Appendix
Computer Formulas

The computer commands most useful in this book are given in both the *Mathematica* and *Maple* systems. More specialized commands appear in the answers to several computer exercises. For each system, we assume a familiarity with how to access the system and type into it.

In recent versions of Mathematica, the core commands have generally remained the same. By contrast, *Maple* has made several fundamental changes; however most older versions are still recognized. For both systems, users should be prepared to adjust for minor changes.

Mathematica

1. Fundamentals

Basic features of *Mathematica* are as follows:

(a) There are no prompts or termination symbols—except that a final semicolon suppresses display of the output. Input (new or old) is activated by the command *Shift-return* (or *Shift-enter*), and the input and resulting output are numbered.
(b) Parentheses (. . .) for algebraic grouping, brackets [. . .] for arguments of functions, and braces {. . .} for lists.
(c) Built-in commands typically spelled in full—with initials capitalized—and then compressed into a single word. Thus it is preferable for user-defined commands to avoid initial capitals.
(d) Multiplication indicated by either $*$ or a blank space; exponents indicated by a caret, e.g., x^2. For an integer n only, $nX = n*X$, where X is not an integer.

(e) Single equal sign for assignments, e.g., $x = 2$; colon-equal (:=) for deferred assignments (evaluated only when needed); double equal signs for mathematical equations, e.g., $x + y == 1$.
(f) Previous outputs are called up by either names assigned by the user or **`%n`** for the *n*th output.
(g) Exact values distinguished from decimal approximations (floating point numbers). Conversion using **`N`** (for "numerical"). For example, **`E^2*Sin[Pi/3]`** returns $e^2\sqrt{3}/2$; then **`N[%]`** gives a decimal approximation.
(h) Substitution by slash-dot. For example, if **`expr`** is an expression involving x, then `expr/.x→u^2 + 1` replaces x everywhere in the expression by $u^2 + 1$.

Mathematica has excellent error notification and online help. In particular, for common terms, **`?term`** will produce a description. Menu items give formats for the built-in commands. The complete general reference book—exposition and examples—is *The Mathematica Book* [W]. For our purposes, the outstanding reference is Alfred Gray's book [G].

> Some basic notation. Functions are given, for example, by

`f[x_]:= x^3-2x+1` or
`g[u_,v_]:= u*Cos[v]-u^2*Sin[v]`

Here, as always, an underscore "_" following a letter (or string) makes it a variable. Thus the function f defined above can be evaluated at u or 3.14 or $a^2 + b^2$.

> Basic calculus operations.
Derivatives (including partial derivatives) by **`D[f[x],x]`** or **`D[g[u,v],v]`**
Definite integrals by **`Integrate[f[x],{x,a,b}]`**. For numerical integration, prefix an **`N`** thus: **`NIntegrate`**.

> Linear algebra. A *vector* is just an n-tuple, that is, a list **`v={v1,...,vn}`**, whose entries can be numbers or expressions. Addition is given by **`v+w`** and scalar multiplication by juxtaposition, with **`sv=s{v1,...,vn}`** yielding **`{s*v1,...,s*vn}`**. The dot product is given by **`v.w`** and, for n = 3, the cross product is **`Cross[v,w]`**.

Mathematica describes a matrix as a list of lists, the latter being its rows. For example, {{a,b},{c,d}} is a matrix and is treated as such in all contexts. To make it look like $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, apply the command **`MatrixForm`**. The determinant of a square matrix **m** is given by **`Det[m]`**.

The full power of the dot operator (**.**) appears only when matrices are involved. First, if **p** and **q** are properly sized matrices, then **p.q** is their product. Next, if **m** is an m × n matrix and **v** is an n-vector, then **m.v** gives the usual operation of **m** on **v**. Taking m = n = 3 for example, if **m1, m2, m3** are the rows of **m** and **v={v1,v2,v3}**, then *Mathematica* defines

m.v to be **{m1.v,m2.v,m3.v}**

This can be seen to be the result of **m** (in 3 × 3 form) matrix-multiplying the column-vector corresponding to **v**, with the resulting column-vector restated as an n-tuple. In this sense, *Mathematica* obeys the "column-vector convention" from the end of Section 3.1, which identifies n-tuples with n × 1 matrices.

If **A** is any *array*—say, a vector or matrix—then for most commands, **cmd[A]** will apply the command **cmd** to each entry of **A**.

2. Curves

A curve in $\mathbf{R}^3$ can be described by giving its components as expressions in a single variable. Example:

```
c[t_]:= {Cos[t],Sin[t],2t}
```

Then the vector derivative (i.e., velocity) is returned by **D[c[t],t]**.

> Curves with parameters. For example, the curve c above can be generalized to

```
helix[a_,b_][t_]:= {a*Cos[t],a*Sin[t],b*t}
```

Then **helix[1,2]=c**.

The following formulas, drawn from Theorem 4.3 of Chapter 2, illustrate aspects of vector calculus in *Mathematica.*

The curvature and torsion functions κ and τ of a curve $c \approx \gamma$ are given by

```
kappa[c_][t_]:=Simplify[Cross[D[c[tt],tt],D[c[tt],
  {tt,2}]].
  Cross[D[c[tt],tt],D[c[tt],{tt,2}]]]^(1/2)/
  Simplify[D[c[tt],tt].D[c[tt],tt]]^(3/2)/.tt->t
```

(Note the description of second derivatives.) The use of the dummy variable **tt** makes **kappa[c]** a real-valued *function* $\mathbf{R} \to \mathbf{R}$. Otherwise, it would be merely an expression in whichever single variable was used.

"Simplify" is the principal *Mathematica* simplification weapon; however, it cannot be expected to give ideal results in every case. ("FullSimplify" is more powerful but slower.) Thus human intervention is often required, either to do hands-on simplification or to use further computer commands such as "Together" or "Factor" or trigonometric simplifications.

```
tau[c_][t_]:=Simplify[
   Det[{D[c[tt],tt],D[c[tt],{tt,2}],D[c[tt],
     {tt,3}]}]]/
   Simplify[Cross[D[c[tt],tt],D[c[tt],{tt,2}]].
   Cross[D[c[tt],tt],D[c[tt],{tt,2}]]]/.tt->t
```

Here the determinant gives a triple scalar product.

Note: The distinction between functions and mathematical expressions is basic. Thus, with notation as above, **tau** applied to a curve, say, **helix[1,2]**, is a real-valued function **tau[helix[1,2]]** whose value on any variable or number **s** is **tau[helix[1,2]][s]**.

The unit tangent, normal, and binormal vector fields T, N, B of a curve with $\kappa > 0$ are given by

```
tang[c_][t_]:=D[c[tt],tt]/
  Simplify[D[c[tt],tt].D[c[tt],tt]]^(1/2)/.tt->t

nor[c_][t_]:=Simplify[Cross[binor[c][t],
   tang[c][t]]]

binor[c_][t_]:=Simplify[Cross[D[c[tt],tt],D[c[tt],
   {tt,2}]]]/
  Simplify[Factor[Cross[D[c[tt],tt],D[c[tt],
   {tt,2}]].
  Cross[D[c[tt],tt],D[c[tt],{tt,2}]]]]^(1/2)/.tt->t
```

Here is how to preserve any such commands for future use: Type (or copy) them into a *Mathematica* notebook, say **frenet**, and use the *Cell* menu to designate the cells containing them as *initialization cells.* When this notebook is saved, a choice will be offered letting you save, not only **frenet**, but also a new file **frenet.m** that contains only the commands. Then these can be read into later work by **<<frenet.m**

3. Surfaces

A coordinate patch, say **x**, is given by listing its components as expressions in two variables. For example,

```
x[u_,v_]:= {u*Cos[v],u*Sin[v],2v}
```

> Parameters can be handled as above for curves. For example, the 2 in this formula can be replaced by an arbitrary parameter using

```
helicoid[b_][u_,v_]:={u*Cos[v],u*Sin[v],b*v}
```

Then **helicoid[2]** gives the original **x**.

For a patch, the following commands return $E, F, G, W = \sqrt{EG - F^2}$, and L, M, N. We elect to represent our capital letters (E) by double lowercase letters (ee), since many capitals have special meaning for *Mathematica* (for example, $E = 2.7183\ldots$).

```
ee[x_][u_,v_]:=
  Simplify[D[x[uu,vv],uu].D[x[uu,vv],uu]]/.
  {uu->u,vv->v}
ff[x_][u_,v_]:=
  Simplify[D[x[uu,vv],uu].D[x[uu,vv],vv]]/.
  {uu->u,vv->v}
gg[x_][u_,v_]:=
  Simplify[D[x[uu,vv],vv].D[x[uu,vv],vv]]/.
  {uu->u,vv->v}
ww[x_][u_,v_]:=
  Simplify[Sqrt[ee[x][u,v]*gg[x][u,v]-
  ff[x][u,v]^2]]
```

The variant command, say **www**, in which **Sqrt[...]** is replaced by **PowerExpand[Sqrt[...]]** will often give decisively simpler square roots. But one must check that its results are positive, since for example, **PowerExpand[Sqrt[x^2]]** yields **x**.

```
ll[x_][u_,v_]:=Simplify[Det[{D[x[uu,vv],uu,uu],
  D[x[uu,vv],uu],D[x[uu,vv],vv]}]/ww[x][u,v]]/.
  {uu->u,vv->v}
```

The formulas for **mm** and **nn** are the same except that the double derivative **uu, uu** is replaced by **uu, vv** and **vv, vv,** respectively.

> Gaussian curvature *K.* When the commands for *E, F, G* and L, M, N have been read in, commands for *K* and *H* follow directly from Corollary 4.1 of Chapter 5 (see Exercise 18 of Section 5.4). However, the fastest way to find *K* for a given patch in $\mathbf{R}^3$ is by the following command, based on Exercise 20 of Section 5.4. In it, "Module" creates an enclave in which temporary definitions can be made that let the final formula be expressed more simply.

```
gaussK[x_][u_,v_]:= Module[{xu,xv,xuu,xuv,xvv},
  xu=D[x[uu,vv],uu];xv= D[x[uu,vv],vv];
```

```
xuu=D[x[uu,vv],uu,uu];
xuv=D[x[uu,vv],uu,vv];
xvv=D[x[uu,vv],vv,vv];
Simplify[(Det[{xuu,xu,xv}]*Det[{xvv,xu,xv}]-
Det[{xuv,xu,xv}]^2)/
(xu.xu*xv.xv-(xu.xv)^2)^2]]/.{uu->u,vv->v}
```

As with other useful commands, this should be saved for future use.

4. Plots

There are four basic types: **Plot** and **Plot3D** plot the graphs of functions of one and two variables respectively. Examples:

```
Plot[f[x]//Evaluate,{x,a,b}]
Plot3D[g[x,y]//Evaluate,{x,a,b},{y,c,d}]
```

Here **//Evaluate** improves the speed of plotting.

ParametricPlot plots the image of a parametrized curve in the plane $\mathbf{R}^2$.

ParametricPlot3D plots the image of a parametrized curve or patch. For example, a parametrized curve $c(t)$ in $\mathbf{R}^3$ is plotted for $a \leqq t \leqq b$ by

```
ParametricPlot3D[c[t]//Evaluate,{t,a,b}]
```

and if **x** is an explicitly defined patch or parametrization, its image on the rectangle $0 \leqq u \leqq 1$, $0 \leqq v \leqq 2\pi$ is plotted by

```
ParametricPlot3D[x[u,v]//Evaluate,{u,0,1},
{v,0,2Pi}]
```

Various refinements are available for plots. For example, if the end of the command above is altered to

```
...{v,0,2Pi},AspectRatio->Automatic]
```

then the same scale is imposed on height and width. Formally, the *option* “AspectRatio” has been reset from its default value. Various adjuncts to a plot can be also be changed. For example, the box surrounding the preceding plot is eliminated by **Boxed->False**. The plot can be made smoother by using **PlotPoints->{m,n}**, where the integers increase the default values governing smoothness in the u and v directions, respectively.

The options available for a command **cmd** are given, along with their default values, by **Options[cmd]**. Then **?opt** will describe a particular option.

Previously drawn plots can be shown on the same page by

```
Show[plot1,plot2,plot3]
```

5. Differential Equations

Explicit solutions in terms of elementary functions are inherently rare, so we describe how to find and plot numerical solutions, which are all that is needed in many contexts. In the command for such a solution, *Mathematica* lumps equations and initial conditions into a single list, then specifies the dependent variables and the interval of the dependent variable.

