Pumping Lemma for Context-free Languages

Take an **infinite** context-free language

Generates an infinite number of different strings

Example:

$$S \to ABE \mid bBd$$

$$A \to Aa \mid a$$

$$B \to bSD \mid cc$$

$$D \to Dd \mid d$$

$$E \to eE \mid e$$

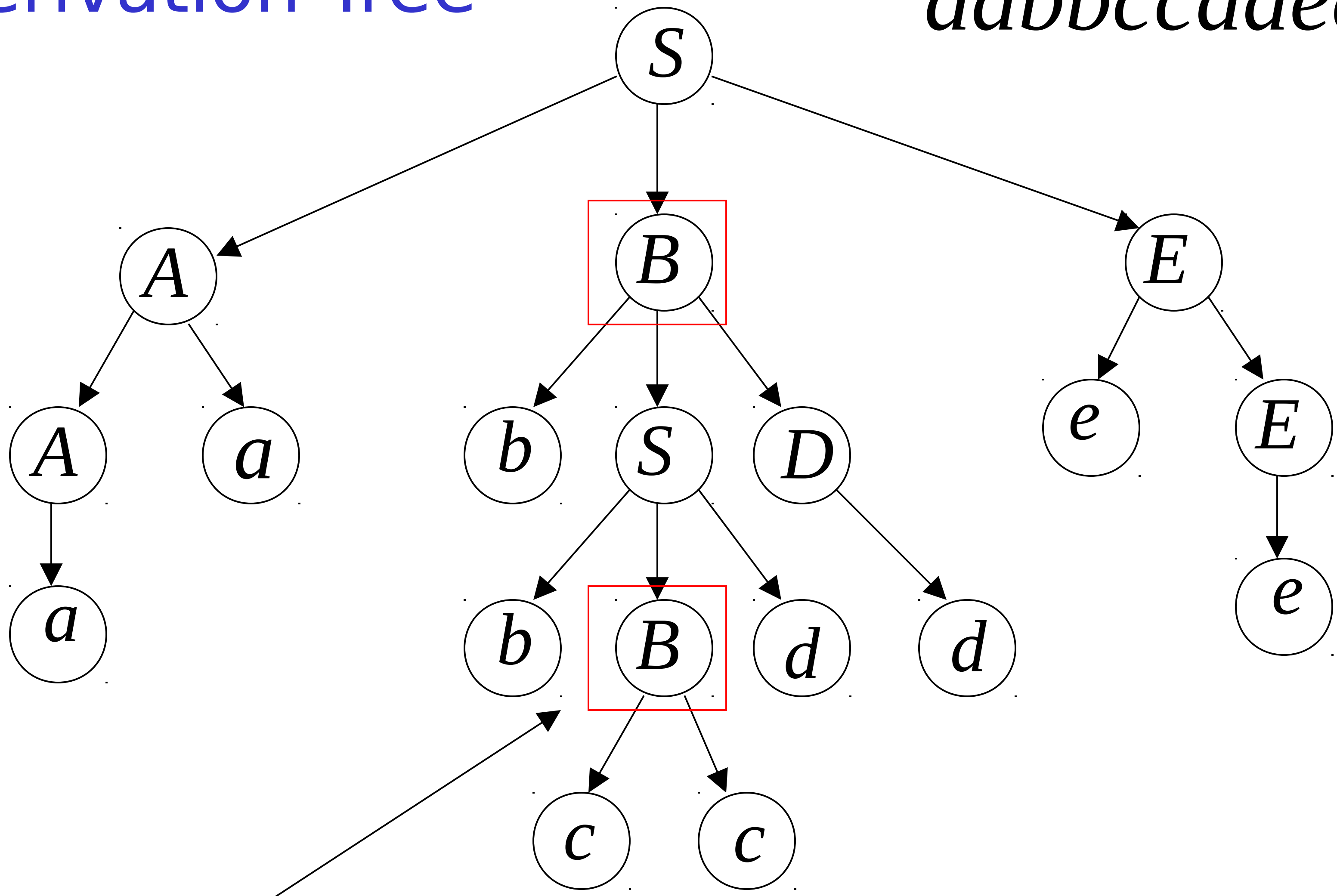
In a derivation of a "long" enough string, variables are repeated

A possible derivation:

$$S \Rightarrow ABE \Rightarrow AaBE \Rightarrow aaBE$$
$$\Rightarrow aabSDE \Rightarrow aabbBdDE \Rightarrow$$
$$\Rightarrow aaabbccdDE \Rightarrow aabbccddE$$
$$\Rightarrow aabbccddeE \Rightarrow aabbccddee$$

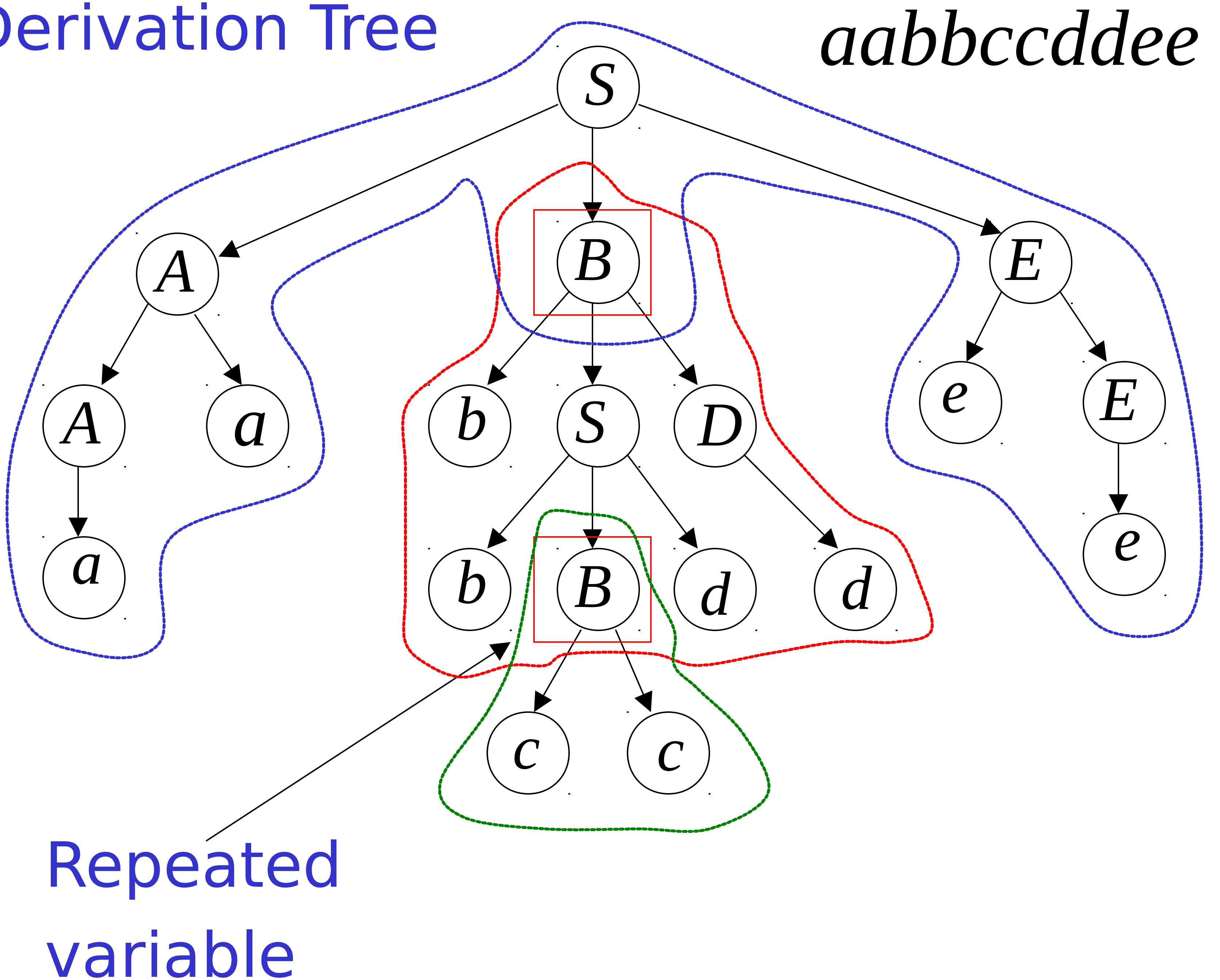
Derivation Tree

$aabbccddee$

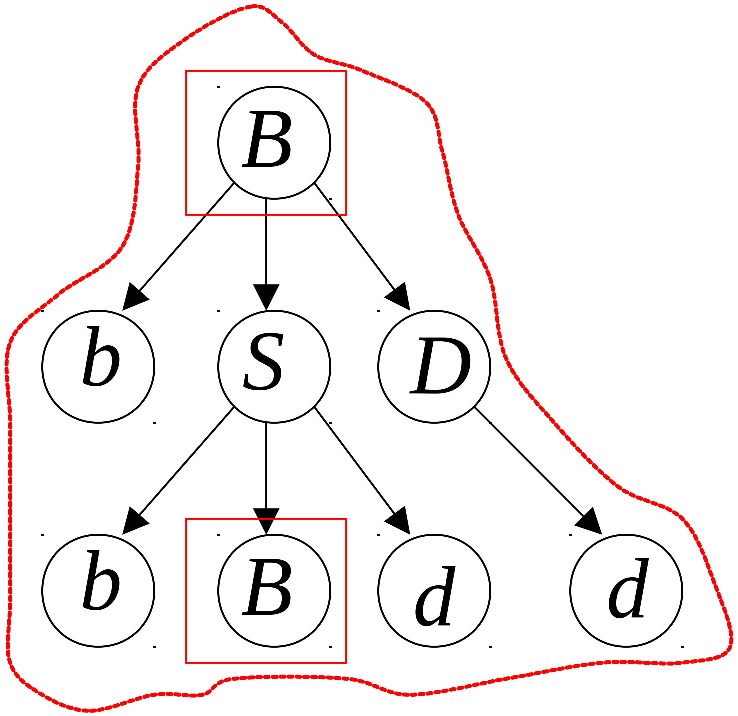


Repeated variable

Derivation Tree

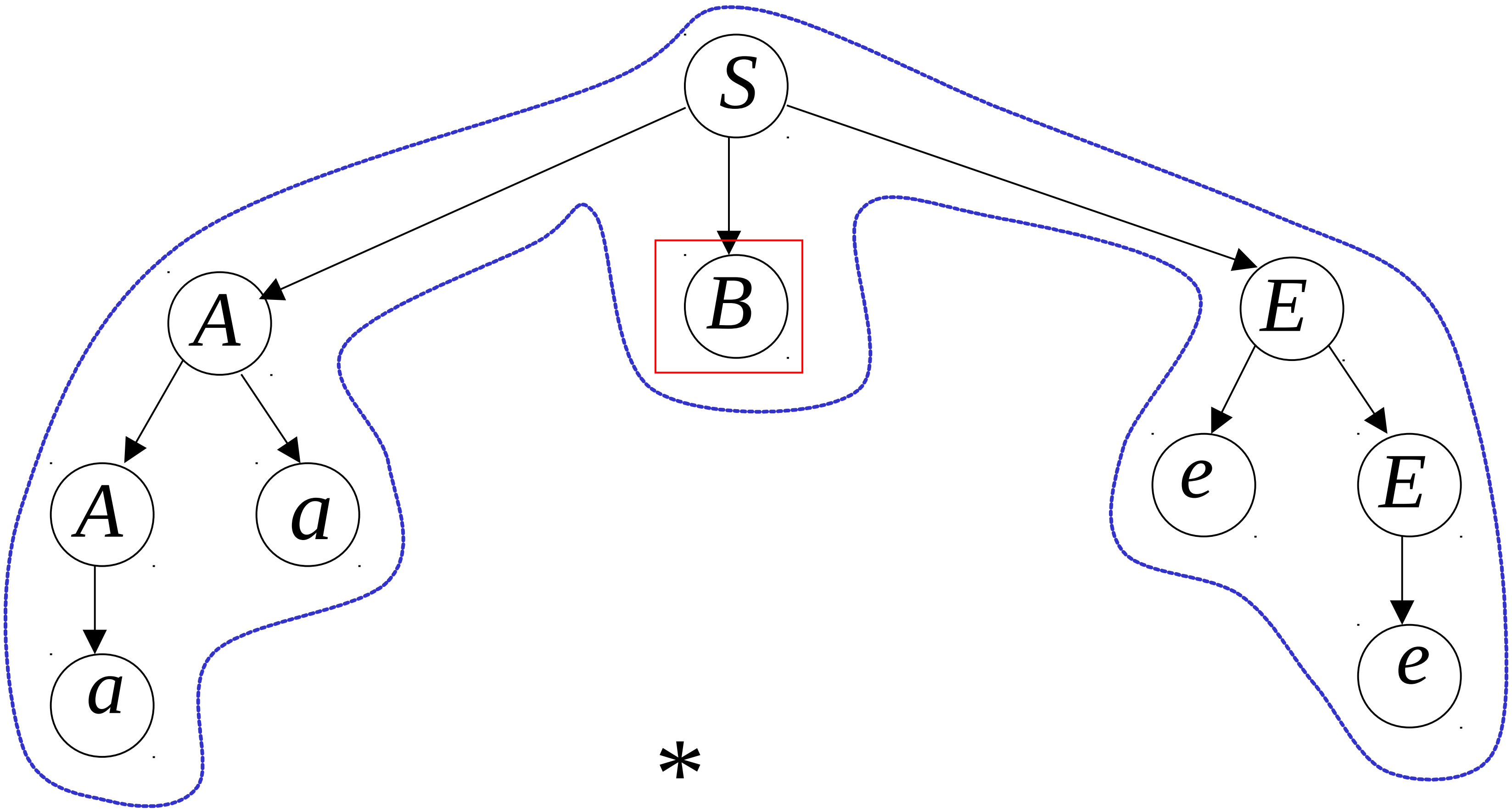


$$B \Rightarrow bSD \Rightarrow bbBdD \Rightarrow bbBdd$$

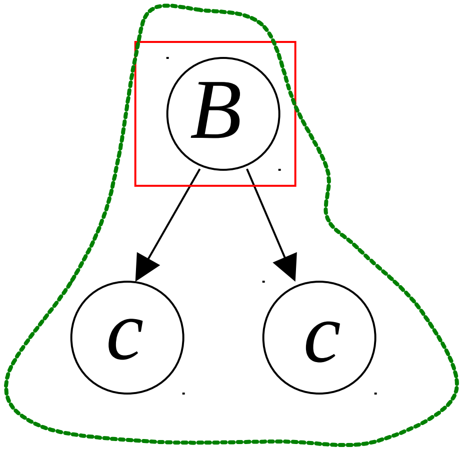


$$B \overset{*}{\Rightarrow} bbBdd$$

$$S \Rightarrow ABE \Rightarrow AaBE \Rightarrow aaBE \Rightarrow aaBeE \Rightarrow aaBee$$

