Chapter 3
Euclidean Geometry

We recall some familiar features of plane geometry. First of all, two triangles are *congruent* if there is a rigid motion of the plane that carries one triangle exactly onto the other. Corresponding angles of congruent triangles are equal, corresponding sides have the same length, the areas enclosed are equal, and so on. Indeed, any geometric property of a given triangle is automatically shared by every congruent triangle. Conversely, there are a number of simple ways in which one can decide whether two given triangles are congruent—for example, if for each the same three numbers occur as lengths of sides.

In this chapter we shall investigate the rigid motions (isometries) of Euclidean space, and see how these remarks about triangles can be extended to other geometric objects.

3.1 Isometries of $\mathbf{R}^3$

An isometry, or rigid motion, of Euclidean space is a mapping that preserves the Euclidean distance d between points (Definition 1.2, Chapter 2).

1.1 Definition An *isometry* of $\mathbf{R}^3$ is a mapping F: $\mathbf{R}^3 \to \mathbf{R}^3$ such that

$$d(F(\mathbf{p}), F(\mathbf{q})) = d(\mathbf{p}, \mathbf{q})$$

for all points $\mathbf{p}$, $\mathbf{q}$ in $\mathbf{R}^3$.

1.2 Example (1) ***Translations***. Fix a point $\mathbf{a}$ in $\mathbf{R}^3$ and let T be the mapping that adds $\mathbf{a}$ to every point of $\mathbf{R}^3$. Thus $T(\mathbf{p}) = \mathbf{p} + \mathbf{a}$ for all

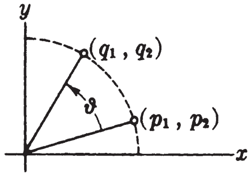

FIG. 3.1

points **p**. T is called *translation* by **a**. It is easy to see that T is an isometry, since

$$\begin{aligned} d(T(\mathbf{p}),\ T(\mathbf{q})) &= d(\mathbf{p}+\mathbf{a},\ \mathbf{q}+\mathbf{a}) \\ &= \|(\mathbf{p}+\mathbf{a})-(\mathbf{q}+\mathbf{a})\| \\ &= \|\mathbf{p}-\mathbf{q}\| = d(\mathbf{p},\mathbf{q}). \end{aligned}$$

(2) ***Rotation around a coordinate axis.*** A rotation of the xy plane through an angle ϑ carries the point (p_1, p_2) to the point (q_1, q_2) with coordinates (Fig. 3.1)

$$\begin{aligned} q_1 &= p_1 \cos\vartheta - p_2 \sin\vartheta, \\ q_2 &= p_1 \sin\vartheta + p_2 \cos\vartheta. \end{aligned}$$

Thus a *rotation* C of three-dimensional Euclidean space $\mathbf{R}^3$ around the z axis, through an angle ϑ, has the formula

$$C(\mathbf{p}) = C(p_1, p_2, p_3) = (p_1 \cos\vartheta - p_2 \sin\vartheta,\ p_1 \sin\vartheta + p_2 \cos\vartheta,\ p_3).$$

Evidently, the mapping C is a linear transformation. A straightforward computation shows that C preserves Euclidean distance, so it is an isometry.

Recall that if F and G are mappings of $\mathbf{R}^3$, the composite function GF is a mapping of $\mathbf{R}^3$ obtained by applying first F, then G.

1.3 Lemma If F and G are isometries of $\mathbf{R}^3$, then the composite mapping GF is also an isometry of $\mathbf{R}^3$.

Proof. Since G is an isometry, the distance from $G(F(\mathbf{p}))$ to $G(F(\mathbf{q}))$ is $d(F(\mathbf{p}), F(\mathbf{q}))$. But since F is an isometry, this distance equals $d(\mathbf{p}, \mathbf{q})$. Thus GF preserves distance; hence it is an isometry. ◆

In short, a composition of isometries is again an isometry.

We also recall that if $F\colon \mathbf{R}^3 \to \mathbf{R}^3$ is both one-to-one and onto, then F has a unique inverse function $F^{-1}\colon \mathbf{R}^3 \to \mathbf{R}^3$, which sends each point $F(\mathbf{p})$ back to $\mathbf{p}$. The relationship between F and F^{-1} is best described by the formulas

$$FF^{-1} = I, \quad F^{-1}F = I,$$

where I is the *identity mapping* of $\mathbf{R}^3$, that is, the mapping such that $I(\mathbf{p}) = \mathbf{p}$ for all $\mathbf{p}$.

Translations of $\mathbf{R}^3$ (as defined in Example 1.2) are the simplest type of isometry.

1.4 Lemma (1) If S and T are translations, then $ST = TS$ is also a translation.

(2) If T is translation by $\mathbf{a}$, then T has an inverse T^{-1}, which is translation by $-\mathbf{a}$.

(3) Given any two points $\mathbf{p}$ and $\mathbf{q}$ of $\mathbf{R}^3$, there exists a unique translation T such that $T(\mathbf{p}) = \mathbf{q}$.

Proof. To prove (3), for example, note that translation by $\mathbf{q} - \mathbf{p}$ certainly carries $\mathbf{p}$ to $\mathbf{q}$. This is the only possibility, since if T is translation by $\mathbf{a}$ and $T(\mathbf{p}) = \mathbf{q}$, then $\mathbf{p} + \mathbf{a} = \mathbf{q}$; hence $\mathbf{a} = \mathbf{q} - \mathbf{p}$. ◆

A useful special case of (3) is that if T is a translation such that for some one point $T(\mathbf{p}) = \mathbf{p}$, then $T = I$.

The rotation in Example 1.2 is an example of an *orthogonal transformation* of $\mathbf{R}^3$, that is, a linear transformation $C\colon \mathbf{R}^3 \to \mathbf{R}^3$ that preserves dot products in the sense that

$$C(\mathbf{p}) \bullet C(\mathbf{q}) = \mathbf{p} \bullet \mathbf{q} \quad \text{for all } \mathbf{p}, \mathbf{q}.$$

1.5 Lemma If $C\colon \mathbf{R}^3 \to \mathbf{R}^3$ is an orthogonal transformation, then C is an isometry of $\mathbf{R}^3$.

Proof. First we show that C preserves norms. By definition, $\|\mathbf{p}\|^2 = \mathbf{p} \bullet \mathbf{p}$; hence

$$\|C(\mathbf{p})\|^2 = C(\mathbf{p}) \bullet C(\mathbf{p}) = \mathbf{p} \bullet \mathbf{p} = \|\mathbf{p}\|^2.$$

Thus $\| C(\mathbf{p}) \| = \| \mathbf{p} \|$ for all points $\mathbf{p}$. Since C is linear, it follows easily that C is an isometry:

$$d(C(\mathbf{p}), C(\mathbf{q})) = \|C(\mathbf{p}) - C(\mathbf{q})\| = \|C(\mathbf{p} - \mathbf{q})\| = \|\mathbf{p} - \mathbf{q}\|$$
$$= d(\mathbf{p}, \mathbf{q}) \quad \text{for all } \mathbf{p}, \mathbf{q}.$$ ◆

Our goal now is Theorem 1.7, which asserts that every isometry can be expressed as an orthogonal transformation followed by a translation. The main part of the proof is the following converse of Lemma 1.5.

1.6 Lemma If F is an isometry of $\mathbf{R}^3$ such that $F(\mathbf{0}) = \mathbf{0}$, then F is an orthogonal transformation.

Proof. First we show that F preserves dot products; then we show that F is a linear transformation. Note that by definition of Euclidean distance, the norm $\| \mathbf{p} \|$ of a point $\mathbf{p}$ is just the Euclidean distance $d(\mathbf{0}, \mathbf{p})$ from the origin to $\mathbf{p}$. By hypothesis, F preserves Euclidean distance, and $F(\mathbf{0}) = \mathbf{0}$; hence

$$\|F(\mathbf{p})\| = d(\mathbf{0}, F(\mathbf{p})) = d(F(\mathbf{0}), F(\mathbf{p})) = d(\mathbf{0}, \mathbf{p}) = \|\mathbf{p}\|.$$

Thus F preserves norms. Now by a standard trick (“polarization”), we shall deduce that it also preserves dot products. Since F is an isometry,

$$d(F(\mathbf{p}), F(\mathbf{q})) = d(\mathbf{p}, \mathbf{q})$$

for any pair of points. Hence

$$\|F(\mathbf{p}) - F(\mathbf{q})\| = \|\mathbf{p} - \mathbf{q}\|.$$

By the definition of norm, this implies

$$(F(\mathbf{p}) - F(\mathbf{q})) \bullet (F(\mathbf{p}) - F(\mathbf{q})) = (\mathbf{p} - \mathbf{q}) \bullet (\mathbf{p} - \mathbf{q}).$$

Hence

$$\|F(\mathbf{p})\|^2 - 2F(\mathbf{p}) \bullet F(\mathbf{q}) + \|F(\mathbf{q})\|^2 = \|\mathbf{p}\|^2 - 2\mathbf{p} \bullet \mathbf{q} + \|\mathbf{q}\|^2.$$

The norm terms here cancel, since F preserves norms, and we find

$$F(\mathbf{p}) \bullet F(\mathbf{q}) = \mathbf{p} \bullet \mathbf{q},$$

as required.

