Chapter 2
Frame Fields

Roughly speaking, geometry begins with the measurement of distances and angles. We shall see that the geometry of Euclidean space can be derived from the *dot product*, the natural inner product on Euclidean space.

Much of this chapter is devoted to the geometry of curves in $\mathbf{R}^3$. We emphasize this topic not only because of its intrinsic importance, but also because the basic method used to investigate curves has proved effective throughout differential geometry. A curve in $\mathbf{R}^3$ is studied by assigning at each point a certain *frame*—that is, set of three orthogonal unit vectors. The rate of change of these vectors along the curve is then expressed in terms of the vectors themselves by the celebrated *Frenet formulas* (Theorem 3.2). In a real sense, the theory of curves in $\mathbf{R}^3$ is merely a corollary of these fundamental formulas.

Later on we shall use this "method of moving frames" to study a *surface* in $\mathbf{R}^3$. The general idea is to think of a surface as a kind of two-dimensional curve and follow the Frenet approach as closely as possible. To carry out this scheme we shall need the generalization (Theorem 7.2) of the Frenet formulas devised by E. Cartan. It was Cartan who, in the early 1900s, first realized the full power of this method not only in differential geometry but also in a variety of related fields.

2.1 Dot Product

We begin by reviewing some basic facts about the natural inner product on the vector space $\mathbf{R}^3$.

1.1 Definition The dot product of points $\mathbf{p} = (p_1, p_2, p_3)$ and $\mathbf{q} = (q_1, q_2, q_3)$ in $\mathbf{R}^3$ is the number

$$\mathbf{p} \bullet \mathbf{q} = p_1q_1 + p_2q_2 + p_3q_3.$$

The dot product is an inner product since it has the following three properties:

(1) Bilinearity:

$$(a\mathbf{p} + b\mathbf{q}) \bullet \mathbf{r} = a\mathbf{p} \bullet \mathbf{r} + b\mathbf{q} \bullet \mathbf{r},$$

$$\mathbf{r} \bullet (a\mathbf{p} + b\mathbf{q}) = a\mathbf{r} \bullet \mathbf{p} + b\mathbf{r} \bullet \mathbf{q}.$$

(2) Symmetry: $\mathbf{p} \bullet \mathbf{q} = \mathbf{q} \bullet \mathbf{p}$.

(3) Positive definiteness: $\mathbf{p} \bullet \mathbf{p} \geqq 0$, and $\mathbf{p} \bullet \mathbf{p} = 0$ if and only if $\mathbf{p} = 0$.

(Here $\mathbf{p}$, $\mathbf{q}$, and $\mathbf{r}$ are arbitrary points of $\mathbf{R}^3$, and a and b are numbers.)

The *norm* of a point $\mathbf{p} = (p_1, p_2, p_3)$ is the number

$$\| \mathbf{p} \| = (\mathbf{p} \bullet \mathbf{p})^{1/2} = (p_1^2 + p_2^2 + p_3^2)^{1/2}.$$

The norm is thus a real-valued function on $\mathbf{R}^3$; it has the fundamental properties $\| \mathbf{p} + \mathbf{q} \| \leqq \| \mathbf{p} \| + \| \mathbf{q} \|$ and $\| a\mathbf{p} \| = | a | \, \| \mathbf{p} \|$, where $| a |$ is the absolute value of the number a.

In terms of the norm we get a compact version of the usual distance formula in $\mathbf{R}^3$.

1.2 Definition If $\mathbf{p}$ and $\mathbf{q}$ are points of $\mathbf{R}^3$, the *Euclidean distance* from $\mathbf{p}$ to $\mathbf{q}$ is the number

$$d(\mathbf{p}, \mathbf{q}) = \| \mathbf{p} - \mathbf{q} \|.$$

In fact, since

$$\mathbf{p} - \mathbf{q} = (p_1 - q_1, p_2 - q_2, p_3 - q_3),$$

expansion of the norm gives the well-known formula (Fig. 2.1)

$$d(\mathbf{p}, \mathbf{q}) = \left((p_1 - q_1)^2 + (p_2 - q_2)^2 + (p_3 - q_3)^2\right)^{1/2}.$$

Euclidean distance may be used to give a more precise definition of open sets (Chapter 1, Section 1). First, if $\mathbf{p}$ is a point of $\mathbf{R}^3$ and $\varepsilon > 0$ is a number, the ε *neighborhood* $\mathcal{N}_\varepsilon$ of $\mathbf{p}$ in $\mathbf{R}^3$ is the set of all points $\mathbf{q}$ of $\mathbf{R}^3$ such that $d(\mathbf{p}, \mathbf{q}) < \varepsilon$. Then a subset $\mathcal{O}$ of $\mathbf{R}^3$ is *open* provided that each point of $\mathcal{O}$ has an ε neighborhood that is entirely contained in $\mathcal{O}$. In short, all points near enough to a point of an open set are also in the set. This definition is valid with $\mathbf{R}^3$ replaced by $\mathbf{R}^n$—or indeed any set furnished with a reasonable distance function.

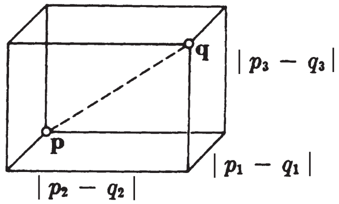


FIG. 2.1

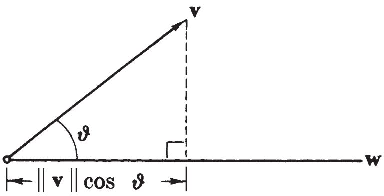


FIG. 2.2

We saw in Chapter 1 that for each point $\mathbf{p}$ of $\mathbf{R}^3$ there is a *canonical isomorphism* $\mathbf{v} \to \mathbf{v}_p$ from $\mathbf{R}^3$ onto the tangent space $\mathbf{T}_p(\mathbf{R}^3)$ at $\mathbf{p}$. These isomorphisms lie at the heart of Euclidean geometry—using them, the dot product on $\mathbf{R}^3$ itself may be transferred to each of its tangent spaces.

1.3 Definition The *dot product* of tangent vectors $\mathbf{v}_p$ and $\mathbf{w}_p$ at the same point of $\mathbf{R}^3$ is the number $\mathbf{v}_p \bullet \mathbf{w}_p = \mathbf{v} \bullet \mathbf{w}$.

For example, $(1, 0, -1)_p \bullet (3, -3, 7)_p = 1(3) + 0(-3) + (-1)7 = -4$. Evidently this definition provides a dot product on each tangent space $T_p(\mathbf{R}^3)$ with the same properties as the original dot product on $\mathbf{R}^3$. In particular, each tangent vector $\mathbf{v}_p$ to $\mathbf{R}^3$ has *norm* (or *length*) $\| \mathbf{v}_p \| = \| \mathbf{v} \|$.

A fundamental result of linear algebra is the Schwarz inequality $| \mathbf{v} \bullet \mathbf{w} | \leqq \| \mathbf{v} \| \, \| \mathbf{w} \|$. This permits us to define the cosine of the angle ϑ between $\mathbf{v}$ and $\mathbf{w}$ by the equation (Fig. 2.2).

$$\mathbf{v} \bullet \mathbf{w} = \|\mathbf{v}\| \, \|\mathbf{w}\| \cos \vartheta.$$

Thus the dot product of two vectors is the product of their lengths times the cosine of the angle between them. (The angle ϑ is not uniquely determined unless further restrictions are imposed, say $0 \leqq \vartheta \leqq \pi$.)

In particular, if $\vartheta = \pi/2$, then $\mathbf{v} \bullet \mathbf{w} = 0$. Thus we shall define two vectors to be *orthogonal* provided their dot product is zero. A vector of length 1 is called a *unit vector.*

1.4 Definition A set $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ of three mutually orthogonal unit vectors tangent to $\mathbf{R}^3$ at $\mathbf{p}$ is called a *frame* at the point $\mathbf{p}$.

Thus $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ is a frame if and only if

$$\mathbf{e}_1 \bullet \mathbf{e}_1 = \mathbf{e}_2 \bullet \mathbf{e}_2 = \mathbf{e}_3 \bullet \mathbf{e}_3 = 1,$$

$$\mathbf{e}_1 \bullet \mathbf{e}_2 = \mathbf{e}_1 \bullet \mathbf{e}_3 = \mathbf{e}_2 \bullet \mathbf{e}_3 = 0.$$

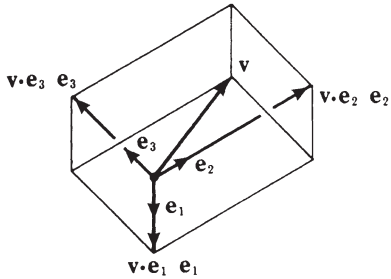


FIG. 2.3

By the symmetry of the dot product, the second row of equations is, of course, the same as

$$\mathbf{e}_2 \bullet \mathbf{e}_1 = \mathbf{e}_3 \bullet \mathbf{e}_1 = \mathbf{e}_3 \bullet \mathbf{e}_2 = 0.$$

Using index notation, all nine equations may be concisely expressed as $\mathbf{e}_i \bullet \mathbf{e}_j = \delta_{ij}$ for $1 \leqq i\, j \leqq 3$, where δ_{ij} is the Kronecker delta (0 if $i \neq j$, 1 if $i = j$). For example, at each point $\mathbf{p}$ of $\mathbf{R}^3$, the vectors $U_1(\mathbf{p})$, $U_2(\mathbf{p})$, $U_3(\mathbf{p})$ of Definition 2.4 in Chapter 1 constitute a frame at $\mathbf{p}$.