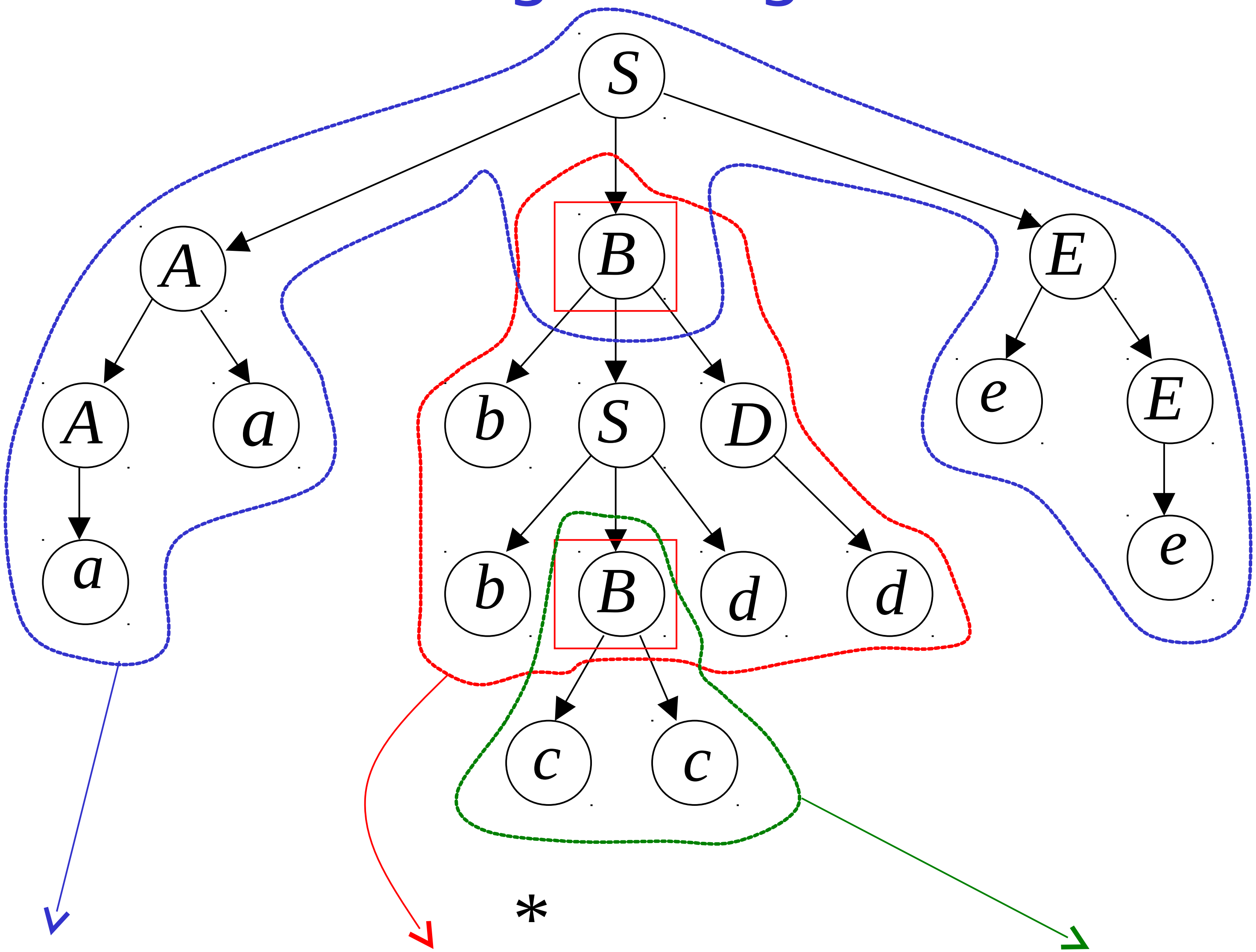


$$S \overset{*}{\Rightarrow} aaBee$$



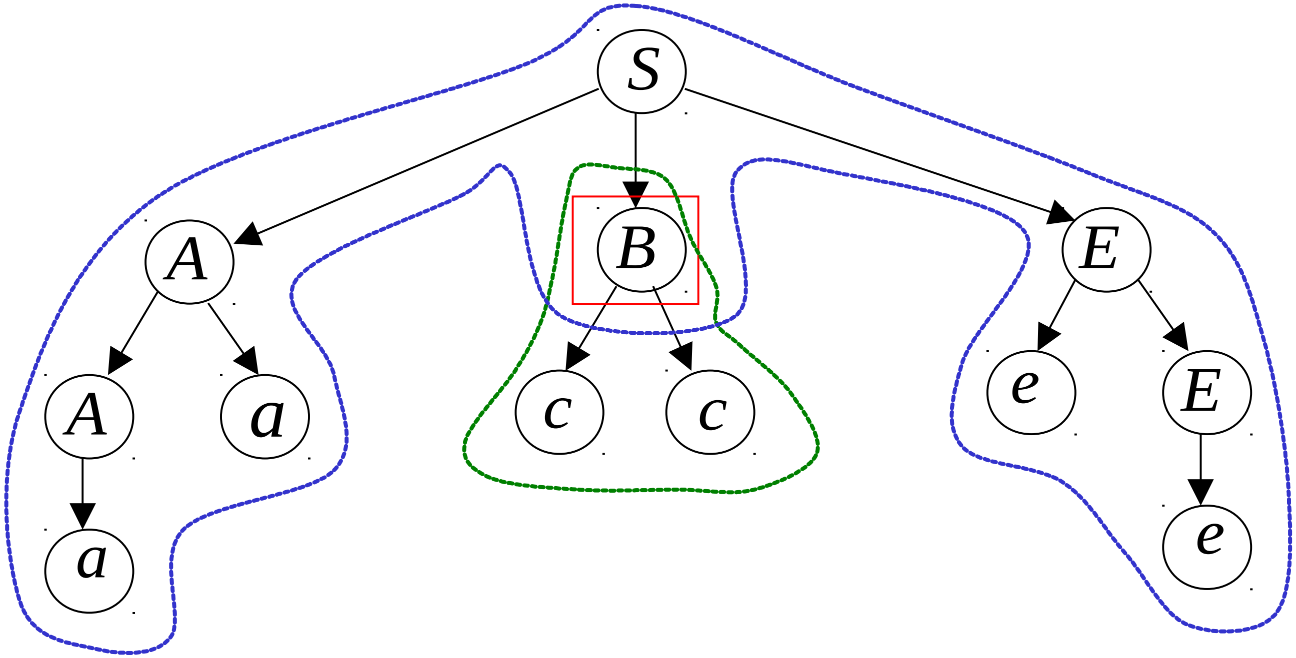
$$B \Rightarrow cc$$

Putting all together



$S \overset{*}{\Rightarrow} aaBee$ $\quad B \overset{*}{\Rightarrow} bbBdd$ $\quad B \Rightarrow cc$

We can remove the middle part



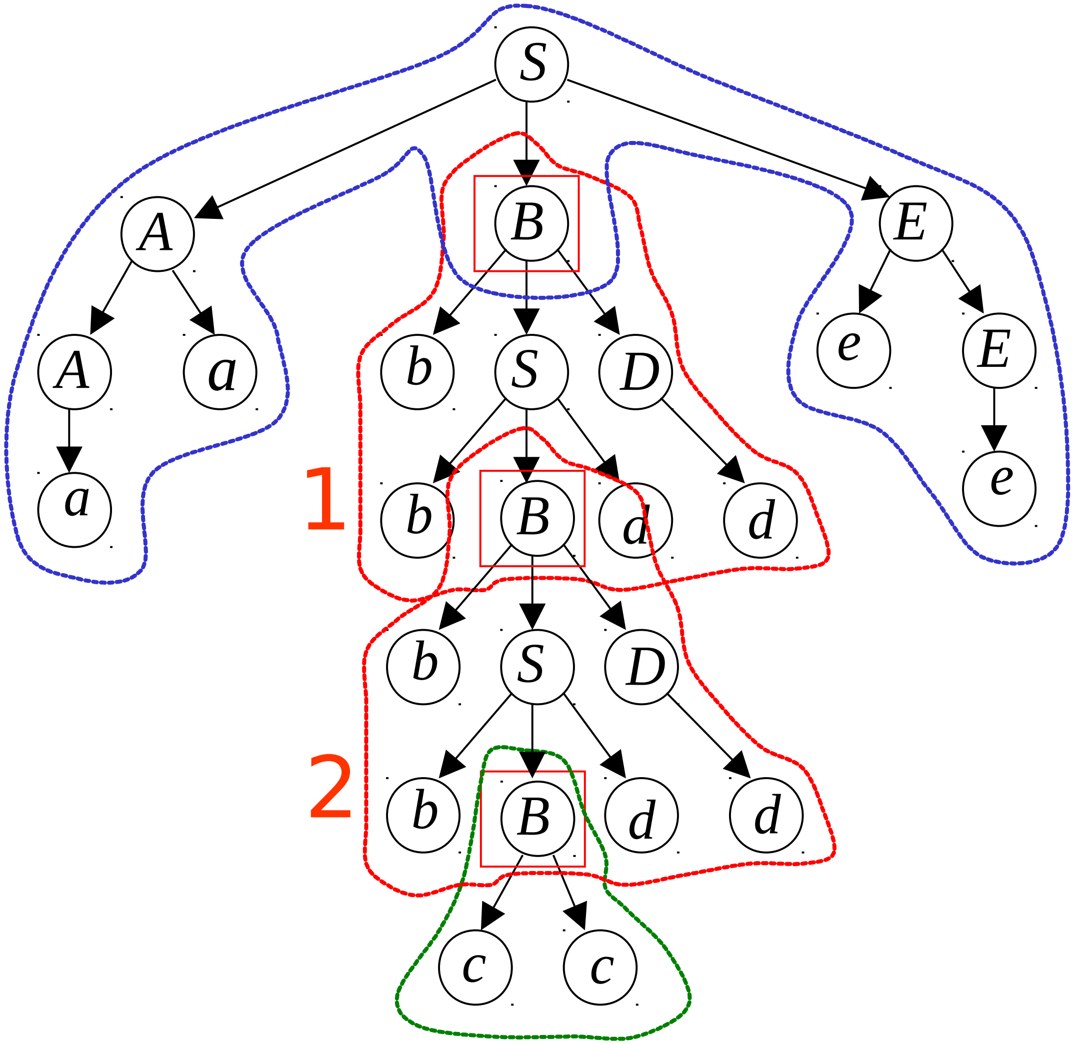
$$S \overset{*}{\Rightarrow} aa(bb)^0 cc(dd)^0 ee$$

$S \overset{*}{\Rightarrow} aaBee \qquad B \overset{*}{\Rightarrow} bbBdd \qquad B \Rightarrow cc$

$$S \overset{*}{\Rightarrow} aaBee \overset{*}{\Rightarrow} aaccee = aa(bb)^0 cc(dd)^0 ee$$

$$aa(bb)^0 cc(dd)^0 ee \in L(G)$$

We can repeated middle part two times



$$S \overset{*}{\Rightarrow} aa(bb)^2 cc(dd)^2 ee$$

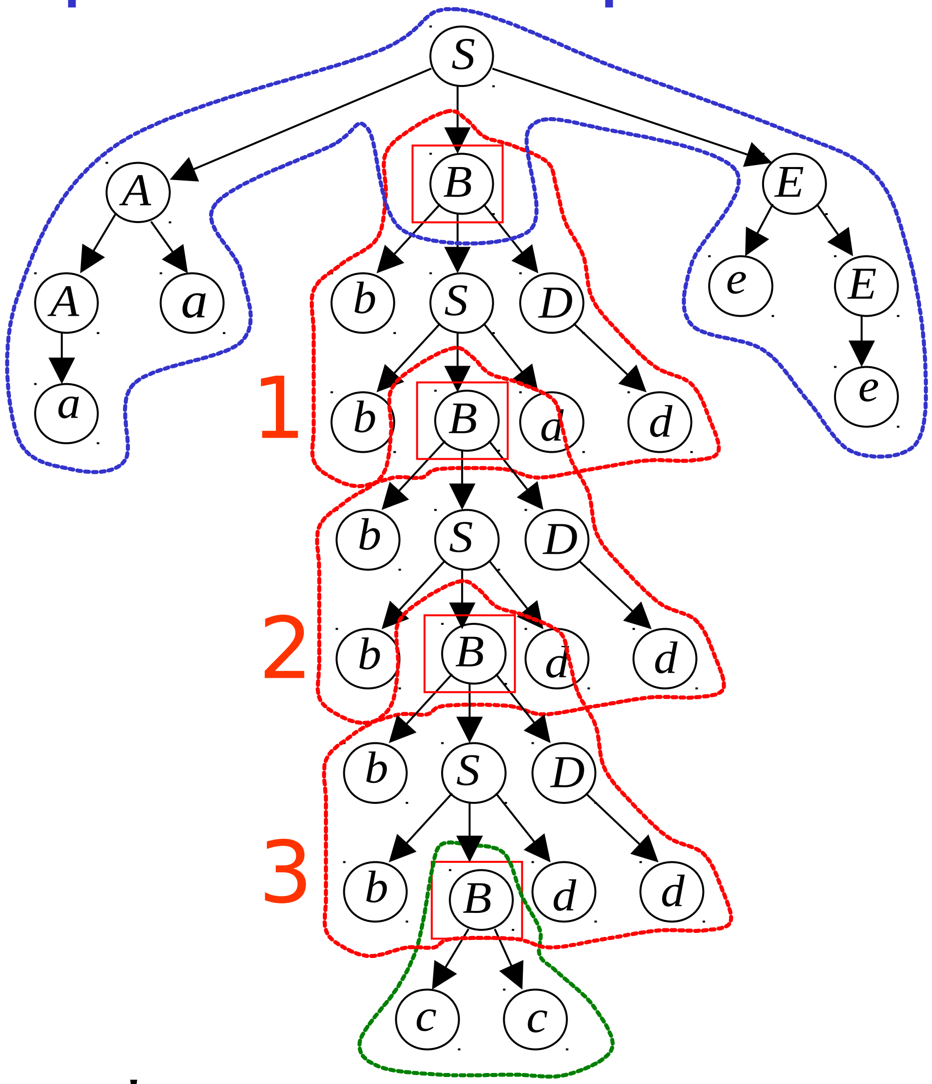
$$S \overset{*}{\Rightarrow} aaBee \qquad B \overset{*}{\Rightarrow} bbBdd \qquad B \Rightarrow cc$$

$$S \overset{*}{\Rightarrow} aaBee \overset{*}{\Rightarrow} aabbBddee$$

$$\overset{*}{\Rightarrow} aa(bb)^2 B(dd)^2 ee \overset{*}{\Rightarrow} aa(bb)^2 cc(dd)^2 ee$$