It remains to prove that F is linear. Let $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$ be the unit points (1, 0, 0), (0, 1, 0), (0, 0, 1), respectively. Then we have the identity

$$\mathbf{p} = (p_1, p_2, p_3) = \sum p_i \mathbf{u}_i.$$

Also, the points $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$ are orthonormal; that is, $\mathbf{u}_i \cdot \mathbf{u}_j = \delta_{ij}$.

We know that F preserves dot products, so $F(\mathbf{u}_1)$, $F(\mathbf{u}_2)$, $F(\mathbf{u}_3)$ must also be orthonormal. Thus orthonormal expansion gives

$$F(\mathbf{p}) = \sum F(\mathbf{p}) \cdot F(\mathbf{u}_i)\ F(\mathbf{u}_i).$$

But

$$F(\mathbf{p}) \cdot F(\mathbf{u}_i) = \mathbf{p} \cdot \mathbf{u}_i = p_i,$$

so

$$F(\mathbf{p}) = \sum p_i F(\mathbf{u}_i).$$

Using this identity, it is a simple matter to check the linearity condition

$$F(a\mathbf{p} + b\mathbf{q}) = aF(\mathbf{p}) + bF(\mathbf{q}). \qquad \blacklozenge$$

We now give a concrete description of an arbitrary isometry.

1.7 Theorem If F is an isometry of $\mathbf{R}^3$, then there exist a unique translation T and a unique orthogonal transformation C such that

$$F = TC.$$

Proof. Let T be translation by $F(\mathbf{0})$. Then Lemma 1.4 shows that T^{-1} is translation by $-F(\mathbf{0})$. But $T^{-1}F$ is an isometry, by Lemma 1.3, and furthermore,

$$(T^{-1}F)(\mathbf{0}) = T^{-1}(F(\mathbf{0})) = F(\mathbf{0}) - F(\mathbf{0}) = \mathbf{0}.$$

Thus by Lemma 1.6, $T^{-1}F$ is an orthogonal transformation, say $T^{-1}F = C$. Applying T on the left, we get $F = TC$.

To prove the required uniqueness, we suppose that F can also be expressed as $\bar{T}\bar{C}$, where $\bar{T}$ is a translation and $\bar{C}$ an orthogonal transformation. We must prove $\bar{T} = T$ and $\bar{C} = C$. Now $TC = \bar{T}\bar{C}$; hence $C = T^{-1}\bar{T}\bar{C}$. Since C and $\bar{C}$ are linear transformations, they of course send the origin to itself. It follows that $(T^{-1}\bar{T})(\mathbf{0}) = \mathbf{0}$. But since $T^{-1}\bar{T}$ is a translation, we conclude that $T^{-1}\bar{T} = I$; hence $\bar{T} = T$. Then the equation $TC = \bar{T}\bar{C}$ becomes $TC = T\bar{C}$. Applying T^{-1} gives $C = \bar{C}$ $\blacklozenge$

Thus every isometry of $\mathbf{R}^3$ *can be* uniquely *described as an orthogonal transformation followed by a translation.* When $F = TC$ as in Theorem 1.7, we call

C the *orthogonal part of F,* and T the *translation part of F.* Note that CT is generally not the same as TC (Exercise 1).

This decomposition theorem is the decisive fact about isometries of $\mathbf{R}^3$ (and its proof holds for $\mathbf{R}^n$ as well). We will use it to find an explicit formula for an arbitrary isometry.

First, recall from linear algebra that if $C\colon \mathbf{R}^3 \to \mathbf{R}^3$ is *any* linear transformation, its *matrix* (relative to the natural basis of $\mathbf{R}^3$) is the 3×3 matrix $\{c_{ij}\}$ such that

$$C(p_1, p_2, p_3) = \left(\sum c_{1j}p_j, \sum c_{2j}p_j, \sum c_{3j}p_j\right).$$

Thus, using the *column-vector* conventions, $\mathbf{q} = C(\mathbf{p})$ can be written as

$$\begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}.$$

By a standard result of linear algebra, *a linear transformation of* $\mathbf{R}^3$ *is orthogonal* (preserves dot products) *if and only if its matrix is orthogonal* (transpose equals inverse).

Returning to the decomposition $F = TC$ in Theorem 1.7, if T is translation by $\mathbf{a} = (a_1, a_2, a_3)$, then

$$F(\mathbf{p}) = TC(\mathbf{p}) = \mathbf{a} + C(\mathbf{p}).$$

Using the above formula for $C(\mathbf{p})$, we get

$$F(\mathbf{p}) = F(p_1, p_2, p_3) = \left(a_1 + \sum c_{1j}p_j, a_2 + \sum c_{2j}p_j, a_3 + \sum c_{3j}p_j\right).$$

Alternatively, using the column-vector conventions, $\mathbf{q} = F(\mathbf{p})$ means

$$\begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}.$$

Exercises

Throughout these exercises, A, B, and C denote orthogonal transformations (or their matrices), and T_a is translation by $\mathbf{a}$.

1. Prove that $CT_a = T_{C(a)}C$.

2. Given isometries $F = T_aA$ and $G = T_bB$, find the translation and orthogonal part of FG and GF.

3. Show that an isometry $F = T_aC$ has an inverse mapping F^{-1}, which is also an isometry. Find the translation and orthogonal parts of F^{-1}.

4. If

$$C = \begin{pmatrix} -2/3 & 2/3 & -1/3 \\ 2/3 & 1/3 & -2/3 \\ 1/3 & 2/3 & 2/3 \end{pmatrix} \quad \text{and} \quad \begin{cases} \mathbf{p} = (3, 1, -6), \\ \mathbf{q} = (1, 0, 3), \end{cases}$$

show that C is orthogonal; then compute $C(\mathbf{p})$ and $C(\mathbf{q})$, and check that $C(\mathbf{p}) \bullet C(\mathbf{q}) = \mathbf{p} \bullet \mathbf{q}$.

5. Let $F = T_aC$, where $\mathbf{a} = (1, 3, -1)$ and

$$C = \begin{pmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix}.$$

If $\mathbf{p} = (2, -2, 8)$, find the coordinates of the point $\mathbf{q}$ for which

(a) $\mathbf{q} = F(\mathbf{p})$.
(b) $\mathbf{q} = F^{-1}(\mathbf{p})$.
(c) $\mathbf{q} = (CT_a)(\mathbf{p})$.

6. In each case decide whether F is an isometry of $\mathbf{R}^3$. If so, find its translation and orthogonal parts.

(a) $F(\mathbf{p}) = -\mathbf{p}$.
(b) $F(\mathbf{p}) = (\mathbf{p} \bullet \mathbf{a})\,\mathbf{a}$, where $\|\mathbf{a}\| = 1$.
(c) $F(\mathbf{p}) = (p_3 - 1, p_2 - 2, p_1 - 3)$.
(d) $F(\mathbf{p}) = (p_1, p_2, 1)$.

A *group* G is a set furnished with an *operation* that assigns to each pair g_1, g_2 of elements of G an element g_1g_2, subject to these rules: (1) associative law: $(g_1g_2)g_3 = g_1(g_2g_3)$, (2) there is a unique *identity element* e such that $eg = ge = g$ for all g in G, and (3) inverses: For each g in G there is an element g^{-1} in G such that $gg^{-1} = g^{-1}g = e$.

Groups occur naturally in many parts of geometry, and we shall mention a few in subsequent exercises. Basic properties of groups may be found in a variety of elementary textbooks.

7. Prove that the set $\mathscr{E}(3)$ of all isometries of $\mathbf{R}^3$ forms a group—with composition of functions as the operation. $\mathscr{E}(3)$ is called the *Euclidean group* of order 3.

A subset H of a group G is a *subgroup* of G provided (1) if g_1 and g_2 are in H, then so is g_1g_2, (2) is g is in H, so is g^{-1}, and hence (3) the identity element e of G is in H. A subgroup H of G is automatically a group.

8. Prove that the set $\mathscr{T}(3)$ of all translations of $\mathbf{R}^3$ and the set $O(3)$ of all orthogonal transformations of $\mathbf{R}^3$ are each subgroups of the Euclidean group $\mathscr{E}(3)$. $O(3)$ is called the *orthogonal group* of order 3. Which isometries of $\mathbf{R}^3$ are in both these subgroups?

It is easy to check that the results of this section, though stated for $\mathbf{R}^3$, remain valid for Euclidean spaces $\mathbf{R}^n$ of any dimension.

9. (a) Give an explicit description of an arbitrary 2×2 orthogonal matrix C. (*Hint:* Use an angle and a sign.)
(b) Give a formula for an arbitrary isometry F of $\mathbf{R} = \mathbf{R}^1$.

3.2 The Tangent Map of an Isometry

In Chapter 1 we showed that an arbitrary mapping $F\colon \mathbf{R}^3 \to \mathbf{R}^3$ has a tangent map F_* that carries each tangent vector $\mathbf{v}$ at $\mathbf{p}$ to a tangent vector $F_*(\mathbf{v})$ at $F(\mathbf{p})$. If F is an isometry, its tangent map is remarkably simple. (Since the distinction between tangent vector and point is crucial here, we temporarily restore the point of application to the notation.)