1.5 Theorem Let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ be a frame at a point $\mathbf{p}$ of $\mathbf{R}^3$. If $\mathbf{v}$ is any tangent vector to $\mathbf{R}^3$ at $\mathbf{p}$, then (Fig. 2.3)

$$\mathbf{v} = (\mathbf{v} \bullet \mathbf{e}_1)\mathbf{e}_1 + (\mathbf{v} \bullet \mathbf{e}_2)\mathbf{e}_2 + (\mathbf{v} \bullet \mathbf{e}_3)\mathbf{e}_3.$$

Proof. First we show that the vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are linearly independent. Suppose $\sum a_i \mathbf{e}_i = 0$. Then

$$0 = \left(\sum a_i \mathbf{e}_i\right) \bullet \mathbf{e}_j = \sum a_i \mathbf{e}_i \bullet \mathbf{e}_j = \sum a_i \delta_{ij} = a_j,$$

where all sums are over $i = 1, 2, 3$. Thus

$$a_1 = a_2 = a_3 = 0,$$

as required. Now, the tangent space $T_p(\mathbf{R}^3)$ has dimension 3, since it is linearly isomorphic to $\mathbf{R}^3$. Thus by a well-known theorem of linear algebra, the three independent vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ form a basis for $T_p(\mathbf{R}^3)$. Hence for each vector $\mathbf{v}$ there are three (unique) numbers c_1, c_2, c_3 such that

$$\mathbf{v} = \sum c_i \mathbf{e}_i.$$

But

$$\mathbf{v} \bullet \mathbf{e}_j = \left(\sum c_i \mathbf{e}_i\right) \bullet \mathbf{e}_j = \sum c_i \delta_{ij} = c_j,$$

and thus

$$\mathbf{v} = \sum (\mathbf{v} \bullet \mathbf{e}_j)\mathbf{e}_j. \qquad \blacklozenge$$

This result (valid in any inner-product space) is one of the great labor-saving devices in mathematics. For to find the coordinates of a vector **v** with respect to an *arbitrary* basis, one must in general solve a set of nonhomogeneous linear equations, a task that even in dimension 3 is not always entirely trivial. But the theorem shows that to find the coordinates of **v** with respect to a frame (that is, an *orthonormal* basis) it suffices merely to compute the three dot products $\mathbf{v} \bullet \mathbf{e}_1$, $\mathbf{v} \bullet \mathbf{e}_2$, $\mathbf{v} \bullet \mathbf{e}_3$. We call this process *orthonormal expansion* of **v** in terms of the frame $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$. In the special case of the natural frame $U_1(\mathbf{p})$, $U_2(\mathbf{p})$, $U_3(\mathbf{p})$, the identity

$$\mathbf{v} = (v_1, v_2, v_3) = \sum v_i U_i(\mathbf{p})$$

is an orthonormal expansion, and the dot product is defined in terms of these *Euclidean coordinates* by $\mathbf{v} \bullet \mathbf{w} = \sum v_i w_i$. If we use instead an arbitrary frame $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, then each vector **v** has new coordinates $a_i = \mathbf{v} \bullet \mathbf{e}_i$ relative to this frame, but *the dot product is still given by the same simple formula*

$$\mathbf{v} \bullet \mathbf{w} = \sum a_i b_i$$

since

$$\begin{aligned} \mathbf{v} \bullet \mathbf{w} &= \left(\sum a_i \mathbf{e}_i\right) \bullet \left(\sum b_i \mathbf{e}_i\right) = \sum_{i,j} a_i b_j \mathbf{e}_i \bullet \mathbf{e}_j \\ &= \sum_{i,j} a_i b_j \delta_{ij} = \sum a_i b_i. \end{aligned}$$

When applied to more complicated geometric situations, the advantage of using frames becomes enormous, and this is why they appear so frequently throughout this book.

The notion of frame is very close to that of orthogonal matrix.

1.6 Definition Let $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ be a frame at a point **p** of $\mathbf{R}^3$. The 3×3 matrix A whose rows are the Euclidean coordinates of these three vectors is called the *attitude matrix* of the frame.

Explicitly, if

$$\begin{aligned} \mathbf{e}_1 &= (a_{11}, a_{12}, a_{13})_p, \\ \mathbf{e}_2 &= (a_{21}, a_{22}, a_{23})_p, \\ \mathbf{e}_3 &= (a_{31}, a_{32}, a_{33})_p, \end{aligned}$$

then

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

Thus A does describe the "attitude" of the frame in $\mathbf{R}^3$, although not its point of application.

Evidently the rows of A are orthonormal, since

$$\sum_k a_{ik}a_{jk} = \mathbf{e}_i \bullet \mathbf{e}_j = \delta_{ij} \quad \text{for } 1 \leqq i,\ j \leqq 3.$$

By definition, this means that A is an *orthogonal* matrix.

In terms of matrix multiplication, these equations may be written $A\,{}^tA = I$, where I is the 3×3 identity matrix and ${}^tA$ is the *transpose* of A:

$${}^tA = \begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}.$$

It follows by a standard theorem of linear algebra that ${}^tAA = I$, so that ${}^tA = A^{-1}$, the *inverse* of A.

There is another product on $\mathbf{R}^3$, closely related to the wedge product of 1-forms and second in importance only to the dot product. We shall transfer it immediately to each tangent space of $\mathbf{R}^3$.

1.7 Definition If $\mathbf{v}$ and $\mathbf{w}$ are tangent vectors to $\mathbf{R}^3$ at the same point $\mathbf{p}$, then the *cross product* of $\mathbf{v}$ and $\mathbf{w}$ is the tangent vector

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} U_1(\mathbf{p}) & U_2(\mathbf{p}) & U_3(\mathbf{p}) \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

This formal determinant is to be expanded along its first row. For example, if $\mathbf{v} = (1, 0, -1)_p$ and $\mathbf{w} = (2, 2, -7)_p$, then

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} U_1(\mathbf{p}) & U_2(\mathbf{p}) & U_3(\mathbf{p}) \\ 1 & 0 & -1 \\ 2 & 2 & -7 \end{vmatrix}$$

$$= 2U_1(\mathbf{p}) + 5U_2(\mathbf{p}) + 2U_3(\mathbf{p}) = (2, 5, 2)_p.$$

Familiar properties of determinants show that the cross product $\mathbf{v} \times \mathbf{w}$ is *linear* in $\mathbf{v}$ and in $\mathbf{w}$, and satisfies the *alternation rule*

$$\mathbf{v} \times \mathbf{w} = -\mathbf{w} \times \mathbf{v}.$$

Hence, in particular, $\mathbf{v} \times \mathbf{v} = 0$. The geometric usefulness of the cross product is based mostly on this fact:

1.8 Lemma The cross product $\mathbf{v} \times \mathbf{w}$ is orthogonal to both $\mathbf{v}$ and $\mathbf{w}$, and has length such that

$$\| \mathbf{v} \times \mathbf{w} \|^2 = (\mathbf{v} \bullet \mathbf{v})(\mathbf{w} \bullet \mathbf{w}) - (\mathbf{v} \bullet \mathbf{w})^2.$$

Proof. Let $\mathbf{v} \times \mathbf{w} = \sum c_i U_i(\mathbf{p})$. Then the dot product $\mathbf{v} \bullet (\mathbf{v} \times \mathbf{w})$ is just $\sum v_i c_i$. But by the definition of cross product, the Euclidean coordinates c_1, c_2, c_3 of $\mathbf{v} \times \mathbf{w}$ are such that

$$\mathbf{v} \bullet (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} v_1 & v_2 & v_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

This determinant is zero, since two of its rows are the same; thus $\mathbf{v} \times \mathbf{w}$ is orthogonal to $\mathbf{v}$, and similarly, to $\mathbf{w}$.

Rather than use tricks to prove the length formula, we give a brute-force computation. Now,

$$\begin{aligned}(\mathbf{v} \bullet \mathbf{v})(\mathbf{w} \bullet \mathbf{w}) - (\mathbf{v} \bullet \mathbf{w})^2 &= \left(\sum v_i^2\right)\left(\sum w_j^2\right) - \left(\sum v_i w_i\right)^2 \\ &= \sum_{i,j} v_i^2 w_j^2 - \left\{\sum v_i^2 w_i^2 + 2\sum_{i<j} v_i w_i v_j w_j\right\} \\ &= \sum_{i \neq j} v_i^2 w_j^2 - 2\sum_{i<j} v_i w_i v_j w_j .\end{aligned}$$

On the other hand,

$$\begin{aligned}\| \mathbf{v} \times \mathbf{w} \|^2 &= (\mathbf{v} \times \mathbf{w}) \bullet (\mathbf{v} \times \mathbf{w}) = \sum c_i^2 \\ &= (v_2 w_3 - v_3 w_2)^2 + (v_3 w_1 - v_1 w_3)^2 + (v_1 w_2 - v_2 w_1)^2,\end{aligned}$$

and expanding these squares gives the same result as above. ◆

A more intuitive description of the length of a cross product is

$$\| \mathbf{v} \times \mathbf{w} \| = \| \mathbf{v} \| \| \mathbf{w} \| \sin \vartheta,$$

where $0 \leqq \vartheta \leqq \pi$ is the smaller of the two angles from $\mathbf{v}$ to $\mathbf{w}$. The direction of $\mathbf{v} \times \mathbf{w}$ on the line orthogonal to $\mathbf{v}$ and $\mathbf{w}$ is given, for practical purposes, by this "right-hand rule": If the fingers of the right hand point in the

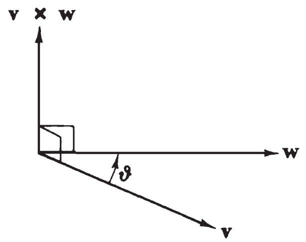


FIG. 2.4

direction of the shortest rotation of **v** to **w**, then the thumb points in the direction of $\mathbf{v} \times \mathbf{w}$ (Fig. 2.4).

Combining the dot and cross product, we get the *triple scalar product*, which assigns to any three vectors **u**, **v**, **w** the number $\mathbf{u} \bullet \mathbf{v} \times \mathbf{w}$ (Exercise 4). Parentheses are unnecessary: $\mathbf{u} \bullet (\mathbf{v} \times \mathbf{w})$ is the only possible meaning.

Exercises

1. Let $\mathbf{v} = (1, 2, -1)$ and $\mathbf{w} = (-1, 0, 3)$ be tangent vectors at a point of $\mathbf{R}^3$. Compute:

(a) $\mathbf{v} \bullet \mathbf{w}$.
(b) $\mathbf{v} \times \mathbf{w}$.
(c) $\mathbf{v}/\|\mathbf{v}\|$, $\mathbf{w}/\|\mathbf{w}\|$.
(d) $\|\mathbf{v} \times \mathbf{w}\|$.
(e) the cosine of the angle between **v** and **w**.

2. Prove that Euclidean distance has the properties
(a) $d(\mathbf{p}, \mathbf{q}) \geqq 0$; $d(\mathbf{p}, \mathbf{q}) = 0$ if and only if $\mathbf{p} = \mathbf{q}$,
(b) $d(\mathbf{p}, \mathbf{q}) = d(\mathbf{q}, \mathbf{p})$,
(c) $d(\mathbf{p}, \mathbf{q}) + d(\mathbf{q}, \mathbf{r}) \geqq d(\mathbf{p}, \mathbf{r})$, for any points **p**, **q**, **r** in $\mathbf{R}^3$.

3. Prove that the tangent vectors

$$\mathbf{e}_1 = \frac{(1, 2, 1)}{\sqrt{6}}, \quad \mathbf{e}_2 = \frac{(-2, 0, 2)}{\sqrt{8}}, \quad \mathbf{e}_3 = \frac{(1, -1, 1)}{\sqrt{3}}$$

constitute a frame. Express $\mathbf{v} = (6, 1, -1)$ as a linear combination of these vectors. (Check the result by direct computation.)