$$aa(bb)^2 cc(dd)^2 ee \in L(G)$$

We can repeat middle part three times

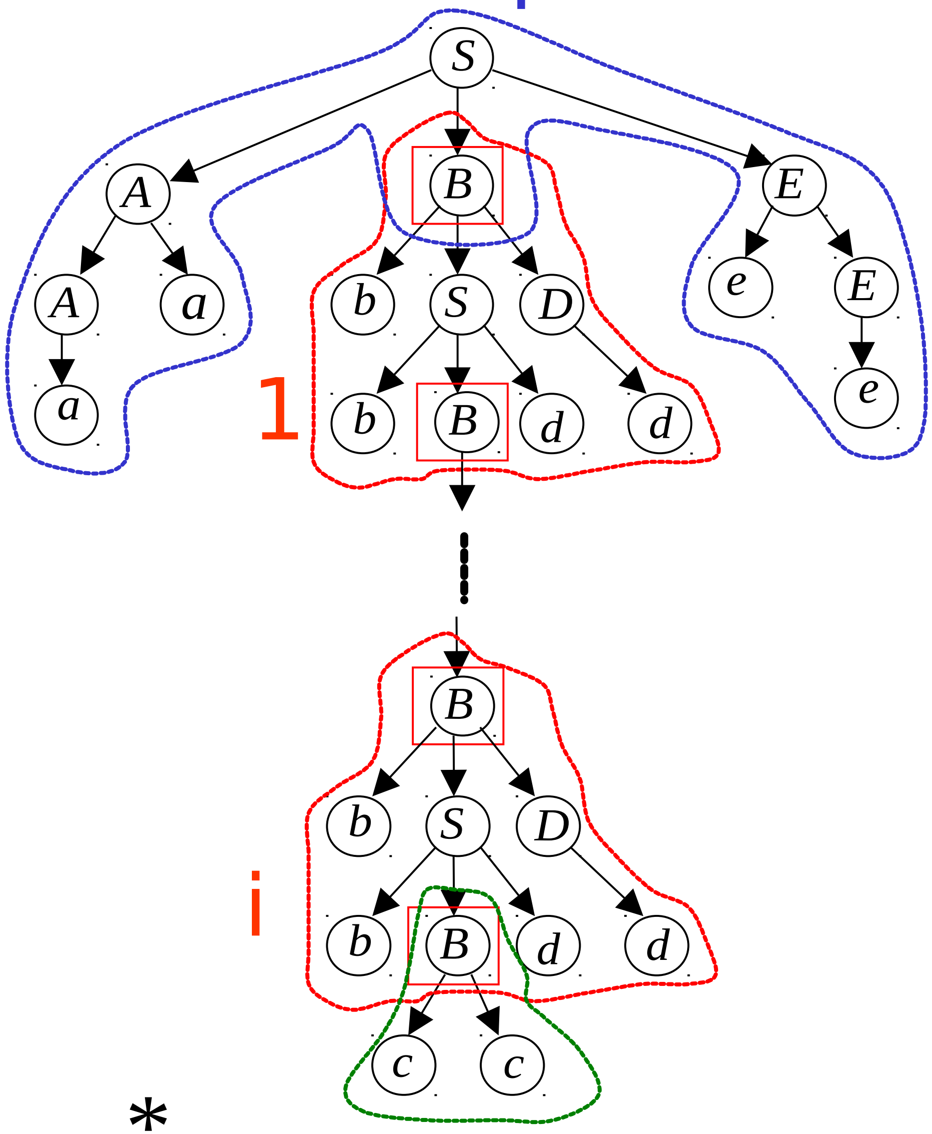


$$S \overset{*}{\Rightarrow} aa(bb)^3cc(dd)^3ee$$

$$S \overset{*}{\Rightarrow} aaBee \qquad B \overset{*}{\Rightarrow} bbBdd \qquad B \Rightarrow cc$$

$$S \overset{*}{\Rightarrow} aa(bb)^3cc(dd)^3ee \quad \in L(G)$$

Repeat middle part i times



$$S \overset{*}{\Rightarrow} aa(bb)^i cc(dd)^i ee$$

$$S \overset{*}{\Rightarrow} aaBee \qquad B \overset{*}{\Rightarrow} bbBdd \qquad B \Rightarrow cc$$

$$S \overset{*}{\Rightarrow} aa(bb)^i cc(dd)^i ee \quad \in L(G)$$

For any $i \geq 0$

From Grammar

$$S \to ABE \mid bBd$$

$$A \to Aa \mid a$$

$$B \to bSD \mid cc$$

$$D \to Dd \mid d$$

$$E \to eE \mid e$$

and given string

$$aabbccddee \quad \in L(G)$$

We inferred that a family of strings is in $L(G)$

$$S \overset{*}{\Rightarrow} aa(bb)^i cc(dd)^i ee \quad \in L(G)$$ for any $i \geq 0$

Arbitrary Grammars

Consider now an arbitrary infinite context-free language L

Let G be the grammar of $L - \{\varepsilon\}$

Take G so that it has no unit-productions and no ε -productions

(remove them)

Let r be the number of variables

Let t be the maximum right-hand size of any production

Example:

$$S \rightarrow ABE \mid bBd$$
$$A \rightarrow Aa \mid a$$
$$B \rightarrow bSD \mid cc$$
$$D \rightarrow Dd \mid d$$
$$E \rightarrow eE \mid e$$

$r = 5$ (variables)

$t = 3$

Claim:

Take string $w \in L(G)$ with $|w| > t^r$.

Then in the derivation tree of w

there is a path from the root to a leaf

where a variable of G is repeated

Proof:

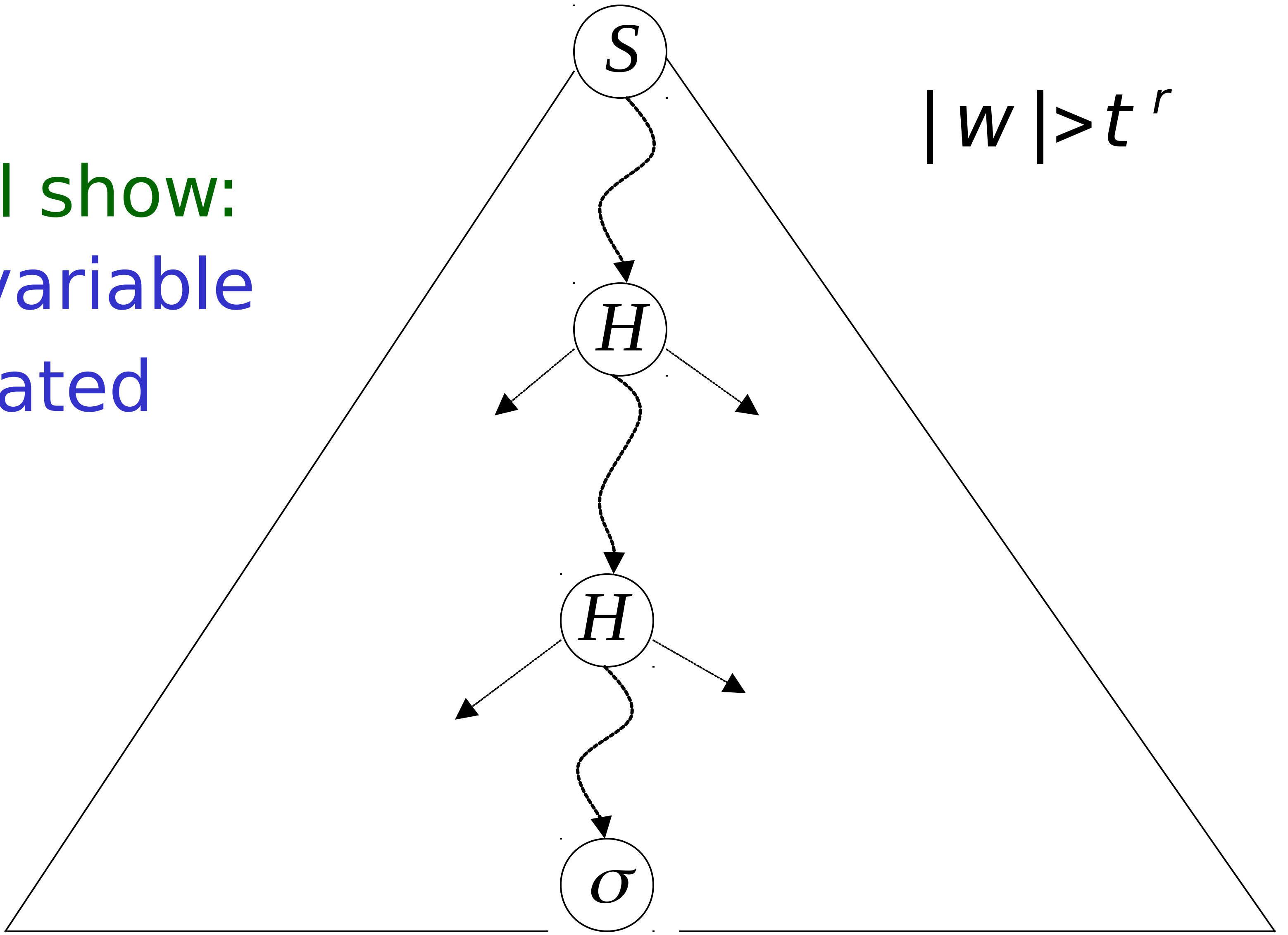
Proof by contradiction

Derivation tree of w

$|w| > t^r$

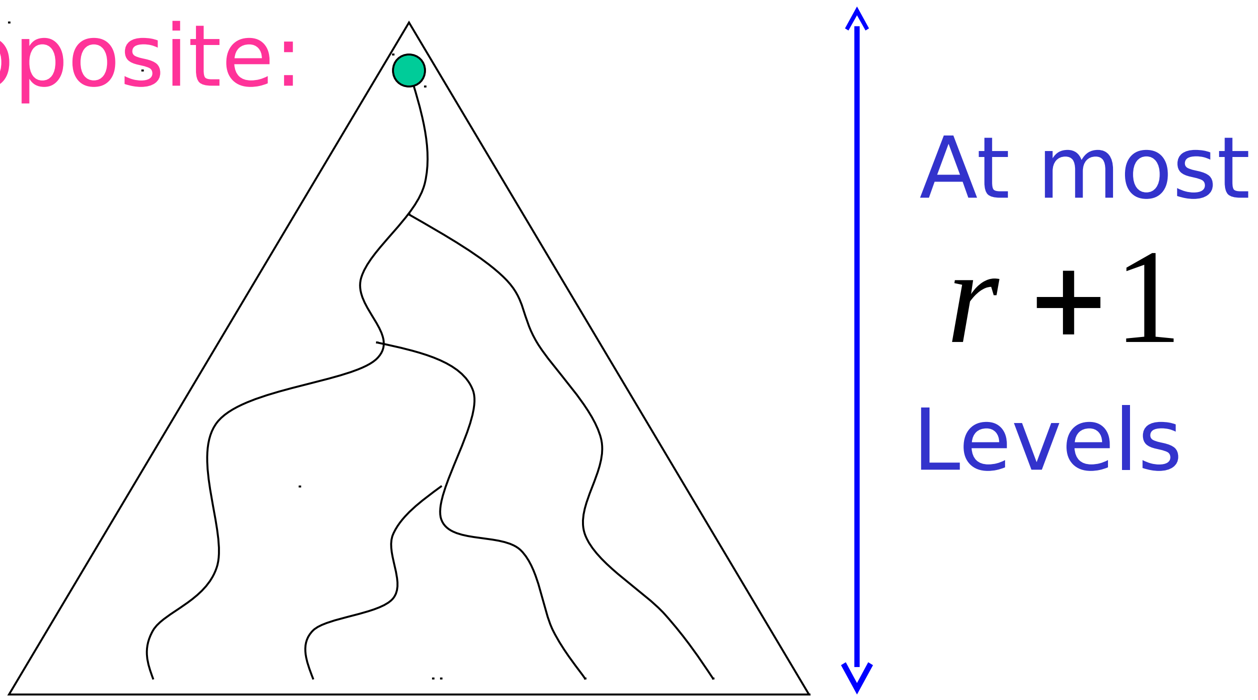
We will show:

some variable is repeated

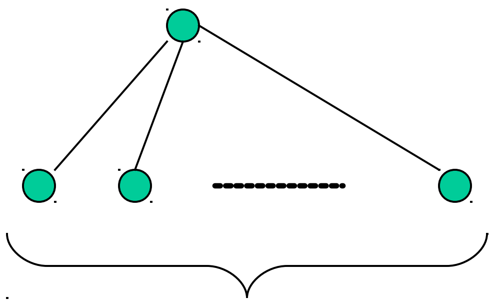


First we show that the tree of w has at least $r+2$ levels of nodes

Suppose the opposite:



Maximum number of nodes per level



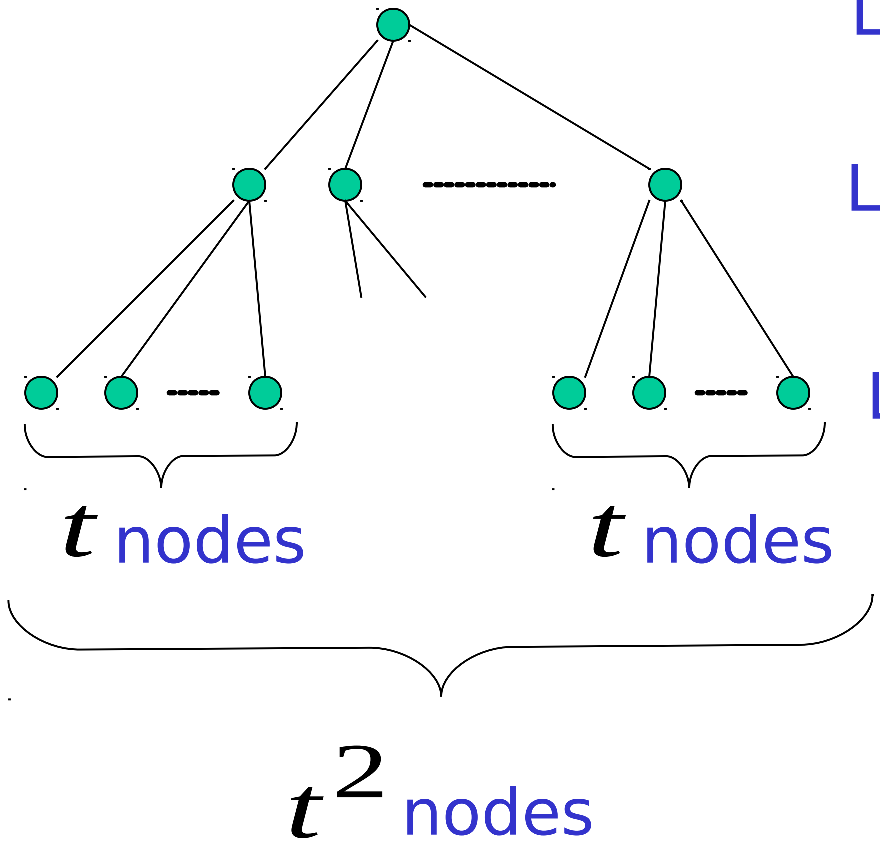
Level 0: 1 nodes

Level 1: t nodes

t nodes

he maximum right-hand side of any productio

Maximum number of nodes per level

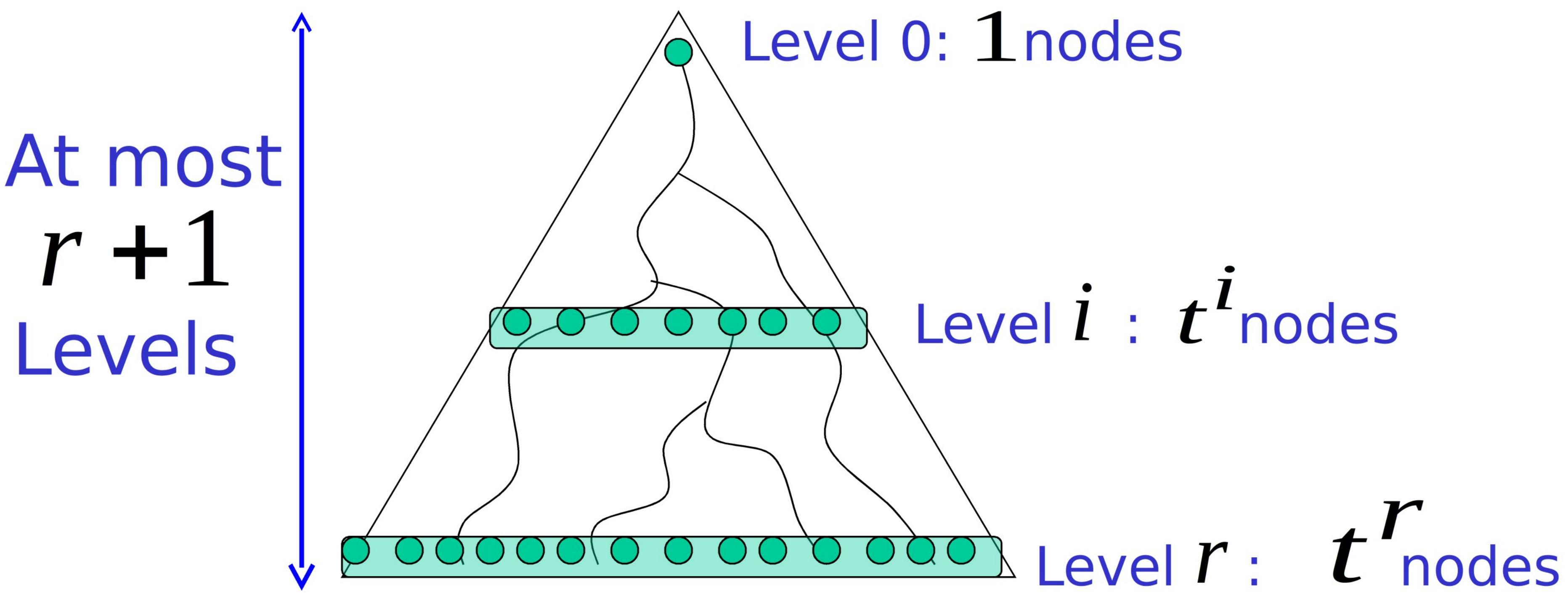


Level 0: 1 nodes

Level 1: t nodes

Level 2: t^2 nodes

Maximum number of nodes per level



Maximum possible string length
= max nodes at level r = t^r

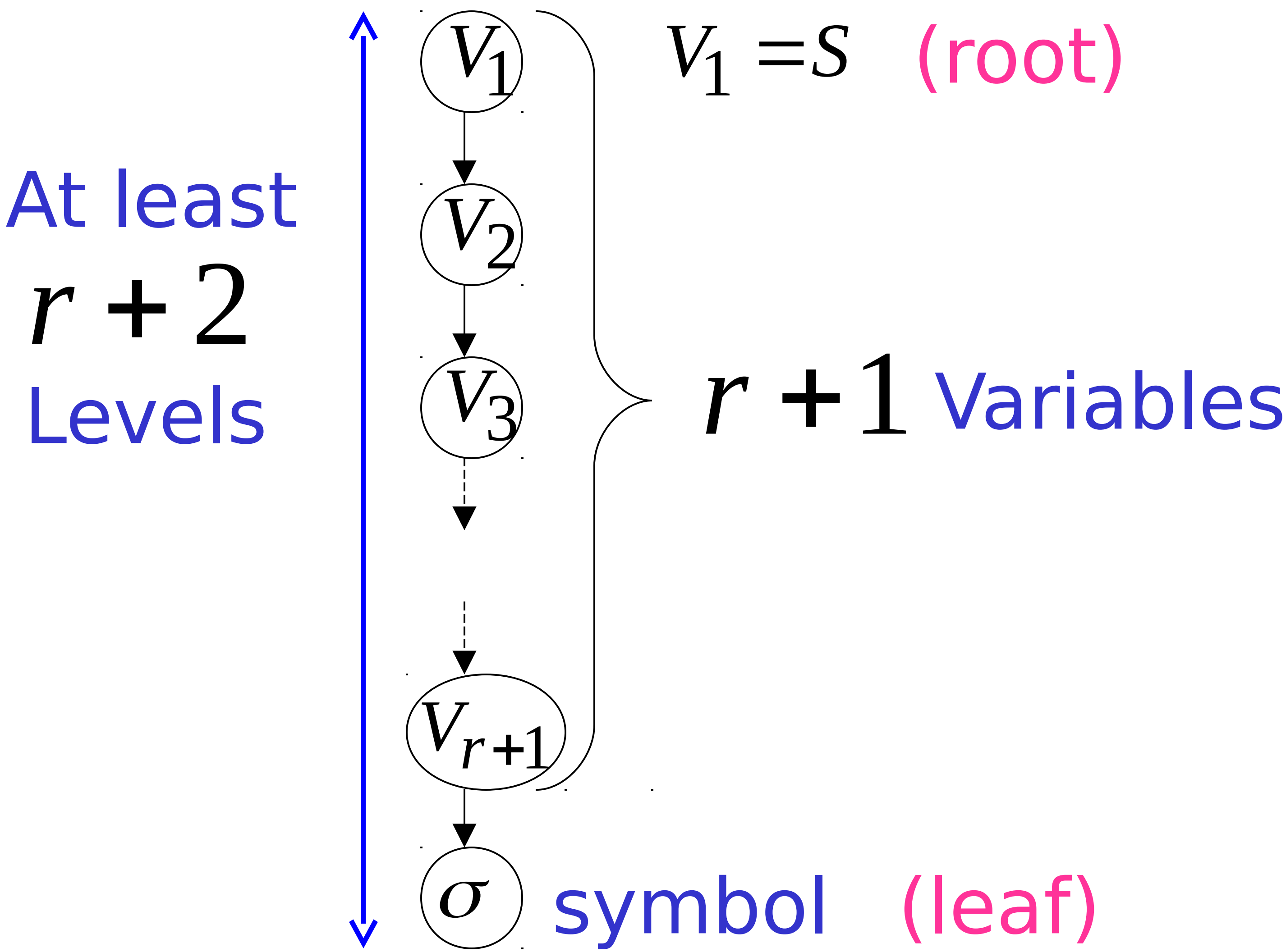
Therefore,
maximum length of string w : $|w| \leq t^r$

However we took, $|w| > t^r$

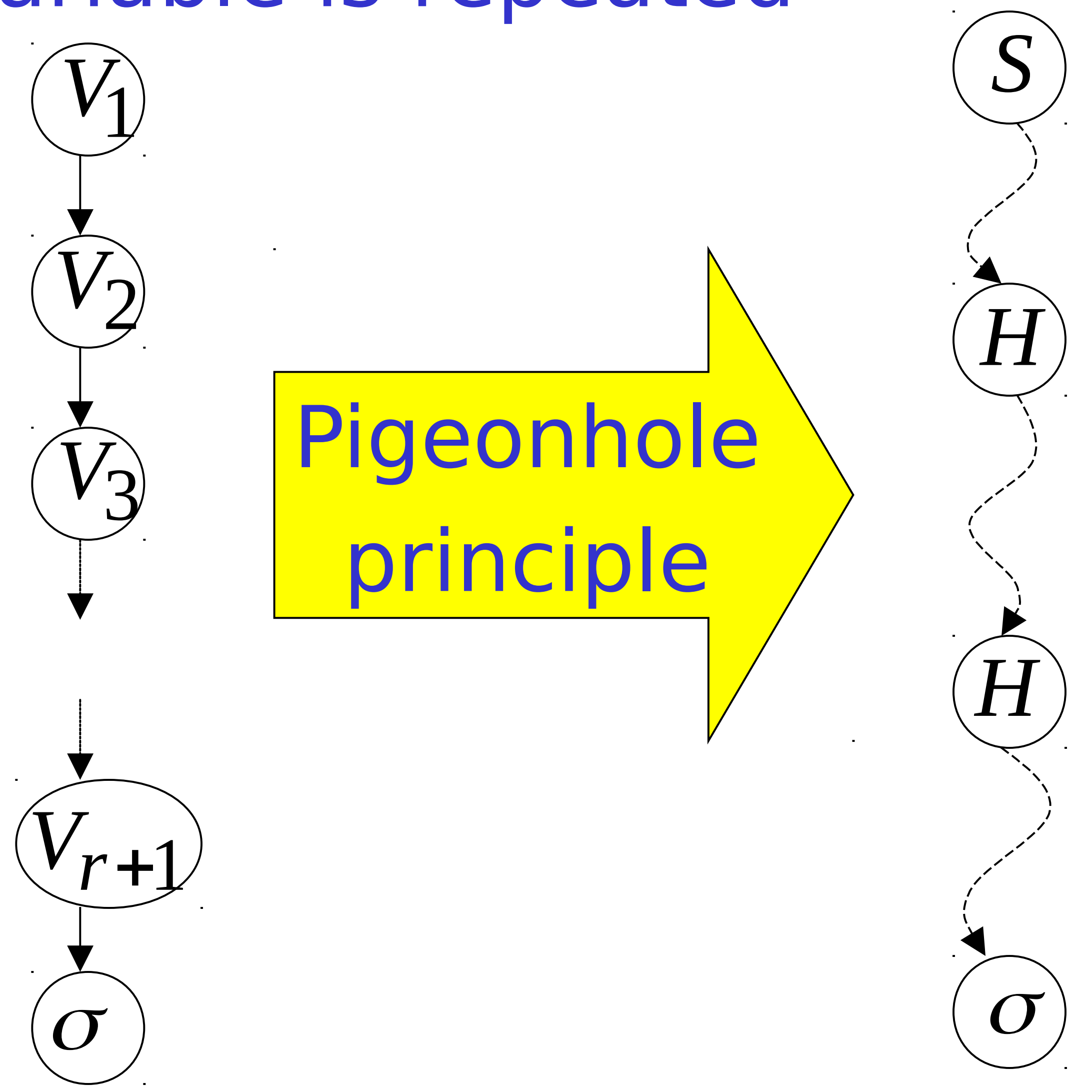
Contradiction!!!

Therefore,
the tree must have at least $r+2$ levels

Thus, there is a path from the root to a leaf with at least $r+2$ nodes



Since there are at most r different variables some variable is repeated

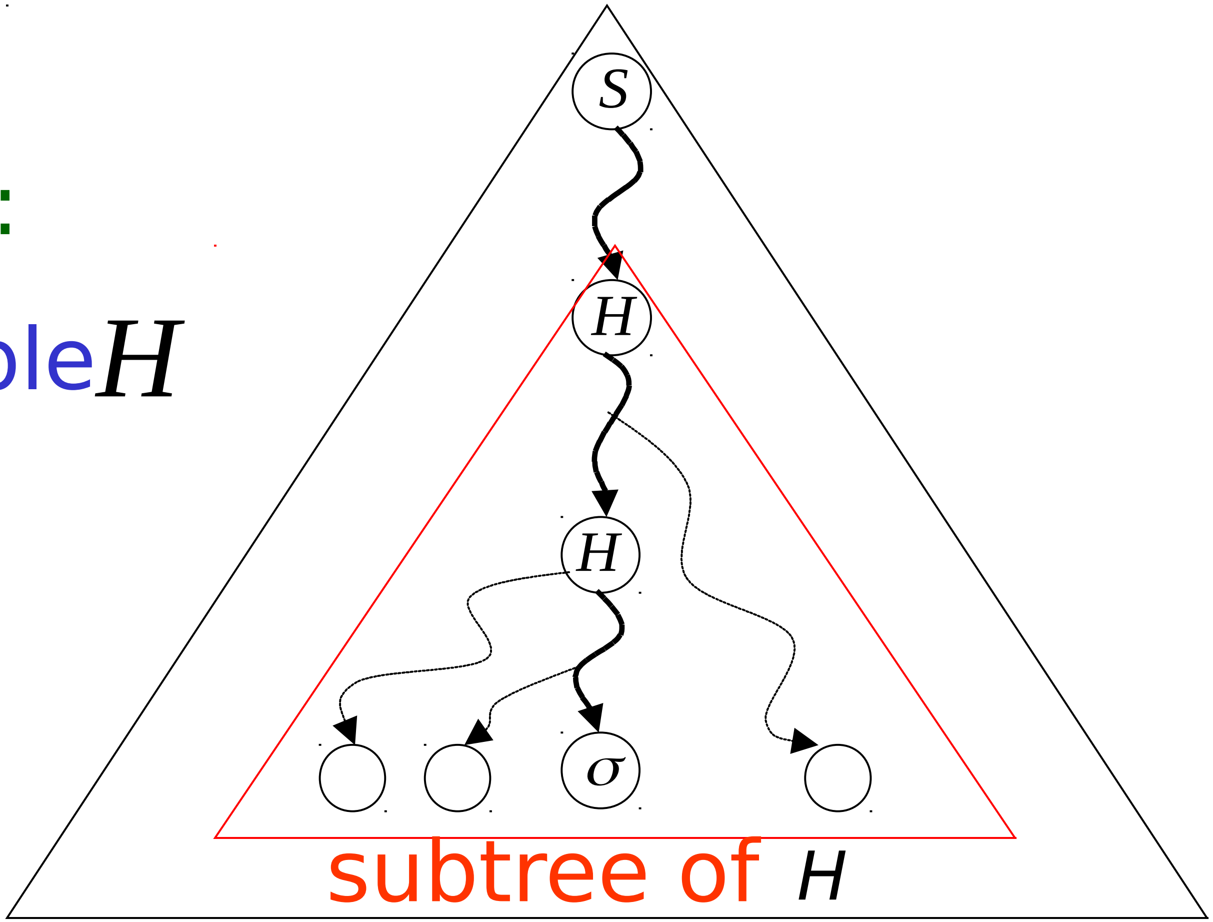


END OF CLAIM PROOF

Take now a string w with $|w| > t^r$

From claim:

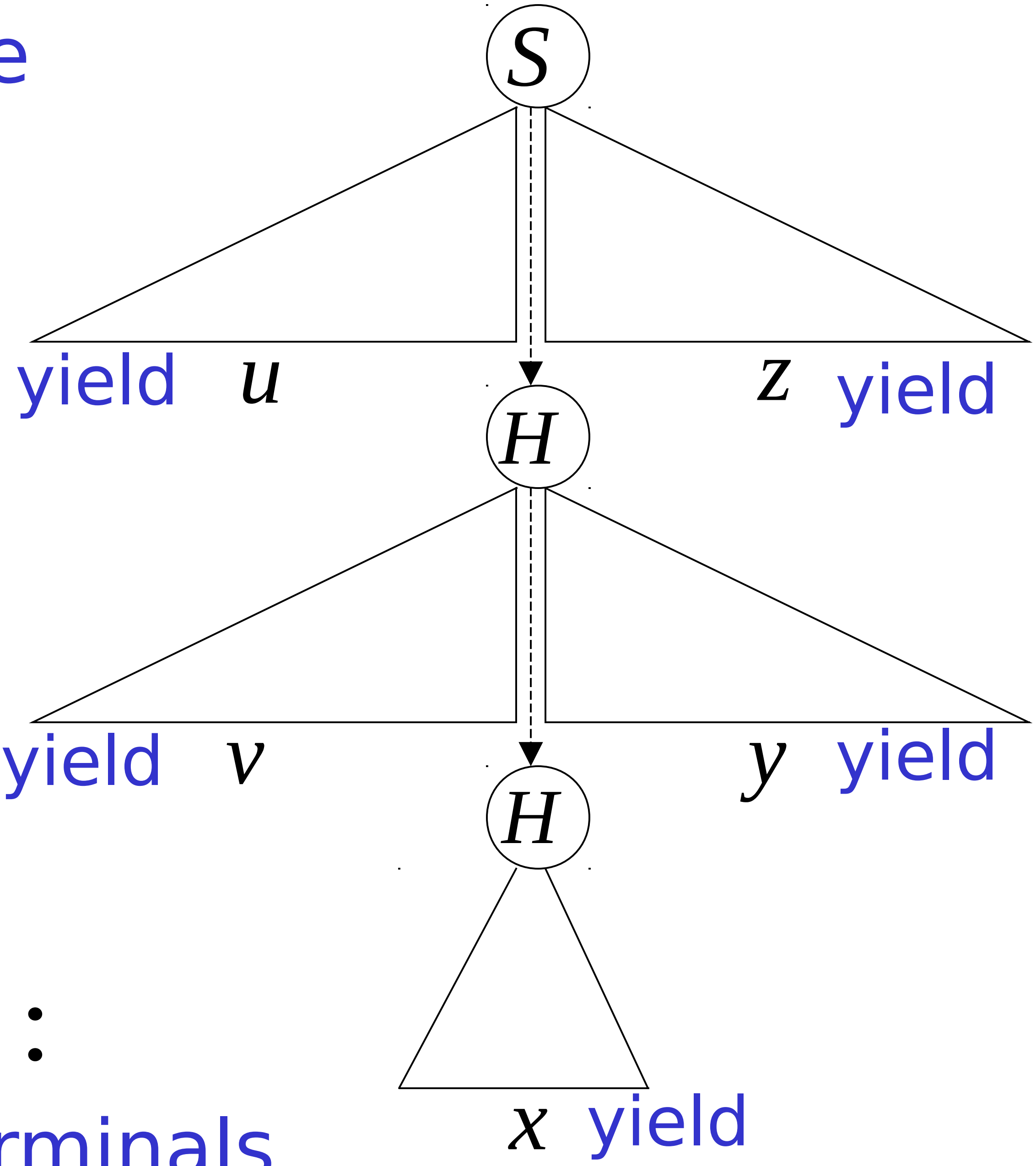
some variable H is repeated



Take H to be the deepest, so that only H is repeated in subtree

We can write

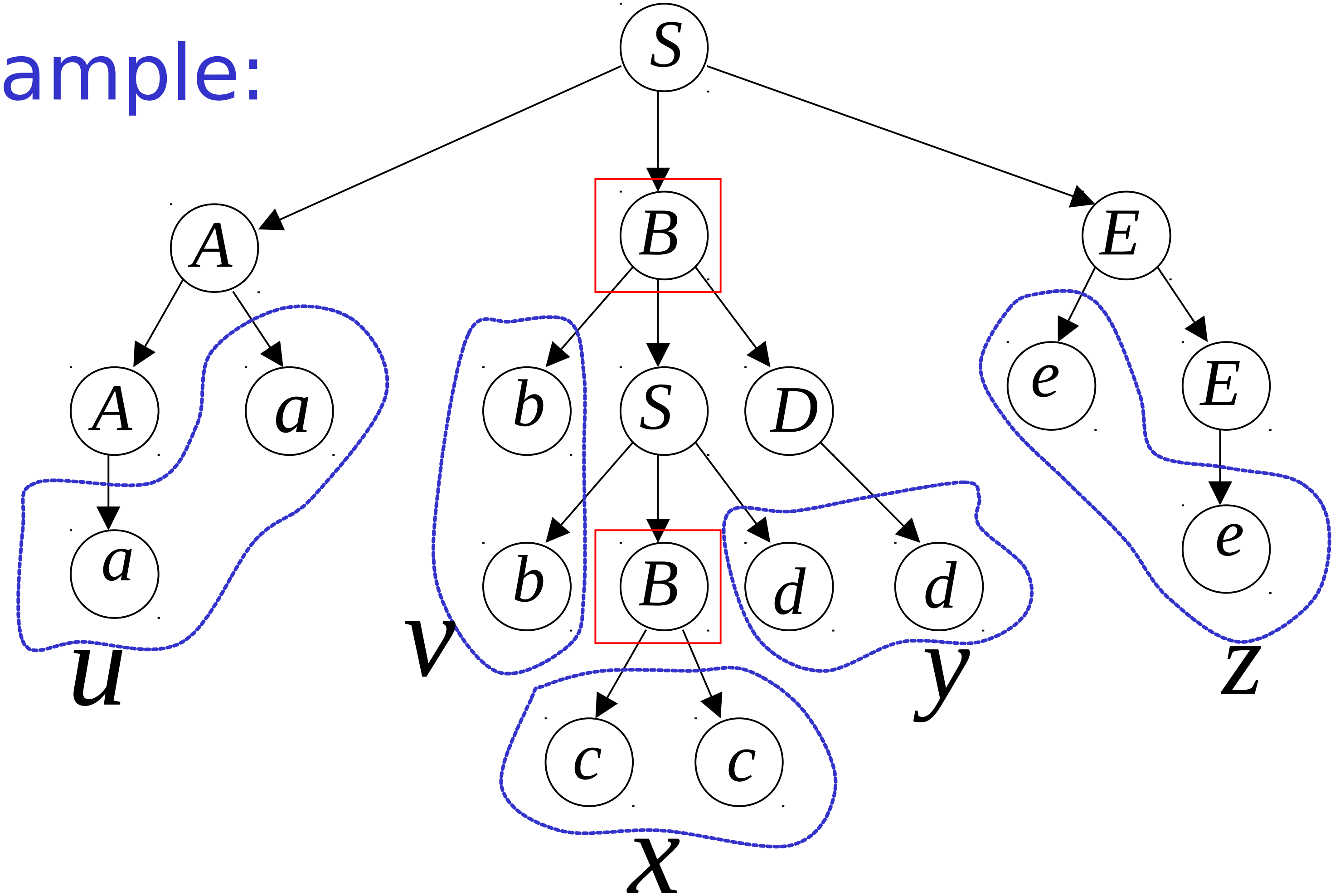
$$w = uvxyz$$



$$u, v, x, y, z:$$

Strings of terminals

Example:



$u = aa$

$v = bb$

$x = cc$

$y = dd$

$z = ee$

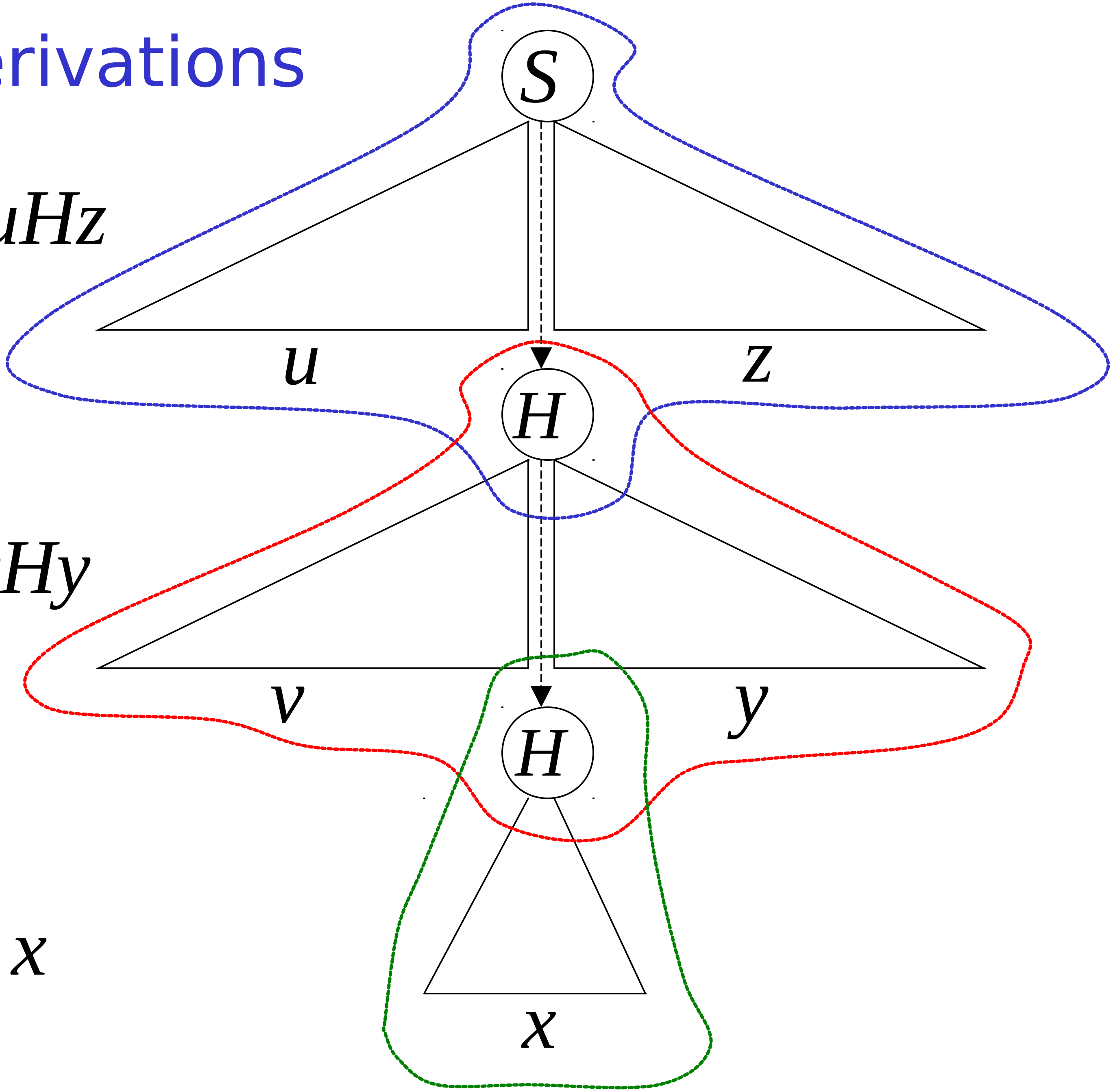
B corresponds to H

Possible derivations

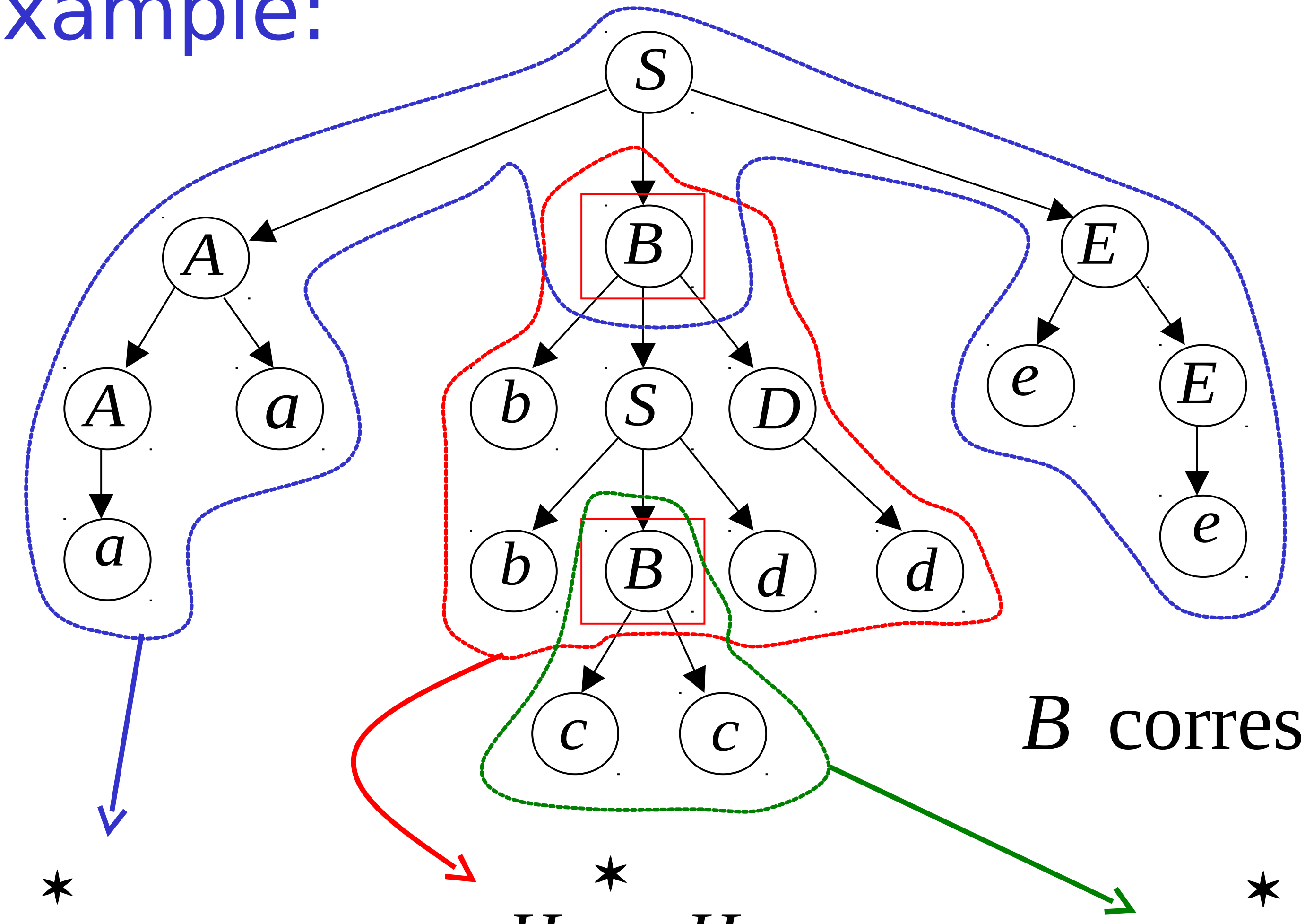
$$S \overset{*}{\Rightarrow} uHz$$

$$H \overset{*}{\Rightarrow} vHy$$

$$H \overset{*}{\Rightarrow} x$$



Example:



$u = aa$

$v = bb$

$x = cc$

$y = dd$

$z = ee$

B corresponds to H

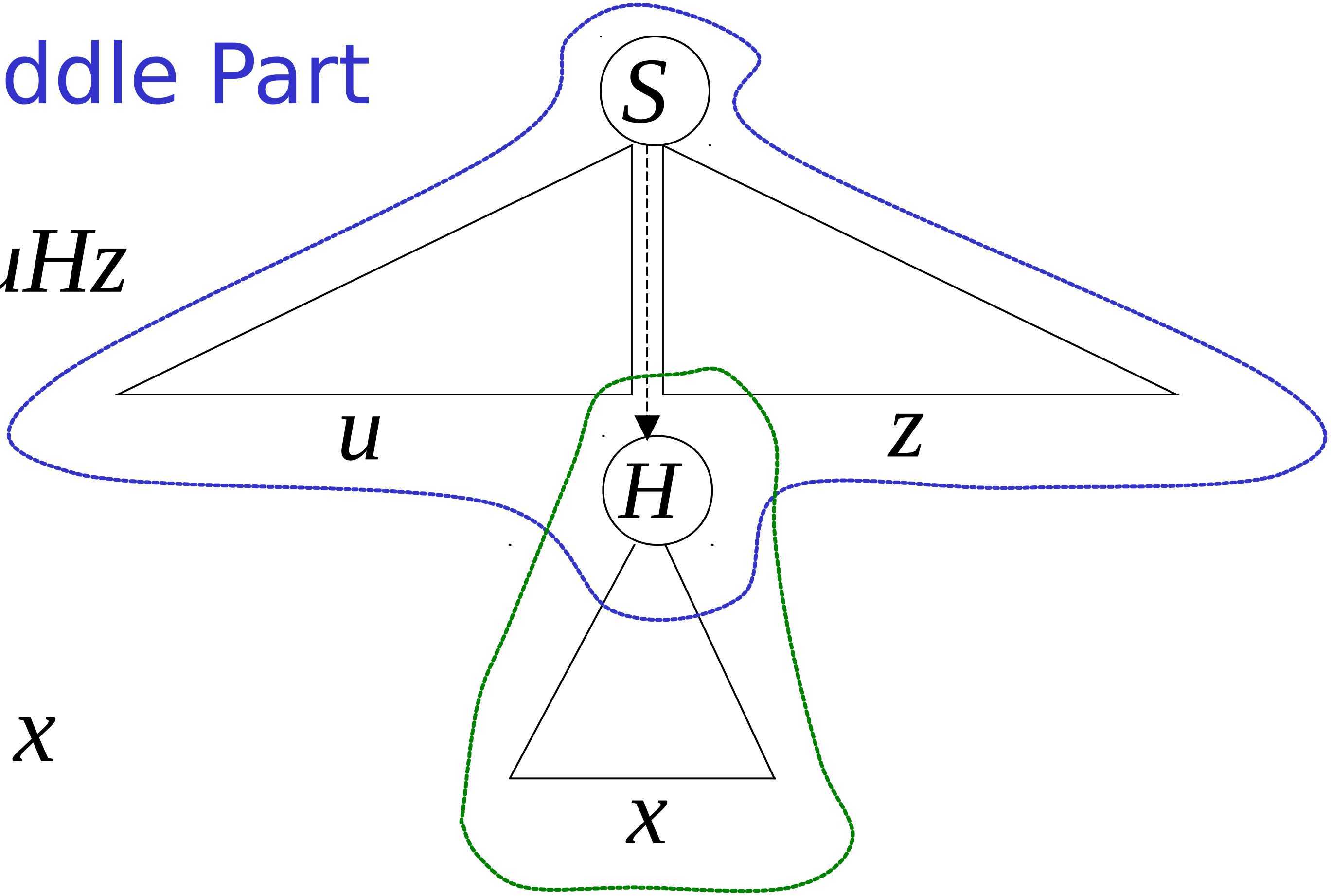
$S \overset{*}{\Rightarrow} uHz$ $\quad H \overset{*}{\Rightarrow} vHy$ $\quad H \overset{*}{\Rightarrow} x$