2.1 Theorem Let F be an isometry of $\mathbf{R}^3$ with orthogonal part C. Then

$$F_*(\mathbf{v}_p) = C(\mathbf{v})_{F(p)}$$

for all tangent vectors $\mathbf{v}_p$ to $\mathbf{R}^3$.

Verbally: To get $F_*(\mathbf{v}_p)$, first shift the tangent vector $\mathbf{v}_p$ to the canonically corresponding point $\mathbf{v}$ of $\mathbf{R}^3$, then apply the orthogonal part C of F, and finally shift this point $C(\mathbf{v})$ to the canonically corresponding tangent vector at $F(\mathbf{p})$ (Fig. 3.2). *Thus all tangent vectors at all points* $\mathbf{p}$ *of* $\mathbf{R}^3$ *are "rotated" in exactly the same way by* F_**—only the new point of application* $F(\mathbf{p})$ *depends on* $\mathbf{p}$.

Proof. Write $F = TC$ as in Theorem 1.7. Let T be translation by $\mathbf{a}$, so $F(\mathbf{p}) = \mathbf{a} + C(\mathbf{p})$. If $\mathbf{v}_p$ is a tangent vector to $\mathbf{R}^3$, then by Definition 7.4 of Chapter 1, $F_*(\mathbf{v}_p)$ is the initial velocity of the curve $t \to F(\mathbf{p} + t\mathbf{v})$. But using the linearity of C, we obtain

$$\begin{aligned} F(\mathbf{p} + t\mathbf{v}) &= TC(\mathbf{p} + t\mathbf{v}) = T(C(\mathbf{p}) + tC(\mathbf{v})) = \mathbf{a} + C(\mathbf{p}) + tC(\mathbf{v}) \\ &= F(\mathbf{p}) + tC(\mathbf{v}). \end{aligned}$$

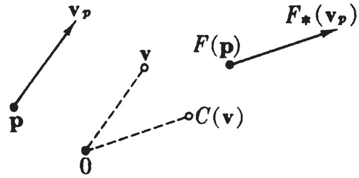

FIG. 3.2

Thus $F_*(\mathbf{v}_p)$ is the initial velocity of the curve $t \to F(\mathbf{p}) + tC(\mathbf{v})$, which is precisely the tangent vector $C(\mathbf{v})_{F(p)}$. ◆

Expressed in terms of Euclidean coordinates, this result becomes

$$F_*\left(\sum_j v_j U_j\right) = \sum_{i,j} c_{ij} v_j \overline{U}_i,$$

where $C = (c_{ij})$ is the orthogonal part of the isometry F, and if U_i is evaluated at $\mathbf{p}$, then $\overline{U}_i$ is evaluated at $F(\mathbf{p})$.

2.2 Corollary Isometries preserve dot products of tangent vectors. That is, if $\mathbf{v}_p$ and $\mathbf{w}_p$ are tangent vectors to $\mathbf{R}^3$ at the same point, and F is an isometry, then

$$F_*(\mathbf{v}_p) \bullet F_*(\mathbf{w}_p) = \mathbf{v}_p \bullet \mathbf{w}_p.$$

Proof. Let C be the orthogonal part of F, and recall that C, being an orthogonal transformation, preserves dot products in $\mathbf{R}^3$. By Theorem 2.1,

$$F_*(\mathbf{v}_p) \bullet F_*(\mathbf{w}_p) = C(\mathbf{v})_{F(p)} \bullet C(\mathbf{w})_{F(p)} = C(\mathbf{v}) \bullet C(\mathbf{w})$$
$$= \mathbf{v} \bullet \mathbf{w} = \mathbf{v}_p \bullet \mathbf{w}_p$$

where we have twice used Definition 1.3 of Chapter 2 (dot products of tangent vectors). ◆

Since dot products are preserved, it follows automatically that derived concepts such as norm and orthogonality are preserved. Explicitly, if F is an isometry, then $\| F_*(\mathbf{v}) \| = \| \mathbf{v} \|$, and if $\mathbf{v}$ and $\mathbf{w}$ are orthogonal, so are $F_*(\mathbf{v})$ and $F_*(\mathbf{w})$. Thus frames are also preserved: if $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ is a frame at some point $\mathbf{p}$ of $\mathbf{R}^3$ and F is an isometry, then $F_*(\mathbf{e}_1), F_*(\mathbf{e}_2), F_*(\mathbf{e}_3)$ is a frame at $F(\mathbf{p})$. (A direct proof is easy: $\mathbf{e}_i \bullet \mathbf{e}_j = \delta_{ij}$, so by Corollary 2.2, $F_*(\mathbf{e}_i) \bullet F_*(\mathbf{e}_j) = \mathbf{e}_i \bullet \mathbf{e}_j = \delta_{ij}$.)

Assertion (3) of Lemma 1.4 shows how two *points* uniquely determine a translation. We now show that two *frames* uniquely determine an isometry.

2.3 Theorem Given any two frames on $\mathbf{R}^3$, say $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ at the point $\mathbf{p}$ and $\mathbf{f}_1$, $\mathbf{f}_2$, $\mathbf{f}_3$ at the point $\mathbf{q}$, there exists a unique isometry F of $\mathbf{R}^3$ such that $F_*(\mathbf{e}_i) = \mathbf{f}_i$ for $1 \leqq i \leqq 3$.

Proof. First we show that there is such an isometry. Let $\hat{\mathbf{e}}_1$, $\hat{\mathbf{e}}_2$, $\hat{\mathbf{e}}_3$, and $\hat{\mathbf{f}}_1$, $\hat{\mathbf{f}}_2$, $\hat{\mathbf{f}}_3$ be the points of $\mathbf{R}^3$ canonically corresponding to the vectors in the two frames. Let C be the unique linear transformation of $\mathbf{R}^3$ such that $C(\hat{\mathbf{e}}_i) = \hat{\mathbf{f}}_i$ for $1 \leqq i \leqq 3$. It is easy to check that C is orthogonal. Then let T be a translation by the point $\mathbf{q} - C(\mathbf{p})$. Now we assert that the isometry $F = TC$ carries the $\mathbf{e}$ frame to the $\mathbf{f}$ frame. First note that

$$F(\mathbf{p}) = T(C(\mathbf{p})) = \mathbf{q} - C(\mathbf{p}) + C(\mathbf{p}) = \mathbf{q}.$$

Then using Theorem 2.1 we get

$$F_*(\mathbf{e}_i) = C(\hat{\mathbf{e}}_i)_{F(p)} = (\hat{\mathbf{f}}_i)_{F(p)} = (\hat{\mathbf{f}}_i)_q = \mathbf{f}_i$$

for $1 \leqq i \leqq 3$.

To prove uniqueness, we observe that by Theorem 2.1 this choice of C is the *only* possibility for the orthogonal part of the required isometry. The translation part is then completely determined also, since it must carry $C(\mathbf{p})$ to $\mathbf{q}$. Thus the isometry $F = TC$ is uniquely determined. ◆

To compute the isometry in the theorem, recall that the attitude matrix A of the $\mathbf{e}$ frame has the Euclidean coordinates of $\mathbf{e}_i$ as its ith row: a_{i1}, a_{i2}, a_{i3}. The attitude matrix B of the $\mathbf{f}$ frame is similar. We claim that C in the theorem (or strictly speaking, its matrix) is ${}^tBA$. To verify this it suffices to check that ${}^tBA(\mathbf{e}_i) = \mathbf{f}_i$, since this uniquely characterizes C. For $i = 1$ we find, using the column-vector conventions,

$$
\begin{aligned}
{}^tBA\begin{pmatrix} a_{11} \\ a_{12} \\ a_{13} \end{pmatrix} &= {}^tB\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} a_{11} \\ a_{12} \\ a_{13} \end{pmatrix} \\
&= {}^tB\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} b_{11} & b_{21} & b_{31} \\ b_{12} & b_{22} & b_{32} \\ b_{13} & b_{23} & b_{33} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} b_{11} \\ b_{12} \\ b_{13} \end{pmatrix},
\end{aligned}
$$

that is, ${}^tBA(\mathbf{e}_1) = \mathbf{f}_1$. The cases $i = 2, 3$ are similar; hence $C = {}^tBA$. As noted above, T is then necessarily translated by $\mathbf{q} - C(\mathbf{p})$.

Exercises

1. If T is a translation, show that for every tangent vector $\mathbf{v}$ the vector $T(\mathbf{v})$ is parallel to $\mathbf{v}$ (same Euclidean coordinates).

2. Prove the general formulas $(GF)* = G*F*$ and $(F^{-1})* = (F*)^{-1}$ in the special case where F and G are isometries of $\mathbf{R}^3$.

3. Given the frame

$$\mathbf{e}_1 = (2, 2, 1)/3, \qquad \mathbf{e}_2 = (-2, 1, 2)/3, \qquad \mathbf{e}_3 = (1, -2, 2)/3$$

at $\mathbf{p} = (0, 1, 0)$ and the frame

$$\mathbf{f}_1 = (1, 0, 1)/\sqrt{2}, \qquad \mathbf{f}_2 = (0, 1, 0), \qquad \mathbf{f}_3 = (1, 0, -1)/\sqrt{2}$$

at $\mathbf{q} = (3, -1, 1)$, find a and C such that the isometry $F = T_aC$ carries the $\mathbf{e}$ frame to the $\mathbf{f}$ frame.