4. Let $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3)$, $\mathbf{w} = (w_1, w_2, w_3)$. Prove that

(a) $$\mathbf{u} \bullet \mathbf{v} \times \mathbf{w} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

(b) $\mathbf{u} \bullet \mathbf{v} \times \mathbf{w} \neq 0$ if and only if $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are linearly independent.
(c) If any two vectors in $\mathbf{u} \bullet \mathbf{v} \times \mathbf{w}$ are reversed, the product changes sign.
(d) $\mathbf{u} \bullet \mathbf{v} \times \mathbf{w} = \mathbf{u} \times \mathbf{v} \bullet \mathbf{w}$.

5. Prove that $\mathbf{v} \times \mathbf{w} \neq 0$ if and only if $\mathbf{v}$ and $\mathbf{w}$ are linearly independent, and show that $\| \mathbf{v} \times \mathbf{w} \|$ is the area of the parallelogram with sides $\mathbf{v}$ and $\mathbf{w}$.

6. If $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ is a frame, show that

$$\mathbf{e}_1 \bullet \mathbf{e}_2 \times \mathbf{e}_3 = \pm 1.$$

Deduce that any 3×3 orthogonal matrix has determinant ±1.

7. If $\mathbf{u}$ is a unit vector, then the *component* of $\mathbf{v}$ in the $\mathbf{u}$ direction is

$$(\mathbf{v} \bullet \mathbf{u})\mathbf{u} = \| \mathbf{v} \| \cos \vartheta \, \mathbf{u}.$$

Show that $\mathbf{v}$ has a unique expression $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$, where $\mathbf{v}_1 \bullet \mathbf{v}_2 = 0$ and $\mathbf{v}_1$ is the component of $\mathbf{v}$ in the $\mathbf{u}$ direction.

8. Prove: The volume of the parallelepiped with sides $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ is $\pm \mathbf{u} \bullet \mathbf{v} \times \mathbf{w}$ (Fig. 2.5). (*Hint:* Use the indicated unit vector $\mathbf{e} = \mathbf{v} \times \mathbf{w} / \| \mathbf{v} \times \mathbf{w} \|$.)

9. Prove, using ε-neighborhoods, that each of the following subsets of $\mathbf{R}^3$ is open:
(a) All points $\mathbf{p}$ such that $\| \mathbf{p} \| < 1$.
(b) All $\mathbf{p}$ such that $p_3 > 0$. (*Hint:* $| p_i - q_i | \leqq d(\mathbf{p}, \mathbf{q})$.)

10. In each case, let S be the set of all points $\mathbf{p}$ that satisfy the given condition. Describe S, and decide whether it is *open.*
(a) $p_1^2 + p_2^2 + p_3^2 = 1$.
(b) $p_3 \neq 0$.
(c) $p_1 = p_2 \neq p_3$.
(d) $p_1^2 + p_2^2 < 9$.

11. If f is a differentiable function on $\mathbf{R}^3$, show that the gradient

$$\nabla f = \sum \frac{\partial f}{\partial x_i} U_i$$

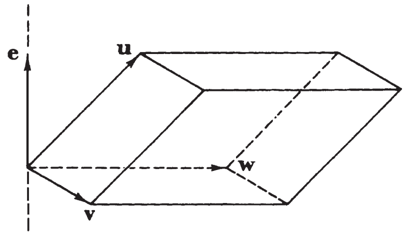


FIG. 2.5

(Ex. 8 of Sec. 1.6) has the following properties:

(a) $\mathbf{v}[f] = (df)(\mathbf{v}) = \mathbf{v} \bullet (\nabla f)(\mathbf{p})$ for any tangent vector at $\mathbf{p}$.

(b) The norm $\| (\nabla f)(\mathbf{p}) \| = \left[\sum (\partial f/\partial x_i)^2(\mathbf{p})\right]^{1/2}$ of (∇f) $(\mathbf{p})$ is the maximum of the directional derivatives $\mathbf{u}[f]$ for all *unit* vectors at $\mathbf{p}$. Furthermore, if $(\nabla f)(\mathbf{p}) \neq 0$, the unit vector for which the maximum occurs is

$$(\nabla f)(\mathbf{p})/\| (\nabla f)(\mathbf{p}) \|.$$

The notations grad f, curl V, and div V (in the exercise referred to) are often replaced by ∇f, $\nabla \times V$, and $\nabla \bullet V$, respectively.

12. *Angle functions.* Let f and g be differentiable real-valued functions on an interval I. Suppose that $f^2 + g^2 = 1$ and that ϑ_0 is a number such that $f(0) = \cos \vartheta_0$, $g(0) = \sin \vartheta_0$. If ϑ is the function such that

$$\vartheta(t) = \vartheta_0 + \int_0^t (fg' - gf')\,du,$$

prove that

$$f = \cos \vartheta, \quad g = \sin \vartheta.$$

Hint: We want $(f - \cos \vartheta)^2 + (g - \sin \vartheta)^2 = 0$, so show that its derivative is zero.

The point of this exercise is that ϑ is a differentiable function, unambiguously defined on the whole interval I.

2.2 Curves

We begin the geometric study of curves by reviewing some familiar definitions. Let $\alpha\colon I \to \mathbf{R}^3$ be a curve. In Chapter 1, Section 4, we defined the velocity vector $\alpha'(t)$ of α at t. Now we define the *speed* of α at t to be the length $v(t) = \| \alpha'(t) \|$ of the velocity vector. Thus speed is a real-valued function on the interval I. In terms of Euclidean coordinates $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, we have

$$\alpha'(t) = \left(\frac{d\alpha_1}{dt}(t), \frac{d\alpha_2}{dt}(t), \frac{d\alpha_3}{dt}(t) \right).$$

Hence the speed function v of α is given by the usual formula

$$v = \| \alpha' \| = \left(\left(\frac{d\alpha_1}{dt}\right)^2 + \left(\frac{d\alpha_2}{dt}\right)^2 + \left(\frac{d\alpha_3}{dt}\right)^2 \right)^{1/2}.$$

In physics, the distance traveled by a moving point is determined by integrating its speed with respect to time. Thus we define the *arc length* of α from $t = a$ to $t = b$ to be the number

$$\int_a^b \| \alpha'(t) \| \, dt.$$

Substituting the formula for $\| \alpha' \|$ given above, we get the usual formula for arc length. This length involves only the restriction of α (defined on some open interval) to the *closed* interval $[a, b]$: $a \leqq t \leqq b$. Such a restriction σ: $[a, b] \rightarrow \mathbf{R}^3$ is called a *curve segment*, and its length is denoted by $L(\sigma)$. Note that the velocity of σ is well defined at the endpoints a and b of $[a, b]$.

Sometimes one is interested only in the route followed by a curve and not in the particular speed at which it traverses its route. One way to ignore the speed of a curve α is to reparametrize to a curve β that has *unit speed* $\| \beta' \| = 1$. Then β represents a "standard trip" along the route of α.

2.1 Theorem If α is a regular curve in $\mathbf{R}^3$, then there exists a reparametrization β of α such that β has unit speed.

Proof. Fix a number a in the domain I of α: $I \rightarrow \mathbf{R}^3$, and consider the *arc length function*

$$s(t) = \int_a^b \| \alpha'(u) \| \, du.$$

(The resulting reparametrization is said to be *based at* $t = a$.) Thus the derivative ds/dt of the function $s = s(t)$ is the speed function $v = \| \alpha' \|$ of α. Since α is regular, by definition α' is never zero; hence $ds/dt > 0$. By a standard theorem of calculus, the function s has an inverse function $t = t(s)$, whose derivative dt/ds at $s = s(t)$ is the reciprocal of ds/dt at $t = t(s)$. In particular, $dt/ds > 0$.

Now let β be the reparametrization $\beta(s) = \alpha(t(s))$ of α. We assert that β has unit speed. In fact, by Lemma 4.5 of Chapter 1,

$$\beta'(s) = \frac{dt}{ds}(s)\alpha'(t(s)).$$

Hence, by the preceding remarks, the speed of β is

$$\| \beta'(s) \| = \frac{dt}{ds}(s) \| \alpha'(t(s)) \| = \frac{dt}{ds}(s)\frac{ds}{dt}(t(s)) = 1. \qquad \blacklozenge$$

We shall use the notation of this proof frequently in later work. The unit-speed curve β is sometimes said to have *arc-length parametrization*, since the arc length of β from $s = a$ to $s = b$ $(a < b)$ is just $b - a$.

For example, consider the helix α in Example 4.2 of Chapter 1. Since $\alpha(t) = (a\cos t, a\sin t, bt)$, the velocity α' is given by the formula

$$\alpha'(t) = (-a \sin t, a \cos t, b).$$

Hence

$$\| \alpha'(t) \|^2 = \alpha'(t) \bullet \alpha'(t) = a^2 \sin^2 t + a^2 \cos^2 t + b^2 = a^2 + b^2.$$

Thus α has *constant* speed $c = \| \alpha' \| = (a^2 + b^2)^{1/2}$. If we measure arc length from $t = 0$, then

$$s(t) = \int_0^t c\, du = ct.$$

Hence, $t(s) = s/c$. Substituting in the formula for α, we get the unit-speed reparametrization

$$\beta(s) = \alpha\left(\frac{s}{c}\right) = \left(a \cos \frac{s}{c}, a \sin \frac{s}{c}, \frac{bs}{c}\right).$$

It is easy to check directly that $\| \beta'(s) \| = 1$ for all s.

A reparametrization $\alpha(h)$ of a curve α is *orientation-preserving* if $h' \geq 0$ and *orientation-reversing* if $h' \leq 0$. In the latter case, $\alpha(h)$ still follows the route of α but in the opposite direction. By definition, a unit-speed reparametrization is always orientation-preserving since $ds/dt > 0$ for a regular curve.

In the *theory* of curves we will frequently reparametrize regular curves to obtain unit speed; however, it is rarely possible to do this in practice. The problem is basic calculus: Even when the coordinate functions of the curve are rather simple, the speed function cannot usually be integrated explicitly—at least in terms of familiar functions.

The general notion of vector field (Definition 2.3 of Chapter 1) can be adapted to curves as follows.

2.2 Definition A *vector field* Y *on curve* α: $I \to \mathbf{R}^3$ is a function that assigns to each number t in I a tangent vector $Y(t)$ to $\mathbf{R}^3$ at the point $\alpha(t)$.

We have already met such vector fields: For any curve α, its velocity α' evidently satisfies this definition. Note that unlike α', arbitrary vector fields on α need not be tangent to α, but may point in any direction (Fig. 2.6).

