to y, L$
$a \to a, L$
q_3
$y \to y, R$
q_0
$a \to x, R$
q_1
$b \to y, L$
q_2
$x \to x, R$

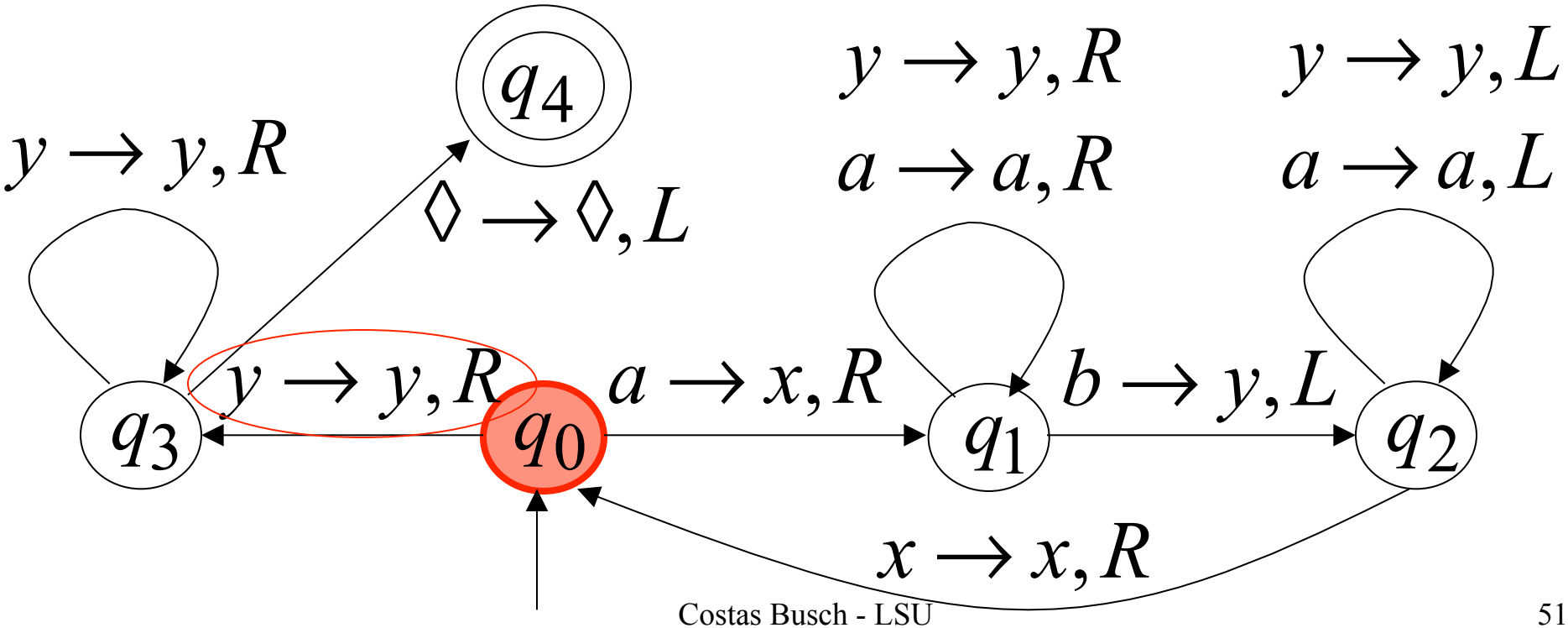
Time 9


◊ x x y y ◊ ◊
q_2

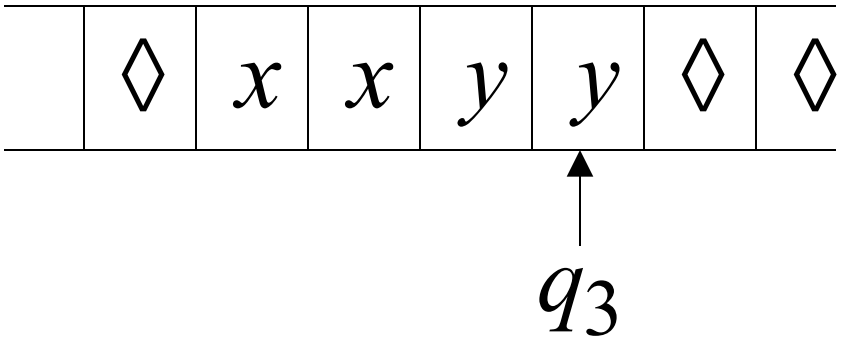

q_4
$y \to y, R$
$\Diamond \to \Diamond, L$
$y \to y, R$
$a \to a, R$
$y \to y, L$
$a \to a, L$
q_3
$y \to y, R$
q_0
$a \to x, R$
q_1
$b \to y, L$
q_2
$x \to x, R$