4. (a) Prove that an isometry $F = TC$ carries the plane through $\mathbf{p}$ orthogonal to $\mathbf{q} \neq 0$ to the plane through $F(\mathbf{p})$ orthogonal to $C(\mathbf{q})$.

(b) If P is the plane through (1/2, −1, 0) orthogonal to (0, 1, 0) find an isometry $F = TC$ such that $F(P)$ is the plane through (1, −2, 1) orthogonal to (1, 0, −1).

5. (*Computer*.)

(a) Verify that both sets of vectors in Exercise 3 form frames by showing that ${}^tAA = I$ for their attitude matrices.

(b) Find the matrix C that carries each $\mathbf{e}_i$ to $\mathbf{f}_i$, and check this for $i = 1, 2, 3$.

3.3 Orientation

We now come to one of the most interesting and elusive ideas in geometry. Intuitively, it is *orientation* that distinguishes between a right-handed glove and a left-handed glove in ordinary space. To handle this concept mathematically, we replace gloves by frames and separate all the frames on $\mathbf{R}^3$ into two classes as follows. Recall that associated with each frame $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ at a point of $\mathbf{R}^3$ is its attitude matrix A. According to the exercises for Section 1 of Chapter 2,

$$\mathbf{e}_1 \bullet \mathbf{e}_2 \times \mathbf{e}_3 = \det A = \pm 1.$$

When this number is +1, we shall say that the frame $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ is *positively oriented* (or right-handed); when it is −1, the frame is *negatively oriented* (or left-handed).

We omit the easy proof of the following facts.

3.1 Remark (1) At each point of $\mathbf{R}^3$ the frame assigned by the natural frame field U_1, U_2, U_3 is positively oriented.

(2) A frame $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ is positively oriented if and only if $\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3$. Thus the orientation of a frame can be determined, for practical purposes, by the "right-hand rule" given at the end of Section 1 of Chapter 2. Pictorially, the frame (P) in Fig. 3.3 is positively oriented, whereas the frame (N) is negatively oriented. In particular, *Frenet frames are always positively oriented*, since by definition, $B = T \times N$.

(3) For a positively oriented frame $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, the cross products are

$$\mathbf{e}_1 = \mathbf{e}_2 \times \mathbf{e}_3 = -\mathbf{e}_3 \times \mathbf{e}_2,$$

$$\mathbf{e}_2 = \mathbf{e}_3 \times \mathbf{e}_1 = -\mathbf{e}_1 \times \mathbf{e}_3,$$

$$\mathbf{e}_3 = \mathbf{e}_1 \times \mathbf{e}_2 = -\mathbf{e}_2 \times \mathbf{e}_1.$$

For a negatively oriented frame, reverse the vectors in each cross product. (One need not memorize these formulas—the right-hand rule will give them all correctly.)

Having attached a sign to each frame on $\mathbf{R}^3$, we next attach a sign to each isometry F of $\mathbf{R}^3$. In Chapter 2 we proved the well-known fact that the determinant of an orthogonal matrix is either +1 or −1. Thus if C is the orthogonal part of the isometry F, we define the *sign* of F to be the determinant of C, with notation

$$\operatorname{sgn} F = \det C.$$

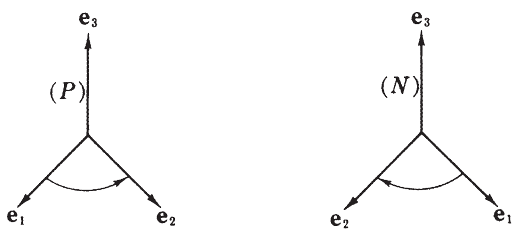

FIG. 3.3

We know that the tangent map of an isometry carries frames to frames. The following result tells what happens to their orientations.

3.2 Lemma If $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ is a frame at some point of $\mathbf{R}^3$ and F is an isometry, then

$$F_*(\mathbf{e}_1) \bullet F_*(\mathbf{e}_2) \times F_*(\mathbf{e}_3) = (\operatorname{sgn} F)\, \mathbf{e}_1 \bullet \mathbf{e}_2 \times \mathbf{e}_3.$$

Proof. If $\mathbf{e}_j = \sum a_{jk} U_k$, then by the coordinate form of Theorem 2.1 we have

$$F_*(\mathbf{e}_j) = \sum_{i,k} c_{ik} a_{jk} \overline{U}_i,$$

where $C = (c_{ij})$ is the orthogonal part of F. Thus the attitude matrix of the frame $F_*(\mathbf{e}_1)$, $F_*(\mathbf{e}_2)$, $F_*(\mathbf{e}_3)$ is the matrix

$$\left(\sum_k c_{ik} a_{jk}\right) = \left(\sum_k c_{ik}\, {}^t a_{kj}\right) = C\,{}^t A.$$

But the triple scalar product of a frame is the determinant of its attitude matrix, and by definition, $\operatorname{sgn} F = \det C$. Consequently,

$$\begin{aligned} F_*(\mathbf{e}_1) \bullet F_*(\mathbf{e}_2) \times F_*(\mathbf{e}_3) &= \det\left(C\,{}^t A\right) \\ &= \det C \cdot \det {}^t A = \det C \cdot \det A \\ &= (\operatorname{sgn} F)\mathbf{e}_1 \cdot \mathbf{e}_2 \times \mathbf{e}_3. \end{aligned}$$ ◆

This lemma shows that if $\operatorname{sgn} F = +1$, then F_* carries positively oriented frames to positively oriented frames and carries negatively oriented frames to negatively oriented frames. On the other hand, if $\operatorname{sgn} F = -1$, positive goes to negative and negative to positive.

3.3 Definition An isometry F of $\mathbf{R}^3$ is said to be

orientation-preserving if $\operatorname{sgn} F = \det C = +1$,

orientation-reversing if $\operatorname{sgn} F = \det C = -1$,

where C is the orthogonal part of F.

3.4 Example (1) ***Translations.*** All translations are orientation-preserving. Geometrically this is clear, and in fact the orthogonal part of a translation T is just the identity mapping I, so $\operatorname{sgn} T = \det I = +1$.

(2) ***Rotations.*** Consider the orthogonal transformation C given in Example 1.2, which rotates $\mathbf{R}^3$ through angle θ around the z axis. Its matrix is

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Hence $\operatorname{sgn} C = \det C = +1$, so C is orientation-preserving (see Exercise 4).

(3) ***Reflections.*** One can (literally) see reversal of orientation by using a mirror. Suppose the yz plane of $\mathbf{R}^3$ is the mirror. If one looks toward that plane, the point $\mathbf{p} = (p_1, p_2, p_3)$ appears to be located at the point

$$R(\mathbf{p}) = (-p_1, p_2, p_3)$$

(Fig. 3.4). The mapping R so defined is called *reflection* in the yz plane. Evidently it is an orthogonal transformation, with matrix

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus R is an orientation-reversing isometry, as confirmed by the experimental fact that the mirror image of a right hand is a left hand.

Both dot and cross product were originally defined in terms of *Euclidean* coordinates. We have seen that the dot product is given by the same formula,

$$\mathbf{v} \bullet \mathbf{w} = \left(\sum v_i \mathbf{e}_i\right) \bullet \left(\sum w_i \mathbf{e}_i\right) = \sum v_i w_i,$$

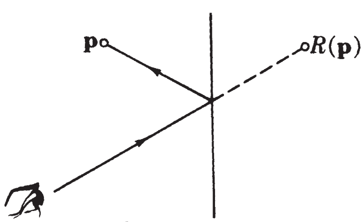

FIG. 3.4

no matter what frame $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ is used to get coordinates for $\mathbf{v}$ and $\mathbf{w}$. Almost the same result holds for cross products, but orientation is now involved.

3.5 Lemma Let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ be a frame at a point of $\mathbf{R}^3$. If $\mathbf{v} = \sum v_i \mathbf{e}_i$ and $\mathbf{w} = \sum w_i \mathbf{e}_i$, then

$$\mathbf{v} \times \mathbf{w} = \varepsilon \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix},$$

where $\varepsilon = \mathbf{e}_1 \bullet \mathbf{e}_2 \times \mathbf{e}_3 = \pm 1$.

Proof. It suffices merely to expand the cross product

$$\mathbf{v} \times \mathbf{w} = (v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3) \times (w_1\mathbf{e}_1 + w_2\mathbf{e}_2 + w_3\mathbf{e}_3)$$

using the formulas (3) of Remark 3.1. For example, if the frame is positively oriented, for the $\mathbf{e}_1$ component of $\mathbf{v} \times \mathbf{w}$ we get

$$v_2\mathbf{e}_2 \times w_3\mathbf{e}_3 + v_3\mathbf{e}_3 \times w_2\mathbf{e}_2 = (v_2w_3 - v_3w_2)\mathbf{e}_1.$$

Since $\varepsilon = 1$ in this case, we get the same result by expanding the determinant in the statement of this lemma. ◆

It follows immediately that the effect of an isometry on cross products also involves orientation.