The properties of vector fields on curves are analogous to those of vector fields on $\mathbf{R}^3$. For example, if Y is a vector field on α: $I \to \mathbf{R}^3$, then for each t in I we can write

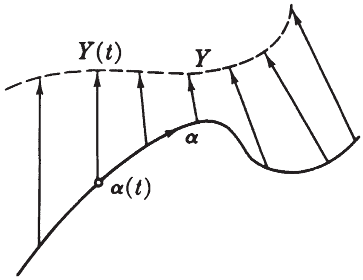


FIG. 2.6

$$Y(t) = (y_1(t), y_2(t), y_3(t))_{\alpha(t)} = \sum y_i(t)U_i(\alpha(t)).$$

We have thus defined real-valued functions y_1, y_2, y_3 on I, called the *Euclidean coordinate functions* of Y. These will always be assumed to be differentiable. Note that the composite function $t \to U_i(\alpha(t))$ is a vector field on α. Where it seems safe to do so, we shall often write merely U_i instead of $U_i(\alpha(t))$.

The operations of addition, scalar multiplication, dot product, and cross product of vector fields (on the same curve) are all defined in the usual pointwise fashion. Thus if

$$Y(t) = t^2U_1 - tU_3, \quad Z(t) = (1 - t^2)U_2 + tU_3,$$

and $f(t) = (t + 1)/t$, we obtain the vector fields

$$(Y + Z)(t) = t^2U_1 + (1 - t^2)U_2,$$

$$(fY)(t) = t(t + 1)U_1 - (t + 1)U_3,$$

$$(Y \times Z)(t) = \begin{vmatrix} U_1 & U_2 & U_3 \\ t^2 & 0 & -t \\ 0 & 1 - t^2 & t \end{vmatrix}$$

$$= t(1 - t^2)U_1 - t^3U_2 + t^2(1 - t^2)U_3$$

and the real-valued function

$$(Y \bullet Z)(t) = -t^2.$$

To differentiate a vector field on α *one simply differentiates its Euclidean coordinate functions*, thus obtaining a new vector field on α. Explicitly, if $Y = \sum y_iU_i$, then $Y' = \sum \frac{dy_i}{dt}U_i$. Thus, for Y as above, we get

$$Y' = 2tU_1 - U_3, \quad Y'' = 2U_1, \quad \text{and} \quad Y''' = 0.$$

In particular, the derivative α'' of the velocity α' of α is called the *acceleration* of α. Thus if $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, the acceleration α'' is the vector field

$$\alpha'' = \left(\frac{d^2\alpha_1}{dt^2}, \frac{d^2\alpha_2}{dt^2}, \frac{d^2\alpha_3}{dt^2}\right)_\alpha$$

on α. By contrast with velocity, acceleration is generally not tangent to the curve.

As we mentioned earlier, in whatever form it appears, differentiation always has suitable linearity and Leibnizian properties. In the case of vector fields on a curve, it is easy to prove the linearity property

$$(aY + bZ)' = aY' + bZ'$$

(a and b numbers) and the Leibnizian properties

$$(fY)' = \frac{df}{dt}Y + fY' \quad \text{and} \quad (Y \bullet Z)' = Y' \bullet Z + Y \bullet Z'.$$

If the function $Y \bullet Z$ is constant, the last formula shows that

$$Y' \bullet Z + Y \bullet Z' = 0.$$

This observation will be used frequently in later work. In particular, if Y has constant length $\| Y \|$, then Y and Y' are orthogonal at each point, since $\| Y \|^2 = Y \bullet Y$ constant implies $2Y \bullet Y' = 0$.

Recall that tangent vectors are parallel if they have the same vector parts. We say that a vector field Y on a curve is *parallel* provided all its (tangent vector) values are parallel. In this case, if the common vector part is (c_1, c_2, c_3), then

$$Y(t) = (c_1, c_2, c_3)_{\alpha(t)} = \sum c_i U_i \quad \text{for all } t.$$

Thus parallelism for a vector field is equivalent to the constancy of its Euclidean coordinate functions.

Vanishing of derivatives is always important in calculus; here are three simple cases.

2.3 Lemma (1) A curve α is constant if and only if its velocity is zero, $\alpha' = 0$.

(2) A nonconstant curve α is a straight line if and only if its acceleration is zero, $\alpha'' = 0$.

(3) A vector field Y on a curve is parallel if and only if its derivative is zero, $Y' = 0$.

Proof. In each case it suffices to look at the Euclidean coordinate functions. For example, we shall prove (2). If $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, then

$$\alpha'' = \left(\frac{d^2\alpha_1}{dt^2}, \frac{d^2\alpha_2}{dt^2}, \frac{d^2\alpha_3}{dt^2} \right).$$

Thus $\alpha'' = 0$ if and only if each $d^2\alpha_i/dt^2 = 0$. By elementary calculus, this is equivalent to the existence of constants p_i and q_i such that

$$\alpha_i(t) = p_i + tq_i, \quad \text{for } i = 1, 2, 3.$$

Thus $\alpha(t) = \mathbf{p} + t\mathbf{q}$, and α is a straight line as defined in Example 4.2 of Chapter 1. (Note that nonconstancy implies $\mathbf{q} \neq 0$.) ◆

Exercises

1. For the curve $\alpha(t) = (2t, t^2, t^3/3)$,
(a) find the velocity, speed, and acceleration for arbitrary t, and at $t = 1$;
(b) find the arc length function $s = s(t)$ (based at $t = 0$), and determine the arc length of α from $t = -1$ to $t = +1$.

2. Show that a curve has constant speed if and only if its acceleration is everywhere orthogonal to its velocity.

3. Show that the curve $\alpha(t) = (\cosh t, \sinh t, t)$ has arc length function $s(t) = \sqrt{2}\sinh t$, and find a unit-speed reparametrization of α.

4. Consider the curve $\alpha(t) = (2t, t^2, \log t)$ on I: $t > 0$. Show that this curve passes through the points $\mathbf{p} = (2, 1, 0)$ and $\mathbf{q} = (4, 4, \log 2)$, and find its arc length between these points.

5. Suppose that β_1 and β_2 are unit-speed reparametrizations of the same curve α. Show that there is a number s_0 such that $\beta_2(s) = \beta_1(s + s_0)$ for all s. What is the geometric significance of s_0?

6. Let Y be a vector field on the helix $\alpha(t) = (\cos t, \sin t, t)$. In each of the following cases, express Y in the form $\sum y_i U_i$:
(a) $Y(t)$ is the vector from $\alpha(t)$ to the origin of $\mathbf{R}^3$.
(b) $Y(t) = \alpha'(t) - \alpha''(t)$.
(c) $Y(t)$ has unit length and is orthogonal to both $\alpha'(t)$ and $\alpha''(t)$.
(d) $Y(t)$ is the vector from $\alpha(t)$ to $\alpha(t + \pi)$.

7. A reparametrization $\alpha(h)$: $[c, d] \to \mathbf{R}^3$ of a curve segment α: $[a, b] \to \mathbf{R}^3$ is *monotone* provided either

(i) $h' \geq 0$, $h(c) = a$, $h(d) = b$ or (ii) $h' \leqq 0$, $h(c) = b$, $h(d) = a$.

Prove that monotone reparametrization does not change arc length.

8. Let Y be a vector field on a curve α. If $\alpha(h)$ is a reparametrization of α, show that the reparametrization $Y(h)$ is a vector field on $\alpha(h)$, and prove the chain rule $Y(h)' = h' Y'(h)$.

9. (*Numerical integration.*) The curve segments

$$\alpha(t) = (\sin t,\ t^2 \cos t,\ \sin 2t), \quad \beta(t) = (t^2 \sin t, t^2, t^2(1 + \cos t)),$$

defined on $0 \leqq t \leqq \pi$, run from the origin 0 to $(0, \pi^2, 0)$. Which is shorter? (See Integration in the Appendix.)

10. Let $\alpha, \beta \colon I \to \mathbf{R}^3$ be curves such that $\alpha'(t)$ and $\beta'(t)$ are parallel (same Euclidean coordinates) at each t. Prove that α and β are *parallel* in the sense that there is a point $\mathbf{p}$ in $\mathbf{R}^3$ such that $\beta(t) = \alpha(t) + \mathbf{p}$ for all t.

11. Prove that a straight line is the shortest distance between two points in $\mathbf{R}^3$. Use the following scheme; let $\alpha\colon [a, b] \to \mathbf{R}^3$ be an arbitrary curve segment from $\mathbf{p} = \alpha(a)$ to $\mathbf{q} = \alpha(b)$. Let $\mathbf{u} = (\mathbf{q} - \mathbf{p})/\| \mathbf{q} - \mathbf{p} \|$.

(a) If σ is a straight line segment from $\mathbf{p}$ to $\mathbf{q}$, say

$$\sigma(t) = (1 - t)\mathbf{p} + t\mathbf{q} \quad (0 \leqq t \leqq 1),$$

show that $L(\sigma) = d(\mathbf{p}, \mathbf{q})$.

(b) From $\| \alpha' \| \geqq \alpha' \bullet \mathbf{u}$, deduce $L(\alpha) \geqq d(\mathbf{p}, \mathbf{q})$, where $L(\alpha)$ is the length of α and d is Euclidean distance.

(c) Furthermore, show that if $L(\alpha) = d(\mathbf{p}, \mathbf{q})$, then (but for parametrization) α is a straight line segment. (*Hint:* write $\alpha' = (\alpha' \bullet \mathbf{u})\mathbf{u} + Y$, where $Y \bullet \mathbf{u} = 0$.)

2.3 The Frenet Formulas

We now derive mathematical measurements of the turning and twisting of a curve in $\mathbf{R}^3$. Throughout this section we deal only with *unit-speed* curves; in the next we extend the results to arbitrary regular curves.

Let $\beta\colon I \to \mathbf{R}^3$ be a unit-speed curve, so $\| \beta'(s) \| = 1$ for each s in I. Then $T = \beta'$ is called the *unit tangent* vector field on β. Since T has constant length 1, its derivative $T' = \beta''$ measures the way the curve is turning in $\mathbf{R}^3$. We call T' the *curvature* vector field of β. Differentiation of $T \bullet T = 1$ gives $2T' \bullet T = 0$, so T' is always orthogonal to T, that is, *normal* to β.

The length of the curvature vector field T' gives a numerical measurement of the turning of β. The real-valued function κ such that $\kappa(s) = \| T'(s) \|$ for

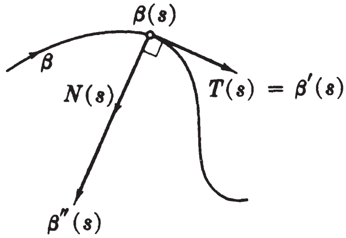


FIG. 2.7

all s in I is called the *curvature* function of β. Thus $\kappa \geqq 0$, and the larger κ is, the sharper the turning of β.