$S \overset{*}{\Rightarrow} aaBee$ $\quad B \overset{*}{\Rightarrow} bbBdd$ $\quad B \Rightarrow cc$

Remove Middle Part

$$S \overset{*}{\Rightarrow} uHz$$

$$H \overset{*}{\Rightarrow} x$$



Yield: $uxz = uv^0xy^0z$

$$S \overset{*}{\Rightarrow} uHz \overset{*}{\Rightarrow} uxz \quad = uv^0xy^0z \quad \in L(G)$$

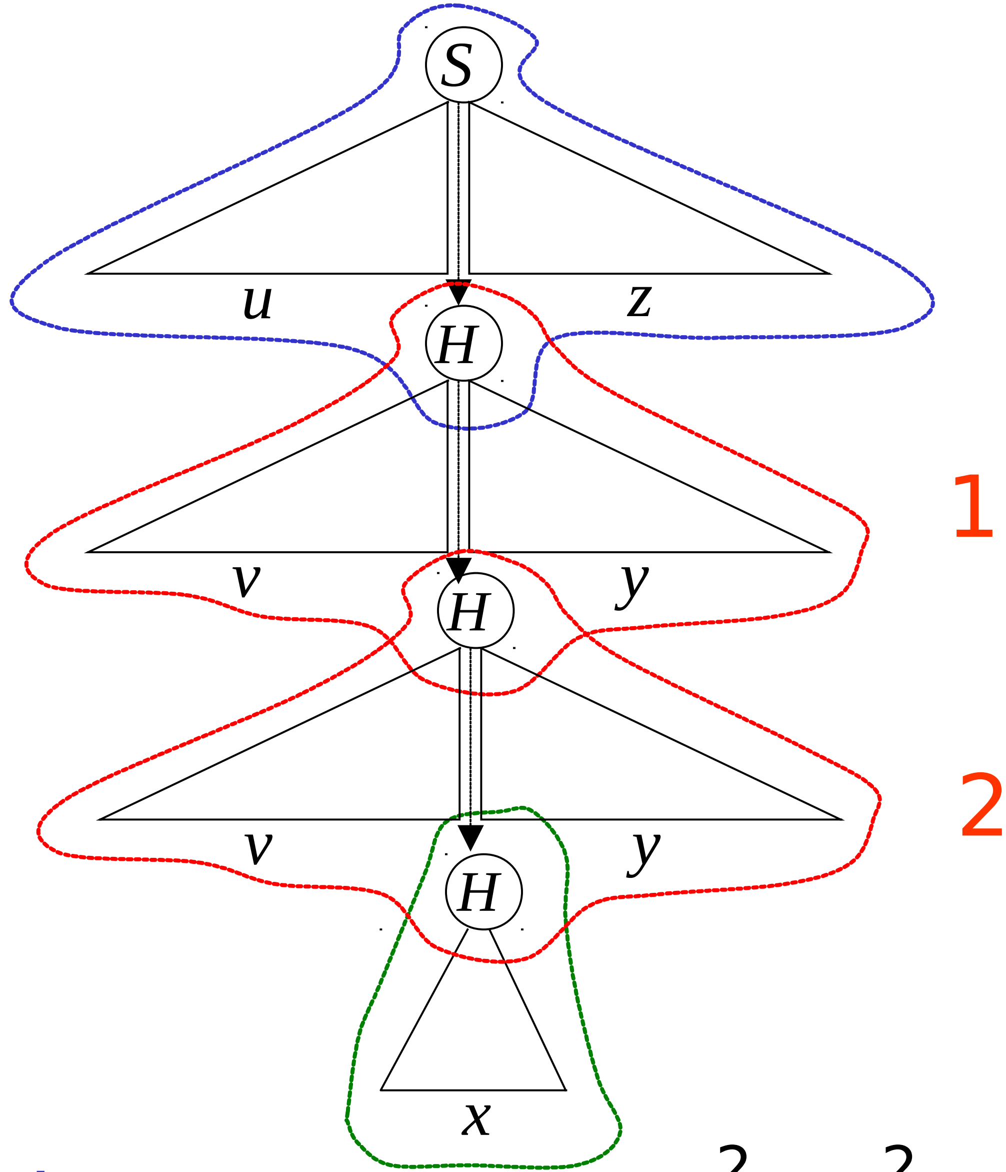
Repeat Middle part two times

$$S \overset{*}{\Rightarrow} uHz$$

$$H \overset{*}{\Rightarrow} vHy$$

$$H \overset{*}{\Rightarrow} vHy$$

$$H \overset{*}{\Rightarrow} x$$



Yield: $uvvxyyz = uv^2xy^2z$

$$S \overset{*}{\Rightarrow} uHz \qquad H \overset{*}{\Rightarrow} vHy \qquad H \overset{*}{\Rightarrow} x$$

$$S \overset{*}{\Rightarrow} uHz \overset{*}{\Rightarrow} uvHyz \overset{*}{\Rightarrow} uvvHyyz$$

$$\overset{*}{\Rightarrow} uvvxyyz \quad = uv^2xy^2z \in L(G)$$

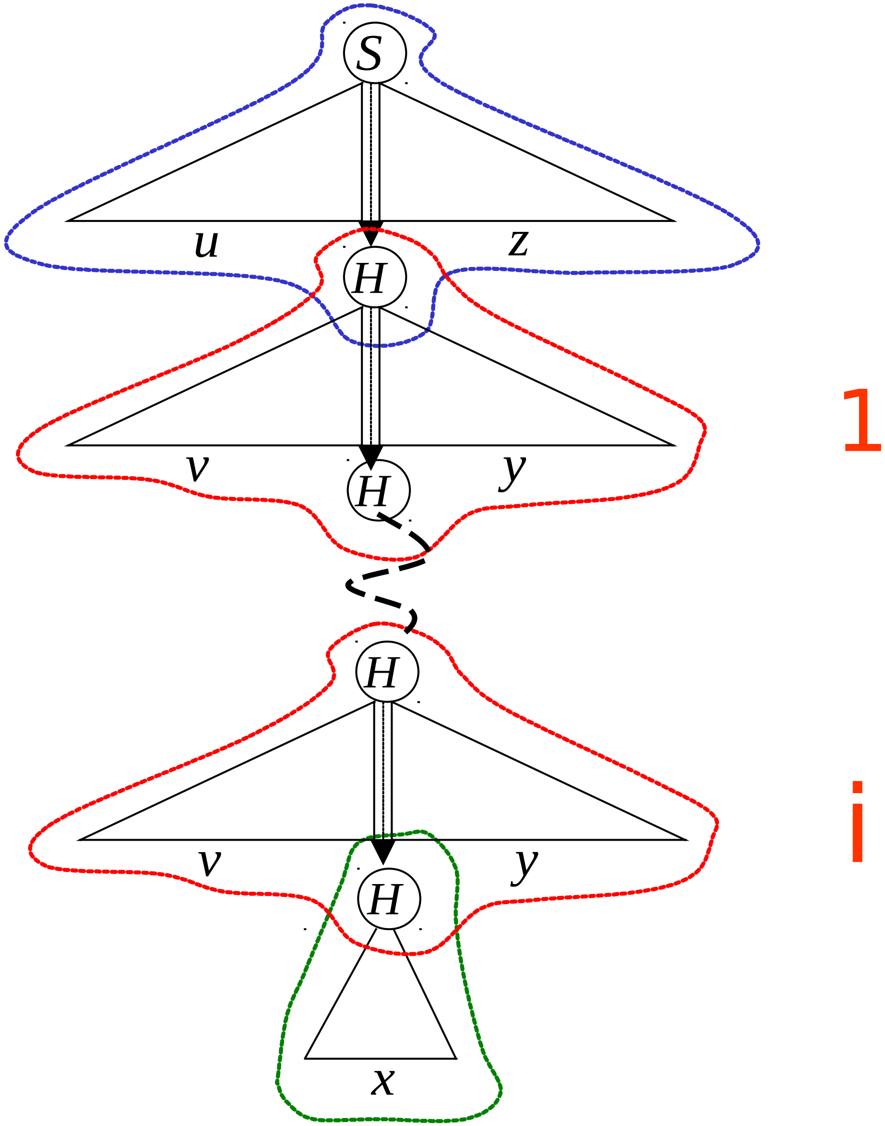
Repeat Middle part i times

$$S \overset{*}{\Rightarrow} uHz$$

$$H \overset{*}{\Rightarrow} vHy$$

$$H \overset{*}{\Rightarrow} vHy$$

$$H \overset{*}{\Rightarrow} x$$



Yield: $uv^i xy^i z$

$$S \overset{*}{\Rightarrow} uHz \qquad H \overset{*}{\Rightarrow} vHy \qquad H \overset{*}{\Rightarrow} x$$

$$S \overset{*}{\Rightarrow} uHz \overset{*}{\Rightarrow} uvHyz \overset{*}{\Rightarrow} uvvHyyz \overset{*}{\Rightarrow}$$

$$\overset{*}{\Rightarrow} \ldots$$

$$\overset{*}{\Rightarrow} uv^iHy^iz \overset{*}{\Rightarrow} uv^ixy^iz \in L(G)$$

Therefore,

$$|w| > t^r$$

If we know that: $w = uvxyz \in L(G)$

then we also know: $uv^ixy^iz \in L(G)$

For all $i \geq 0$

since

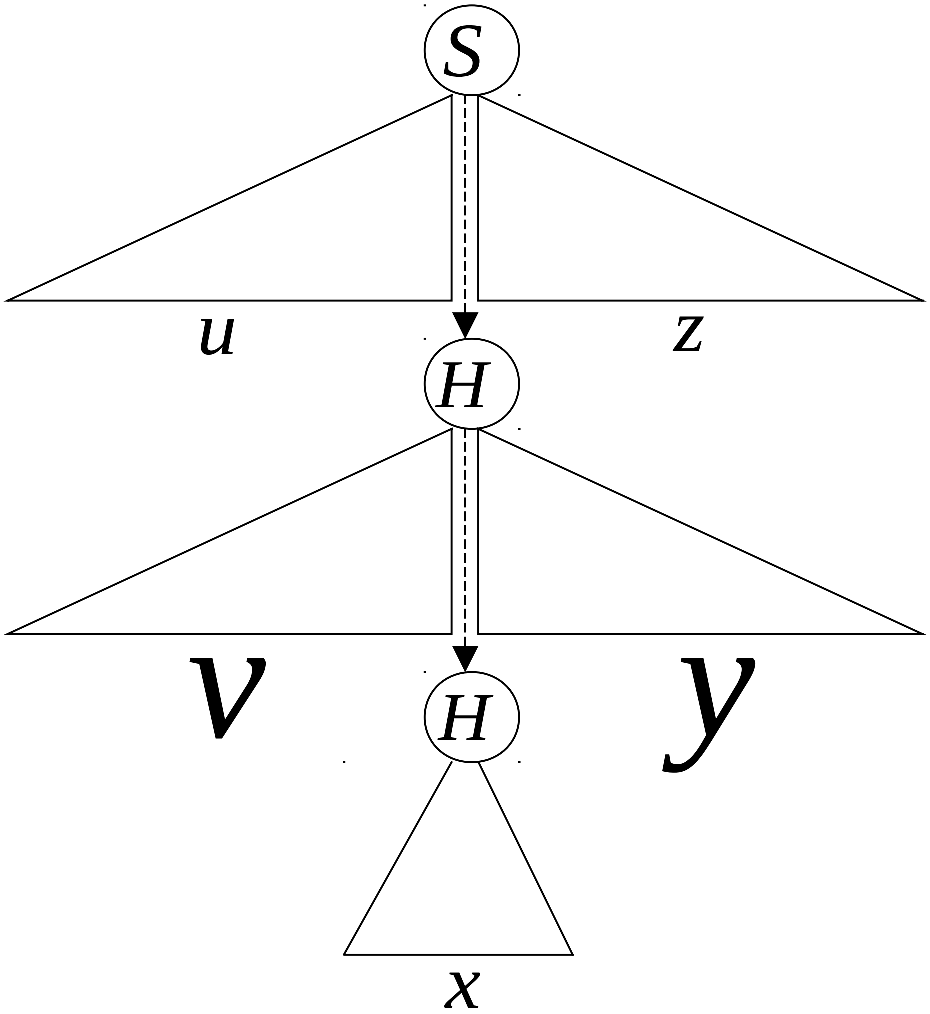
$L(G) = L - \{\lambda\}$

$$uv^ixy^iz \in L$$

Observation 1:

$$|vy| \geq 1$$

Since G has no unit and ε-productions

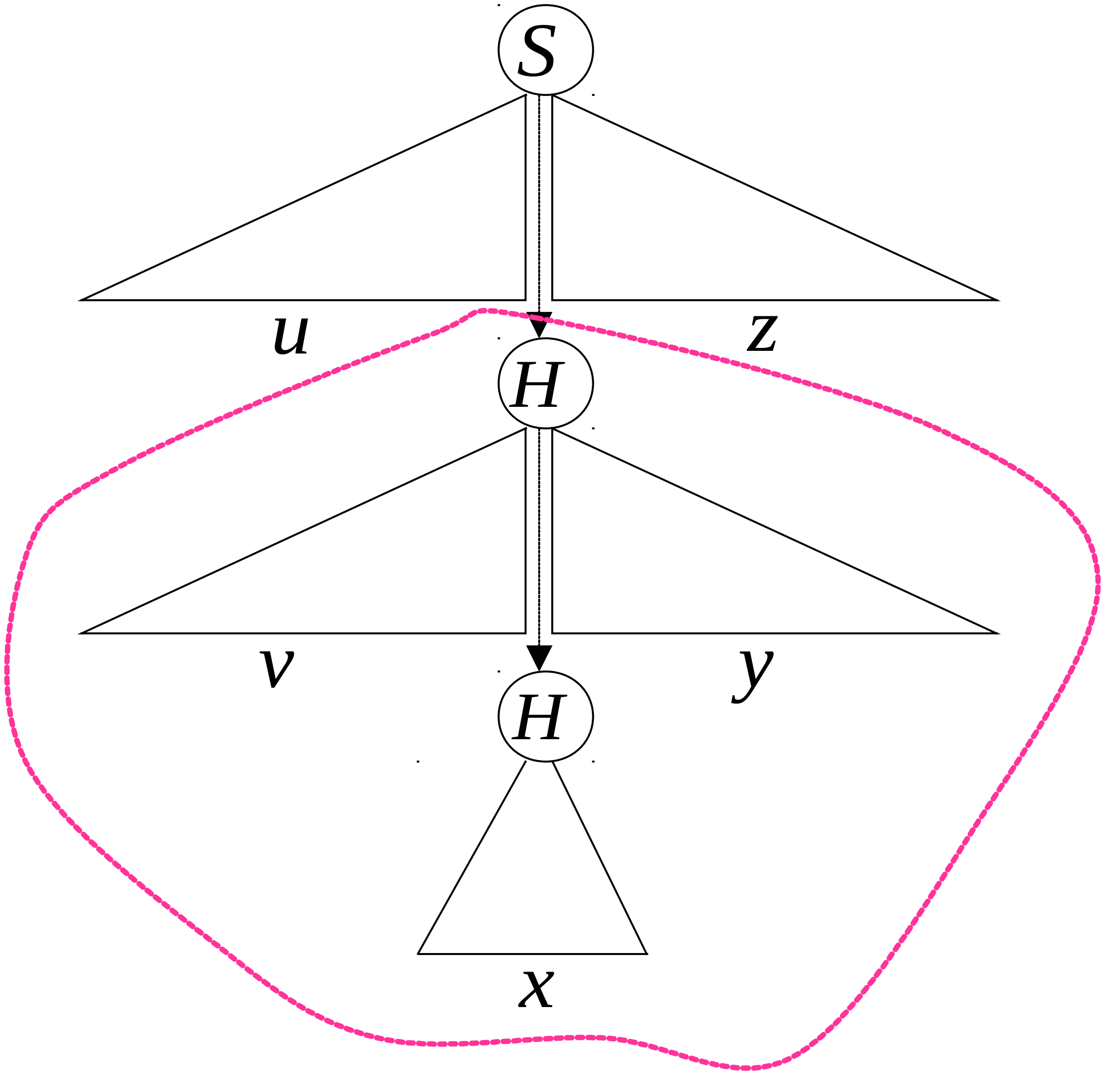


At least one of v or y is not ε

Observation 2:

$$|vxy| \leq t^{r+1}$$

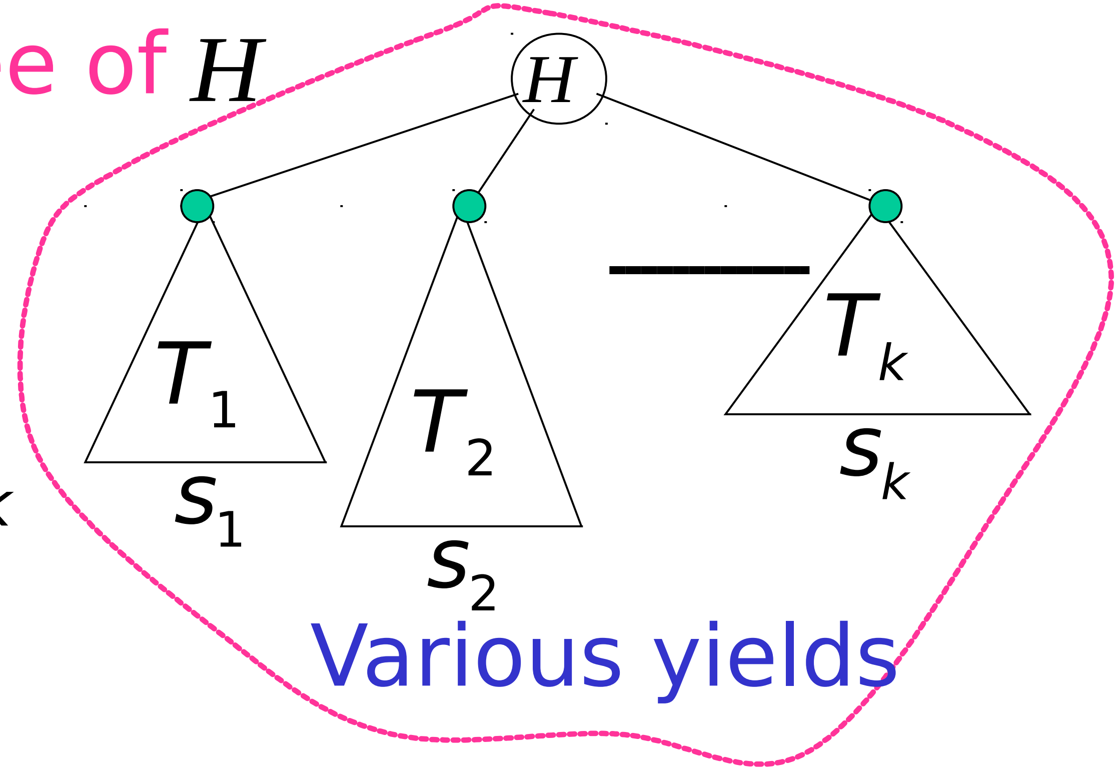
since in subtree only variable H is repeated



subtree of H

Explanation follows....

subtree of H



$$vxy = s_1 s_2 \cdots s_k$$

$|s_j| \leq t^r$ since no variable is repeated in T_j

$$|vxy| = \sum_{j=1}^{k} |s_j| \leq k \cdot t^r \leq t \cdot t^r = t^{r+1}$$

Maximum right-hand side of any production

Thus, if we choose critical length

$$p = t^{r+1} > t^r$$

then, we obtain the pumping lemma for context-free languages

he Pumping Lemma:

For any infinite context-free language L

there exists an integer p such that

for any string $w \in L, \quad |w| \geq p$

we can write $w = uvxyz$

with lengths $|vxy| \leq p$ and $|vy| \geq 1$

and it must be that:

$$uv^i xy^i z \in L, \quad \text{for all } i \geq 0$$

Applications of The Pumping Lemma

Non-context free languages

$$\{a^n b^n c^n : n \geq 0\}$$

Context-free languages

$$\{a^n b^n : n \geq 0\}$$

Theorem: The language

$$L = \{a^n b^n c^n : n \geq 0\}$$

is **not** context free

Proof: Use the Pumping Lemma for context-free languages

$$L = \{a^n b^n c^n : n \geq 0\}$$

Assume for contradiction that L

is context-free

Since L is context-free and infinite

we can apply the pumping lemma

$$L = \{a^n b^n c^n : n \geq 0\}$$

Let p be the critical length of the pumping lemma

Pick any string $w \in L$ with length $|w| \geq p$

We pick: $w = a^p b^p c^p$

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^p b^p c^p$$

From pumping lemma:

we can write: $w = uvxyz$

with lengths $|vxy| \leq p$ and $|vy| \geq 1$

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^p b^p c^p$$

$$w = uvxyz \qquad |vxy| \leq p \qquad |vy| \geq 1$$

Pumping Lemma says:

$$uv^i xy^i z \in L$$ for all $i \geq 0$

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^p b^p c^p$$

$$w = uvxyz \qquad |vxy| \leq p \qquad |vy| \geq 1$$

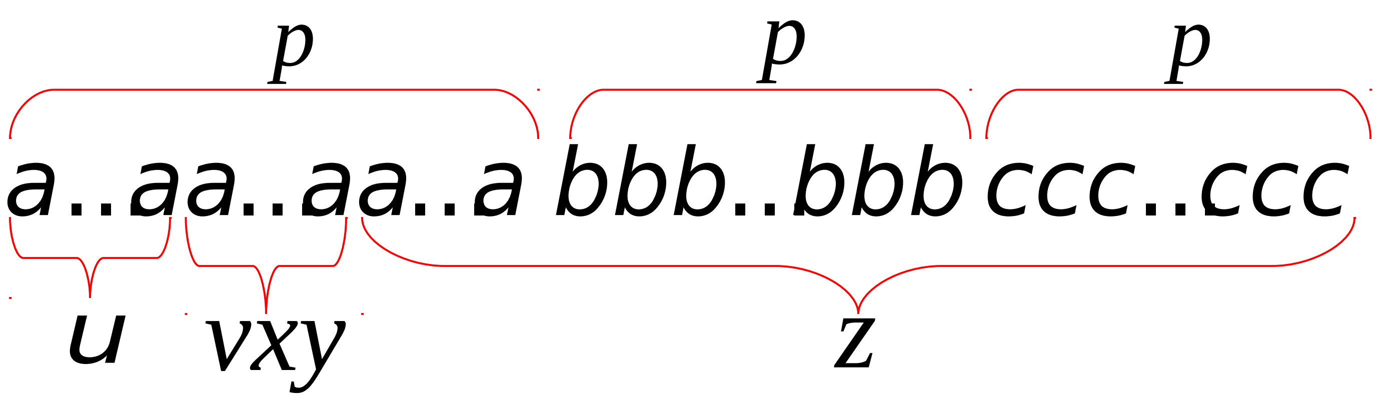
We examine **all** the possible locations of string vxy in w

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^p b^p c^p$$

$$w = uvxyz \qquad |vxy| \leq p \qquad |vy| \geq 1$$

Case 1: vxy is in a^p



$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^p b^p c^p$$

$$w = uvxyz \qquad |vxy| \leq p \qquad |vy| \geq 1$$

$$v = a^{k_1} \qquad y = a^{k_2} \qquad k_1 + k_2 \geq 1$$

$$\overbrace{\underbrace{a..a}_{u}\underbrace{a..a}_{v}\underbrace{a..a}_{x}\underbrace{a..a}_{y}a..a}^{p}\;\overbrace{bbb..bbb}^{p}\;\overbrace{ccc..ccc}^{p}$$

with z = the final $a..a\,bbb..bbb\,ccc..ccc$

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^p b^p c^p$$

$$w = uvxyz \qquad |vxy| \leq p \qquad |vy| \geq 1$$

$$v = a^{k_1} \qquad y = a^{k_2} \qquad k_1 + k_2 \geq 1$$

$$\overbrace{\underbrace{a..a}_{u}\underbrace{a..a}_{v^2}\underbrace{a..a}_{x}\underbrace{a..a}_{y^2}\underbrace{a..a\ \overbrace{bbb..bbb}^{p}\ \overbrace{ccc..ccc}^{p}}_{z}}^{p+k_1+k_2}$$

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^p b^p c^p$$

$w = uvxyz$ $\quad |vxy| \leq p$ $\quad |vy| \geq 1$

From Pumping Lemma: $uv^2xy^2z \in L$

$$k_1 + k_2 \geq 1$$

However: $uv^2xy^2z = a^{p+k_1+k_2} b^p c^p \notin L$

Contradiction!!!

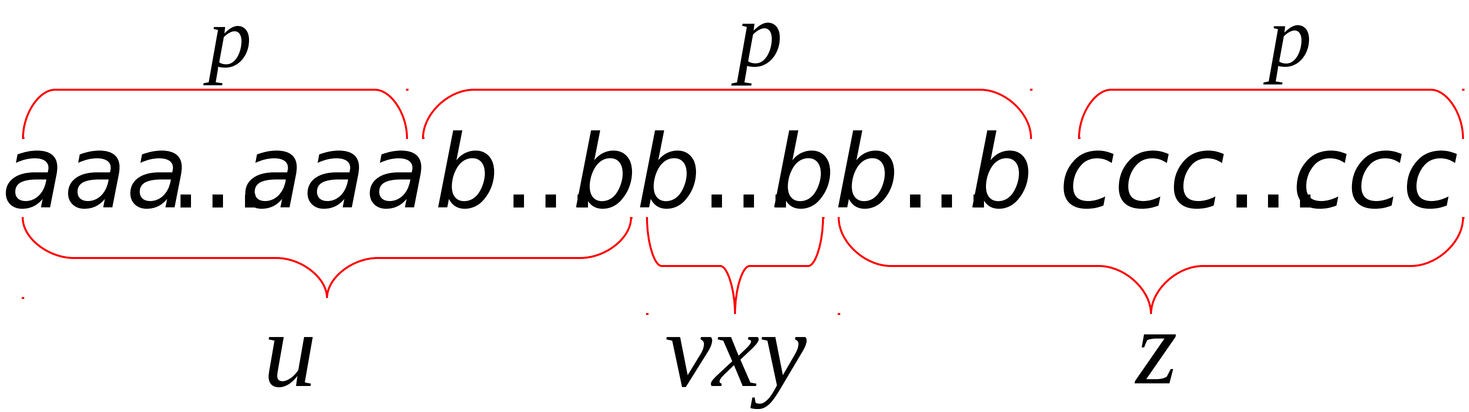
$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^p b^p c^p$$

$$w = uvxyz \qquad |vxy| \leq p \qquad |vy| \geq 1$$

Case 2: vxy is in b^p

Similar to case 1



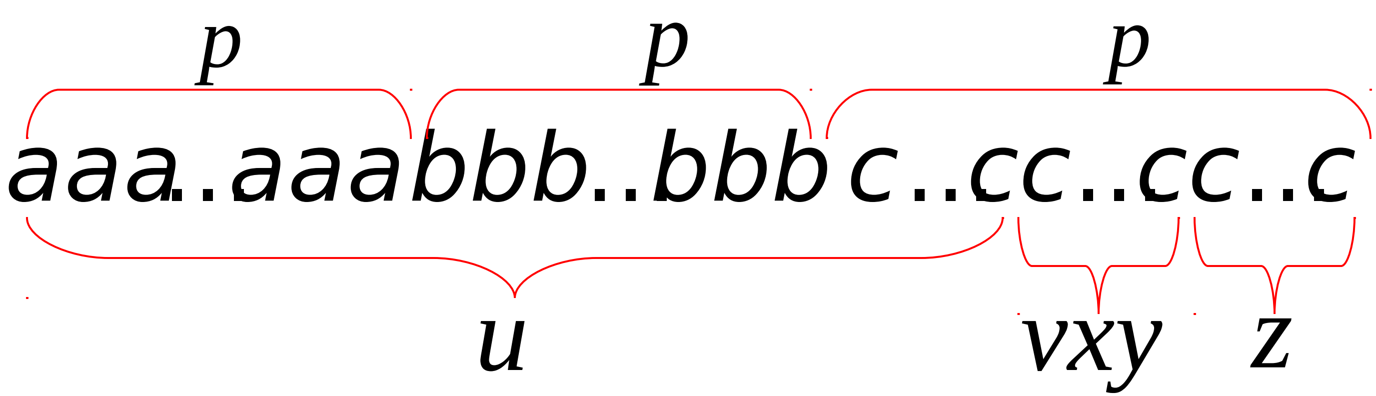
$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^p b^p c^p$$