3.6 Theorem Let $\mathbf{v}$ and $\mathbf{w}$ be tangent vectors to $\mathbf{R}^3$ at $\mathbf{p}$. If F is an isometry of $\mathbf{R}^3$, then

$$F_*(\mathbf{v} \times \mathbf{w}) = (\operatorname{sgn} F) F_*(\mathbf{v}) \times F_*(\mathbf{w}).$$

Proof. Write $\mathbf{v} = \sum v_i U_i(\mathbf{p})$ *and* $\mathbf{w} = \sum w_i U_i(\mathbf{p})$. Now let

$$\mathbf{e}_i = F_*(U_i(\mathbf{p})).$$

Since F_* is linear,

$$F_*(\mathbf{v}) = \sum v_i \mathbf{e}_i \quad \text{and} \quad F_*(\mathbf{w}) = \sum w_i \mathbf{e}_i.$$

A straightforward computation using Lemma 3.5 shows that

$$F_*(\mathbf{v}) \times F_*(\mathbf{w}) = \varepsilon F_*(\mathbf{v} \times \mathbf{w}),$$

where

$$\varepsilon = \mathbf{e}_1 \bullet \mathbf{e}_2 \times \mathbf{e}_3 = F_*(U_1(\mathbf{p})) \bullet F_*(U_2(\mathbf{p})) \times F_*(U_3(\mathbf{p})).$$

But U_1, U_2, U_3 is positively oriented, so by Lemma 3.2, $\varepsilon = \operatorname{sgn} F$. ◆

Exercises

1. Prove

$$\operatorname{sgn}(FG) = \operatorname{sgn} F \cdot \operatorname{sgn} G = \operatorname{sgn}(GF).$$

Deduce that $\operatorname{sgn} F = \operatorname{sgn}(F^{-1})$.

2. If H_0 is an orientation-reversing isometry of $\mathbf{R}^3$, show that *every* orientation-reversing isometry has a unique expression H_0F, where F is orientation-preserving.

3. Let $\mathbf{v} = (3, 1, -1)$ and $\mathbf{w} = (-3, -3, 1)$ be tangent vectors at some point. If C is the orthogonal transformation given in Exercise 4 of Section 1, check the formula

$$C_*(\mathbf{v} \times \mathbf{w}) = (\operatorname{sgn} C)C_*(\mathbf{v}) \times C_*(\mathbf{w}).$$

4. A *rotation* is an orthogonal transformation C such that $\det C = +1$. Prove that C does, in fact, rotate $\mathbf{R}^3$ around an axis. Explicitly, given a rotation C, show that there exists a number ϑ and points $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ with $\mathbf{e}_i \bullet \mathbf{e}_j = \delta_{ij}$ such that (Fig. 3.5)

$$C(\mathbf{e}_1) = \cos\vartheta\, \mathbf{e}_1 + \sin\vartheta\, \mathbf{e}_2,$$

$$C(\mathbf{e}_2) = -\sin\vartheta\, \mathbf{e}_1 + \cos\vartheta\, \mathbf{e}_2,$$

$$C(\mathbf{e}_3) = \mathbf{e}_3.$$

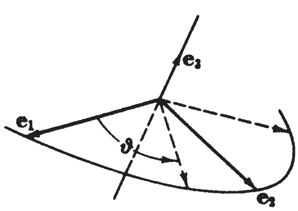

FIG. 3.5

(*Hint:* The fact that the dimension of $\mathbf{R}^3$ is odd means that C has an eigenvalue +1, so there is a point $\mathbf{p} \neq \mathbf{0}$ such that $C(\mathbf{p}) = \mathbf{p}$.)

5. Let $\mathbf{a}$ be a point of $\mathbf{R}^3$ such that $\|\mathbf{a}\| = 1$. Prove that the formula

$$C(\mathbf{p}) = \mathbf{a} \times \mathbf{p} + (\mathbf{p} \bullet \mathbf{a})\,\mathbf{a}$$

defines an orthogonal transformation. Describe its general effect on $\mathbf{R}^3$.

6. Prove
(a) The set $O^+(3)$ of all rotations of $\mathbf{R}^3$ is a subgroup of the orthogonal group $O(3)$ (see Ex. 8 of Sec. 3.1).
(b) The set $\mathscr{E}^+(3)$ of all orientation-preserving isometries of $\mathbf{R}^3$ is a subgroup of the Euclidean group $\mathscr{E}(3)$.

3.4 Euclidean Geometry

In the discussion at the beginning of this chapter, we recalled a fundamental feature of plane geometry: If there is an isometry carrying one triangle onto another, then the two (congruent) triangles have exactly the same geometric properties. A close examination of this statement will show that it does not admit a proof—it is, in fact, just the definition of "geometric property of a triangle." More generally, *Euclidean geometry* can be defined as the totality of concepts that are preserved by isometries of Euclidean space. For example, Corollary 2.2 shows that the notion of dot product on tangent vectors belongs to Euclidean geometry. Similarly, Theorem 3.6 shows that the cross product is preserved by isometries (except possibly for sign).

This famous definition of Euclidean geometry is somewhat generous, however. In practice, the label "Euclidean geometry" is usually attached only to those concepts that are preserved by isometries, but *not* by arbitrary mappings, or even the more restrictive class of mappings (diffeomorphisms) that possess inverse mappings. An example should make this distinction clearer. If $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ is a curve in $\mathbf{R}^3$, then the various derivatives

$$\alpha' = \left(\frac{d\alpha_1}{dt}, \frac{d\alpha_2}{dt}, \frac{d\alpha_3}{dt}\right), \qquad \alpha'' = \left(\frac{d^2\alpha_1}{dt^2}, \frac{d^2\alpha_2}{dt^2}, \frac{d^2\alpha_3}{dt^2}\right), \ \ldots$$

look pretty much alike. Now, Theorem 7.8 of Chapter 1 asserts that *velocity is preserved by arbitrary mappings* F: $\mathbf{R}^3 \to \mathbf{R}^3$, that is, if $\beta = F(\alpha)$, then $\beta' = F_*(\alpha')$. But it is easy to see that *acceleration is not preserved by arbitrary mappings*. For example, if $\alpha(t) = (t, 0, 0)$ and $F = (x^2, y, z)$, then $\alpha'' = 0$; hence $F_*(\alpha'') = 0$. But $\beta = F(\alpha)$ has the formula $\beta(t) = (t^2, 0, 0)$, so $\beta'' = 2U_1$. Thus in this case, $\beta = F(\alpha)$, but $\beta'' \neq F*(\alpha'')$. We shall see in a moment, however, that acceleration is preserved by *isometries*.

For this reason, the notion of velocity belongs to the *calculus* of Euclidean space, while the notion of acceleration belongs to Euclidean *geometry*. In this section we examine some of the concepts introduced in Chapter 2 and prove that they are, in fact, preserved by isometries. (We leave largely to the reader the easier task of showing that they are not preserved by diffeomorphisms.)

Recall the notion of vector field on a curve (Definition 2.2 of Chapter 2). If Y is a vector field on $\alpha\colon I \to \mathbf{R}^3$ and $F\colon \mathbf{R}^3 \to \mathbf{R}^3$ is any mapping, then $\bar{Y} = F*(Y)$ is a vector field on the image curve $\bar{\alpha} = F(\alpha)$. In fact, for each t in I, $Y(t)$ is a tangent vector to $\mathbf{R}^3$ at the point $\alpha(t)$. But then $\bar{Y}(t) = F*(Y(t))$ is a tangent vector to $\mathbf{R}^3$ at the point $F(\alpha(t)) = \bar{\alpha}(t)$.

(These relationships are illustrated in Fig. 3.6.) Isometries preserve the *derivatives* of such vector fields.

4.1 Corollary Let Y be a vector field on a curve α in $\mathbf{R}^3$, and let F be an isometry of $\mathbf{R}^3$. Then $\bar{Y} = F*(Y)$ is a vector field on $\bar{\alpha} = F(\alpha)$, and

$$\bar{Y}' = F*(Y').$$

Proof. To differentiate a vector field $Y = \sum y_j U_j$, one simply differentiates its Euclidean coordinate functions, so

$$Y' = \sum \frac{dy_j}{dt} U_j.$$

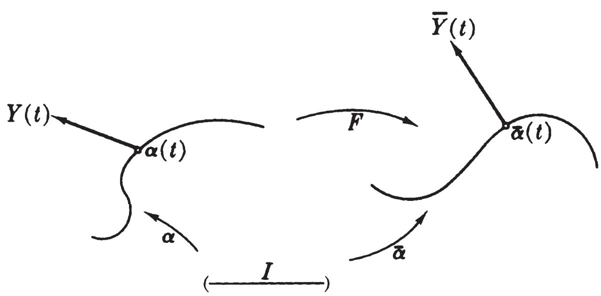

FIG. 3.6

Thus by the coordinate version of Theorem 2.1, we get

$$F_*(Y') = \sum c_{ij} \frac{dy_j}{dt} \overline{U}_i.$$

On the other hand,

$$\overline{Y} = F_*(Y) = \sum c_{ij} y_j \overline{U}_i.$$

But each c_{ij} is constant, being by definition an entry in the matrix of the orthogonal part of the isometry F. Hence

$$\overline{Y}' = \sum \frac{d}{dt}(c_{ij} y_j)\overline{U}_i = \sum c_{ij} \frac{dy_j}{dt} \overline{U}_i.$$