To carry this analysis further, we impose the restriction that κ is never zero so $\kappa > 0$. The unit-vector field $N = T'/\kappa$ on β then tells the *direction* in which β is turning at each point. N is called the *principal normal* vector field of β (Fig. 2.7). The vector field $B = T \times N$ on β is called the *binormal* vector field of β.

3.1 Lemma Let β be a unit-speed curve in $\mathbf{R}^3$ with $\kappa > 0$. Then the three vector fields T, N, and B on β are unit vector fields that are mutually orthogonal at each point. We call T, N, B the *Frenet frame field* on β.

Proof. By definition $\| T \| = 1$. Since $\kappa = \| T' \| > 0$,

$$\|N\| = (1/\kappa)\,\|T'\| = 1.$$

We saw above that T and N are orthogonal—that is, $T \bullet N = 0$. Then by applying Lemma 1.8 at each point, we conclude that $\| B \| = 1$, and B is orthogonal to both T and N. ◆

In summary, we have $T = \beta'$, $N = T'/\kappa$, and $B = T \times N$, satisfying $T \bullet T = N \bullet N = B \bullet B = 1$, with all other dot products zero.

The key to the successful study of the geometry of a curve β is to use its Frenet frame field T, N, B whenever possible, instead of the natural frame field U_1, U_2, U_3. The Frenet frame field of β is full of information about β, whereas the natural frame field contains none at all.

The first and most important use of this idea is to express the *derivatives* T', N', B' *in terms of* T, N, B. Since $T = \beta'$, we have $T' = \beta'' = \kappa N$. Next consider B'. We claim that B' is, at each point, a scalar multiple of N. To prove this, it suffices by orthonormal expansion to show that $B' \bullet B = 0$ and $B' \bullet T = 0$. The former holds since B is a unit vector. To prove the latter, differentiate $B \bullet T = 0$, obtaining $B' \bullet T + B \bullet T' = 0$; then

$$B' \bullet T = -B \bullet T' = -B \bullet \kappa N = 0.$$

Thus we can now define the *torsion* function τ of the curve β to be the real-valued function on the interval I such that $B' = -\tau N$. (The minus sign is traditional.) By contrast with curvature, there is no restriction on the values of τ—it may be positive, negative, or zero at various points of I. We shall presently show that τ does measure the torsion, or twisting, of the curve β.

3.2 Theorem (Frenet formulas). If $\beta\colon I \to \mathbf{R}^3$ is a unit-speed curve with curvature $\kappa > 0$ and torsion τ, then

$$\begin{aligned} T' &= \kappa N, \\ N' &= -\kappa T + \tau B, \\ B' &= -\tau N. \end{aligned}$$

Proof. As we saw above, the first and third formulas are essentially just the definitions of curvature and torsion. To prove the second, we use orthonormal expansion to express N' in terms of T, N, B:

$$N' = N' \bullet T\, T + N' \bullet N\, N + N' \bullet B\, B.$$

These coefficients are easily found. Differentiating $N \bullet T = 0$, we get $N' \bullet T + N \bullet T' = 0$; hence

$$N' \bullet T = -N \bullet T' = -N \bullet \kappa N = -\kappa.$$

As usual, $N' \bullet N = 0$, since N is a unit vector field. Finally,

$$N' \bullet B = -N \bullet B' = -N \bullet (-\tau N) = \tau. \qquad \blacklozenge$$

3.3 Example We compute the Frenet frame T, N, B and the curvature and torsion functions of the unit-speed helix

$$\beta(s) = \left(a \cos \frac{s}{c}, a \sin \frac{s}{c}, \frac{bs}{c}\right),$$

where $c = (a^2 + b^2)^{1/2}$ and $a > 0$. Now

$$T(s) = \beta'(s) = \left(-\frac{a}{c} \sin \frac{s}{c}, \frac{a}{c} \cos \frac{s}{c}, \frac{b}{c}\right).$$

Hence

$$T'(s) = \left(-\frac{a}{c^2} \cos \frac{s}{c}, -\frac{a}{c^2} \sin \frac{s}{c}, 0\right).$$

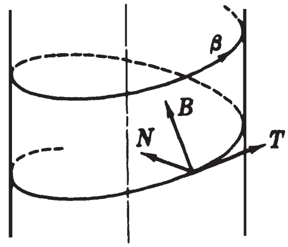


FIG. 2.8

Thus

$$\kappa(s) = \|T'(s)\| = \frac{a}{c^2} = \frac{a}{a^2 + b^2} > 0.$$

Since $T' = \kappa N$, we get

$$N(s) = \left(-\cos\frac{s}{c}, -\sin\frac{s}{c}, 0\right).$$

Note that regardless of what values a and b have, N always points straight in toward the axis of the cylinder on which β lies (Fig. 2.8).

Applying the definition of cross product to $B = T \times N$ gives

$$B(s) = \left(\frac{b}{c}\sin\frac{s}{c}, -\frac{b}{c}\cos\frac{s}{c}, \frac{a}{c}\right).$$

It remains to compute torsion. Now,

$$B'(s) = \left(\frac{b}{c^2}\cos\frac{s}{c}, \frac{b}{c^2}\sin\frac{s}{c}, 0\right),$$

and by definition, $B' = -\tau N$. Comparing the formulas for B' and N, we conclude that

$$\tau(s) = \frac{b}{c^2} = \frac{b}{a^2 + b^2}.$$

So the torsion of the helix is also constant.

Note that when the parameter b is zero, the helix reduces to a circle of radius a. The curvature of this circle is $\kappa = 1/a$ (so the smaller the radius, the larger the curvature), and the torsion is identically zero.

This example is a very special one—in general (as the examples in the exercises show) neither the curvature nor the torsion functions of a curve need be constant.

3.4 Remark We have emphasized all along the distinction between a tangent vector and a point of $\mathbf{R}^3$. However, Euclidean space has, as we have seen, the remarkable property that given a point $\mathbf{p}$, there is a natural one-to-one correspondence between points (v_1, v_2, v_3) and tangent vectors $(v_1, v_2, v_3)_p$ at $\mathbf{p}$. Thus one can transform points into tangent vectors (and vice versa) by means of this canonical isomorphism. In the next two sections particularly, it will often be convenient to switch quietly from one to the other without change of notation. Since *corresponding objects have the same Euclidean coordinates*, this switching can have no effect on scalar multiplication, addition, dot products, differentiation, or any other operation defined in terms of Euclidean coordinates.

Thus a vector field $Y = (y_1, y_2, y_3)_\beta$ on a curve β becomes itself a curve (y_1, y_2, y_3) in $\mathbf{R}^3$. In particular, if Y is parallel, its Euclidean coordinate functions are constant, so Y is identified with a single point of $\mathbf{R}^3$.

A *plane* in $\mathbf{R}^3$ can be described as the union of all the perpendiculars to a given line at a given point. In vector language then, the *plane through* $\mathbf{p}$ *orthogonal to* $\mathbf{q} \neq 0$ consists of all points $\mathbf{r}$ in $\mathbf{R}^3$ such that $(\mathbf{r} - \mathbf{p}) \bullet \mathbf{q} = 0$. By the remark above, we may picture $\mathbf{q}$ as a tangent vector at $\mathbf{p}$ as shown in Fig. 2.9.

We can now give an informative approximation of a given curve near an arbitrary point on the curve. The goal is to show how curvature and torsion influence the shape of the curve. To derive this approximation we use a Taylor approximation of the curve—and express this in terms of the Frenet frame at the selected point.

For simplicity, we shall consider the unit-speed curve $\beta = (\beta_1, \beta_2, \beta_3)$ near the point $\beta(0)$. For s small, each coordinate $\beta_i(s)$ is closely approximated by the initial terms of its Taylor series:

$$\beta_i(s) \sim \beta_i(0) + \frac{d\beta_i}{ds}(0)s + \frac{d^2\beta_i}{ds^2}(0)\frac{s^2}{2} + \frac{d^3\beta_i}{ds^3}(0)\frac{s^3}{6}.$$

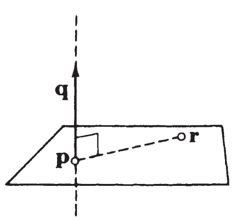


FIG. 2.9

Hence

$$\beta(s) \sim \beta(0) + s\beta'(0) + \frac{s^2}{2}\beta''(0) + \frac{s^3}{6}\beta'''(0).$$

But $\beta'(0) = T_0$, and $\beta''(0) = \kappa_0 N_0$, where the subscript indicates evaluation at $s = 0$, and we assume $\kappa_0 \neq 0$. Now

$$\beta''' = (\kappa N)' = \frac{d\kappa}{ds} N + \kappa N'.$$

Thus by the Frenet formula for N', we get

$$\beta'''(0) = -\kappa_0^2 T_0 + \frac{d\kappa}{ds}(0) N_0 + \kappa_0 \tau_0 B_0.$$

Finally, substitute these derivatives into the approximation of $\beta(s)$ given above, and keep only the dominant term in each component (that is, the one containing the smallest power of s). The result is

$$\beta(s) \sim \beta(0) + sT_0 + \kappa_0 \frac{s^2}{2} N_0 + \kappa_0 \tau_0 \frac{s^3}{6} B_0.$$

Denoting the right side by $\hat{\beta}(s)$, we obtain a curve $\hat{\beta}$ called the *Frenet approximation* of β near $s = 0$. We emphasize that β has a different Frenet approximation near each of its points; if 0 is replaced by an arbitrary number s_0, then s is replaced by $s - s_0$, as usual in Taylor expansions.

Let us now examine the Frenet approximation given above. The first term in the expression for $\hat{\beta}$ is just the point $\beta(0)$. The first two terms give the *tangent line* $s \to \beta(0) + sT_0$ of β at $\beta(0)$—the best linear approximation of β near $\beta(0)$. The first three terms give the parabola

$$s \to \beta(0) + sT_0 + \kappa_0 (s^2/2) N_0,$$

which is the best quadratic approximation of β near $\beta(0)$. Note that this parabola lies in the plane through $\beta(0)$ orthogonal to B_0, the *osculating plane* of β at $\beta(0)$. This parabola has the same shape as the parabola $y = \kappa_0 x^2/2$ in the xy plane, and is completely determined by the curvature κ_0 of β at $s = 0$.

Finally, the torsion τ_0, which appears in the last and smallest term of $\hat{\beta}$, controls the motion of β orthogonal to its osculating plane at $\beta(0)$, as shown in Fig. 2.10.