Example: Solve numerically the differential equations

$$x' = f(x, y, t), \quad y' = g(x, y, t),$$

subject to the initial conditions

$$x(t_0) = x_0, \quad y(t_0) = y_0,$$

on the interval $t_{\min} \leqq t \leqq t_{\max}$. The format is

```
soln = NDSolve[{x'[t]==f[x[t],y[t],t],
  y'[t]==g[x[t],y[t],t],
  x[t0]==x0,y[t0]==y0},{x,y},{t,tmin,tmax}]
```

Note the double equal signs. Without the N for "numerical," an exact solution would be sought.

`NDSolve` expresses x and y in terms of *Interpolating Functions*, data sufficient for subsequent plots. If **`soln`** is an explicit result from the preceding command, the solution is plotted by

```
ParametricPlot[Evaluate[{x[t],y[t]}/.soln],
  {t,tmin,tmax}]
```

Here "**`/.`**" substitutes **`soln`** into the coordinates. Note the general equivalence: **`Evaluate[X]`** is the same as **`X//Evaluate`**.

Maple

1. Fundamentals

Basic features of *Maple* are as follows:

(a) Input is typed after a prompt and *must* be terminated by a semicolon—or colon, to suppress display of the output. **We do not show these below.** Then press ENTER (or RETURN).
(b) Parentheses used for algebraic grouping and arguments of functions; braces {. . .} for sets; brackets [. . .] for lists.

(c) Built-in commands are abbreviated, with multiword commands compressed into a single word; most are written in lower case.
(d) Multiplication *always* indicated by $*$, exponents by a caret, e.g., **`x^2`**.
(e) Assignments indicated by colon-equal, e.g., **`x:=2`**; equations by single equal, e.g., **`x+y=1`**.
(f) Previous outputs are called up by names assigned by the user. (Naming is important since input/outputs are not numbered.) Also, the percent symbol (%) gives the immediately preceding output, and two of these give the one before that.
(g) Exact values distinguished from decimal approximations (floating point numbers). Conversion is accomplished by the "evalf" command. For example, **`exp(2)*sin(Pi/3)`** returns $e^2\sqrt{3}/2$; then **`evalf(%)`** gives a decimal approximation.
(h) Substitution by the "subs" command. If *expr* is an expression involving x, then **`subs(x=u^2+1,`** *expr***`)`** replaces every x in the expression by $u^2 + 1$.
(i) If A is an *array*—say a matrix or vector—then to apply an operation F to each entry of A, use the command "map" thus: **`map`**(F,A).

Maple has a distinctive command "unapply" that converts mathematical expressions into functions. For example, if *expr* is an expression involving u and v, then **`unapply(`***expr***`,u,v)`** is the corresponding function of u and v.

Many specialized *Maple* commands are collected in *packages*, which are loaded, for example, by **`with(plots)`**. A list of the commands in the package appears unless output is suppressed. We rarely use packages other than *plots* and *LinearAlgebra* (which is replacing *linalg*).

Maple has reasonable error notification and excellent on-line help. For common terms, **`?term`** will produce a detailed description (no semicolon required).

The *Maple Learning Guide* is a good introduction to the most recent version of *Maple*; it may be obtained from the website *maplesoft.com.* Of course, there are a variety of more advanced books.

Some basic notations.

Functions can be produced by the arrow notation. Examples:

```
f:= x->x^3-2*x+1  or
g:=(u,v)->u*cos(v)-u^2*sin(v)
```

Derivatives (including partials):

`diff(f(x),x)` or **`diff(g(u,v),v)`**

Definite integral:

`int(f(x),x=a..b)` or
`int(g(x,y),x=a..b,y=c..d)`

If an explicit integral cannot be found, then **`evalf(%)`** gives a numerical result. Direct numerical integration is given by **`evalf(Int(f(x),x=a..b))`**.

Linear algebra. Recent versions of *Maple* have changed considerably (though it still recognizes many old forms). Currently, its commands, whether new or not, are often signalled by new names. Typically, the new command begins with a capital letter and is not abbreviated. These changes are most evident in the package *LinearAlgebra* that is replacing *linalg*.

Maple has always made a fundamental distinction between an n-tuple **`[v1,..,vn]`**—which is a *list*—and a *vector*, in any notation. The two types cannot directly interact. In the new version, *vector* is replaced by *Vector* (capital V).

Lists are the easiest to deal with. For instance, the usual sum of n-tuples **`v=[v1,..,vn]`** and **`w=[w1,..,wn]`** is given by **`v+w`**, and scalar multiplication of an n-tuple by a number **`s`** uses an asterisk, with **`s*v`** giving **`[s*v1,..,s*vn]`**.

A matrix is produced by applying the command **`Matrix`** to a list whose entries are lists, the latter being the rows of the matrix. Thus

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is described by *Maple* as **`Matrix([[a,b],[c,d]])`**.

With the package *LinearAlgebra* loaded, the determinant of a square matrix **`m`** is given by **`Determinant(m)`**.

When an n × n matrix **`C`** is considered as a linear transformation on R^n, it cannot directly attack **`[v1,..,vn]`** to give the image **`[w1,..,wn]`**. The list **`[v1,..,vn]`** must first be stood on end as **`Vector([v1,..,vn])`**, which is, in fact, an n × 1 matrix. Now matrix multiplication is valid, and, with *LinearAlgebra* installed, **`Multiply(C,Vector([v1,..,vn])`** is the n × 1 matrix that **`convert(%,list)`** turns into **`[w1,..,wn]`**. This identification of an n-tuple with a column vector is just the "column vector convention" at the end of Section 3.1.

Since curves and surfaces are described in terms of lists, we can largely avoid the list/Vector conflict by defining three basic vector operations directly in terms of lists. First, note that the entries of a list **`p:=[p1,p2,...,pn]`** can be any expressions, and the i^{th} entry is displayed by the command **`p[i]`**.

An operation applied to a list is automatically applied to each entry. (By contrast, other *arrays* require the command **map**.)

Dot product: **dot:=(p,q)-> simplify(p[1]*q[1]+ p[2]*q[2]+p[3]*q[3])**

Cross product: **cross:=(p,q)-> simplify ([p[2]*q[3]-p[3]*q[2],p[3]*q[1]- p[1]*q[3],p[1]*q[2]-p[2]*q[1]])**

Triple scalar product: **tsp:=(p,q,r)-> dot(p,cross(q,r))**

The built-in *simplify* above will reduce the number needed in later commands. Note that **tsp(p,q,r)** is just the determinant of the matrix with rows **p,q,r**, so reversal of any two entries gives (only) a sign change.

The three commands can be saved in Maple's concise machine language by:

save dot,cross,tsp,"dotcrosstsp.m"

(Any name ending in "**.m**" will do as well.) These commands can then be introduced into later sessions by

read "dotcrosstsp.m"

(Formerly, *save* and *read* were expressed by **save(cmd1,cmd2,'filename.m')** and **read('filename.m')**, using backquotes.)

> Differential forms. The package *difforms* provides the essentials, including the exterior derivative operator **d**. The command **defform** is used to specify the degree of the forms involved. For example, **defforms(x=0,y=0)** tells *Maple* that x and y are 0-forms, that is, real-valued functions. Then the command **d(x^2*sin(y))** yields **2x sin(y)d(x)+x^2 cos(y)d(y)**.

2. Curves

A curve in $\mathbf{R}^3$ is described by giving its components as expressions in a single variable, for example, **c:= t->[3*cos(t),3*sin(t),2*t]**. Then the vector derivative (i.e., velocity) of **c** is returned by **diff(c(t),t)**, which differentiates each component of the curve by **t.**

> Curves with parameters. For example, using the *unapply* command, the curve **c** can be generalized to

helix:=(a,b)-> unapply([a*cos(t),a*sin(t),b*t]

Then **helix(3,2)** gives **c** as above.

> Frenet apparatus. We now show how the Frenet formulas in Theorem 4.3 of Chapter 2 can be expressed in terms of *Maple*.

The curvature function κ of a curve $c \sim \gamma$ is given by

```
kappa := c –> unapply(simplify(
  dot(cross(diff(c(t),t),diff(c(t),t,t)),
  cross(diff(c(t),t),diff(c(t),t,t)))^(1/2)/
  dot(diff(c(t),t),diff(c(t),t))^(3/2)),t)
```

Here "unapply" makes **kappa(c)** a real-valued function on the domain of c. Otherwise, it would merely be an expression in t and could not be evaluated on real numbers or other variables.

The command "simplify" is the principal *Maple* simplification weapon, but it not a panacea. It can be augmented by related commands such as "factor" or "expand." Use **?simplify** for information about these.

No set pattern of commands will give good results in every case, and human intervention is often required to get reasonable simplification.

The torsion function **tau** of a curve **c** is given by

```
tau := c -> unapply(simplify(
  tsp(diff(c(t),t),diff(c(t),t,t),
    diff(c(t),t,t,t))/factor(
  dot(cross(diff(c(t),t),diff(c(t),t,t)),
  cross(diff(c(t),t),diff(c(t),t,t))))),t)
```

The distinction between functions and mathematical expressions is always important. Thus, with notation as above, **tau**, applied to a curve, say **helix(3,2)**, is a real-valued function whose value at a number or variable **s** is given by **tau(helix(3,2))(s)**.
Maple has several varieties of scalar multiplication when *LinearAlgebra* is installed, however, since we are working with lists, **s*v** suffices.

The Frenet frame of a curve. The unit tangent, normal, and binormal vector fields T, N, B of a curve c are given by

```
tang := c->unapply(
  dot(diff(c(t),t),diff(c(t),t))^(-1/2)
    *diff(c(t),t),t)

nor := c->unapply(cross(binor(c)(t),tang(c)(t)),t)

binor := c->unapply(simplify(factor(
dot(cross(diff(c(t),t),diff(c(t),t,t)),
  cross(diff(c(t),t),diff(c(t),t,t)))))^(-1/2)*
  cross(diff(c(t),t),diff(c(t),t,t)),t)
```

The presence of square roots in these formulas means that we cannot expect simple results unless the curve itself is quite simple. However, individual values of the vector fields are usually readable.

Once the Frenet commands have been typed, they can be saved in a *Maple* dot-m file by

```
save kappa,tau,tang,nor,binor,"frenet.m"
```

and, as usual, these commands can be installed in later work by **read "frenet.m"**.

3. Surfaces

A coordinate patch, say **x**, in $\mathbf{R}^3$ is defined as a list-valued function whose entries are expressions in two variables. For example,

```
x:=(u,v)->[3*u*cos(v),3*u*sin(v),2*v]
```

Parameters in a patch can be handled as above for curves. For example, the 3 and 2 in this formula can be replaced by an arbitrary parameters **a** and **b** using

```
helicoid:=(a,b)-> unapply([a*u*cos(v),a*u*sin(v),
  b*v],u,v)
```

Then **helicoid(3,2)** gives the original patch **x**.

The following commands, applied to a patch **x**, return $E, F, G, W = EG - F^2$, and L, M, N. We elect to represent these capital letters (E) by double lowercase letters (ee) since some capitals have special meaning for *Maple* (for example, $I = -1$).

```
ee := x-> unapply(dot(diff(x(u,v),u),
  diff(x(u,v),u)),u,v)
ff := x-> unapply(dot(diff(x(u,v),u),
  diff(x(u,v),v)),u,v)
gg := x-> unapply(dot(diff(x(u,v),v),
  diff(x(u,v),v)),u,v)
```

(Recall that *simplify* is built into the **dot** command, defined earlier.)

```
ww := x-> unapply(simplify(
    ee(x)(u,v)*gg(x)(u,v)-ff(x)(u,v)^2)^
      (1/2),u,v)
```

```
ll := x-> unapply(tsp(diff(x(u,v),u,u),
    diff(x(u,v),u),diff(c(u,v),v))/
    ww(x)(u,v),u,v)
```

The formulas for **mm** and **nn** are the same, except that the double derivative **u,u** is replaced by **u,v** and **v,v**, respectively.

As before, these commands can be saved by

```
save ee,ff,gg,ww,ll,mm,nn,"efgwlmn.m"
```

> Gaussian and mean curvature. When the commands above for *E, F, G* and L, M, N have been read in, commands for *K* and *H* follow immediately from Corollary 4.1 of Chapter 5. However, a faster way to find *K* for a given patch in $\mathbf{R}^3$ is to use the following command, based on Exercise 20 of Section 5.4. In it, **proc**, for "procedure", begins an enclave—terminated by **end proc**—within which definitions can be made that do not escape to the outside. These temporary definitions allow the final formula to be expressed more concisely.

```
gaussK := proc(x) local xu,xv,xuu,xuv,xvv;
    xu := diff(x(u,v),u);xv := diff(x(u,v),v);
   xuu := diff(x(u,v),u,u);
   xuv := diff(x(u,v),u,v);
   xvv := diff(x(u,v),v,v);
 unapply(simplify(factor(
 tsp(xuu,xu,xv)*tsp(xvv,xu,xv)-
   tsp(xuv,xu,xv)^2)/
 (dot(xu,xu)*dot(xv,xv)-dot(xu,xv)^2)^2),u,v)
 end proc
```

Here **tsp** is the *triple scalar product*, defined earlier. As usual, **gaussK** can be saved for future use.