Thus the vector fields $F_*(Y')$ and $\overline{Y}'$ are the same. ◆

We claimed earlier that isometries preserve acceleration: If $\overline{\alpha} = F(\alpha)$, where F is an isometry, then $\overline{\alpha}'' = F_*(\alpha'')$. This is an immediate consequence of the preceding result, for if we set $Y = \alpha'$, then by Theorem 7.8 of Chapter 1, $\overline{Y} = \overline{\alpha}'$; hence

$$\overline{\alpha}'' = \overline{Y}' = F_*(Y') = F_*(\alpha'').$$

Now we show that the Frenet apparatus of a curve is preserved by isometries. This is certainly to be expected on intuitive grounds, since a rigid motion ought to carry one curve into another that turns and twists in exactly the same way. And this is what happens *when the isometry is orientation-preserving.*

4.2 Theorem Let β be a unit-speed curve in $\mathbf{R}^3$ with positive curvature, and let $\overline{\beta} = F(\beta)$ be the image curve of β under an isometry F of $\mathbf{R}^3$. Then

$$\overline{\kappa} = \kappa, \qquad \overline{T} = F_*(T),$$

$$\overline{\tau} = (\operatorname{sgn} F)\tau, \qquad \overline{N} = F_*(N),$$

$$\overline{B} = (\operatorname{sgn} F) F_*(B),$$

where $\operatorname{sgn} F = \pm 1$ is the sign of the isometry F.

Proof. Note that $\overline{\beta}$ is also a unit-speed curve, since

$$\|\overline{\beta}'\| = \|F_*(\beta')\| = \|\beta'\| = 1.$$

Thus the definitions in Section 3 of Chapter 2 apply to both β and $\bar{\beta}$, so

$$\bar{T} = \bar{\beta}' = F_*(\beta') = F_*(T).$$

Since F_* preserves both acceleration and norms, it follows from the definition of curvature that

$$\bar{\kappa} = \|\bar{\beta}''\| = \|F_*(\beta'')\| = \| \beta'' \| = \kappa.$$

To get the full Frenet frame, we now use the hypothesis $\kappa > 0$ (which implies $\bar{\kappa} > 0$, since $\bar{\kappa} = \kappa$). By definition, $N = \beta''/\kappa$; hence using preceding facts, we find

$$\bar{N} = \frac{\bar{\beta}''}{\bar{\kappa}} = \frac{F_*(\beta'')}{\kappa} = F_*\left(\frac{\beta''}{\kappa}\right) = F_*(N).$$

It remains only to prove the interesting cases B and τ. Since the definition $B = T \times N$ involves a cross product, we use Theorem 3.6 to get

$$\bar{B} = \bar{T} \times \bar{N} = F_*(T) \times F_*(N) = (\operatorname{sgn} F)F_*(T \times N) = (\operatorname{sgn} F)F_*(B).$$

The definition of torsion is essentially $\tau = -B' \bullet N = B \bullet N'$. Thus, using the results above for B and N, we get

$$\bar{\tau} = \bar{B} \bullet \bar{N}' = (\operatorname{sgn} F)F_*(B) \bullet F_*(N') = (\operatorname{sgn} F)B \bullet N' = (\operatorname{sgn} F)\tau. \qquad \blacklozenge$$

The presence of sgn F in the formula for the torsion of $F(\beta)$ shows that the torsion of a curve gives a more subtle description of the curve than has been apparent so far. *The sign of τ measures the orientation of the twisting of the curve.* If F is orientation-reversing, the formula $\bar{\tau} = -\tau$ proves that the twisting of the image of curve $F(\beta)$ is exactly opposite to that of β itself.

A simple example will illustrate this reversal.

4.3 Example Let β be the unit-speed helix

$$\beta(s) = \left(\cos\frac{s}{c}, \sin\frac{s}{c}, \frac{s}{c}\right),$$

gotten from Example 3.3 of Chapter 2 by setting $a = b = 1$; hence $c = \sqrt{2}$. We know from the general formulas for helices that $\kappa = \tau = 1/2$. Now let R be reflection in the xy plane, so R is the isometry $R(x, y, z) = (x, y, -z)$. Thus the image curve $\bar{\beta} = \mathbf{R}(\beta)$ is the mirror image

$$\bar{\beta}(s) = \left(\cos\frac{s}{c}, \sin\frac{s}{c}, -\frac{s}{c}\right)$$

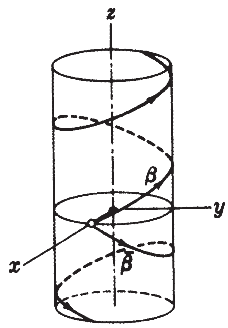

FIG. 3.7

of the original curve. One can see in Fig. 3.7 that the mirror has its usual effect: β and $\bar{\beta}$ twist in opposite ways—if β is "right-handed," then $\bar{\beta}$ is "left-handed." (The fact that β is going up and $\bar{\beta}$ down is, in itself, irrelevant.) Formally: The reflection R is orientation-reversing; hence the theorem predicts $\bar{\kappa} = \kappa = \frac{1}{2}$ and $\bar{\tau} = -\tau = -\frac{1}{2}$. Since $\bar{\beta}$ is just the helix gotten in Example 3.3 of Chapter 2 by taking $a = 1$ and $b = -1$, this may be checked by the general formulas there.

Exercises

1. Let $F = TC$ be an isometry of $\mathbf{R}^3$, β a unit speed curve in $\mathbf{R}^3$. Prove
(a) If β is a cylindrical helix, then $F(\beta)$ is a cylindrical helix.
(b) If β has spherical image σ, then $F(\beta)$ has spherical image $C(\sigma)$.

2. Let $Y = (t, 1 - t^2, 1 + t^2)$ be a vector field on the helix

$$\alpha(t) = (\cos t, \sin t, 2t),$$

and let C be the orthogonal transformation

$$C = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}.$$

Compute $\bar{\alpha} = C(\alpha)$ and $\bar{Y} = C_*(Y)$, and check that

$$C_*(Y') = \bar{Y}', \quad C_*(\alpha'') = \bar{\alpha}'', \quad Y' \bullet \alpha'' = \bar{Y}' \bullet \bar{\alpha}''.$$

3. Sketch the triangles in $\mathbf{R}^2$ that have vertices

$$\Delta_1:\ (3,1),\ (7,1),\ (7,4),\quad \Delta_2:\ (2,0),\ (2,5),\ (-2/5, 16/5).$$

Show that these triangles are congruent by exhibiting an isometry $F = TC$ that carries Δ_1 to Δ_2. (*Hint*: the orthogonal part C is not altered if the triangles are translated.)

4. If F: $\mathbf{R}^3 \to \mathbf{R}^3$ is a diffeomorphism such that $F*$ preserves dot products, show that F is an isometry. (*Hint:* Show that F preserves lengths of curve segments and deduce that F^{-1} does also.)

5. Let F be an isometry of $\mathbf{R}^3$. For each vector field V let $\overline{V}$ be the vector field such that $F*(V(\mathbf{p})) = \overline{V}(F(\mathbf{p}))$ for all $\mathbf{p}$. Prove that isometries preserve covariant derivatives; that is, show $\overline{\nabla_V W} = \nabla_{\overline{V}}\overline{W}$.

3.5 Congruence of Curves

In the case of curves in $\mathbf{R}^3$, the general notion of congruence takes the following form.

5.1 Definition Two curves α, β: $I \to \mathbf{E}^3$ are *congruent* provided there exists an isometry F of $\mathbf{R}^3$ such that $\beta = F(\alpha)$; that is, $\beta(t) = F(\alpha(t))$ for all t in I.

Intuitively speaking, congruent curves are the same except for position in space. They represent *trips at the same speed along routes of the same shape.* For example, the helix $\alpha(t) = (\cos t, \sin t, t)$ spirals around the z axis in exactly the same way the helix $\beta(t) = (t, \cos t, \sin t)$ spirals around the x axis. Evidently these two curves are congruent, since if F is the isometry such that

$$F(p_1, p_2, p_3) = (p_3, p_1, p_2),$$

then $F(\alpha) = \beta$.

To decide whether given curves α and β are congruent, it is hardly practical to try all the isometries of $\mathbf{R}^3$ to see whether there is one that carries α to β. What we want is a description of the shape of a unit-speed curve so accurate that if α and β have the same description, then they must be congruent. The proper description, as the reader will doubtless suspect, is given by curvature and torsion. To prove this we need one preliminary result.

Curves whose congruence is established by a translation are said to be *parallel.* Thus, curves α, β: $I \to \mathbf{E}^3$ are parallel if and only if there is a point

p in $\mathbf{R}^3$ such that $\beta(s) = \alpha(s) + \mathbf{p}$ for all s in I, or, in functional notation, $\beta = \alpha + \mathbf{p}$.