On the basis of this discussion, it is a reasonable guess that *if a unit-speed curve has curvature identically zero, then it is a straight line*. In fact, this follows immediately from (2) of Lemma 2.3, since $\kappa = \| T' \| = \| \beta'' \|$, so that $\kappa = 0$ if and only if $\beta'' = 0$. Thus curvature does measure deviation from straightness.

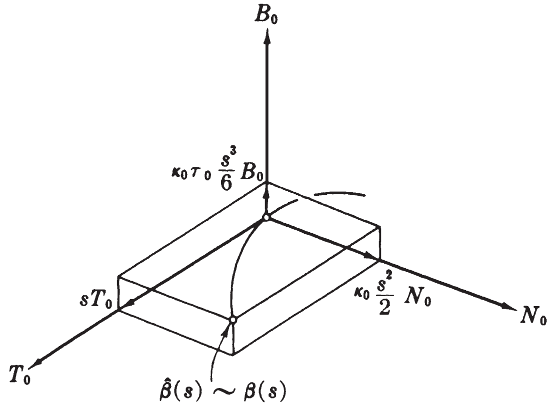


FIG. 2.10

A *plane curve* in $\mathbf{R}^3$ is a curve that lies in a single plane of $\mathbf{R}^3$. Evidently a plane curve does not twist in as interesting a way as even the simple helix in Example 3.3. The discussion above shows that for s small the curve β tends to stay in its osculating plane at $\beta(0)$; it is $\tau_0 \neq 0$ that causes β to twist out of the osculating plane. Thus if the torsion of β is identically zero, we may well suspect that β never leaves this plane.

3.5 Corollary Let β be a unit-speed curve in $\mathbf{R}^3$ with $\kappa > 0$. Then β is a plane curve if and only if $\tau = 0$.

Proof. Suppose β is a plane curve. Then by the remarks above, there exist points $\mathbf{p}$ and $\mathbf{q}$ such that $(\beta(s) - \mathbf{p}) \bullet \mathbf{q} = 0$ for all s. Differentiation yields

$$\beta'(s) \bullet \mathbf{q} = \beta''(s) \bullet \mathbf{q} = 0 \quad \text{for all } s.$$

Thus $\mathbf{q}$ is always orthogonal to $T = \beta'$ and $N = \beta''/\kappa$. But B is also orthogonal to T and N, so, since B has unit length, $B = \pm\mathbf{q}/\|\mathbf{q}\|$. Thus $B' = 0$, and by definition $\tau = 0$ (Fig. 2.11).

Conversely, suppose $\tau = 0$. Thus $B' = 0$; that is, B is parallel and may thus be identified (by Remark 3.4) with a *point* of $\mathbf{R}^3$. We assert that β lies in the plane through $\beta(0)$ orthogonal to B. To prove this, consider the real-valued function

$$f(s) = (\beta(s) - \beta(0)) \bullet B \quad \text{for all } s.$$

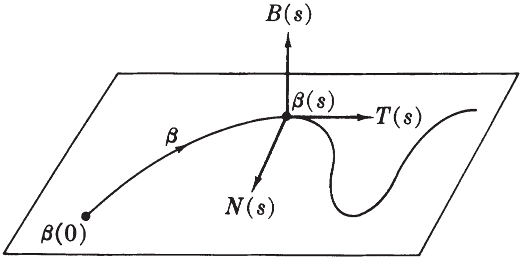


FIG. 2.11

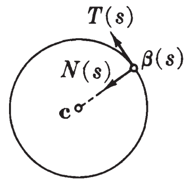


FIG. 2.12

Then

$$\frac{df}{ds} = \beta' \bullet B = T \bullet B = 0.$$

But obviously, $f(0) = 0$, so f is identically zero. Thus

$$(\beta(s) - \beta(0)) \bullet B \quad \text{for all } s,$$

which shows that β lies entirely in this plane orthogonal to the (parallel) binormal of β. ◆

We saw at the end of Example 3.3 that a circle of radius a has curvature $1/a$ and torsion zero. Furthermore, the formula given there for the principal normal shows that for a circle, N always points toward its center. This suggests how to prove the following converse.

3.6 Lemma If β is a unit-speed curve with constant curvature $\kappa > 0$ and torsion zero, then β is part of a circle of radius $1/\kappa$.

Proof. Since $\tau = 0$, β is a plane curve. What we must now show is that every point of β is at distance $1/\kappa$ from some fixed point—which will thus be the center of the circle. Consider the curve $\gamma = \beta + (1/\kappa)N$. Using the hypothesis on β, and (as usual) a Frenet formula, we find

$$\gamma' = \beta' + \frac{1}{\kappa}N' = T + \frac{1}{\kappa}(-\kappa T) = 0.$$

Hence the curve γ is constant; that is, $\beta(s) + (1/\kappa)N(s)$ has the same value, say $\mathbf{c}$, for all s (see Fig. 2.12). But the distance from $\mathbf{c}$ to $\beta(s)$ is

$$d(\mathbf{c}, \beta(s)) = \|\mathbf{c} - \beta(s)\| = \left\|\frac{1}{\kappa}N(s)\right\| = \frac{1}{\kappa}. \quad ◆$$

In principle, every geometric problem about curves can be solved by means of the Frenet formulas. In simple cases it may be just enough to record the data of the problem in convenient form, differentiate, and use the Frenet formulas. For example, suppose β is a unit-speed curve that lies entirely in the sphere $\sum$ of radius a centered at the origin of $\mathbf{R}^3$. To stay in the sphere, β must curve; in fact it is a reasonable guess that the minimum possible curvature occurs when β is on a great circle of $\sum$. Such a circle has radius a, so we conjecture that *a spherical curve* β *has curvature* $\kappa \geqq 1/a$, *where* a *is the radius of its sphere*.

To prove this, observe that since every point of $\sum$ has distance a from the origin, we have $\beta \bullet \beta = a^2$. Differentiation yields $2\beta' \bullet \beta = 0$, that is, $\beta \bullet T = 0$. Another differentiation gives $\beta' \bullet T + \beta \bullet T' = 0$, and by using a Frenet formula we get $T \bullet T + \kappa\beta \bullet N = 0$; hence

$$\kappa\beta \bullet N = -1.$$

By the Schwarz inequality,

$$|\beta \bullet N| \leqq \|\beta\| \, \|N\| = a,$$

and since $\kappa \geqq 0$ we obtain the required result:

$$\kappa = |\kappa| = \frac{1}{|\beta \bullet N|} \geqq \frac{1}{a}.$$

Continuation of this procedure leads to a necessary and sufficient condition (expressed in terms of curvature and torsion) for a curve to be *spherical*, that is, lie on some sphere in $\mathbf{R}^3$ (Exercise 10).

Exercises

1. Compute the *Frenet apparatus* κ, τ, T, N, B of the unit-speed curve $\beta(s) = (\frac{4}{5}\cos s,\ 1 - \sin s,\ -\frac{3}{5}\cos s)$. Show that this curve is a circle; find its center and radius.

2. Consider the curve

$$\beta(s) = \left(\frac{(1+s)^{3/2}}{3}, \frac{(1-s)^{3/2}}{3}, \frac{s}{\sqrt{2}} \right)$$

defined on I: $-1 < s < 1$. Show that β has unit speed, and compute its Frenet apparatus.

3. For the helix in Example 3.3, check the Frenet formulas by direct substitution of the computed values of κ, τ, T, N, B.

4. Prove that

$$\begin{aligned} T &= N \times B = -B \times N, \\ N &= B \times T = -T \times B, \\ B &= T \times N = -N \times T. \end{aligned}$$

(A formal proof uses properties of the cross product established in the Exercises of Section 1—but one can recall these formulas by using the right-hand rule given at the end of that section.)

5. If A is the vector field $\tau T + \kappa B$ on a unit-speed curve β, show that the Frenet formulas become

$$\begin{aligned} T' &= A \times T, \\ N' &= A \times N, \\ B' &= A \times B. \end{aligned}$$

6. A unit-speed parametrization of a circle may be written

$$\gamma(s) = \mathbf{c} + r \cos \frac{s}{r}\, \mathbf{e}_1 + r \sin \frac{s}{r}\, \mathbf{e}_2,$$

where $\mathbf{e}_i \bullet \mathbf{e}_j = \delta_{ij}$.

If β is a unit-speed curve with $\kappa(0) > 0$, prove that there is one and only one circle γ that approximates β near $\beta(0)$ in the sense that

$$\gamma(0) = \beta(0), \quad \gamma'(0) = \beta'(0), \quad \text{and} \quad \gamma''(0) = \beta''(0).$$

Show that γ lies in the osculating plane of β at $\beta(0)$ and find its center $\mathbf{c}$ and radius r (see Fig. 2.13). The circle γ is called the *osculating circle* and $\mathbf{c}$ the *center of curvature* of β at $\beta(0)$. (The same results hold when 0 is replaced by any number s.)

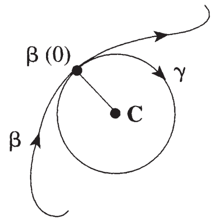


FIG. 2.13

7. If α and a reparametrization $\overline{\alpha} = \alpha(h)$ are both unit-speed curves, show that

(a) $h(s) = \pm s + s_0$ for some number s_0;

(b) $\overline{T} = \pm T(h)$,

$\overline{N} = N(h), \quad \overline{\kappa} = \kappa(h), \quad \overline{\tau} = \tau(h)$,

$\overline{B} = \pm B(h)$,

where the sign ($\pm$) is the same as that in (a), and we assume $\kappa > 0$. Thus even in the orientation-reversing case, the principal normals N and $\overline{N}$ still point in the same direction.

8. *Curves in the plane.* For a unit-speed curve $\beta(s) = (x(s), y(s))$ in $\mathbf{R}^2$, the *unit tangent* is $T = \beta' = (x', y')$ as usual, but the *unit normal* N is defined by rotating T through $+90°$, so $N = (-y', x')$. Thus T' *and* N are collinear, and the *plane curvature* $\tilde{\kappa}$ of β is defined by the Frenet equation $T' = \tilde{\kappa}N$.

(a) Prove that $\tilde{\kappa} = T' \bullet N$ and $N' = -\tilde{\kappa}T$.

(b) The *slope angle* $\varphi(s)$ of β is the differentiable function such that

$$T = (\cos\varphi, \sin\varphi) = \cos\varphi\, U_x + \sin\varphi\, U_y.$$

(The existence of φ derives from Ex. 12 of Sec. 1.) Show that $\tilde{\kappa} = \varphi'$.

(c) Find the curvature $\tilde{\kappa}$ of the following plane curves.

(i) $(r\cos\frac{t}{r}, r\sin\frac{t}{r})$, counterclockwise circle.