4. Plots

Maple has three basic plot commands.

(1) The command **plot** has two uses:
   (i) Graphs. If *f* is a real-valued function defined on $a \leqq t \leqq b$, then **plot(f(t),t=a..b)** draws its graph.
   (ii) Parametric plots. If *g* is another such function, then the curve with $c(t) = [f(t), g(t)]$ is plotted in $\mathbf{R}^2$ by **plot(c(t), t=a..b)**. Alternatively, **plot([f(t),g(t)],t=a..b)** gives the same result.

Plots can be modified by options, thus: **plot([c(t),t=a..b], <*option*>)**, where, for example, the option **numpoints=200** would increase the smoothness of the plot, and **scaling=constrained** imposes the same scale on the axes. Use **?plot[options]** to get many others.

(2) The command **plot3d** also has two uses. Let D be a region $a \leqq u \leqq b$, $c \leqq v \leqq d$ in $\mathbf{R}^2$. Then
   (i) Graphs. If f is a real-valued function defined on D, its graph is plotted by **plot3d(f(u,v),u=a..b,v=c..d)**.
   (ii) Parametric plots. If **x**$:D \to \mathbf{R}^3$ is a list-valued patch or parametrization, its image is plotted by **plot3d(x(u,v),u=a..b, v=c..d)**.

Again, **?plot3d** describes a number of ways to specify plot style.

(3) Parametrized curves in $\mathbf{R}^3$ are plotted using the command "spacecurve" from the *plots* package. As an example: **spacecurve(c(t), t=–2..4)**

To show more than one plot on the same page, each plot should be named, say, **A := plot3d(x(u,v),u=0..1,v=0..Pi):** with terminal colon to avoid a flood of numbers. Then use "display" from the *plots* package: **display([A,B,C]).**

5. Differential Equations

Explicit solutions in terms of elementary functions are rare, so we describe how to find and plot numerical solutions, which are just as useful in many contexts. In the command for a numerical solution, *Maple* lumps equations and initial conditions into a single *set*, then gives the dependent variables (as follows).

For example, suppose we want to solve numerically the equations

$$x' = f(x, y, t), \quad y' = g(x, y, t)$$

subject to the initial conditions

$$x(t_0) = x_0, \quad y(t_0) = y_0$$

on the interval $a \leqq t \leqq b$. The format is

```
numsol := dsolve(
  {diff(x(t),t)=f(x(t),y(t),t),
  diff(y(t),t)=g(x(t),y(t),t),
  x(t0)=x0,y(t0)=y0},{x(t),y(t)},type=numeric)
```

This solution is plotted by a command from the *plots* package:

```
odeplot(numsol,[x(t),y(t)],a..b)
```

Only now is the domain $a \leqq t \leqq b$ of the solution specified.

Computer Exercises

Chapter 2: 2.2/9, 2.4/11, 14, 15, 19, 20, 2.7/7
Chapter 3: 3.2/5, 3.5/4, 5, 9, 10
Chapter 4: 4.2/5, 6, 11, 4.3/6, 11, 4.6/6, 4.8/10
Chapter 5: 5.4/16, 18–21, 5.6/16, 18, 5.7/8, 9
Chapter 6: 6.5/6, 6.8/11, 13
Chapter 7: 7.2/13, 7.5/9–12, 7.7/12, 13
Chapter 8: 8.1/8

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The book by do Carmo is a clearly written exposition of differential geometry with a viewpoint similar to this one, but at a more advanced level. Gray's book is recommended for readers interested in the use of computers, especially for differential geometry. Both these books have extensive bibliographies.

Answers to Odd-Numbered Exercises

These answers are not complete; and in some cases where a proof is required, we give only a hint.

Chapter 1

Section 1.1

1. (a) $x^2y^3 \sin^2 z$.
   (c) $2x^2y \cos z$.
3. (b) $2xe^h \cos(e^h)$, $h = x^2 + y^2 + z^2$.

Section 1.2

1. (a) $-6U_1(\mathbf{p}) + U_2(\mathbf{p}) - 9U_3(\mathbf{p})$.
3. (a) $V = (2z^2/7)U_1 - (xy/7)U_3$.
   (c) $V = xU_1 + 2yU_2 + xy^2U_3$.
5. (b) Use Cramer's rule.

Section 1.3

1. (a) 0.
   (b) $7 \cdot 2^7$.
   (c) $2e^2$.

3. (a) y^3.
   (c) $yz^2(y^2z - 3x^2)$.
   (e) $2x(y^4 - 3z^5)$.
5. Use Exercise 4.

Section 1.4

1. $\alpha'(\pi/2) = \left(-1, 0, 1/\sqrt{2}\right)$ at $\left(1, 1, \sqrt{2}\right)$.
3. $\beta(s) = \left(1 + s, \sqrt{1 - s^2}, \sqrt{2}\sqrt{1 - s}\right)$.
5. The lines meet at (11, 7, 3).
7. $\mathbf{v}_p = (1, 0, 1)_p$ at $\mathbf{p} = (0, 1, 0)$.

Section 1.5

1. (a) 4.
   (b) –4.
   (c) –2.
3. Use Exercise 2 and $\phi((1/x)V + (1/y)W) = \phi(V)/x + \phi(W)/y$.
5. (b) $(x\,dy - y\,dx)/(x^2 + y^2)$.
7. (a) $dx - dz$.
   (b) not a 1-form.
   (c) $z\,dx + x\,dy$.
9. $\pm(0, 1, 1/2)$.
11. (a) Consider the Taylor series for $t \to f(\mathbf{p} + t\mathbf{v})$.
    (b) Exact: –.420, approximate: –.500.

Section 1.6

1. (a) $\phi \wedge \psi = yz \cos z\, dx\, dy - \sin z\, dx\, dz - \cos z\, dy\, dz$.
   (b) $d\phi = -z\, dx\, dy - y\, dx\, dz$, since $d(dz) = 0$.
7. Apply this definition to the formula following Definition 6.3.
9. For the alternation rule, set $f = y$, $g = x$.

Section 1.7

1. (c) (0, 0), (1, 0).
3. $F_*(\mathbf{v}) = F(\mathbf{p} + t\mathbf{v})'(0) = 2(p_1v_1 - p_2v_2, v_1p_2 + v_2p_1)$ at $F(\mathbf{p})$.
5. $F_*(\mathbf{v}_p) = F(\mathbf{p} + t\mathbf{v})'(0) = (F(\mathbf{p}) + tF(\mathbf{v}))'(0) = F(\mathbf{v})_{F(p)}$.

7. Using Lemma 4.6 gives $\mathbf{v}_p[g(F)] = (d/dt)|_0\, g(F(\mathbf{p} + t\mathbf{v})) = F(\mathbf{p} + t\mathbf{v})'(0)[g] = F_*(\mathbf{v}_p)[g]$.
9. (a) $GF = (g_1(f_1, f_2), g_2(f_1, f_2))$.
   (b) $(GF)_*(\alpha'(0)) = (GF(\alpha))'(0) = G_*(F(\alpha)'(0)) = G_*F_*(\alpha'(0))$.
   (c) F^{-1} is one-to-one and onto. To show it is regular, start from $F(F^{-1}) = I$, the identity map. Hence $F_*(F^{-1})_* = I_* =$ identity map on tangent vectors. So $(F^{-1})_*$ cannot carry a nonzero vector to zero.

Chapter 2

Section 2.1

1. (a) -4.
   (b) $(6, -2, 2)$.
   (c) $(1, 2, -1)/\sqrt{6}, (-1, 0, 3)/\sqrt{10}$.
   (d) $2\sqrt{11}$.
   (e) $-2/\sqrt{15}$.
5. If $\mathbf{v} \times \mathbf{w} = 0$, then $\mathbf{u} \bullet \mathbf{v} \times \mathbf{w} = 0$ for all $\mathbf{u}$; use Exercise 4.
7. $\mathbf{v}_2 = \mathbf{v} - (\mathbf{v} \bullet \mathbf{u})\,\mathbf{u}$.

Section 2.2

1. (b) $s(t) = 2t + t^3/3$.
3. $\beta(s) = \left(\sqrt{1 + s^2/2},\, s/\sqrt{2}, \sinh^{-1}\left(s/\sqrt{2}\right)\right)$.
7. For (ii), $|h'| = -h' \geqq 0$, so the change of variables formula in an integral gives $L(\alpha(h)) = \int_c^d \|\alpha(h)'\| ds = \int_c^d \|\alpha'(h)\|(-h')ds = -\int_c^d \|\alpha'\| h' ds = -\int_b^a \|\alpha'\| dt = \int_a^b \|\alpha'\| dt = L(\alpha)$.
9. $L(\alpha) \approx 12.9153 < 14.1438 \approx L(\beta)$.

Section 2.3

1. $\kappa = 1$, $\tau = 0$, $B = -(3, 0, 4)/5$, center $(0, 1, 0)$, radius 1.
7. (a) $1 = \|\alpha(h)'\| = \|\alpha'(h)h'\| = |h'|$, hence $h' = \pm 1$.
   (b) Let $\varepsilon = \pm 1$. Then $\bar{\alpha} = \alpha(h)$ implies $\bar{T} = \alpha'(h)h' = \varepsilon T(h)$. Hence $\bar{\kappa}\bar{N} = \kappa(h)N(h)$, and so on.

9. For the rectifying plane. From the formula for $\tilde{\beta}$ in the text, delete $\beta(0)$ and the N_0 term. The remaining terms give the same general shape as the curve $(s, \pm s^3)$.
11. (b) First differentiate $B = \overline{B}$; consider the two ± cases and differentiate again.

Section 2.4

1. (a) Let $f = t^2 + 2$. Then $\kappa = \tau = 2/f^2$ and $B = (t^2, -2t, 2)/f$.
   (c) All the limits are natural unit vectors, ±(1, 0, 0), . . .
3. (a) $N = (0, -1, 0)$, $\tau(0) = 3/4$.
7. (a) $(\gamma(t) - \alpha(t_0)) \bullet \mathbf{u} = 0$.
   (b) γ has constant speed, so use Exercise 5.
9. Evidently, α is a cylindrical helix. By Exercise 7 its cross-sectional curve γ is a plane curve with constant curvature, hence γ lies in a circle.
11. (c) (*Mathematica*):

```
helix[a_,b_][t_]:={a*Cos[t],a*Sin[t],b*t}
ParametricPlot3D[{helix[2,1][t],helix[-.5,1]
[t]}//Evaluate,{t,0,6Pi}]
```

   (*Maple*): With the *plots* package installed,

```
helix:=(a,b)->[a*cos(t),a*sin(t),b*t]
spacecurve({helix(2,1)(t),helix(-.5,1)(t)},
t=0..6*Pi,numpoints=100)
```

   Recall that we do not show *Maple*'s mandatory terminal semicolon.
13. (b) $\lambda_t(s) = \alpha(t) + s(\alpha'(t) \bullet \alpha'(t)/\alpha''(t) \bullet J(\alpha'(t))\, J(\alpha'(t))$ for $0 \leqq s \leqq 1$.
   (c) For α unit speed, $\lambda_t(s) = \alpha + s(1/\tilde{\kappa})N$. Hence $d\lambda_t/ds = (1/\tilde{\kappa})N$ (independent of s). Evidently this is normal to α at $\alpha(t)$. Since $\alpha^* = \alpha + (1/\tilde{\kappa})N$, we get $(\alpha^*)' = T + (1/\tilde{\kappa})'N - T = (1/\tilde{\kappa})'N$, in agreement with $d\lambda_t/ds$ at α^* (1).
15. (a) For the rectifying plane (orthogonal to N):
   (*Mathematica*):

```
viewN[a_,eps_]:=ParametricPlot[{(a[t]-a[0]).
tang[a][0],
  (a[t]-a[0]).binor[a][0]}//Evaluate,
    {t,-eps,eps}]
```

   (*Maple*)

```
viewN:=(a,eps)->plot([dot((a(t)-a(0)),
tang(a)(0)),
  dot((a(t)-a(0)),binor(a)(0)),t=-eps..eps])
```

(b) (iii) For all curves with $\tau(0) \neq 0$ there are essentially only two cases, depending on the sign of τ.