5.2 Lemma Two curves $\alpha, \beta: I \to \mathbf{R}^3$ are parallel if their velocity vectors $\alpha'(s)$ and $\beta'(s)$ are parallel for each s in I. In this case, if $\alpha(s_0) = \beta(s_0)$ for some one s_0 in I, then $\alpha = \beta$.

Proof. By definition, if $\alpha'(s)$ and $\beta'(s)$ are parallel, they have the same Euclidean coordinates. Thus

$$\frac{d\alpha_i}{ds}(s) = \frac{d\beta_i}{ds}(s) \quad \text{for } 1 \leqq i \leqq 3,$$

where α_i and β_i are the Euclidean coordinate functions of α and β. But by elementary calculus, the equation $d\alpha_i/ds = d\beta_i/ds$ implies that there is a constant p_i such that $\beta_i = \alpha_i + p_i$. Hence $\beta = \alpha + \mathbf{p}$. Furthermore, if $\alpha(s_0) = \beta(s_0)$, we deduce that $\mathbf{p} = \mathbf{0}$; hence $\alpha = \beta$. ◆

5.3 Theorem If $\alpha, \beta: I \to \mathbf{R}^3$ are unit-speed curves such that $\kappa_\alpha = \kappa_\beta$ and $\tau_\alpha = \pm\tau_\beta$, then α and β are congruent.

Proof. There are two main steps:

(1) Replace α by a suitably chosen congruent curve $F(\alpha)$.

(2) Show that $F(\alpha) = \beta$ (Fig. 3.8).

Our guide for the choice in (1) is Theorem 4.2. Fix a number, say 0, in the interval I. If $\tau_\alpha = \tau_\beta$, then let F be the (orientation-preserving) isometry that carries the Frenet frame $T_\alpha(0)$, $N_\alpha(0)$, $B_\alpha(0)$ of α at $\alpha(0)$ to the

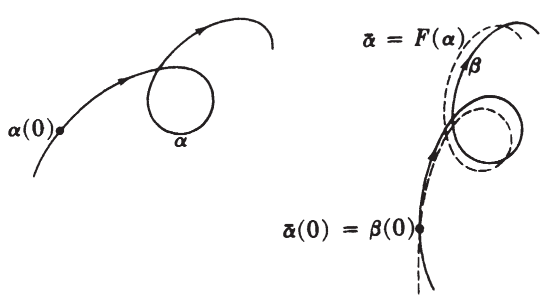

FIG. 3.8

Frenet frame $T_\beta(0)$, $N_\beta(0)$, $B_\beta(0)$, of β at $\beta(0)$. (The existence of this isometry is guaranteed by Theorem 2.3.) Denote the Frenet apparatus of $\bar{\alpha} = F(\alpha)$ by $\bar{\kappa}$, $\bar{\tau}$, $\bar{T}$, $\bar{N}$, $\bar{B}$; then it follows immediately from Theorem 4.2 and the information above that

$$\begin{aligned} \bar{\alpha}(0) &= \beta(0), & \bar{T}(0) &= T_\beta(0), \\ \bar{\kappa} &= \kappa_\beta, & \bar{N}(0) &= N_\beta(0), \\ \bar{\tau} &= \tau_\beta, & \bar{B}(0) &= B_\beta(0). \end{aligned} \qquad (\ddagger)$$

On the other hand, if $\tau_\alpha = -\tau_\beta$, we choose F to be the (orientation-reversing) isometry that carries $T_\alpha(0)$, $N_\alpha(0)$, $B_\alpha(0)$ at $\alpha(0)$ to the frame $T_\beta(0)$, $N_\beta(0)$, $B_\beta(0)$ at $\beta(0)$. (Frenet frames are positively oriented; hence this last frame is negatively oriented: This is why F is orientation-reversing.) Then it follows from Theorem 4.2 that the equations (‡) hold also for $\bar{\alpha} = F(\alpha)$ and β. For example,

$$\bar{B}(0) = -F_*(B_\alpha(0)) = B_\beta(0).$$

For step (2) of the proof, we shall show $\bar{T} = T_\beta$; that is, the unit tangents of $\bar{\alpha} = F(\alpha)$ and β are parallel at each point. Since $\bar{\alpha}(0) = \beta(0)$, it will follow from Lemma 5.2 that $F(\alpha) = \beta$. On the interval I, consider the real-valued function $f = \bar{T} \cdot T_\beta + \bar{N} \cdot N_\beta + \bar{B} \cdot B_\beta$. Since these are *unit* vector fields, the Schwarz inequality (Sec. 1, Ch. 2) shows that

$$\bar{T} \cdot T_\beta \leqq 1;$$

furthermore, $\bar{T} \cdot T_\beta = 1$ if and only if $\bar{T} = T_\beta$. Similar remarks hold for the other two terms in f. Thus it *suffices to show that f has constant value* 3. By (‡), $f(0) = 3$. Now consider

$$f' = \bar{T}' \cdot T_\beta + \bar{T} \cdot T_\beta' + \bar{N}' \cdot N_\beta + \bar{N} \cdot N_\beta' + \bar{B}' \cdot B_\beta + \bar{B} \cdot B_\beta'$$

A simple computation completes the proof. Substitute the Frenet formulas in this expression and use the equations $\bar{\kappa} = \kappa_\beta$, $\bar{\tau} = \tau_\beta$ from (‡). The resulting eight terms cancel in pairs, so $f' = 0$, and f has, indeed, constant value 3. ◆

Thus, *a unit-speed curve is determined but for position in* $\mathbf{R}^3$ *by its curvature and torsion.*

Actually the proof of Theorem 5.3 does more than establish that α and β are congruent; it shows how to compute *explicitly* an isometry carrying α to β. We illustrate this in a special case.

5.4 Example Consider the unit-speed curves α, β: $\mathbf{R} \to \mathbf{R}^3$ such that

$$\alpha(s) = \left(\cos\frac{s}{c}, \sin\frac{s}{c}, \frac{s}{c}\right),$$

$$\beta(s) = \left(\cos\frac{s}{c}, \sin\frac{s}{c}, -\frac{s}{c}\right),$$

where $c = \sqrt{2}$. Obviously, these curves are congruent by means of a reflection—they are the helices considered in Example 4.3—but we shall ignore this in order to describe a general method for computing the required isometry. According to Example 3.3 of Chapter 2, α and β have the same curvature, $\kappa_\alpha = 1/2 = \kappa_\beta$; but torsions of opposite sign, $\tau_\alpha = 1/2 = -\tau_\beta$. Thus the theorem predicts congruence by means of an orientation-reversing isometry F. From its proof we see that F must carry the Frenet frame

$$T_\alpha(0) = (0, a, a),$$

$$N_\alpha(0) = (-1, 0, 0),$$

$$B_\alpha(0) = (0, -a, a),$$

where $a = 1/\sqrt{2}$, to the frame

$$T_\beta(0) = (0, a, -a),$$

$$N_\beta(0) = (-1, 0, 0),$$

$$-B_\beta(0) = (0, -a, -a),$$

where the minus sign will produce orientation reversal. (These explicit formulas also come from Example 3.3 of Chapter 2.) By the remark following Theorem 2.3, the isometry F has orthogonal part $C = {}^tBA$, where A and B are the attitude matrices of the two frames above. Thus

$$C = \begin{pmatrix} 0 & -1 & 0 \\ a & 0 & -a \\ -a & 0 & -a \end{pmatrix} \begin{pmatrix} 0 & a & a \\ -1 & 0 & 0 \\ 0 & -a & a \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

since $a = 1/\sqrt{2}$. These two frames have the same point of application $\alpha(0) = \beta(0) = (1, 0, 0)$. But C does not move this point, so the translation part of F is just the identity map. Thus we have (correctly) found that the reflection $F = C$ carries α to β.

From the viewpoint of Euclidean geometry, two curves in $\mathbf{R}^3$ are "the same" if they differ only by an isometry of $\mathbf{R}^3$. What, for example, is a helix?

It is not just a curve that spirals around the z axis as in Example 3.3 of Chapter 2, but any curve congruent to one of these special helices. One can give general formulas, but the best characterization follows.

5.5 Corollary Let α be a unit speed curve in $\mathbf{R}^3$. Then α is a helix if and only if both its curvature and torsion are nonzero constants.

Proof. For any numbers $a > 0$ and $b \neq 0$, let $\beta_{a,b}$ be the special helix given in Example 3.3 of Chapter 2. If α is congruent to $\beta_{a,b}$, then (changing the sign of b if necessary) we can assume the isometry is orientation-preserving. Thus, α has curvature and torsion

$$\kappa = \frac{a}{a^2 + b^2}, \qquad \tau = \frac{b}{a^2 + b^2}.$$

Conversely, suppose α has constant nonzero κ and τ. Solving the preceding equations, we get

$$a = \frac{\kappa}{\kappa^2 + \tau^2}, \qquad b = \frac{\tau}{\kappa^2 + \tau^2}.$$

Thus α and $\beta_{a,b}$ have the same curvature and torsion; hence they are congruent. ◆

Our results so far demand unit speed, but it is easy to weaken this restriction.

5.6 Corollary Let $\alpha, \beta\colon I \to \mathbf{R}^3$ arbitrary-speed curves. If

$$v_\alpha = v_\beta > 0, \quad \kappa_\alpha = \kappa_\beta > 0, \quad \text{and} \quad \tau_\alpha = \pm\tau_\beta,$$

then the curves α and β are congruent.