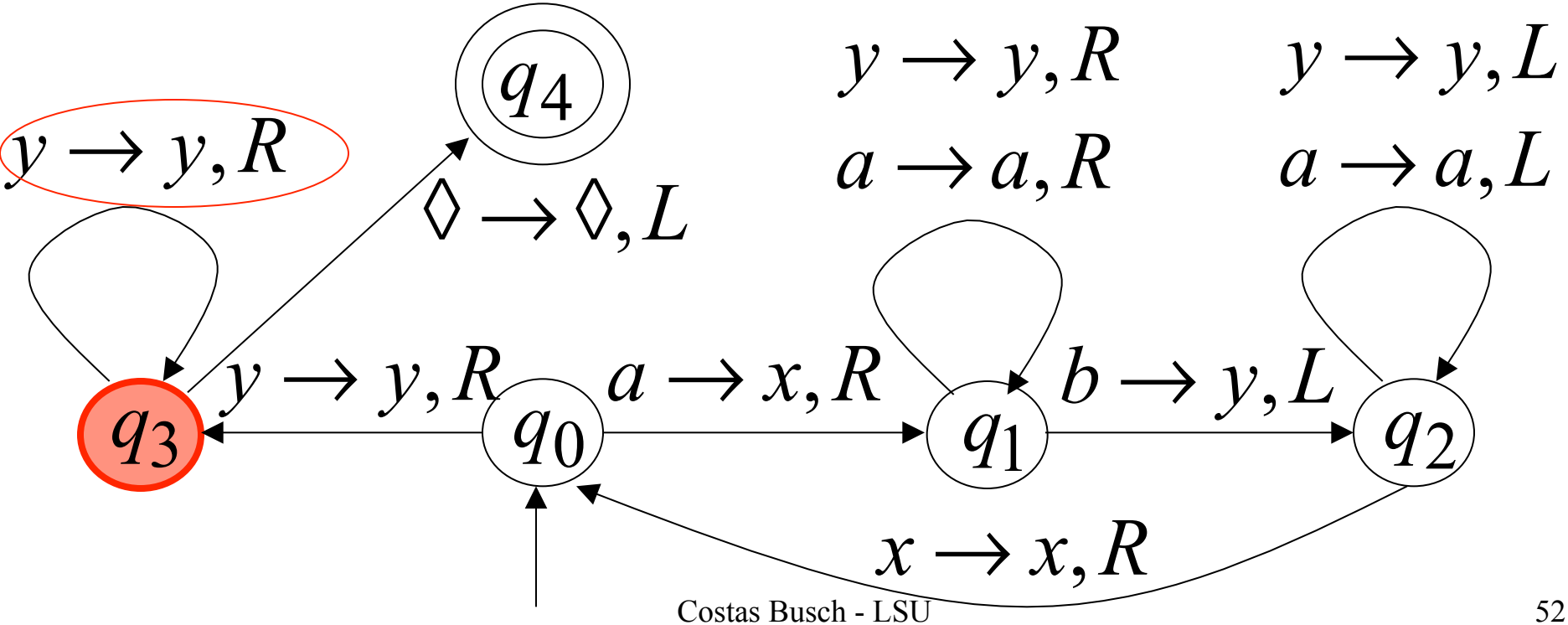
Time 10


◊ x x y y ◊ ◊
q_0

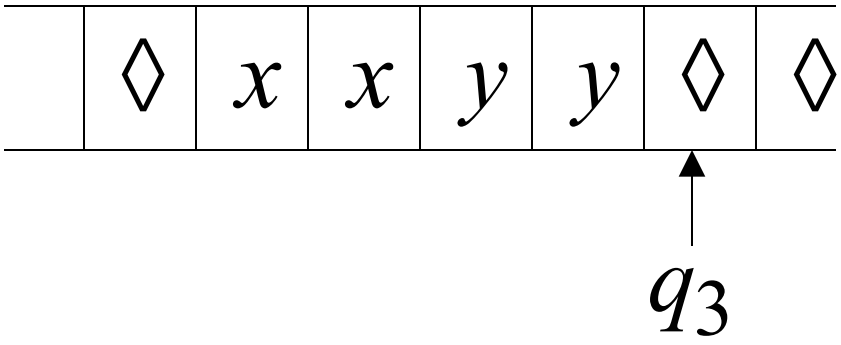

q_4
$y \to y, R$
$y \to y, R$
$a \to a, R$
$y \to y, L$
$a \to a, L$
$\Diamond \to \Diamond, L$
$y \to y, R$
$a \to x, R$
$b \to y, L$
q_3
q_0
q_1
q_2
$x \to x, R$