17. (a) $\pi/\sqrt{2}$.
    (b) ∞.
    (c) $\pi/\sqrt{2}$.
    (d) 2π (see Exercise 18).
19. (c) For a suitable n, let τ_n be τ with new z-component $(1/n)\sin 3t$. Here $\tilde{\kappa} = \kappa$, and in the notation of Exercise 12, $ds/dt = \sqrt{x'^2 + y'^2}$.
21. Use Theorem 4.6. By hand computation (easy, if κ and τ are first found by computer), we get $\tau/\kappa = (3ac/2b^2)(P/Q)^{3/2}$, where
    $$P = 9c^2t^4 + 4b^2t^2 + a^2 \quad \text{and} \quad Q = 9c^2t^4 + (9a^2c^2/b^2)t^2 + a^2$$
    Thus τ/κ is constant if and only if $4b^2 = 9a^2c^2/b^2$, that is, $3ac = \pm 2b^2$. (Hence $\tau/\kappa = \pm 1$).

Section 2.5

1. (a) $2U_1(\mathbf{p}) - U_2(\mathbf{p})$.
   (b) $U_1(\mathbf{p}) + 2U_2(\mathbf{p}) + 4U_3(\mathbf{p})$.
5. $\nabla_{\alpha'(t)} W = \Sigma \alpha'(t)[w_i]U_i = \Sigma(d/dt)(w_i(\alpha))(t)U_i = (W_\alpha)'(t)$.

Section 2.6

1. Show that $V \cdot \tilde{W} = 0$, and use Lemma 1.8.
3. For instance, $E_2 = -\sin z\, U_2 + \cos z\, U_3$ and $E_3 = E_1 \times E_2$.

Section 2.7

1. $\omega_{12} = 0$, $\omega_{13} = \omega_{23} = df/\sqrt{2}$.
3. $\omega_{12} = -df$, $\omega_{13} = \cos f\, df$, $\omega_{23} = \sin f\, df$.
5. By Corollary 5.4(3), $\nabla_V(\Sigma f_i E_i) = \Sigma V[f_i]E_i + f_i \nabla_V E_i$
7. (*Mathematica*):
   (a) **`connform[A_]:=Simplify[Dt[A].Transpose[A]]`**
   (b) In A, write q for ϑ and f for φ. Then in **`MatrixForm[connform[A]]`**, read **`Dt[q]`** as **`dq`**.

   (*Maple*): Install the packages *LinearAlgebra* and *difforms*. With q and f as above, write **`defform(q=0,f=0)`** to identify them as real-valued functions.
   (a) **`connform:=A->simplify(Multiply(map(d,A), Transpose(A)))`**

Section 2.8

3. (a) Compute $\theta = A\,d\xi$, as in the text. (A was found in Section 7.)
   (b) For example, $E_1[r] = dr[E_1] = \theta_1(E_1) = 1$.
   (c) Use the appropriate form of the chain rule.

Chapter 3

Section 3.1

3. $(T_a)^{-1} = T_{-a}$, and since C is orthogonal, $C^{-1} = {}^tC$. Thus $F^{-1} = (T_aC)^{-1} = C^{-1}(T_a)^{-1} = {}^tCT_{-a}$. By Exercise 1, this equals $T_{{}^tC(-a)}{}^tC = T_{-{}^tC(a)}{}^tC$.
5. (b) Using Exercise 3 we find $F^{-1}(\mathbf{p}) = \left(5\sqrt{2}, -2, 3\sqrt{2}\right)$
7. Use Exercises 2 and 3.
9. (a) For ϑ such that $C(1, 0) = (\cos\vartheta, \sin\vartheta)$, C has matrix
$$\begin{pmatrix} \cos\vartheta & \mp\sin\vartheta \\ \sin\vartheta & \pm\cos\vartheta \end{pmatrix}.$$
   (b) $O(1)$ consists of $+1$ and -1, so $F(t) = a \pm t$ for any number a.

Section 3.2

1. $T(\mathbf{v}_p) = \mathbf{v}_{T(p)}$.
3. The middle row of C is $(-2, 1, 2)/3$, and T is translation by $\left(3, -4/3, 1-2\sqrt{2}/3\right)$
5. (*Mathematica*):
   Let **ame={e1,e2,e3}** and **amf={f1,f2,f3}** be the attitude matrices of the frames in Exercise 3.
   (b) Set **cc:=Simplify[Transpose[amf].ame]** Then **Simplify[cc.e1]** is **f1**, etc.
   (*Maple*):
   Install the package *LinearAlgebra*, and let **ame=Matrix([e1,e2, e3])** and **amf=Matrix([f1,f2,f3])** be the attitude matrices of the frames in Exercise 3.
   (b) Set **cc:=simplify(Multiply(Transpose(amf), ame)).** Then **simplify(Multiply(cc,Vector(e1)))** is **Vector(f1)**, etc.

Section 3.3

1. If the orthogonal parts of F and G are A and B, then by Exercise 2 of Section 1, $\operatorname{sgn}(FG) = \det AB = (\det A)(\det B) = \det BA = \operatorname{sgn}(GF)$. Then $+1 = \operatorname{sgn} I = \operatorname{sgn}(FF^{-1}) = \operatorname{sgn}(F)\operatorname{sgn}(F^{-1})$.
5. C is rotation through angle $\pi/2$ about the axis given by $\mathbf{a}$.

Section 3.4

1. (b) By definition, $\sigma(s)$ is the point canonically corresponding to $T(s)$; hence by Exercise 1 of Section 2, $C(\sigma)$ corresponds to $F_*(T)$, the unit tangent of $F(\beta)$.
3. Translate each triangle so that its new first vertex is at the origin. A sketch will show that the required C is orientation-reversing, and we find $C =$

$$\begin{pmatrix} -3/5 & 4/5 \\ 4/5 & 3/5 \end{pmatrix}$$

5. For a tangent vector $\mathbf{v}$ at $\mathbf{p}$,

$$F_*(\nabla_v W) = F_*(W(\mathbf{p}+t\mathbf{v})'(0)) = (\overline{W}(F(\mathbf{p})+tC(\mathbf{v}))'(0) = \nabla_{F_*(v)}\overline{W}.$$

Section 3.5

3. Take $a = 2$, $b = \pm 2$.
5. Yes, since c has constant speed, curvature, and torsion.
7. $\beta(s) = \left(\int \cos\varphi(s)ds, \int \sin\varphi(s)ds\right)$, where $\varphi(s) = \int f(s)ds$
9. For simplicity, assume $a \leqq 0 \leqq b$; then:
   (*Mathematica*):
   (a)
```
kdetc[f_,a_,b_]:=
    NDSolve[{x'[s]==Cos[phi[s]],
    y'[s]==Sin[phi[s]],
    phi'[s]==f[s],x[0]==0,y[0]==0,
    phi[0]==0},{x,y,phi},{s,a,b}]
```
   (b)
```
draw[f_,a_,b_]:=ParametricPlot[Evaluate
[{x[s],y[s]}/.kdetc[f,a,b]],{s,a,b},
AspectRatio->Automatic]
```
   (*Maple*):
   (a)
```
kdetc:=f->dsolve({diff(x(s),s)=cos(phi(s)),
diff(y(s),s)=sin(phi(s)),diff(phi(s),s)=f(s),
x(0)=0,y(0)=0,phi(0)=0},{x(s),y(s),
phi(s)},type=numeric)
```

(b) Install *plots*. Define **`draw:=(f,a,b)->odeplot(kdetc(f),[x(s),y(s)],a..b,scaling=constrained)`**.

Chapter 4

Section 4.1

1. (a) The vertex.
   (b) All points on the circle $x^2 + y^2 = 1$.
   (c) All points on the z axis.
5. (b) $c \neq -1$.
9. Use Exercise 7.
11. **q** is in $F(M)$ if and only if $F^{-1}(\mathbf{q})$ is in M, that is, $g(F^{-1}(\mathbf{q})) = c$. Use the Hint to apply Theorem 1.4.

Section 4.2

1. (c) The Monge patch $\mathbf{x}(u, v) = (u, v, u^2 + v^2)$ covers the entire surface; a parametrization based on Example 2.4 omits the point (0, 0, 0).
3. $\mathbf{x}_u \times \mathbf{x}_v = v\delta'(u) \times \delta(u)$.
5. (a) $EG - F^2 = b^2 + u^2$ is never zero.
   (b) Helices and straight lines (rulings).
   (c) H: $x\sin(z/b) - y\cos(z/b)$.
   (d) For **x** as given:
   (*Mathematica*): **`ParametricPlot3D[x[u,v]//Evaluate, {u,-1,1},{v,0,2Pi}]`**
   (*Maple*): **`plot3d(x(u,v),u=-1..1,v=0..2*Pi)`**
7. (b) $\mathbf{x}(u, v) = (\cos u - v\sin u, \sin u + v\cos u, v)$.
9. In all cases, (i) check that the three partial derivatives of the defining function g are never zero simultaneously on M: $g = 1$ (Theorem 1.4), and (ii) First, check that the components of **x** satisfy the equation $g = 1$.
11. (c) $\mathbf{x}_{\pm}(u, v) = (a\cos u, b\sin u, 0) \pm v(-a\sin u, b\cos u, c)$.
    (d) (*Mathematica*):

```
xplus[u_,v_]:={1.5*(Cos[u]-v*Sin[u]),
  Sin[u]+v*Cos[u],2v}
ParametricPlot3D[xplus[u,v]//Evaluate,
  {u,0,2Pi},{v,-1,1}]
```

(*Maple*):

```
xplus:=(u,v)→[1.5*(cos(u)-v*sin(u)),
sin(u)+v*cos(u),2*v]
  plot3d(xplus(u,v),u=0..2*Pi,v=-1..1)
```

Section 4.3

1. (a) $r^2\cos^2 v$.
   (b) $r^2(1 - 2\cos^2 v \cos u \sin u)$.
3. (a) $\bar{u}$ and $\bar{v}$ are the Euclidean coordinate functions of $\mathbf{x}^{-1}\mathbf{y}$.
   (b) Express $\mathbf{y} = \mathbf{x}(\bar{u}, \bar{v})$ in terms of Euclidean coordinates, and differentiate.
5. (a) M is given by $g = z - f(x, y) = 0$, with $\nabla g = (-f_x, -f_y, 1)$, and $\mathbf{v}$ is tangent to M at $\mathbf{p}$ if and only if $\mathbf{v} \bullet \nabla g(\mathbf{p}) = 0$.
7. $\nabla g = (-y, -x, 1)$ is a normal vector field; V is a tangent vector field if and only if $V \bullet \nabla g = 0$, for example, $V = (0, 1, x)$.
9. (a) $\overline{T}_p(M)$ consists of all points $\mathbf{r}$ such that $(\mathbf{r} - \mathbf{p}) \bullet \mathbf{z} = 0$; hence $\mathbf{v}_p$ is in $T_p(M)$ (that is, $\mathbf{v} \bullet \mathbf{z} = 0$) if and only if $\mathbf{p} + \mathbf{v}$ is in $\overline{T}_p(M)$.
11. (a) If $a/b = m/n$ for integers m, n, consider $\Delta t = 2\pi m/a = 2\pi n/b$.
    (b) Assume $\alpha(s) = \alpha(t)$ for $s \neq t$, so $\mathbf{x}(as, bs) = \mathbf{x}(at, bt)$. Equality for z components and for $x^2 + y^2$ implies $as - at = 2\pi m$ and $bs - bt = 2\pi n$ for some integers m, n. Thus $a/b = m/n$, a contradiction.

Section 4.4

1. $$d(f\phi)(\mathbf{x}_u, \mathbf{x}_v) = \frac{\partial(f(\mathbf{x}))}{\partial u}\phi(\mathbf{x}_v) - \frac{\partial(f(\mathbf{x}))}{\partial v}\phi(\mathbf{x}_u) + f(\mathbf{x})\left[\frac{\partial}{\partial u}\phi(\mathbf{x}_v) - \frac{\partial}{\partial v}\phi(\mathbf{x}_u)\right]$$
   $$= (df \wedge \phi + f d\phi)(\mathbf{x}_u, \mathbf{x}_v).$$
3. If α is a curve with initial velocity $\mathbf{v}$ at $\mathbf{p}$, then
   $$\mathbf{v}_p[g(f)] = (gf\alpha)'(0) = g'(f\alpha)(0)(f\alpha)'(0) = g'(f(\mathbf{p}))\mathbf{v}_p[f].$$
5. On the overlap of $\mathscr{U}_i$ and $\mathscr{U}_j$, $df_i - df_j = d(f_i - f_j) = 0$.
7. (b) $d\tilde{u}(\mathbf{x}_u) = \mathbf{x}_u[\tilde{u}] = \dfrac{\partial(\tilde{u}(\mathbf{x}))}{\partial u} = \dfrac{\partial u}{\partial u} = 1.$

Section 4.5

1. If $\mathbf{x}\colon D \to M$ is a patch, then $F(\mathbf{x})\colon D \to N$ is (by Theorem 3.2) a differentiable mapping. Hence $\mathbf{y}^{-1}F\mathbf{x}$ is differentiable for any patch $\mathbf{y}$ in N.
3. If $\bar{\mathbf{x}}$ and $\bar{\mathbf{y}}$ are patches in M and N, respectively, note that $\bar{\mathbf{y}}^{-1}F\bar{\mathbf{x}} = (\bar{\mathbf{y}}^{-1}\mathbf{y})(\mathbf{x}^{-1}\bar{\mathbf{x}})$ is differentiable, being a composition of differentiable functions.
5. By Exercise 1, A is differentiable. Since $A^2 = I$, $A^{-1} = A$, so A is a diffeomorphism. For A_*, consider its effect on a curve $t \to \cos t\,\mathbf{p} + \sin t\,\mathbf{u}$ in Σ.
7. Theorem 5.4.