The proof is immediate, for the data ensures that the unit speed parametrizations of α and β have the same curvature and torsion—hence they are congruent. But then the original curves are congruent under the same isometry since their speeds are the same.

The theory of curves we have presented applies only to regular curves with positive curvature $\kappa > 0$, because only for such curves is it possible to define the Frenet frame field. However, an arbitrary curve α in $\mathbf{R}^3$ can be studied by means of an *arbitrary* frame field on α, that is, three unit-vector fields E_1, E_2, E_3 on α that are orthogonal at each point.

At a critical point later on, we will need this generalization of the congruence theorem (5.3):

5.7 Theorem Let α, β: $I \to \mathbf{R}^3$ be curves defined on the same interval. Let E_1, E_2, E_3 be a frame field on α, and F_1, F_2, F_3 a frame field on β. If

(1) $\alpha' \bullet E_i = \beta' \bullet F_i \quad (1 \leqq i \leqq 3)$,

(2) $E_i' \bullet E_j = F_i' \bullet F_j \quad (1 \leqq i, j \leqq 3)$,

then α and β are congruent.

Explicitly, for any t_0 in I, if $\mathbf{F}$ is the unique Euclidean isometry that sends each $E_i(t_0)$ to $F_i(t_0)$, then $\mathbf{F}(\alpha) = \beta$.

Proof. Let $\mathbf{F}$ be the specified isometry. Since $\mathbf{F}_*$ preserves dot products, it follows that the vector fields $\bar{E}_i = F_*(E_i)$ for $1 \le i \le 3$ form a frame field on $\bar{\alpha} = \mathbf{F}(\alpha)$. And since $\mathbf{F}_*$ preserves velocities of curves and derivatives of vector fields, by using condition (1) in the theorem, we find

$$\bar{\alpha}(t_0) = \beta(t_0) \quad \text{and} \quad \bar{\alpha}' \bullet \bar{E}_i = \beta' \bullet F_i \quad \text{for } 1 \leqq i \leqq 3. \tag{$*$}$$

Similarly, from condition (2), we get

$$\bar{E}_i(t_0) = F_i(t_0) \quad \text{and} \quad \bar{E}_i' \bullet \bar{E}_j = F_i' \cdot F_j \quad \text{for } 1 \leqq i,\ j \leqq 3. \tag{$**$}$$

In view of this last equation, orthonormal expansion yields

$$\bar{E}_i' = \sum_j a_{ij} \bar{E}_j \quad \text{and} \quad F_i' = \sum_j a_{ij} F_j,$$

with the *same* coefficient functions a_{ij}. Note that $a_{ij} + a_{ji} = 0$; hence $a_{ii} = 0$. (*Proof:* Differentiate $\bar{E}_i \bullet \bar{E}_j = \delta_{ij}$.)

Now let $f = \sum \bar{E}_j \bullet F_i$. We prove $f = 3$ as before: $f(t_0) = 3$, and

$$f' = \sum (\bar{E}_i' \bullet F_i + \bar{E}_i \bullet F_i') = \sum_{i,j} (a_{ij} + a_{ji}) \bar{E}_j \bullet F_i = 0.$$

Thus each $\bar{E}_i \bullet F_i = 1$, that is, $\bar{E}_i$ and F_i are parallel at each point. By ($*$) the same is true for

$$\bar{\alpha}' = \sum (\bar{\alpha}' \bullet \bar{E}_i) \bar{E}_i \quad \text{and} \quad \beta' = \sum (\beta' \bullet F_i) F_i.$$

Since $\alpha(t_0) = \beta(t_0)$, Lemma 5.2 gives the required result, $F(\alpha) = \bar{\alpha} = \beta$. ◆

5.8 Remark Existence theorem for curves in $\mathbf{R}^3$. Curvature and torsion tell whether two unit-speed curves are isometric, but they do more than that:

Given any two continuous functions $\kappa > 0$ and τ on an interval I, there exists a unit-speed curve α: $I \to \mathbf{R}^3$ that has these functions as its curvature and torsion. (As we know, any two such curves are congruent.) Thus the natural description of curves in $\mathbf{R}^3$ is devoid of geometry, consisting of a pair of real-valued functions.

The proof of the existence theorem requires advanced methods, so we have preferred to illustrate it by the corresponding result for plane curves (Exercises 7–10). Though simpler, this 2-dimensional version has the advantage that plane curvature $\tilde{\kappa}$ is not required to be positive.

Exercises

1. Given a curve $\alpha = (\alpha_1, \alpha_2, \alpha_3)$: $I \to \mathbf{R}^3$, prove that β: $I \to \mathbf{R}^3$ is congruent to α if and only if β can be written as

$$\beta(t) = \mathbf{p} + \alpha_1(t)\mathbf{e}_1 + \alpha_2(t)\mathbf{e}_2 + \alpha_3(t)\mathbf{e}_3,$$

where $\mathbf{e}_i \bullet \mathbf{e}_j = \delta_{ij}$.

2. Let E_1, E_2, E_3, be a frame field on $\mathbf{R}^3$ with dual forms θ_i and connection forms ω_{ij}. Prove that two curves α, β: $I \to \mathbf{R}^3$ are congruent if $\theta_i(\alpha') = \theta_i(\beta')$ and $\omega_{ij}(\alpha') = \omega_{ij}(\beta')$ for $1 \leqq i, j \leqq 3$ (*Hint:* Use Thm. 5.7.)

3. Show that the curve

$$\beta(t) = \left(t + \sqrt{3}\sin t,\ 2\cos t,\ \sqrt{3}t - \sin t\right)$$

is a helix by finding its curvature and torsion. Find a helix of the form $\alpha(t) = (a\cos t, a\sin t, bt)$ and an isometry F such that $F(\alpha) = \beta$.

4. (*Computer; see Appendix.*) (a) Show that the curves

$$\alpha(t) = \left(t + t^2, t - t^2, 1 + \sqrt{2}t^3\right), \quad \beta(t) = \left(t^2 + t^3, 1 - \sqrt{2}t, t^2 - t^3\right),$$

defined on the entire real line, have the same speed, curvature, and torsion. (b) Find formulas for T and C such that the isometry $F = TC$ carries α to β and verify explicitly that $F(\alpha) = \beta$. (*Hint:* Use Ex. 5 of Sec. 2.)

5. (*Computer optional.*) Is the following curve a helix? Prove your answer.

$$c(t) = (-2\cos t + 2\sin t + 2t, 2\cos t + \sin t + 4t, \cos t + 2\sin t - 4t).$$

6. Congruence of curves.

(a) Prove that curves α, β: $I \to \mathbf{R}^2$ are congruent if $\tilde{\kappa}_\alpha = \tilde{\kappa}_\beta$ and they have the same speed.

(b) Show that the space curves

$$\alpha(t) = \left(\sqrt{2}t, t^2, 0\right) \quad \text{and} \quad \beta(t) = (-t, t, t^2)$$

are congruent. Find an isometry that carries α to β.

7. Given a continuous function f on an interval I, prove—using ordinary integration of functions—that there exists a unit-speed curve $\beta(s)$ in $\mathbf{R}^2$ for which $f(s)$ is the plane curvature. (*Hint:* Reverse the logic in Ex. 8 of Sec. 2.3.)

8. Show that $\beta(s) = (x(s), y(s))$ in the preceding exercise is given by the solutions of the differential equations

$$x'(s) = \cos\varphi(s), \quad y'(s) = \sin\varphi(s), \quad \varphi'(s) = f(s),$$

with initial conditions $x(0) = y(0) = \varphi(0) = 0$. (These initial conditions suffice, since any other β differs at most by a Euclidean isometry and a reparametrization $s \to s + c$.)

Explicit integration is rarely possible; the following exercises use numerical integration.

9. (*Numerical integration, computer graphics.*) Write computer commands that (a) given $f(s)$, produce a numerical description of the solution curve $\beta(s)$ in the preceding exercise, and (b) given $f(s)$, plot the solution curve.

10. (*Continuation.*) Plot unit-speed plane curves with the given plane curvature function f on at least the given interval.

(a) $f(s) = 1 + e^s$, on $-6 \le s \le 3$.
(b) $f(s) = 2 + 3\cos 3s$, on $0 \le s \le 2\pi$.
(c) $f(s) = 3 - 2s^2 + s^3$, on $-2.5 \le s \le 3.5$.

Adjust scales on axes as needed.

3.6 Summary

The basic result of this chapter is that an arbitrary isometry of Euclidean space can be uniquely expressed as an orthogonal transformation followed by a translation. A consequence is that the tangent map of an isometry F is, at every point, essentially just the orthogonal part of F. Then it is a routine matter to test the concepts introduced earlier to see which belong to *Euclidean geometry*, that is, which are preserved by isometries of Euclidean space.

Finally, we proved an analogue for curves of the various criteria for congruence of triangles in plane geometry; namely, we showed that a necessary and sufficient condition for two curves in $\mathbf{R}^3$ to be congruent is that they have the same curvature and torsion (and speed). Furthermore, the sufficiency proof shows how to find the required isometry explicitly.