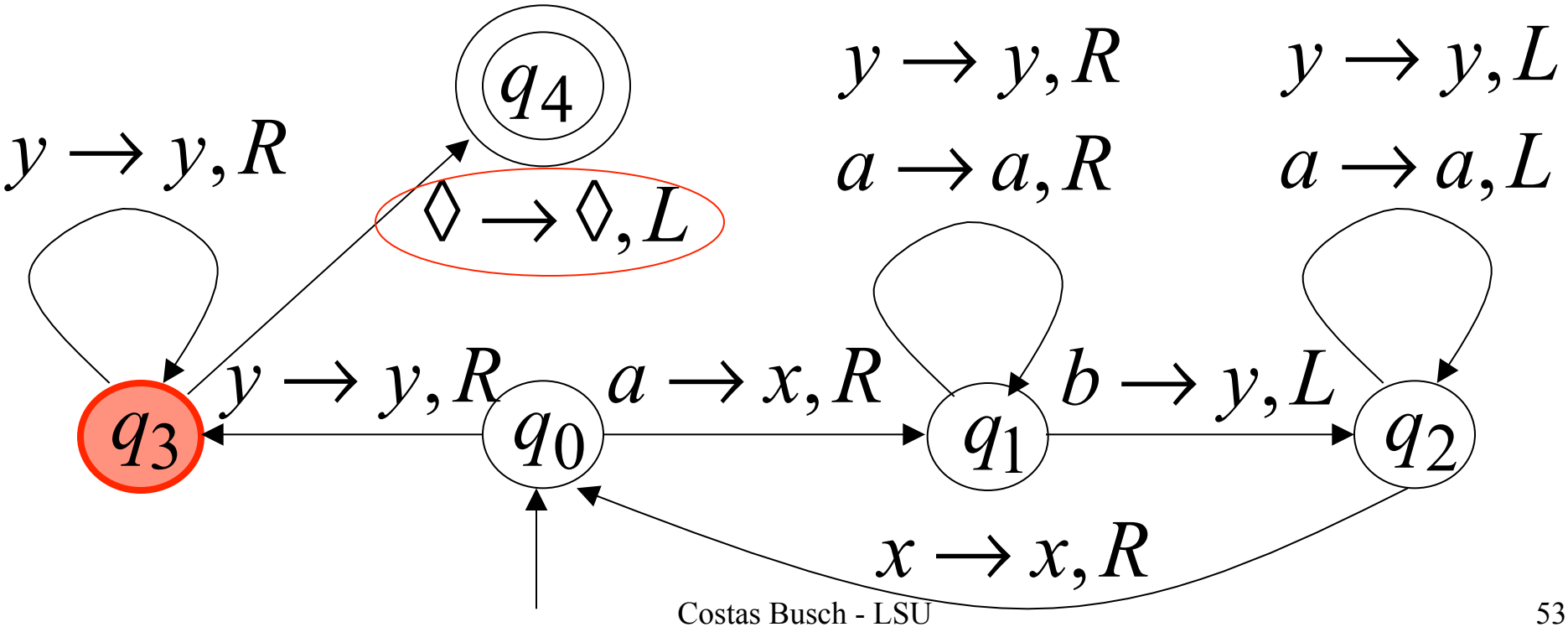
Time 11


◊ x x y y ◊ ◊
q_3

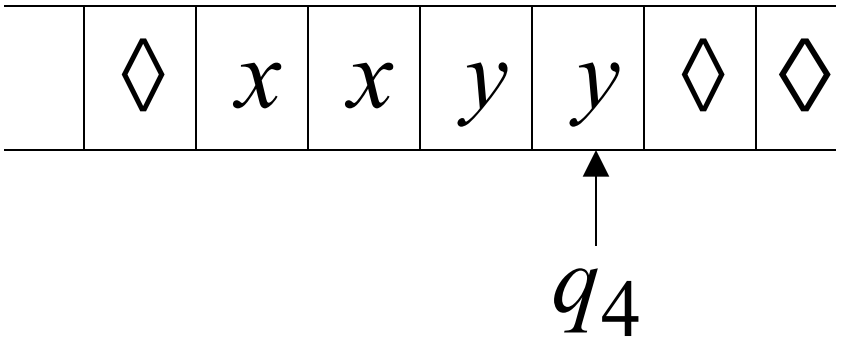

q_4
$y \to y, R$
$a \to a, R$
$y \to y, L$
$a \to a, L$
$y \to y, R$
$\Diamond \to \Diamond, L$
$y \to y, R$
$a \to x, R$
$b \to y, L$
q_3
q_0
q_1
q_2
$x \to x, R$