(ii) $(r\cos(-\frac{t}{r}), r\sin(-\frac{t}{r}))$, clockwise circle.

(d) Show that if $\tilde{\kappa}$ does not change sign, then $|\tilde{\kappa}|$ is the usual $\mathbf{R}^3$ curvature κ. (For such comparisons we can always regard $\mathbf{R}^2$ as, say, the xy plane in $\mathbf{R}^3$.)

9. Let $\tilde{\beta}$ be the Frenet approximation of a unit-speed curve β with $\tau \neq 0$ near $s = 0$.

If, say, the B_0 component of $\tilde{\beta}$ is removed, the resulting curve is the *orthogonal projection* of $\tilde{\beta}$ in the T_0N_0 plane. It is the view of $\beta \approx \tilde{\beta}$ that one gets by looking toward $\beta(0) = \tilde{\beta}(0)$ directly along the vector B_0.

Sketch the general shape of the orthogonal projections of $\tilde{\beta}$ near $s = 0$ in each of the planes T_0N_0 (*osculating plane*), T_0B_0 (*rectifying plane*), and N_0B_0 (*normal plane*). These views of $\beta \approx \tilde{\beta}$ can be confirmed experimentally using a bent piece of wire. For computer views, see Exercise 15 of Section 4.

10. *Spherical curves.* Let α be a unit-speed curve with $\kappa > 0$, $\tau \neq 0$.

(a) If α lies on a sphere of center $\mathbf{c}$ and radius r, show that

$$\alpha - \mathbf{c} = -\rho N - \rho'\sigma B,$$

where $\rho = 1/\kappa$ and $\sigma = 1/\tau$. Thus $r^2 = \rho^2 + (\rho'\sigma)^2$.

(b) Conversely, if $\rho^2 + (\rho'\sigma)^2$ has constant value r^2 and $\rho' \neq 0$, show that α lies on a sphere of radius r.
(*Hint:* For (b), show that the "center curve" $\gamma = \alpha + \rho N + \rho'\sigma B$—suggested by (a)—is constant.)

11. Let β, $\bar{\beta}$: $I \to \mathbf{R}^3$ be unit-speed curves with nonvanishing curvature and torsion. If $T = \bar{T}$, then β and $\bar{\beta}$ are parallel (Ex. 10 of Sec. 2). If $B = \bar{B}$, prove that $\bar{\beta}$ is parallel to either β or the curve $s \to -\beta(s)$.

2.4 Arbitrary-Speed Curves

It is a simple matter to adapt the results of the previous section to the study of a regular curve α: $I \to \mathbf{R}^3$ that does not necessarily have unit speed. We merely transfer to α the Frenet apparatus of a unit-speed reparametrization $\bar{\alpha}$ of α. Explicitly, if s is an arc length function for α as in Theorem 2.1, then

$$\alpha(t) = \bar{\alpha}(s(t)) \quad \text{for all } t,$$

or, in functional notation, $\alpha = \bar{\alpha}(s)$, as suggested by Fig. 2.14. Now if $\bar{\kappa} > 0$, $\bar{\tau}$, $\bar{T}$, $\bar{N}$, and $\bar{B}$ are defined for $\bar{\alpha}$ as in Section 3, we define for α the

curvature function: $\kappa = \bar{\kappa}(s)$,
torsion function: $\tau = \bar{\tau}(s)$,
unit tangent vector field: $T = \bar{T}(s)$,
principal normal vector field: $N = \bar{N}(s)$,
binormal vector field: $B = \bar{B}(s)$.

In general κ and $\bar{\kappa}$ are different functions, defined on different intervals. But *they give exactly the same description of the turning of the common route of α and $\bar{\alpha}$*, since at any point $\alpha(t) = \bar{\alpha}(s(t))$ the numbers $\kappa(t)$ and $\bar{\kappa}(s(t))$ are

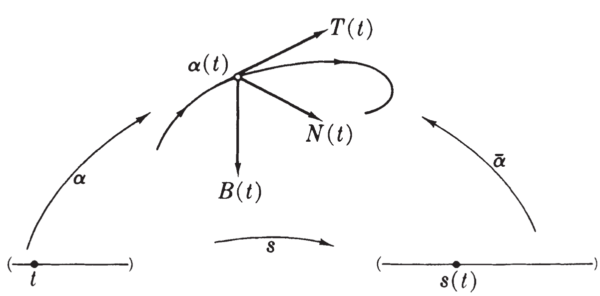


FIG. 2.14

by definition the same. Similarly with the rest of the Frenet apparatus; since only a change of parametrization is involved, its fundamental geometric meaning is the same as before. In particular, T, N, B is again a frame field on α linked to the shape of α as indicated in the discussion of Frenet approximations.

For purely theoretical work, this simple transference is often all that is needed. Data about α converts into data about the unit-speed reparametrization $\bar{\alpha}$; results about $\bar{\alpha}$ convert to results about α. For example, if α is a regular curve with $\tau = 0$, then by the definition above $\bar{\alpha}$ has $\bar{\tau} = 0$; by Corollary 3.5, $\bar{\alpha}$ is a plane curve, so obviously α is too.

However, for explicit numerical computations—and occasionally for the theory as well—this transference is impractical, since it is rarely possible to find explicit formulas for $\bar{\alpha}$. (For example, try to find a unit-speed parametrization for the curve $\alpha(t) = (t, t^2, t^3)$.)

The Frenet formulas are valid only for unit-speed curves; they tell the rate of change of the frame field *T, N, B with respect to arc length.* However, the speed v of the curve is the proper correction factor in the general case.

4.1 Lemma If α is a regular curve in $\mathbf{R}^3$ with $\kappa > 0$, then

$$\begin{aligned} T' &= \qquad \kappa v N, \\ N' &= -\kappa v T \qquad\qquad + \tau v B, \\ B' &= \qquad -\tau v N. \end{aligned}$$

Proof. Let $\bar{\alpha}$ be a unit-speed reparametrization of α. Then by definition, $T = \bar{T}(s)$, where s is an arc length function for α. The chain rule as applied to differentiation of vector fields (Exercise 7 of Section 2) gives

$$T' = \bar{T}'(s)\frac{ds}{dt}.$$

By the usual Frenet equations, $\bar{T}' = \bar{\kappa}\bar{N}$. Substituting the function s in this equation yields

$$\bar{T}'(s) = \bar{\kappa}(s)\bar{N}(s) = \kappa N$$

by the definition of κ and N in the arbitrary-speed case. Since ds/dt is the speed function v of α, these two equations combine to yield $T' = \kappa v N$. The formulas for N' and B' are derived in the same way. ◆

There is a commonly used notation for the calculus that completely ignores change of parametrization. For example, the same letter would designate both a curve α and its unit-speed parametrization $\bar{\alpha}$, and similarly with the

Frenet apparatus of these two curves. Differences in derivatives are handled by writing, say, dT/dt for T', but dT/ds for either $\bar{T}'$ or its reparametrization $\bar{T}'(s)$. If these conventions were used, the proof above would combine the chain rule $dT/dt = (dT/ds)\,(ds/dt)$ and the Frenet formula $dT/ds = \kappa N$ to give $dT/dt = \kappa v N$.

Only for a *constant-speed* curve is acceleration always orthogonal to velocity, since $\beta' \bullet \beta'$ constant is equivalent to $(\beta' \bullet \beta')' = 2\beta' \bullet \beta'' = 0$. In the general case, we analyze velocity and acceleration by expressing them in terms of the Frenet frame field.

4.2 Lemma If α is a regular curve with speed function v, then the velocity and acceleration of α are given by (Fig. 2.15.)

$$\alpha' = vT, \quad \alpha'' = \frac{dv}{dt}T + \kappa v^2 N.$$

Proof. Since $\alpha = \bar{\alpha}(s)$, where s is the arc length function of α, we find, using Lemma 4.5 of Chapter 1, that

$$\alpha' = \bar{\alpha}'(s)\frac{ds}{dt} = v\bar{T}(s) = vT.$$

Then a second differentiation yields

$$\alpha'' = \frac{dv}{dt}T + vT' = \frac{dv}{dt}T + \kappa v^2 N,$$

where we use Lemma 4.1. ◆

The formula $\alpha' = vT$ is to be expected since α' and T are each tangent to the curve and T has a unit length, while $\|\alpha'\| = v$. The formula for acceleration is more interesting. By definition, α'' is the rate of change of the

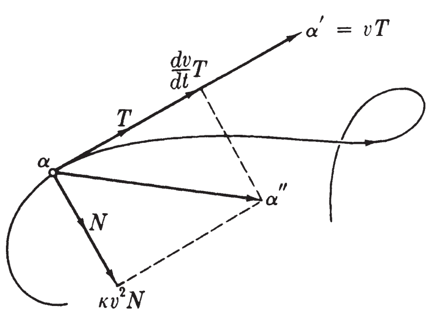


FIG. 2.15

velocity α', and in general both the length and the direction of α' are changing. The *tangential component* $(dv/dt)T$ of α'' measures the rate of change of the length of α' (that is, of the speed of α). The *normal component* $\kappa v^2 N$ measures the rate of change of the direction of α'. Newton's laws of motion show that these components may be experienced as forces. For example, in a car that is speeding up or slowing down on a straight road, the only force one feels is due to $(dv/dt)T$. If one takes an unbanked curve at speed v, the resulting sideways force is due to $\kappa v^2 N$. Here κ measures how sharply the *road* turns; the effect of speed is given by v^2, so 60 miles per hour is four times as unsettling as 30.

We now find effectively computable expressions for the Frenet apparatus.

4.3 Theorem Let α be a regular curve in $\mathbf{R}^3$. Then

$$T = \alpha'/\|\alpha'\|,$$

$$N = B \times T, \qquad \kappa = \|\alpha' \times \alpha''\| / \|\alpha'\|^3,$$
$$B = \alpha' \times \alpha'' / \|\alpha' \times \alpha''\|, \quad \tau = (\alpha' \times \alpha'') \bullet \alpha''' / \|\alpha' \times \alpha''\|^2.$$

Proof. Since $v = \|\alpha'\| > 0$, the formula $T = \alpha'/\|\alpha'\|$ is equivalent to $\alpha' = vT$. From the preceding lemma we get

$$\alpha' \times \alpha'' = (vT) \times \left(\frac{dv}{dt}T + \kappa v^2 N\right)$$

$$= v\frac{dv}{dt}T \times T + \kappa v^3 T \times N = \kappa v^3 B,$$

since $T \times T = 0$. Taking norms we find

$$\|\alpha' \times \alpha''\| = \|\kappa v^3 B\| = \kappa v^3$$

because $\|B\| = 1$, $\kappa \geqq 0$, and $v > 0$. Indeed, *this equation shows that for* regular *curves*, $\|\alpha' \times \alpha''\| > 0$ *is equivalent to the usual condition* $\kappa > 0$. (Thus for $\kappa > 0$, α' and α'' are linearly independent and determine the osculating plane at each point, as do T and N.) Then

$$B = \frac{\alpha' \times \alpha''}{\kappa v^3} = \frac{\alpha' \times \alpha''}{\|\alpha' \times \alpha''\|}.$$

Since $N = B \times T$ is true for any Frenet frame field (Exercise 4 of Section 3), only the formula for torsion remains to be proved.