9. (a) Use Exercise 8.
   (b) $F_*(a\mathbf{x}_u + b\mathbf{x}_v) = a\mathbf{y}_u + b\mathbf{y}_v$ implies linearity.
11. M is diffeomorphic to a torus if the profile curve α of M is closed, and to a cylinder if α is one-to-one. With parametrizations as suggested, $F(\mathbf{x}(u, v)) = \mathbf{y}(u, v)$ is a diffeomorphism.
13. (a) If $\mathbf{p}$ in M, there is a $\mathbf{q}$ in $\tilde{M}$ such that $\mathbf{p} = G(\mathbf{q})$. By consistency, $F(\mathbf{p}) = \tilde{F}(\mathbf{q})$ is a valid definition. G is regular, hence locally has differentiable inverse mappings. Thus, locally $F = \tilde{F}G^{-1}$ so F is differentiable.
   (b) If $F(\mathbf{p}_1) = F(\mathbf{p}_2)$, then for $\mathbf{q}_1, \mathbf{q}_2$ in $\tilde{M}$ such that $G(\mathbf{q}_1) = \mathbf{p}_1$, $G(\mathbf{q}_2) = \mathbf{p}_2$, we have $F(G(\mathbf{q}_1) = F(G(\mathbf{q}_2))$. Thus $\tilde{F}(\mathbf{q}_1) = \tilde{F}(\mathbf{q}_2)$. Then the hypothesis gives $G(\mathbf{q}_1) = G(\mathbf{q}_2)$, that is, $\mathbf{p}_1 = \mathbf{p}_2$.

Section 4.6

3. (b) Use Theorem 6.2.
5. (a) Let $r(t) = \|\alpha(t)\|$. Then let $f = U_1 \bullet \alpha/\|\alpha\|$ and $g = U_2 \bullet \alpha/\|\alpha\|$. Apply Exercise 12 of Section 2.1 to get ϑ.
   (b) $\vartheta(a)$ and $\vartheta(b)$ measure the same angle; hence they differ by some integer multiple of 2π.
   (c) Use Exercise 1 to evaluate ψ on the polar expression for α in (a).
   (d) $\dfrac{\det(\alpha, \alpha')}{\alpha \bullet \alpha} = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} \Big/ (f^2 + g^2) = \dfrac{fg' - gf'}{f^2 + g^2}.$
7. (a) Since $(F_*(\phi))(\alpha') = \phi((F_*)(\alpha')) = \phi(F(\alpha)')$, we get
$$\int_\alpha F_*(\phi) = \int_a^b \phi\Big(F(\alpha)'\Big)\,dt = \int_{F(\alpha)} \phi.$$
9. (a) 2πm, (b) 2πn.
13. The text shows that if ϕ is the dual of V, then $\int V \bullet ds = \int \phi$. The dual of curl V is $d\phi$, and $dA \approx W\,du\,dv$. It follows that
$$U \bullet \text{curl}\,V\,dA = \text{curl}\,V \bullet \frac{\mathbf{x}_u \times \mathbf{x}_v}{W} W\,du\,dv = \phi(\mathbf{x}_u, \mathbf{x}_v)\,du\,dv.$$

Section 4.7

1. (a) Connected, not compact.
   (c) Not connected and not compact.
   (e) Connected and compact.
3. If v is nonvanishing on N, show that $F^*(v)$ is nonvanishing on M.
5. (a) All—by Definition 7.1.
   (b) Sphere, torus—by Lemma 7.2.

(c) All—by Proposition 7.5
(d) Plane, sphere (see text).

9. (c) If M is connected, then path-connectedness (Definition 7.1) follows using parts (a) and (b). If M is path-connected, let $\mathcal{U}$ and $M - \mathcal{U}$ be open sets of M such that $\mathcal{U}$ contains a point **p**.
Assume that $M - \mathcal{U}$ contains a point **q**. There is a curve segment α:[a, b] $\to M$ from **p** to **q**. Since α is continuous, $\alpha^{-1}(\mathcal{U})$ and α^{-1} $(M - \mathcal{U})$ are disjoint open sets filling [a, b]. This contradicts the stated connectedness of [a, b].

11. Fix **q** in $M - \mathcal{R}$; then by the Hausdorff axiom, for each **p** in $\mathcal{R}$, there are disjoint neighborhoods $\mathcal{U}_p$ of **p** and $\mathcal{U}_{q,p}$ of **q**. By compactness, a finite number of the neighborhoods $\mathcal{U}_p$ cover $\mathcal{R}$. Then the intersection of the corresponding neighborhoods $\mathcal{U}_{p,q}$ is a neighborhood of **q** that does not meet $\mathcal{R}$.

Section 4.8

1. If M is orientable it has a nonvanishing 2-form μ. Then $f(t) = \mu(\alpha'(t), Y(t))$ is a differentiable function on [a, b]. By (ii), $f(a)f(b) < 0$; hence f is somewhere zero on $a < t < b$. This contradicts (i).

5. (a) The function $\mathbf{p} \to d(\mathbf{0}, \mathbf{p})$ is continuous on M, hence takes on a maximum.

7. (i) Since M is nonorientable, there is a reversing loop (as in the hint) at some point **q**. Fix U_q. Then every point U_p in $\hat{M}$ can be connected to U_q by a curve in $\hat{M}$. *Proof:* Move U_p along a curve from **p** to **q**. If the result is $-U_q$, move it around the reversing loop.

9. (b) $B - \beta$ is diffeomorphic to an ordinary band.

11. (a) Recall that a neighborhood in a surface is the image under a coordinate patch of a neighborhood in $\mathbf{R}^2$. Evidently every neighborhood $\mathbf{x}(\mathcal{U})$ of $\mathbf{0}$ meets every neighborhood $\mathbf{y}(\mathcal{V})$ of $\mathbf{0}^*$.
(b) The sequence $\{(1/n, 0)\}$ converges to $\mathbf{0}$ when expressed in terms of **x**, and to $\mathbf{0}^*$ in terms of **y**.
(c) Relative to **x** and **y**, the coordinate form of F is the identity map.

13. (a) In terms of the natural coordinates, $\alpha'(t) = V(\alpha(t))$ becomes

$$u'U_1 + v'U_2 = f_1(u, v)U_1 + f_2(u, v)U_2.$$

(b) The differential equations are $u' = -u^2$, $v' = uv$, and the initial conditions are $u(0) = 1$, $v(0) = -1$. The first differential equation integrates to $1/u = t + A$. But $u(0) = 1$, so $u = 1/(t + 1)$. Thus we get $v' = v/(t + 1)$, which integrates to $v = B(t + 1)$. Then $v(0) = -1$ implies $v = -(t + 1)$.

15. Smooth overlap follows from the identity

$$(\mathbf{x}\times\mathbf{y})(\overline{\mathbf{x}}\times\overline{\mathbf{y}})^{-1} = (\mathbf{x}\overline{\mathbf{x}}^{-1})\times(\mathbf{y}\overline{\mathbf{y}}^{-1}).$$

Chapter 5

Section 5.1

1. Use Method 1 in the text.
3. (a) 2.
   (c) 1.
5. Meridians go to meridians (great circles through the poles), parallels to parallels—except for the top and bottom circles of the torus.
7. Use Method 1 and the definition of tangent map in Chapter 1.

Section 5.2

1. (b) If $\mathbf{e}_1,\ \mathbf{e}_2 = (\mathbf{u}_1 \pm \mathbf{u}_2)/\sqrt{2}$, then $S(\mathbf{e}_1) = \mathbf{e}_1$ and $S(\mathbf{e}_2) = -\mathbf{e}_2$.

Section 5.3

1. $k_1k_2 \leqq 0$ and $k_1 = k_2$ imply $k_1 = k_2 = 0$.
5. (b) $K > 0$: an ellipse on one side and no points on the other. $K < 0$: the two branches of a hyperbola. $K = 0$, nonplanar: two parallel lines on one side, no points on the other.
7. (a) If α is a curve with initial velocity $\mathbf{v}$ at $\mathbf{p}$, then $F_*(\mathbf{v}) = F(\alpha)'(0) = (\alpha + \varepsilon U_\alpha)'(0) = \mathbf{v} - \varepsilon S(v)$ at $F(\mathbf{p})$.

Section 5.4

1. $W = r^2 \cos v > 0$, $U = \mathbf{x}/r$, $K = 1/r^2$, $H = -1/r$.
5. Use $\alpha' = a_1'\mathbf{x}_u + a_2'\mathrm{x}_v$ to find speed.
7. $K = -36r^2/(1 + 9r^4)^2$.
9. Expand $S(\mathbf{v}) \times \mathbf{v}$. This vector is zero if and only if its dot product with $\mathbf{x}_u \times \mathbf{x}_v$ is zero. Use the Lagrange identity (Exercise 6 of Section 3).
11. $k(\mathbf{u}) = S(\mathbf{v}) \bullet \mathbf{v}/\mathbf{v} \bullet \mathbf{v}$. Substitute $\mathbf{v} = v_1\mathbf{x}_u + v_2\mathbf{x}_v$.
15. (a) K is negative except at the origin, but this is a planar point, hence an umbilic with $k = 0$.

(b) The hint leads to $\left(0, \pm(b/2)\sqrt{a^2-b^2}, (a^2-b^2)/4\right)$. These two umbilics reduce to one for the paraboloid of rotation, $a = b$, where (by symmetry) we expect **0** to be umbilic.

17. (b) Since $\kappa < B$, if $\varepsilon < 1/B$, then $\mathbf{x}_u \times \mathbf{x}_v \neq 0$.
    (c) $S(\mathbf{x}_u) \times S(\mathbf{x}_v) = -\kappa \cos v\, T \times \mathbf{x}_v/\varepsilon$.

19. (*Mathematica*):
    (b)
    ```
    hyperboloid[a_,b_,e_][u_,v_]:=
    {u,v,u^2/a^2+e*v^2/b^2}
    ```
    (c)
    ```
    monkeypolar[r_,q_]:=monkey[r*Cos[q],
    r*Sin[q]]
    ```
    (*Maple*):
    (b)
    ```
    hyperboloid:=(a,b,e)->unapply([u,v,u^2/a^2+
    e*v^2/b^2],u,v)
    ```
    (c)
    ```
    monkeypolar:=(r,q)->monkey(r*cos(q),
    r*sin(q))
    ```

21. *Maple* has a built-in tube command in the *plots* package. For (c), with τ defined as in the exercise referred to, the tube is plotted by
    ```
    tubeplot(τ(t),t=0..2*Pi,radius=0.5)
    ```
    (*Mathematica*):
    (a) With the commands for unit normal and binormal installed (see Appendix), a tube formula is
    ```
    tube[c_,r_][t_,phi_]:=c[t]+r*(Cos[phi]*
    nor[c][t]+Sin[phi]*binor[c][t])
    ```
    This is plotted—in (b), for example—by
    ```
    ParametricPlot3D[tube[helix,1/2][t,phi]//
    Evaluate,{t,0,4Pi},{phi,0,2Pi},
    PlotPoints->{40,20},Axes->None,Boxed->
    False]
    ```
    (c) If the general approach in (a) is slow in this case, a faster way is to copy the *outputs* of `binor[τ][t,phi]` and `nor[τ][t,phi]` into an explicit definition of the tube function of τ.

Section 5.5

3. (a) The critical points of K are those of h. They occur at the intercepts of M with the coordinate axes.
   (b) For the ellipsoid, $c^2/(a^2b^2) \leqq K \leqq a^2/(b^2c^2)$. (Note again the effect of $a = b = c$.)