$$w = uvxyz \qquad |vxy| \leq p \qquad |vy| \geq 1$$

Case 3: vxy is in c^p

Similar to case 1

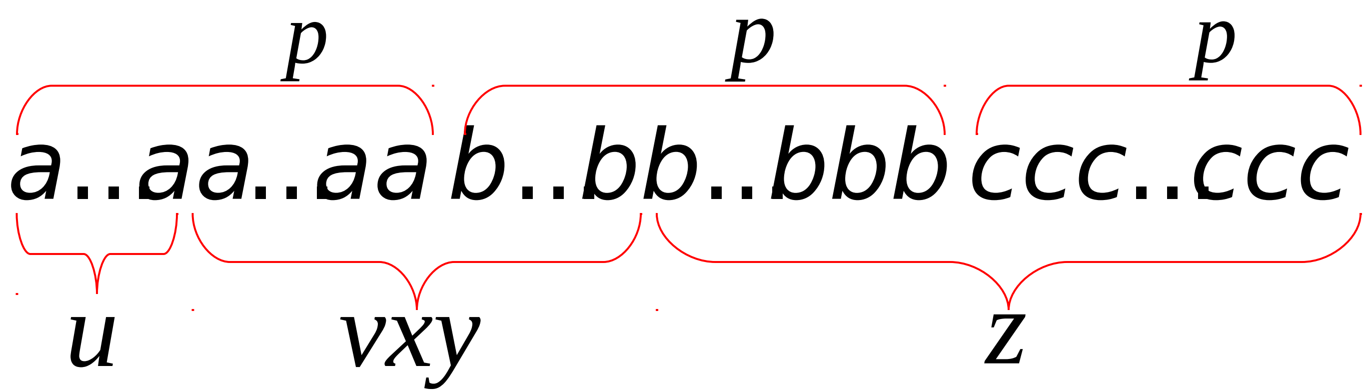


$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^p b^p c^p$$

$$w = uvxyz \qquad |vxy| \leq p \qquad |vy| \geq 1$$

Case 4: vxy overlaps a^p and b^p



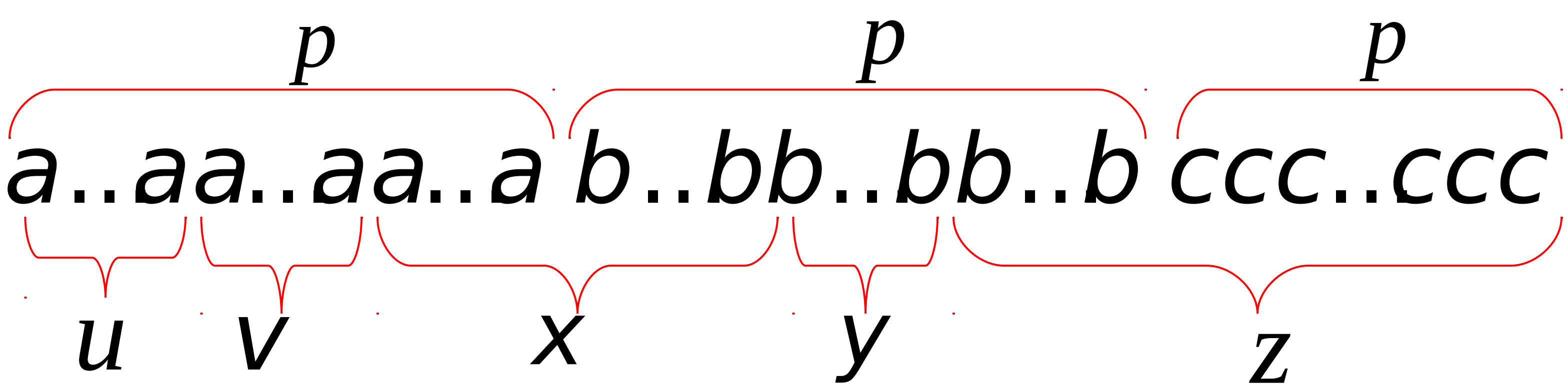
$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^p b^p c^p$$

$$w = uvxyz \qquad |vxy| \leq p \qquad |vy| \geq 1$$

Sub-case 1: v contains only a

y contains only b



$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^p b^p c^p$$

$$w = uvxyz \qquad |vxy| \leq p \qquad |vy| \geq 1$$

$$v = a^{k_1} \qquad y = b^{k_2} \qquad k_1 + k_2 \geq 1$$

$$\overbrace{a..aa..aa..a}^{p}\;\overbrace{b..bb..bb..b}^{p}\;\overbrace{ccc..ccc}^{p}$$

$$\underbrace{a..a}_{u}\,\underbrace{a..a}_{v}\,\underbrace{a..a\;b..b}_{x}\,\underbrace{b..b}_{y}\,\underbrace{b..b\;ccc..ccc}_{z}$$

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^p b^p c^p$$

$$w = uvxyz \qquad |vxy| \leq p \qquad |vy| \geq 1$$

$$v = a^{k_1} \qquad y = b^{k_2} \qquad k_1 + k_2 \geq 1$$

$$\overbrace{a..aa..aa..a}^{p+k_1}\;\overbrace{b..bb..bb..b}^{p+k_2}\;\overbrace{ccc..ccc}^{p}$$

$$\underbrace{a..a}_{u}\underbrace{a..a}_{v^2}\underbrace{a..a\;b..b}_{x}\underbrace{b..b}_{y^2}\underbrace{b..b\;ccc..ccc}_{z}$$

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^p b^p c^p$$

$$w = uvxyz \qquad |vxy| \leq p \qquad |vy| \geq 1$$

From Pumping Lemma: $uv^2xy^2z \in L$

$$k_1 + k_2 \geq 1$$

However: $uv^2xy^2z = a^{p+k_1} b^{p+k_2} c^p \notin L$

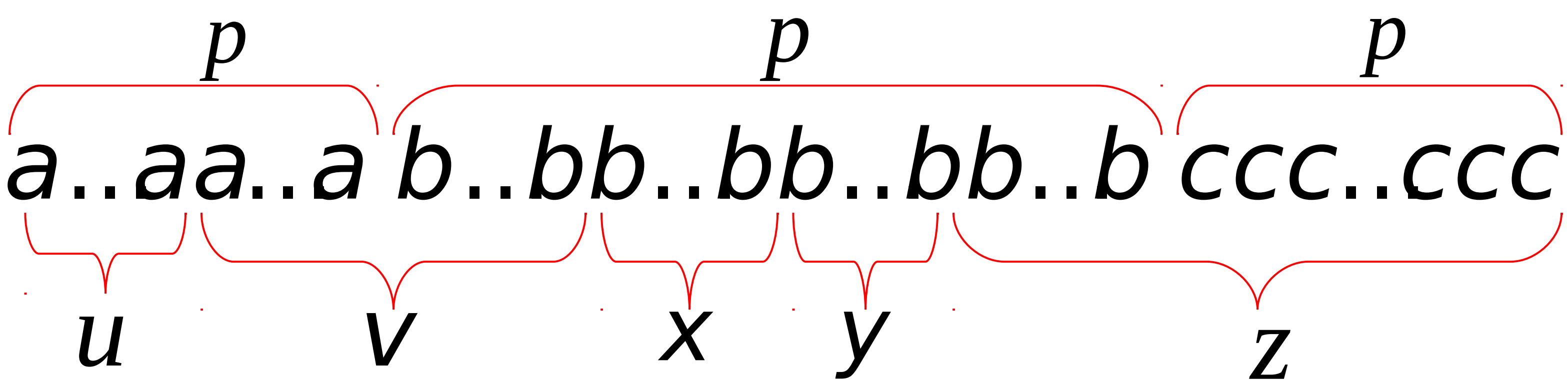
Contradiction!!!

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^p b^p c^p$$

$w = uvxyz$ $\qquad |vxy| \leq p$ $\qquad |vy| \geq 1$

Sub-case 2: v contains a and b
y contains only b



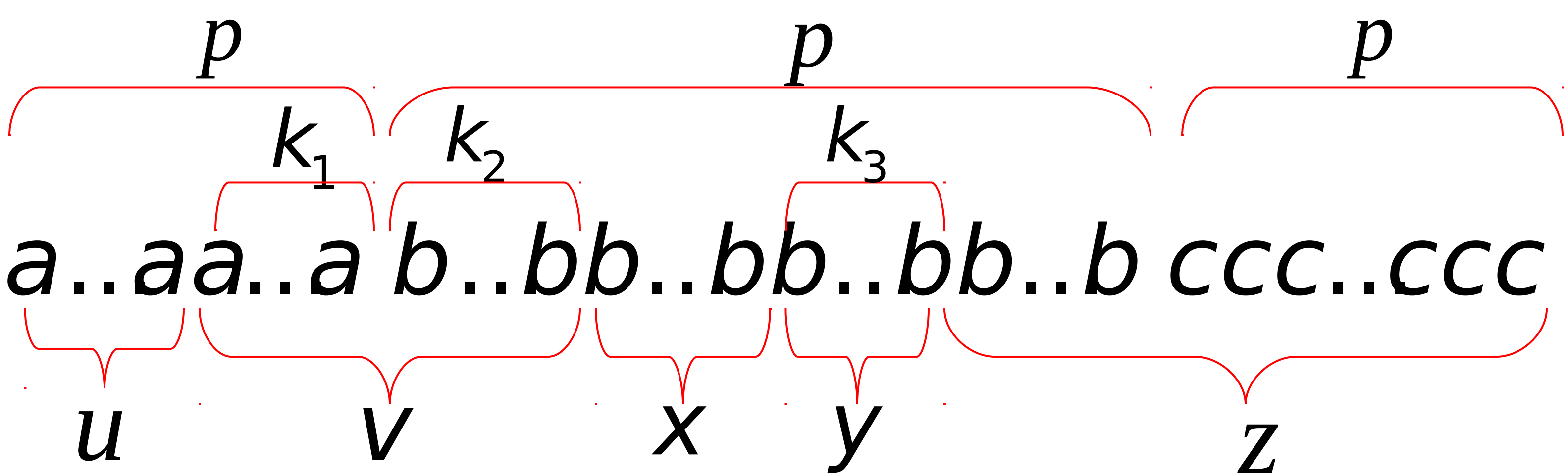
$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^p b^p c^p$$

$$w = uvxyz \qquad |vxy| \leq p \qquad |vy| \geq 1$$

By assumption

$$v = a^{k_1} b^{k_2} \qquad y = b^{k_3} \qquad k_1, k_2 \geq 1$$



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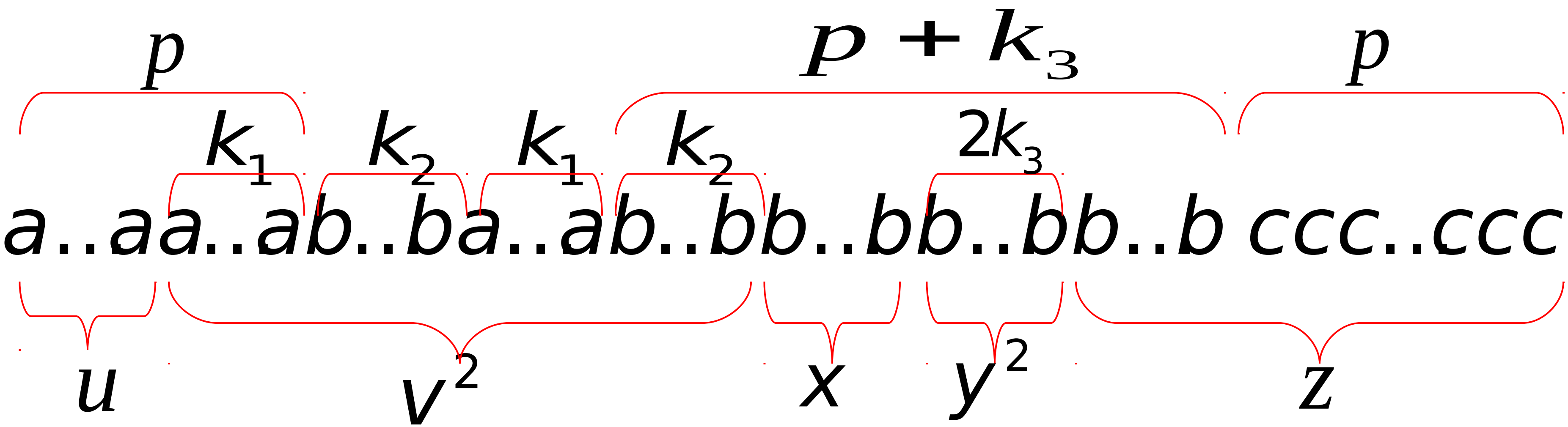


$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^p b^p c^p$$

$$w = uvxyz \qquad |vxy| \leq p \qquad |vy| \geq 1$$

From Pumping Lemma: $uv^2xy^2z \in L$

$$k_1, k_2 \geq 1$$

However: $uv^2xy^2z = a^p b^{k_2} a^{k_1} b^{p+k_3} c^p \notin L$

Contradiction!!!

$$L = \{a^n b^n c^n : n \geq 0\}$$

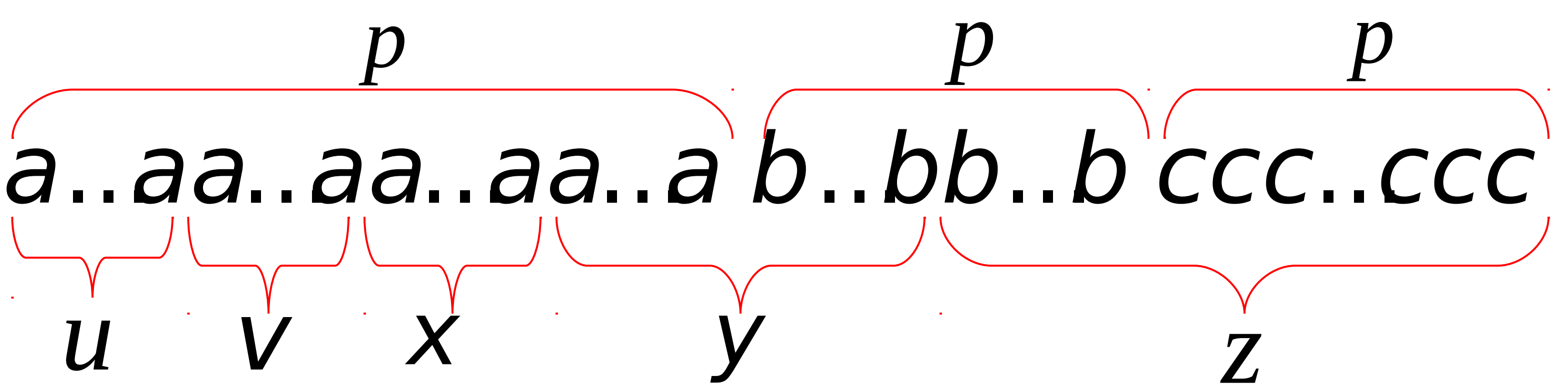
$$w = a^p b^p c^p$$

$$w = uvxyz \qquad |vxy| \leq p \qquad |vy| \geq 1$$

Sub-case 3: v contains only a

y contains a and b

Similar to sub-case 2



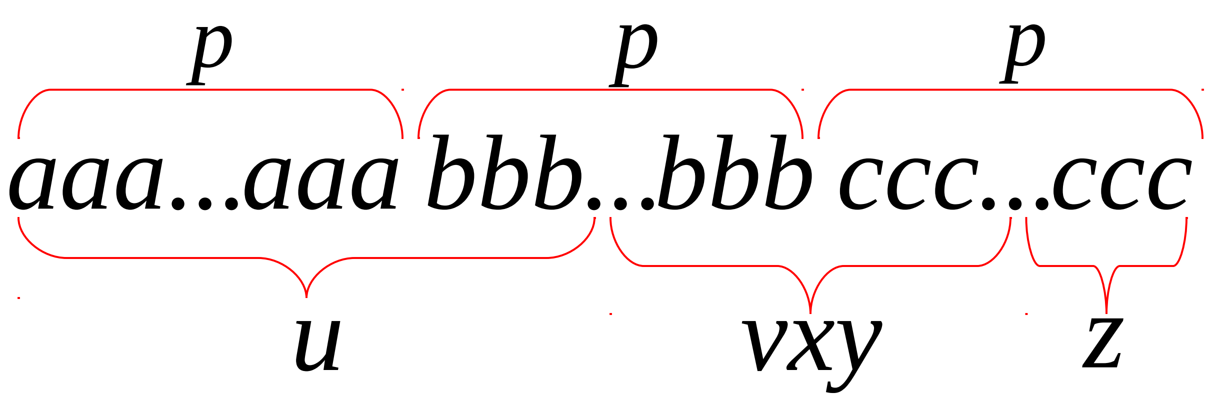
$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^p b^p c^p$$

$$w = uvxyz \qquad |vxy| \leq p \qquad |vy| \geq 1$$

Case 5: vxy overlaps b^p and c^p

Similar to case 4



$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^p b^p c^p$$

$$w = uvxyz \qquad |vxy| \leq p \qquad |vy| \geq 1$$

Case 6: vxy overlaps a^p, b^p and c^p

Impossible!

$$\overbrace{aaa...aaa}^{p}\ \overbrace{bbb...bbb}^{p}\ \overbrace{ccc...ccc}^{p}$$

u $\qquad vxy \qquad$ z

In all cases we obtained a contradiction

Therefore: the original assumption that

$$L = \{a^n b^n c^n : n \geq 0\}$$

is context-free must be wrong

Conclusion: L is not context-free