Time 12

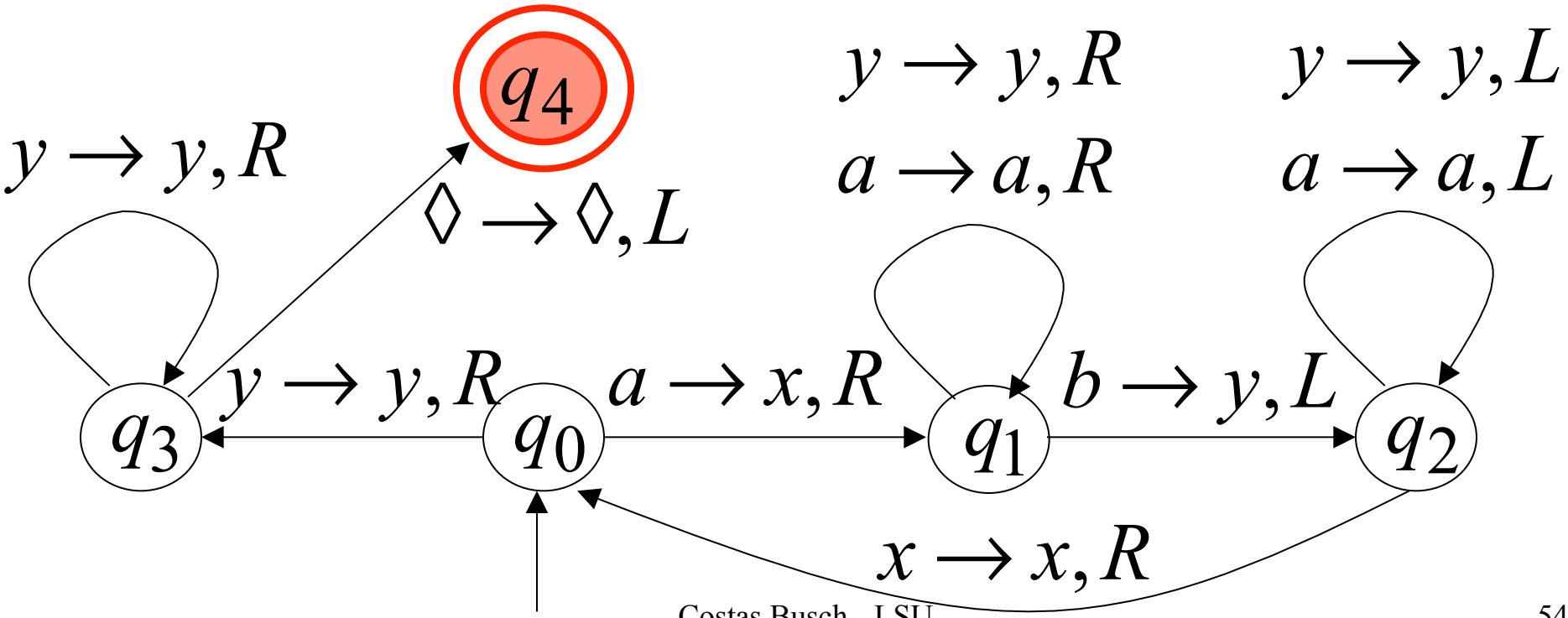

◊ x x y y ◊ ◊
q_3


q_4
$y \to y, R$
$a \to a, R$
$y \to y, L$
$a \to a, L$
$y \to y, R$
$\Diamond \to \Diamond, L$
q_3
$y \to y, R$
q_0
$a \to x, R$
q_1
$b \to y, L$
q_2
$x \to x, R$

Time 13



Halt & Accept



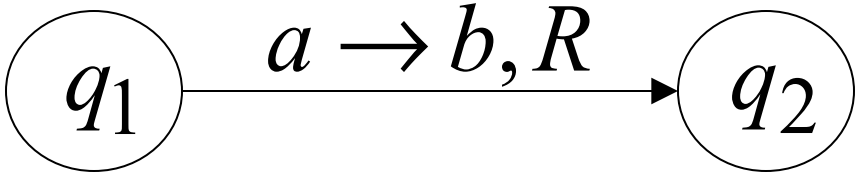
Observation:

If we modify the
machine for the language $\{a^n b^n\}$

we can easily construct
a machine for the language $\{a^n b^n c^n\}$

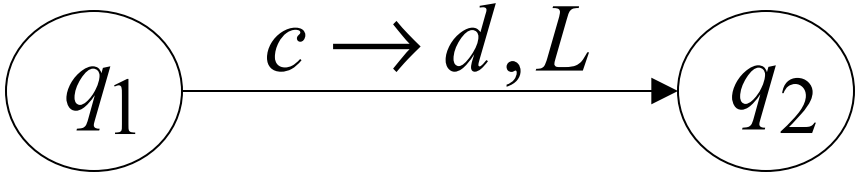
Formal Definitions for Turing Machines

Transition Function



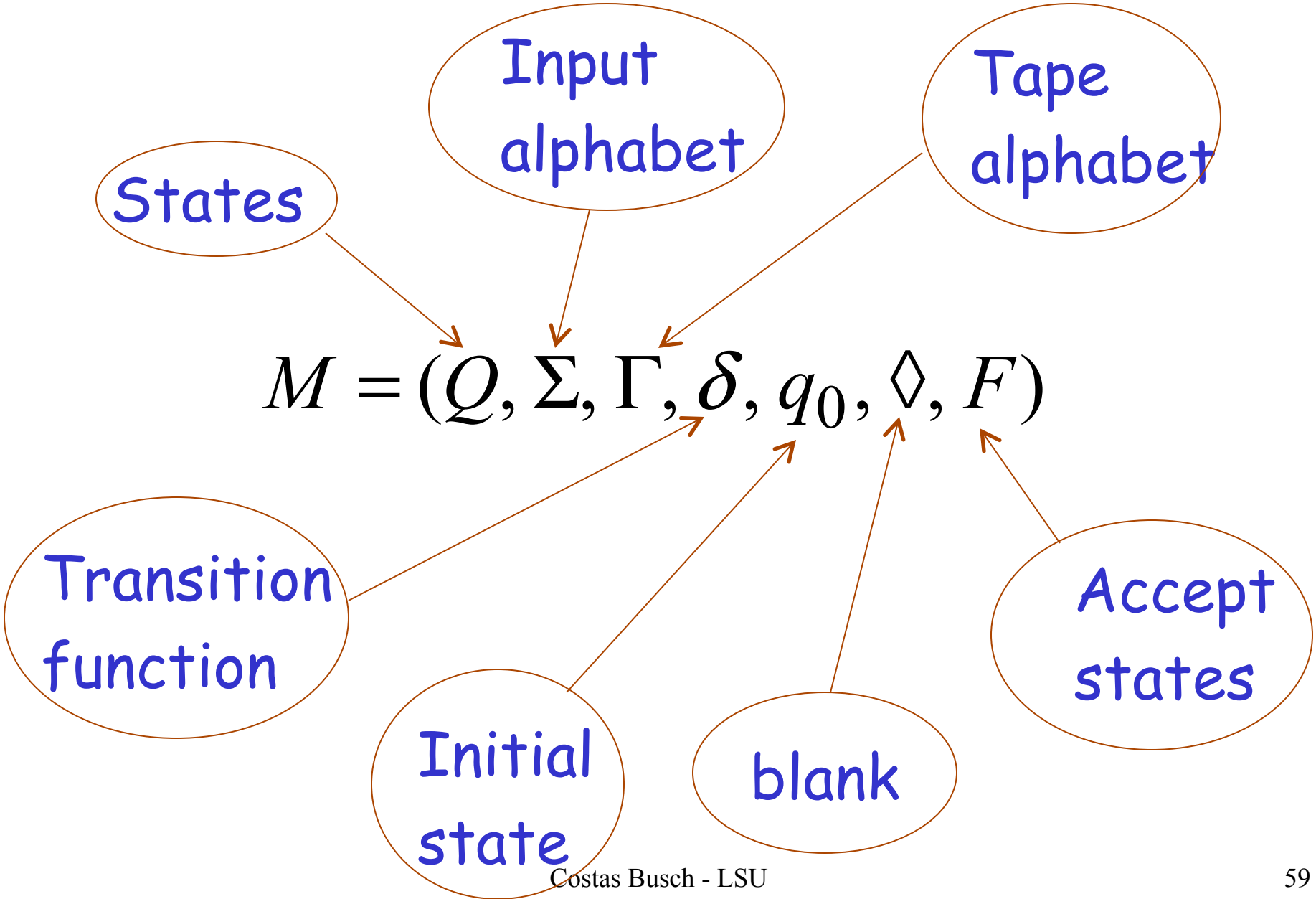
$$\delta(q_1, a) = (q_2, b, R)$$

Transition Function

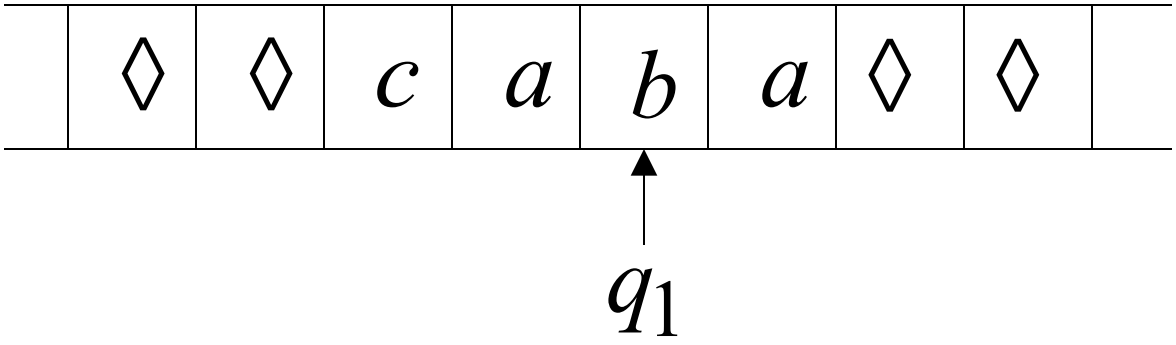


$$\delta(q_1, c) = (q_2, d, L)$$

Turing Machine:

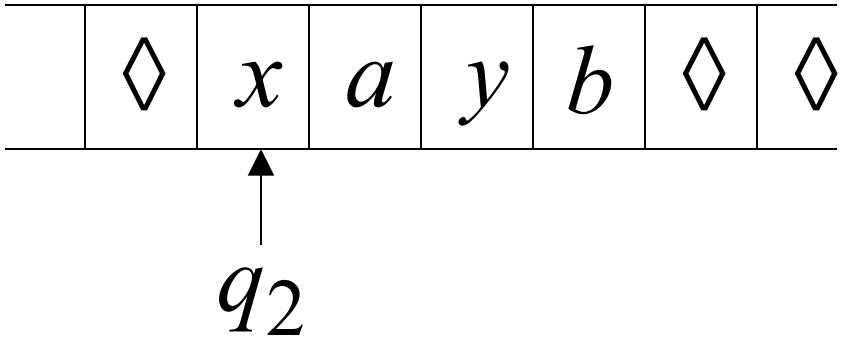


Configuration

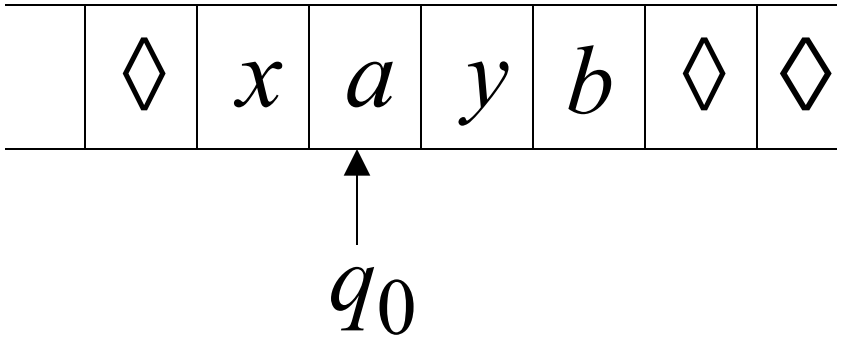


Instantaneous description: $ca\ q_1\ ba$

Time 4



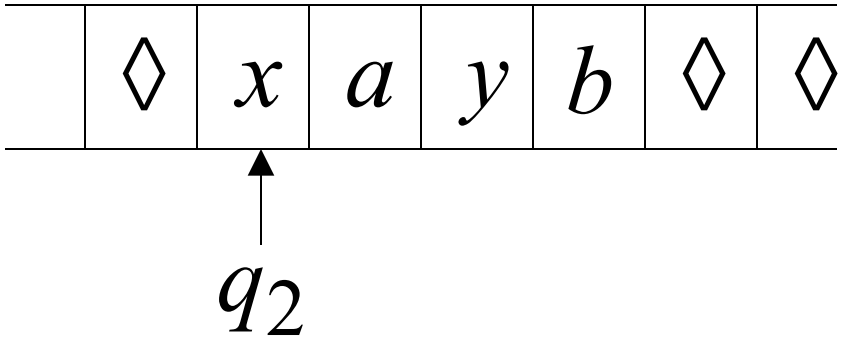
Time 5



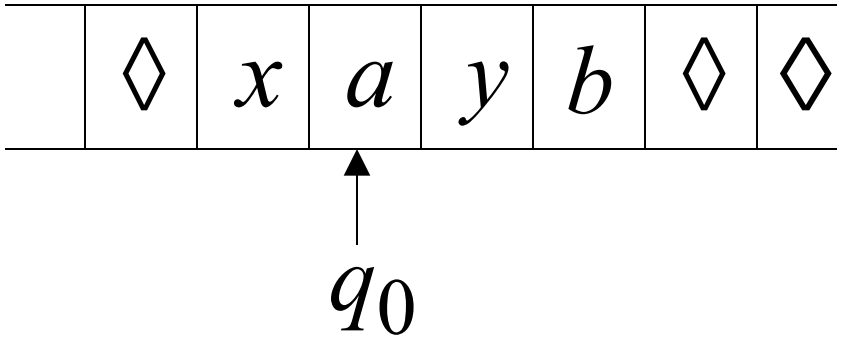
A Move: $q_2\ xayb \succ x\ q_0\ ayb$

(yields in one move)