5. (c) Use $Z = \operatorname{grad}(e^z \cos x - \cos y)$ and $W = Z \times V$. Then $\nabla_V Z \times W + V \times \nabla_W Z = 0$ and $V \bullet \nabla_V Z \times \nabla_W Z = -e^{2z}$.

7. (a) Use $Z = \sum(x_i/a_i)U_i$.
   (b) The tangency condition for a vector **v** at **p** is $\sum v_i p_i/a_i^2 = 0$.

Section 5.6

3. Use Remark 6.10.
5. Since $U \bullet V$ is constant, $U' \bullet V + U \bullet V' = 0$. If α is principal in M, then using Lemma 6.2, $U' \bullet V = 0$, hence $V \bullet U' = 0$. Continue as for Lemma 6.3.
7. $S(T) = -U'$; hence by orthonormal expansion, $U' = -S(T) \bullet T\,T - S(T) \bullet V\,V$. Continue as in the proof of the Frenet formulas.
11. (a) Set $\sigma = \alpha + f\delta$. Then f is determined using the equation $\sigma' \bullet \delta' = 0$.
    (b) $\delta' \perp \delta, \alpha'$ implies that $\alpha' \times \delta$ and δ' are collinear. Then $\alpha' \times \delta = p\delta'$. Hence $\mathbf{x}_u \times \mathbf{x}_v = p\delta' + v\delta' \times \delta$, so $W^2 = (p^2 + v^2)\delta' \bullet \delta'$. Now use Exercise 12 of Section 4.
    (c) On each ruling, K has a unique minimum point; the striction curve meets the ruling at this point.
13. (a) Since $\sigma(u + \varepsilon) - \sigma(u) \approx \varepsilon\sigma'(u)$, the Hint gives $d_\varepsilon = \varepsilon\sigma'(u) \bullet \delta(u) \times \delta'(u)/\|\delta(u) \times \delta'(u)\|$. However $\|\delta(u) \times \delta'(u)\|\varepsilon \approx \|\delta(\mathrm{u}) \times \delta(u + \varepsilon)\| = \sin\vartheta_\varepsilon \approx \vartheta_\varepsilon$. Since $\|\delta(u) \times \delta'(u)\|^2 = \delta' \bullet \delta'$, we see that $\lim_{\mathrm{e}\to 0} d_\varepsilon/\vartheta_\varepsilon = \sigma' \bullet \delta \times d'/\delta' \bullet \delta' = p$.
15. Compute E, F, G and L, M, N. (Computer formulas for these are given in the Appendix.) Then $EG - F^2 \neq 0$ proves (a), and $F = \text{M} = 0$ proves (b).
17. (a) $K = -h'^2\vartheta'^2/W^4$, $H = u(h'\vartheta'' - \vartheta'h'')/(2W^3)$, where $W^2 = h'^2 + u^2\vartheta'^2$.
    (b) $\delta \times \delta' = \vartheta' U_3$. Since K is a minimum when $u = 0$, the z axis is the striction curve, and $p = h'/\vartheta'$, reciprocal of turn rate (Exercise 13 of Section 6).
19. Use $W = \|\mathbf{x}_u \times \mathbf{x}_v\|$.

Section 5.7

1. $K = (1 - x^2)(1 + x^2\exp(-x^2))^{-2}$. Hence $K > 0 \Leftrightarrow -1 < x < 1$.
3. In a canonical parametrization, if g is constant, the profile curve is orthogonal to the axis, so the surface M is part of a plane. Otherwise, $K = 0 \Leftrightarrow h'' = 0 \Leftrightarrow h'$ is constant. If $h' = 0$, the profile curve lies in a line parallel to the axis, so M is part of a cylinder. If $h' \neq 0$, the profile curve is a slanting line, so M is part of a cone.
5. M has parametrization $\mathbf{x}(r, v) = (r\cos v, r\sin v, f(r))$. Then $E = 1 + f'^2$, $F = 0$, $G = r^2$, and $W\text{L} = rf''$, $W\text{M} = 0$, $W\text{N} = r^2 f'$, with $W^2 = EG - F^2 = r^2(1 + f'^2)$.

7. (a) $h(u) = a\sinh(u/c)$ satisfies the given differential equation with $K = -1/c^2$. Use the integral formula for $g(u)$. Then as u $\to 0$, the slope angle $\tan\varphi = h'/g'$ approaches $(a/c)/\sqrt{1-a^2/c^2} = a/\sqrt{c^2-a^2}$. The curve becomes vertical when $g' = 0$, hence the integrand of g vanishes. There $\cos h^2(u^*/c) = c^2/a^2$, so $h_{\max} = a\sinh(u^*/c) = \sqrt{c^2-a^2}$.
   (b) $h(u) = c\,e^{-u/c}$ satisfies the differential equation and initial condition in Example 7.6.

Chapter 6

Section 6.1

1. (a) $\alpha'' = \omega_{12}(T)E_2 + \omega_{13}(T)E_3$. Hence α'' is normal to M if and only if $\omega_{12}(T) = 0$.
3. Apply the symmetry equation to E_1, E_2. Then use Corollary 1.5.

Section 6.2

1. (a) $\theta_1 = dz$, $\theta_2 = r\,d\vartheta$.
   (d) $K = 0$ and $H = -1/2r$.

Section 6.3

1. If $K = H = 0$, then $k_1k_2 = k_1 + k_2 = 0$. Thus $k_1 = k_2 = 0$, so $S = 0$.
3. In the proof of Liebmann's theorem, replace the constancy of $K = k_1k_2$ by that of $2H = k_1 + k_2$.
5. In the case $k_1 \neq k_2$, use Theorem 2.6 to show that, say, $k_1 = 0$. By Exercise 2 the k_1 principal curves are straight lines. Show that the k_2 principal curves are circles and that the (k_1) straight lines are parallel in $\mathbf{R}^3$.

Section 6.4

1. (d) $\Rightarrow$ (b): If $\mathbf{z}$ is an arbitrary tangent vector at $\mathbf{p}$, write $\mathbf{z} = a\mathbf{v} + b\mathbf{w}$. Then

$$\begin{aligned}\|F_*\mathbf{z}\|^2 &= a^2\|F_*\mathbf{v}\|^2 + 2abF_*\mathbf{v}\bullet F_*\mathbf{w} + b^2\|F_*\mathbf{w}\|^2 \\ &= a^2\|\mathbf{v}\|^2 + 2ab\mathbf{v}\bullet\mathbf{w} + b^2\|\mathbf{w}\|^2 = \|\mathbf{z}\|^2.\end{aligned}$$

3. (b) Monotone reparametrization does not affect length of curves.
   (c) By the definition of ρ, given any $\varepsilon > 0$ there is a curve segment α from $\mathbf{p}$ to $\mathbf{q}$ of length $< \rho(\mathbf{p}, \mathbf{q}) + \varepsilon$, and an analogous β for $\mathbf{q}$ and $\mathbf{r}$. Combining α and β gives a piecewise differentiable curve segment from $\mathbf{p}$ to $\mathbf{r}$. (If only everywhere-differentiable curves are allowed, there is no change in ρ, but proofs are harder.)
5. (a) Define $F(\alpha(u) + vT(u)) = \beta(u) + vT(u)$.
   (b) Choose β in $\mathbf{R}^2$ with plane curvature equal to κ.
7. By the exercise mentioned, a shortest curve in $\mathbf{R}^2$ joining the points parametrizes a straight line segment. Thus any curve in M joining the points has length $L > 2$.
9. $F_*((F^{-1})_*\mathbf{v}) = (FF^{-1})_*\mathbf{v} = I_*\mathbf{v} = \mathbf{v}$. Since F is an isometry, $\|(F^{-1})_*\mathbf{v}\| = \|\mathbf{v}\|$.
11. Write $F(\mathbf{x}(u, v)) = \tilde{\mathbf{x}}(f(u), g(v))$ for suitable parametrizations.
13. For $\mathbf{y}$, show that the conditions $E = G$ and $F = 0$ are equivalent to $g' = \cos g$, which has solution $g(v) = 2\tan^{-1}(e^v) - \pi/2$ such that $g(0) = 0$. Use criteria suggested by Exercise 8.
15. $F(\mathbf{x}(u, v)) = (f(u)\cos v, f(u)\sin v)$, where $\mathbf{x}$ is a canonical parametrization and $f(u) = \exp\left(\int_1^u dt/h(t)\right)$.

Section 6.5

1. First show that α is a geodesic if and only if $\omega_{12}(\alpha') = 0$. Let $\overline{E}_1$, $\overline{E}_2$ be the transferred frame field, with connection form $\overline{\omega}_{12}$. Since $\overline{E}_1 = F_*(\alpha') = F(\alpha)'$, Lemma 5.3 gives

$$0 = \omega_{12}(\alpha') = F_*(\overline{\omega}_{12})(\alpha') = \overline{\omega}_{12}(F_*(\alpha')) = \overline{\omega}_{12}\left(F(\alpha)'\right).$$

3. There is no local isometry of the saddle surface M ($-1 \leqq K < 0$) onto a catenoid with $-1 \leqq \overline{K} < 0$—or vice versa—since K has an isolated minimum point, at $\mathbf{0}$, while $\overline{K}$ takes on each of its values on entire circles. Many other examples are possible.
5. (b) Follows from Lemma 4.5, since computation for $\mathbf{x}_t$ shows $E_t = \cosh^2 u = G_t$ and $F_t = 0$.
   (d) For M_t, $U_t = (s, -c, S)/C$, so the Euclidean coordinates of U_t are independent of t.
7. A local isometry must carry minimum points of K_H to minimum points of K_C, and also preserve orthogonality and geodesics.

Section 6.6

1. (b) $\theta_1 = \sqrt{1 + u^2}\,du$, $\theta_2 = u\,dv$, $\omega_{12} = dv/\sqrt{1 + u^2}$, $K = 1/(1 + u^2)^2$.

3. (b) Substitution into $d\omega_{13} = \omega_{12} \wedge \omega_{23}$ leads to

$$\mathrm{L}_v = \frac{E_v}{2}\left(\frac{\mathrm{L}}{E} + \frac{\mathrm{N}}{G}\right) = HE_v.$$

Section 6.7

1. $1 + f_u^2 + f_v^2 \geqq 1$.
3. (a) $\iint_T \nu = \int_0^{2\pi} dv \int_0^{2\pi} (R^2 + r^2 + 2Rr \cos u) du = 4\pi^2(R^2 + r^2)$.
   (b) $\mathbf{x}_u \times \mathbf{x}_v$ points inward, and thus $U \bullet \mathbf{x}_u \times \mathbf{x}_v = -\|\mathbf{x}_u \times \mathbf{x}_v\| = -\sqrt{EG - F^2}$. Hence $\int_T \nu = -\text{area}(T) = -4\pi^2 Rr$.
5. F carries positively oriented pavings of M to positively oriented pavings of N. Apply the suggested exercise to each 2-segment.

Section 6.8

1. (a) $F^*(du \wedge dv) = \mathbf{F}^*(du) \wedge F^*(dv) = df \wedge dg = (f_u du + f_v dv) \wedge (g_u du + g_v dv) = (f_u g_v - f_v g_u) du \wedge dv$.
   (b) $\mathbf{x}^*(dM) = dM(\mathbf{x}_u, \mathbf{x}_v) du \wedge dv = \pm\sqrt{EG - F^2}\, du \wedge dv$.
3. (a) Recall that $G_* \approx -S$. Let $\mathbf{e}_1, \mathbf{e}_2$ be a principal frame at a point of M. Then $G_*(\mathbf{e}_1) \bullet G_*(\mathbf{e}_2) = 0$. Thus G is conformal if and only if $\|G_*(\mathbf{e}_1)\|^2 = \|G_*(\mathbf{e}_2)\|^2 > 0$ at every point.
5. Using a canonical parametrization,

$$\begin{aligned} \iint KdM &= \int_0^{2\pi} dv \int_{a_1}^{b_1} (-h''/h) h\, ds \\ &= -2\pi(h'(b_1) - h'(a_1)) \\ &= 2\pi(\sin\varphi_a - \sin\varphi_b). \end{aligned}$$

7. (a) For a small patchlike 2-segment,

$$A(F(\mathbf{x})) = \iint_{F(\mathbf{x})} dN = \pm \iint_{\mathbf{x}} J_F dM.$$

If this always equals $A(\mathbf{x}) = \iint_x dM$, then taking limits as $\mathbf{x}$ shrinks to a point $\mathbf{p}$ gives $J_F(\mathbf{p}) = \pm 1$. F must be one-to-one, for otherwise two small regions of total area 2ε could map to a single region of area ε.

Conversely, we can suppose F is orientation-preserving; hence $J_F = 1$. Then use Exercise 5 of Section 7.

   (b) An isometry carries frames to frames. We have seen that cylindrical projection of a sphere is area-preserving (Exercise 6 of Section 7).

9. (a) See text.
   (b) See Example 4.3(1) of Chapter 5.
   (c) First show that on one of the vertical lines, exactly four directions are omitted by U. Total curvatures: -4π, $-\infty$, $-\infty$.

13. $TC = 2\pi\int_0^\infty K(r)W(r)dr = -4\pi.$

Section 6.9

5. (a)
$$\mathbf{F} = C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

7. (a) Example 4.3(2) of Chapter 5 shows that K has a unique minimum at **0**. Hence every Euclidean symmetry **F** must carry **0** to **0**, so **F** is an orthogonal transformation C.
   (b) C must carry asymptotic unit vectors to asymptotic unit vectors, and carry U_z to $\pm U_z$. One such C is $\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$.

Chapter 7

Section 7.1

1. (a) The speed squared is $\langle\alpha', \alpha'\rangle = \alpha' \bullet \alpha'/h^2(\alpha)$.
   (b) $\langle hU_i, hU_j\rangle = U_i \bullet U_j = \delta_{ij}$.
3. (a) The definition $J(\mathbf{e}_1) = \mathbf{e}_2$, $J(\mathbf{e}_2) = -\mathbf{e}_1$ is independent of the choice of positively oriented frame field $\mathbf{e}_1, \mathbf{e}_2$, since for another positively oriented frame field,
$$\hat{\mathbf{e}}_1 = \cos\vartheta\, \mathbf{e}_1 + \sin\vartheta\, \mathbf{e}_2, \quad \hat{\mathbf{e}}_2 = -\sin\vartheta\, \mathbf{e}_1 + \cos\vartheta\, \mathbf{e}_2,$$
   and this implies $J(\hat{\mathbf{e}}_1) = \hat{\mathbf{e}}_2$, $J(\hat{\mathbf{e}}_2) = -\hat{\mathbf{e}}_1$. Then for $\mathbf{v} \neq 0$, choose $\mathbf{e}_2$ so that $\mathbf{e}_1 = \mathbf{v}/|\mathbf{v}|$, $\mathbf{e}_2$ is positively oriented.
   (b) $V = f_1E_1 + f_2E_2$, with f_1, f_2 differentiable. For the other two relations, first replace arbitrary vectors by $\mathbf{e}_1, \mathbf{e}_2$.
   (c) If E_1, E_2 is positively oriented for dM, then $E_1, -E_2$ is positively oriented for $-dM$.

5. (a) Expand $\|\mathbf{v} \pm \mathbf{w}\|^2 = \langle \mathbf{v} \pm \mathbf{w}, \mathbf{v} \pm \mathbf{w} \rangle$.
   (b) Compute $\langle \mathbf{v}, \mathbf{w} \rangle$ with the vectors expressed in terms of $\mathbf{x}_u$ and $\mathbf{x}_v$.
   (c) Direct computation with $\alpha' = a_1'\mathbf{x}_u + a_2'\mathbf{x}_v$, yields the same result as applying ds^2 to α', since $du(\alpha') = a_1'$, $dv(\alpha') = a_2'$.
7. We have $F*(U_1) = f_u U_1 + g_u U_2$, $F*(U_2) = f_v U_1 + g_v U_2$. If F is conformal and orientation-preserving, then using Exercise 6,

$$\begin{aligned} f_v U_1 + g_v U_2 &= F*(U_2) \\ &= F*(JU_1) \\ &= J(F*U_1) \\ &= J(f_u U_1 + g_u U_2) \\ &= -g_u U_1 + f_u U_2. \end{aligned}$$

So the Cauchy-Riemann equations hold. Conversely, if the Cauchy-Riemann equations hold, then

$$\langle F*(U_1), F*(U_2) \rangle = f_u f_v + g_u g_v = f_u f_v - f_u f_v = 0, \text{ and}$$

$$\langle F*(U_1), F*(U_1) \rangle = f_u^2 + g_u^2 = |dF/dz|^2 = f_v^2 + g_v^2 = \langle F*(U_2), F*(U_2) \rangle.$$

This proves F is conformal (and shows that $|dF/dz|$ is the scale factor). F is orientation preserving since $J_F = f_u g_v - f_v g_u = f_u^2 + g_u^2 > 0$.
9. $(F*(\mathbf{v}) \bullet F*(\mathbf{w}))/h^2F(\mathbf{p}) = \mathbf{v} \bullet \mathbf{w}/h^2(\mathbf{p})$.

Section 7.2

3. $A = \pi a^2/(1 - a^2/4)$; hence total area is infinite.
5. Since $\mathbf{x}$ is an isometry, the area of T_0 is the same as the area of a Euclidean rectangle with sides $2\pi R$ and $2\pi r$. Hence $A(T_0) = 4\pi^2 Rr$, the same as $A(T)$.
7. (c) Evidently, $\overline{\theta}_i = c\theta_i$, and hence $\overline{\omega}_{12} = \omega_{12}$ follows by uniqueness in the first structural equations.
   (d) $d\overline{M} = \overline{\theta}_1 \wedge \overline{\theta}_2 = c^2\theta_1 \wedge \theta_2 = c^2 dM$.
   (e) Theorem 2.1 defines K.
9. (b) Since $\theta_i = \theta_i(\mathbf{x}_u)\,du + \theta_i(\mathbf{x}_v)\,dv = \langle E_i, \mathbf{x}_u \rangle du + \langle E_i, \mathbf{x}_v \rangle dv$, we find

$$\theta_1 = \sqrt{E}\ du + F/\sqrt{E}dv, \quad \theta_2 = W/\sqrt{E}dv.$$

   (c) Substitute $\omega_{12} = P\,du + Q\,dv$ and preceding results into the first structural equations.
   (d) Substitute into the second structural equation.

11. (b) $K = -2/\cosh^3(2u)$.
13. (a) To define **tensorK**, first simplify the square root of $E(u, v)G(u, v) - F(u, v)^2$ to get $W(u, v)$.
    (b) The formulas for E, F, G in the Appendix are valid for arbitrary n, so evaluate **tensorK** on the *functions* **ee[x],ff[x],gg[x]** for *Mathematica*; **ee(x),ff(x),gg(x)** for *Maple*.

Section 7.3

1. (a) First find the dual 1-forms.
   (b) $\alpha'' = -\cot t\,\alpha'$.
   (c) $\beta' = c/(st)\,E_1 + 1/t\,E_2$, and $\langle \beta', \beta' \rangle' = -2/(s^2t^3)$.
3. From the proof of Lemma 3.8, $\omega_{12}(Y)E_3 = -\nabla_y E_3 \cdot E_1\, E_3 = S(Y) \cdot E_1\, E_3$
5. (a) Let ω_{12} be the connection form of a frame field on $\mathscr{D}$. Since $d\omega_{12} = -K\, dM$, Stokes' theorem gives $\int_\alpha \omega_{12} = -\iint_{\mathscr{G}} K\, dM$. From the text, $\int_\alpha \omega_{12} = -\psi_\alpha$.
7. (a) If $W = fE_1$, then $\nabla_V(W) = V[f]E_1 + f\omega_{12}(V)E_2$, hence
$$\overline{\nabla_v W} = F_*(\nabla_V(W)) = V[f]\bar{E}_1 + f(F^{-1})\omega_{12}(V)\bar{E}_2.$$
   On the other hand,
$$\nabla_{\bar{V}}(\bar{W}) = \nabla_{\bar{V}}(f(F^{-1})\bar{E}_1) = \bar{V}[f(F^{-1})]\bar{E}_1 + f(F^{-1})\bar{\omega}_{12}(\bar{V})\bar{E}_2.$$
   But $\bar{V}[f(F^{-1})] = (F_*V)[f(F^{-1})] = V[f(F^{-1}F)] = V[f]$, and
$$\bar{\omega}_{12}(\bar{V}) = \bar{\omega}_{12}(F_*(V)) = F^*(\bar{\omega}_{12})(V) = \omega_{12}(V).$$
   where the last (crucial) step uses Lemma 5.3 of Chapter 6. This completes the proof.

Section 7.4

1. Since $\alpha'' = 0$, we get $\alpha(h)'' = \alpha'(h)h''$, which is 0 if and only if $h'' = 0$.
3. If L is a line in the xy plane, consider the Euclidean plane passing through both L and the north pole $\mathbf{n}$ of Σ_0; then use stereographic projection.
5. (a) Use Exercise 5 of Section 3. Since α' is parallel on α, $\angle_\alpha\,(\alpha'(a), \alpha'(b))$ is the holonomy angle ψ_α.
   (b) (ii) The image of the Gauss map of a paraboloid is an open hemisphere of Σ, hence any (finite) simple region in it has total curvature $<2\pi$.
7. (a) Fix $\mathbf{p}_0 \in M$, and let $\mathscr{U}$ consist of all points that can be joined to $\mathbf{p}_0$ by a broken geodesic—include $\mathbf{p}_0$ in $\mathscr{U}$. If $\mathbf{p} \in \mathscr{U}$, then by the given fact, $\mathscr{U}$ contains an ε-neighborhood of $\mathbf{p}$. Thus $\mathscr{U}$ is open. In a similar way, $M - \mathscr{U}$ is open. Since $\mathscr{U}$ is not empty, $M = \mathscr{U}$.

Section 7.5

1. The coordinates u, v have $E = G = 1/v^2$, $F = 0$, hence are Clairaut. With the suggested reversals, geodesics are given by

$$\frac{du}{dv} = \frac{\pm c\sqrt{G}}{\sqrt{E}\sqrt{E - c^2}} = \frac{\pm cv}{\sqrt{1 - c^2v^2}}.$$

Set $w = 1 - c^2v^2$ and integrate to get $u - u_0 = \mp\sqrt{w}/c$. Consequently, $(u - u_0)^2 + v^2 = 1/c^2$.

3. At the meeting point, $u_1 = a_1(t_1)$. Since $c = \sqrt{G(a_1)}\sin\varphi$, the condition $G(u) = c^2$ implies $\sin\varphi = \pm 1$. Thus $a_1'(t_1)$ is tangent to the barrier curve, so $a_1'(t_1) = 0$.

   The geodesic equation $A_1 = 0$ in Theorem 4.2 reduces to $a_1'' = G_u\, a_2'^2/(2E)$. At the meeting point, $G_u \neq 0$ since barriers are not geodesic, and $a_2' \neq 0$ since $a_1' = 0$. Thus $a_2''(t_1) \neq 0$. This means that α leaves the barrier curve instantly, remaining on the same side of it.

5. (a) By Exercise 4, tangency to the top circle implies slant $c = R$ (larger of the radii of T). Except for the inner and outer equators, no parallel is geodesic. Hence α leaves the top circle, necessarily entering the outer half of T. As h increases, $\sin\varphi$ decreases; hence α meets and crosses the outer equator. By symmetry, it returns to tangency with the top circle.

   (b) Crossing the inner equator implies slant $c < R - r$.

7. Evidently all meridians approach the rim on a finite parameter interval. In view of the exercises above, so do all other geodesics; even if initially moving away from the rim, they will be turned back by a barrier curve. They cannot asymptotically approach a parallel, since no parallels are geodesic.

9. (a) $E(u) = ee(u)$, $G(u) = gg(u)$ will be given (for abstract surfaces) or computed (for surfaces in $\mathbf{R}^3$).

   (*Mathematica*):

```
clair[u0_,v0_,c_,tmin_,tmax_]:=
NDSolve[{u'[t]==Sqrt[gg[u[t]]-c^2]/Sqrt[ee[u[
t]]*gg[u[t]]],v'[t]==c/gg[u[t]],u[0]==u0,v[0]
==v0},{u,v},{t,tmin,tmax}]
```

   (b)
```
ParametricPlot3D[Evaluate[x[u[t],v[t]]/.
nsol],{t,tmin,tmax}]
```

   where **nsol** is an explicit return from **clair**. (Delete "**3D**" in the abstract case.)

(*Maple*)

(a) `clair:=(u0,v0,c)->dsolve({diff(u(t),t)=(gg(u(t))-c^2)^(1/2)/(ee(u(t))*gg(u(t))^(1/2),diff(v(t),t)=c/gg(u(t)),u(0)=u0,v(0)=v0},{u(t),v(t)},type=numeric)`.

(b) With *plots* installed, if `nsol` is an explicit return from `clair`, `odeplot(nsol,x(u(t),v(t)),tmin..tmax)`

11. (b) Since $G(0) = f(0)^2 = (3/4)^2$, the slant of this geodesic is $\pm 3/4$.