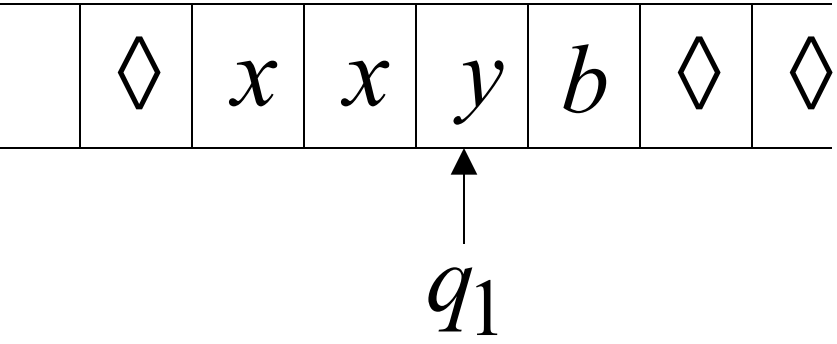
Time 4



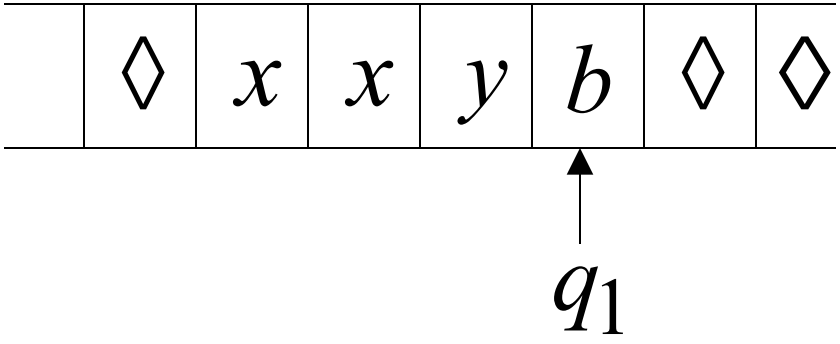
Time 5



Time 6



Time 7



A computation

$$q_2\ xayb \succ x\ q_0\ ayb \succ xx\ q_1\ yb \succ xxy\ q_1\ b$$

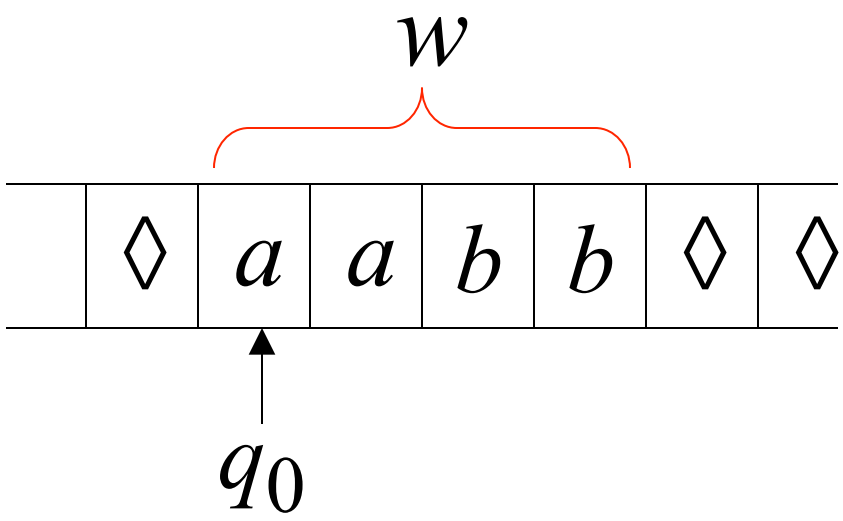
$$q_2\ xayb \succ x\ q_0\ ayb \succ xx\ q_1\ yb \succ xxy\ q_1\ b$$

Equivalent notation:

$$q_2\ xayb \overset{*}{\succ} xxy\ q_1\ b$$

Initial configuration: $q_0\ w$

Input string



The Accepted Language

For any Turing Machine M

$$L(M) = \{w : \quad q_0\, w \overset{*}{\succ} x_1\, q_f\, x_2\}$$

Initial state

Accept state

If a language L is accepted
by a Turing machine M
then we say that L is:

- Turing Recognizable

Other names used:

- Turing Acceptable
- Recursively Enumerable