13. Since $\alpha' = a_1'\mathbf{x}_u + a_2'\mathbf{x}_v$, we have $\cos\varphi = \langle\alpha', \mathbf{x}_u\rangle/\sqrt{E} = \sqrt{E}a_1'$. Hence $\cos^2\varphi = (U(a_1) + V(a_2))a_1'^2$, and $\sin^2\varphi$ is similar. Thus we must show that the function $f = (U(a_1) + V(a_2)(U(a_1)a_2'^2 - V(a_2)a_1'^2)$ is constant. Compute f'. The geodesic equations from Theorem 4.2 then give $f' = 0$.

Section 7.6

1. In (a) and (c) the surface is diffeomorphic to a sphere, so $TC = 4\pi$. In (b), there are four handles, so $TC = -12\pi$.
3. If $h = 0$, then M is a sphere, so $TC > 0$. If $h = 1$, then M is diffeomorphic to a torus; hence $TC = 0$. If $h \geqq 2$, then $TC < 0$.
5. (c) For each polygon, draw lines from a central point to each vertex. Thus each original n-sided face is replaced by n faces, and there are n new edges and one new vertex. Thus for each polygon, the effect on $\chi(M)$ is $1 \to 1 - n + n$, so there is no change.
7. The area of $\mathbf{x}(R)$ is $\pi r^2/\left(4\sqrt{2}\right)$. Three of the four edges are geodesics.
9. We count $e = 6f/2 = 3f$ and $v = 6f/3 = 2f$; hence $\chi = 0$. So this is impossible on the sphere, but a suitable diagram shows that the torus has such a decomposition.

Section 7.7

1. Follows from Theorem 7.5 since a polygon has Euler characteristic +1.
3. (a) By the Gauss-Bonnet theorem, M is diffeomorphic to a sphere; hence if two simply closed geodesics do not meet, they bound a region.
5. (a) The angle function from any X to V_t depends continuously on t; hence the index depends continuously on t. But a continuous integer-valued function on an interval is constant.

   (b) Use (a).
7. (a) Approximate closely by a genuine polygon. In the limit, the interior angles will all be π. Hence by Exercise 1, $-A_n/r^2 = (2 - n)\pi$, so $A_n = (n - 2)\pi r^2$.

(b) As $n \to \infty$, $A_n \to \infty$, so $H(r)$ has infinite area.

9. (a) Let $h = \|V_\alpha\| > 0$. Then $f = h\cos\varphi$, $g = h\sin\varphi$, so the integrand reduces to φ'.

11. (a) The equations $u' = -u$, $v' = v$ have general solutions $u = Ae^{-t}$, $v = Be^t$, so $A = a$, $B = b$.
(b) Since $uv = ab$, the integral curves parametrize hyperbolas (when $ab \neq 0$); this is a meeting of two streams, with index -1.
(c) For the circle $\alpha(t) = (\cos t, \sin t)$, the integrand reduces to -1.

13. (a) (*Mathematica*):
```
numsol[u0_,v0_,tmin_,tmax_]:=NDSolve
[{u'[t]==2u[t]^2-v[t]^2,v'[t]==-3u[t]*v[t],
u[0]==u0,v[0]==v0},{u,v},{t,tmin,tmax}]
draw[u0_,v0_,tmin_,tmax_]:=ParametricPlot
[Evaluate[{u[t],v[t]}/.numsol[u0,v0,tmin,
tmax]],{t,tmin,tmax}]
```
(b) (*Maple*): Take $X = (1, 0)$; hence $J(X) = (0, 1)$. Now apply Exercise 9. Evaluation on the circle $\alpha(t) = (\cos t, \sin t)$ gives
```
f:=t->2*cos(t)^2-sin(t)^2,
g:=t->-3*cos(t)*sin(t)
```
The integrand is
```
wint:=t->(f(t)*diff(g(t),t)-
g(t)*diff(f(t),t))/(f(t)^2+g(t)^2)
```
and `int(wint(t),t = 0..2*Pi)` is -4π, so the index is -2.

Chapter 8

Section 8.1

1. (a) If **q** is in a normal ε-neighborhood $\mathcal{N}$ of **p**, then by Theorem 1.8, the radial geodesic from **p** to **q** has length $\rho(\mathbf{p}, \mathbf{q}) < \varepsilon$. If **q** is not in $\mathcal{N}$, then any curve from **p** to **q** meets every polar circle of $\mathcal{N}$; hence $\rho(\mathbf{p}, \mathbf{q}) \geqq \varepsilon$.

3. $\mathbf{n}(x, y) = (r\cos(x/r), r\sin(x/r), y)$. To get the largest normal ε-neighborhood, fold an open Euclidean disk of radius πr around the cylinder.

5. (a) Any geodesic starting at **p** is initially tangent to a meridian; hence (by the uniqueness of geodesics) parametrizes that meridian. It follows that the entire surface is a normal neighborhood of **p**.

7. (a) By the triangle inequality, $\rho(\mathbf{p}, \mathbf{q}) > \rho(\mathbf{p}_0, \mathbf{q}) - \rho(\mathbf{p}_0, \mathbf{p})$. Reversing **p** and **q**, we conclude that $\rho(\mathbf{p}, \mathbf{q}) > |\rho(\mathbf{p}_0, \mathbf{q}) - \rho(\mathbf{p}_0, \mathbf{p})|$.

(b) Show that if $\rho(\mathbf{p}_0, \mathbf{p}) < \varepsilon$ and $\rho(\mathbf{q}_0, \mathbf{q}) < \varepsilon$, then it follows that $|\rho(\mathbf{p}_0, \mathbf{q}_0) - \rho(\mathbf{p}, \mathbf{q})| < 2\varepsilon$.

Section 8.2

1. Let M be an open disk in $\mathbf{R}^2$.
3. We can assume that C is parametrized by $\alpha(u) + vU_3$, with α a unit-speed curve. If α is (smoothly) closed, let σ have the same arc length and parametrize a circle in $\mathbf{R}^2$. Then $\alpha(u) + vU_3 \to \sigma(u) + vU_3$ is an isometry. Circular cylinders of different radii are not isometric since their closed geodesics have different lengths.

   If α is one-to-one, then since it is a geodesic of C it is defined on the entire real line. Then $\alpha(u) + vU_3 \to (u, v)$ is an isometry onto $\mathbf{R}^2$.
5. The profile curves all approach either a singularity of the curve or the axis of rotation. Only for the sphere was the axis met *orthogonally*, thus giving Σ as an augmented surface of revolution.

Section 8.3

1. For $k = -1/r^2$, the general solution of the Jacobi equation $g'' - g/r^2 = 0$ can be written as $g(u) = A\cosh u/r + B\sinh u/r$. The initial conditions then determine A and B.
3. (a) $L(\varepsilon) = 2\pi\sinh\varepsilon$.
5. (a) $\mathbf{x}_u(0, v) = X(v)$, and since $\mathbf{x}(0, v) = \gamma_{X(v)}(0) = \beta(v)$, we have $\mathbf{x}_v(0, v) = \beta'(v)$. Thus $EG - F^2$ is nonzero when $u = 0$, hence also for $|u|$ small.

   (b) (iii) β as base curve, $X = \delta$.
7. (a) The u-parameter curves of $\mathbf{x}$ are meridians of longitude.

   (b) Since $K = 0$, the Jacobi equation becomes $(\sqrt{G})_{uu} = 0$. Hence $\sqrt{G}$ is linear in u, and it follows that $\sqrt{G}(u, v) = 1 - \kappa_g(v)u$.

Section 8.4

1. Let E be the due-east unit vector field on the sphere Σ (undefined at the poles). If A is the antipodal map, then $A_*(E) = E$, so E transfers to P via the projection $\Sigma \to P$. The unique singularity has index 1.
3. The condition implies $F(M) = N$. If $\mathbf{q}$ is in N, then each point of $F^{-1}(\mathbf{q})$ has a neighborhood mapped diffeomorphically onto a neighborhood of $\mathbf{q}$. The intersection $\mathscr{V}$ of all these neighborhoods of $\mathbf{q}$ is evenly covered; the condition prevents its lifts from meeting.

5. (a) Since covering maps are local diffeomorphisms and T is orientable, T cannot be covered by a nonorientable surface (Exercise 3 of Section 4.7). Thus any compact connected covering surface M of T must also have $\chi(M) = 0$. Hence by Theorem 6.8 of Chapter 7, M is a torus.
   (b) For the usual parametrization of T, let $F(\mathbf{x}(u, v)) = \mathbf{x}(nu, v)$.

Section 8.5

1. If F: $M \to N$ is an isometry, define ϕ: $I(M) \to I(N)$ by $\phi(G) = FGF^{-1}$. Show that ϕ is a homomorphism and is one-to-one and onto.
3. Suppose $\mathbf{p} \neq \mathbf{q}$ in M. Then any geodesic segment σ from $\mathbf{p}$ to $\mathbf{q}$ has nonzero speed, so $F(\sigma)$ is a nonconstant geodesic of N. If $F(\mathbf{p}) = F(\mathbf{q})$, there are two geodesics from this point to the midpoint of $F(\sigma)$.
5. (a) Given points $\mathbf{p}$ and $\mathbf{q}$ in M, if F and G are isometries such that $F(\mathbf{p}_0) = \mathbf{p}$ and $G(\mathbf{p}_0) = \mathbf{q}$, then the isometry GF^{-1} carries $\mathbf{p}$ to $\mathbf{q}$.
   (c) Given frames $\mathbf{e}_1$, $\mathbf{e}_2$ at $\mathbf{p}$ and $\mathbf{f}_1$, $\mathbf{f}_2$ at $\mathbf{q}$, let F and G be isometries such that $F(\mathbf{p}) = \mathbf{p}_0$ and $G(\mathbf{q}) = \mathbf{p}_0$. By hypothesis, there is an isometry H that carries the frame $F*(\mathbf{e}_1)$, $F*(\mathbf{e}_2)$ to $G*(f_1)$, $G*(f_2)$. Then the isometry $G^{-1}HF$ carries $\mathbf{e}_1$, $\mathbf{e}_2$ to $\mathbf{f}_1$, $\mathbf{f}_2$.
7. (a) Since C on $\mathbf{R}^3$ is linear, $C(-\mathbf{p}) = -C(\mathbf{p})$. Then the mapping $\{\mathbf{p}, -\mathbf{p}\} \to \{C(\mathbf{p}), -C(\mathbf{p})\}$ has the required properties.
   (b) Because F is a local isometry, any two frames on P can be written as $F*(\mathbf{e})$ and $F*(\mathbf{f})$, where $\mathbf{e}$ and $\mathbf{f}$ are frames on Σ. By Exercise 6 there is an orthogonal transformation C of $\mathbf{R}^3$ such that $C*(\mathbf{e}) = \mathbf{f}$. Now use $FC = C_pF$.
9. (a) $\mathbf{x}(u, v) = \left(u, \sinh^{-1} v, \sqrt{1 + v^2}\right)$ has $E = 1$, $F = 0$, $G = 1$.
   (b) Use Exercise 1. The only derived isometries are those of the form $\mathrm{F}(\mathbf{x}(u, v)) = \mathbf{x}(\pm u + a, \pm v)$.
11. Calculate $\|F(\mathbf{p})\|$.

Section 8.6

1. (a) One handle implies $\chi(M) = 0$, but by Gauss-Bonnet, $K < 0$ implies $\chi < 0$.
   (b) By the Gauss-Bonnet formula, the angle sum for a $k = -1$ rectangle can never be 2π.
3. Only the projective plane satisfies all three axioms; the others fail on axiom (ii), and Σ also fails (i).
5. For Δ, let $\mathbf{e}_1$, $\mathbf{e}_2$ be the frame at the common vertex of α and β such that $\mathbf{e}_1$ is tangent to α, and $\cos\vartheta\mathbf{e}_1 + \sin\vartheta\mathbf{e}_2$ is tangent to β. Let F be the isometry carrying the frame $\mathbf{e}_1$, $\mathbf{e}_2$ to the corresponding frame on Δ'.

Section 8.7

1. In the proof of assertion (3) in Lemma 7.4, the Jacobi equation now reduces to $g'' = 0$, so the initial conditions then give $g(u) = u$.
3. For a point $\mathbf{p}_0$ in M, the functions $\mathbf{p} \to \rho(\mathbf{p}_0, \mathbf{p})$ and (when relevant) $\mathbf{p} \to \mathrm{d}(\mathbf{p}_0, \mathbf{p})$ are both continuous, hence take on maximum values. Then use the triangle inequality.

Index

R

S

W