To find the dot product $(\alpha' \times \alpha'') \bullet \alpha'''$ we express everything in terms of T, N, B. We already know that $\alpha' \times \alpha'' = \kappa v^3 B$. Thus, since $0 = T \bullet B = N \bullet B$, *we need only find the B component of* α'''. But

$$\alpha''' = \left(\frac{dv}{dt}T + \kappa v^2 N\right)' = \kappa v^2 N' + \cdots$$
$$= \kappa v^3 \tau B + \cdots,$$

where we use Lemma 4.1. Consequently, $(\alpha' \times \alpha'') \bullet \alpha''' = \kappa^2 v^6 \tau$, and since $\| \alpha' \times \alpha'' \| = \kappa v^3$, we have the required formula for τ. ◆

The triple scalar product in this formula for τ could (by Exercise 4 of Section 1) also be written $\alpha' \bullet \alpha'' \times \alpha'''$. But we need $\alpha' \times \alpha''$ anyway, so it is more efficient to find $(\alpha' \times \alpha'') \bullet \alpha'''$.

4.4 Example We compute the Frenet apparatus of the 3-*curve*

$$\alpha(t) = (3t - t^3, 3t^2, 3t + t^3).$$

The derivatives are

$$\alpha'(t) = 3(1 - t^2, 2t, 1 + t^2),$$
$$\alpha''(t) = 6(-t, 1, t),$$
$$\alpha'''(t) = 6(-1, 0, 1).$$

Now,

$$\alpha'(t) \bullet \alpha'(t) = 18(1 + 2t^2 + t^4),$$

so

$$v(t) = \| \alpha'(t) \| = \sqrt{18}(1 + t^2).$$

Applying the definition of cross product yields

$$\alpha'(t) \times \alpha''(t) = 18\begin{vmatrix} U_1 & U_2 & U_3 \\ 1 - t^2 & 2t & 1 + t^2 \\ -t & 1 & t \end{vmatrix} = 18(-1 + t^2, -2t, 1 + t^2).$$

Dotting this vector with itself, we get

$$(18)^2\left[(-1 + t^2)^2 + 4t^2 + (1 + t^2)^2\right] = 2(18)^2(1 + t^2)^2.$$

Hence

$$\| \alpha'(t) \times \alpha''(t) \| = 18\sqrt{2}(1 + t^2).$$

The expressions above for $\alpha' \times \alpha''$ and α''' yield

$$(\alpha' \times \alpha'') \bullet \alpha''' = 6 \bullet 18 \bullet 2.$$

It remains only to substitute this data into the formulas in Theorem 4.3, with N being computed by another cross product. The final results are

$$T = \frac{(1-t^2, 2t, 1+t^2)}{\sqrt{2}(1+t^2)},$$

$$N = \frac{(-2t, 1-t^2, 0)}{1+t^2},$$

$$B = \frac{(-1+t^2, -2t, 1+t^2)}{\sqrt{2}(1+t^2)},$$

$$\kappa = \tau = \frac{1}{3(1+t^2)^2}.$$

Alternatively, we could use the identity in Lemma 1.8 to compute $\| \alpha' \times \alpha'' \|$ and express

$$(\alpha' \times \alpha'') \bullet \alpha''' = \alpha' \bullet (\alpha'' \times \alpha''')$$

as a determinant by Exercise 4 of Section 1.

To summarize, we now have the Frenet apparatus for an arbitrary regular curve α, namely, its curvature, torsion, and Frenet frame field. This apparatus satisfies the extended Frenet formulas with speed factor v and can be computed by Theorem 4.3. If $v = 1$, that is, if α is a unit-speed curve, the results of Section 3 are recovered.

Let us consider some applications of the Frenet formulas. There are a number of natural ways in which a given curve β gives rise to a new curve $\tilde{\beta}$ whose geometric properties illuminate some aspect of the behavior of β.

For example, the *spherical image* of a unit-speed curve β is the curve $\sigma \approx T$ with the same Euclidean coordinates as $T = \beta'$. Geometrically, σ is gotten by moving each $T(s)$ to the origin of $\mathbf{R}^3$, as suggested in Fig. 2.16. Thus σ lies on the unit sphere Σ, and the *motion* of σ represents the *turning* of β.

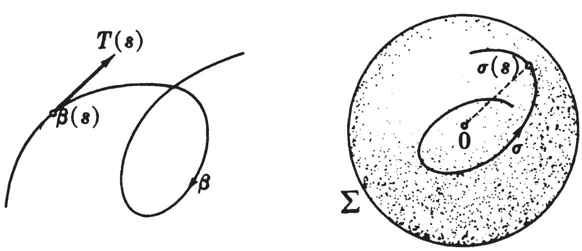


FIG. 2.16

For instance, if β is the helix in Example 3.3, the formula there for T shows that

$$\sigma(s) = \left(-\frac{a}{c}\sin\frac{s}{c}, \frac{a}{c}\cos\frac{s}{c}, \frac{b}{c}\right).$$

So the spherical image of a helix lies on the circle cut from $\sum$ by the plane $z = b/c$.

Although the original curve β has unit speed, we cannot expect that σ does also. In fact, $\sigma = T$ implies $\sigma' = T' = \kappa N$, so the *speed* of σ equals the *curvature* κ of β. Thus to compute the curvature of σ, we must use the extended Frenet formulas in Theorem 4.3. From

$$\sigma'' = (\kappa N)' = \frac{d\kappa}{ds}N + \kappa N' = -\kappa^2 T + \frac{d\kappa}{ds}N + \kappa\tau B,$$

we get

$$\sigma' \times \sigma'' = -\kappa^3 N \times T + \kappa^2\tau N \times B = \kappa^2(\kappa B + \tau T).$$

By Theorem 4.3 the curvature of the spherical image σ is

$$\kappa_\sigma = \frac{\|\sigma' \times \sigma''\|}{v^3} = \frac{\sqrt{\kappa^2 + \tau^2}}{\kappa} = \left(1 + \left(\frac{\tau}{\kappa}\right)^2\right)^{1/2} \geq 1$$

and thus depends only on the ratio of torsion to curvature for the original curve β.

Here is a closely related application in which this ratio τ/κ turns out to be decisive.

4.5 Definition A regular curve α in $\mathbf{R}^3$ is a *cylindrical helix* provided the unit tangent T of α has constant angle ϑ with some fixed unit vector $\mathbf{u}$; that is, $T(t) \bullet \mathbf{u} = \cos\vartheta$ for all t.

This condition is not altered by reparametrization, so for theoretical purposes we need only deal with a cylindrical helix β that has unit speed. So suppose β is a unit-speed curve with $T \bullet \mathbf{u} = \cos\vartheta$. If we pick a reference point, say $\beta(0)$, on β, then the real-valued function

$$h(s) = (\beta(s) - \beta(0)) \bullet \mathbf{u}$$

tells how far $\beta(s)$ has "risen" in the $\mathbf{u}$ direction since leaving $\beta(0)$ (Fig. 2.17). But

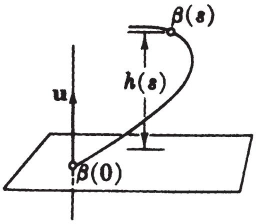


FIG. 2.17

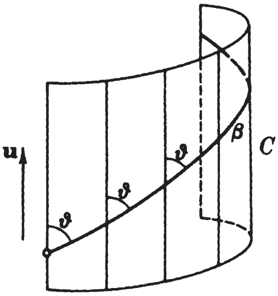


FIG. 2.18

$$\frac{dh}{ds} = \beta' \bullet \mathbf{u} = T \bullet \mathbf{u} = \cos \vartheta,$$

so β is rising at a constant rate *relative to arc length*, and $h(s) = s \cos \vartheta$. If we shift to an arbitrary parametrization, this formula becomes

$$h(t) = s(t) \cos \vartheta,$$

where s is the arc length function.

By drawing a line through each point of β in the $\mathbf{u}$ direction, we construct a cylinder C on which β moves in such a way as to cut each such line at constant angle ϑ, as in Fig. 2.18. In the special case when this cylinder is circular, β is evidently a helix of the type defined in Example 3.3.

It turns out to be quite easy to identify cylindrical helices.

4.6 Theorem A regular curve α with $\kappa > 0$ is a cylindrical helix if and only if the ratio τ/κ is constant.

Proof. It suffices to consider the case where α has unit speed. If α is a cylindrical helix with $T \bullet \mathbf{u} = \cos \vartheta$, then

$$0 = (T \bullet \mathbf{u})' = T' \bullet \mathbf{u} = \kappa N \bullet \mathbf{u}.$$

Since $\kappa > 0$, we conclude that $N \bullet \mathbf{u} = 0$. Thus for each s, $\mathbf{u}$ lies in the plane determined by $T(s)$ and $B(s)$. Orthonormal expansion yields

$$\mathbf{u} = \cos \vartheta \, T + \sin \vartheta \, B.$$

As usual we differentiate and apply Frenet formulas to obtain

$$0 = (\kappa \cos \vartheta - \tau \sin \vartheta)N.$$

Hence $\tau \sin \vartheta = \kappa \cos \vartheta$, so that τ/κ has constant value $\cot \vartheta$.

Conversely, suppose that τ/κ is constant. Choose an angle ϑ such that $\cot \vartheta = \tau/\kappa$. If

$$U = \cos \vartheta \, T + \sin \vartheta \, B,$$

we find

$$U' = (\kappa \cos \vartheta - \tau \sin \vartheta)N = 0.$$

This parallel vector field U then determines (as in Remark 3.4) a unit vector $\mathbf{u}$ such that $T \bullet \mathbf{u} = \cos \vartheta$, so α is a cylindrical helix. ◆

In Exercise 9 this information about cylindrical helices is used to show that *circular* helices are characterized by constancy of curvature and torsion (see also Corollary 5.5 of Chapter 3).

Simple hypotheses on a regular curve in $\mathbf{R}^3$ thus have the following effects ($\Leftrightarrow$ means "if and only if"):

$\kappa = 0$	$\Leftrightarrow$	straight line,
$\tau = 0$	$\Leftrightarrow$	plane curve,
κ const > 0 and $\tau = 0$	$\Leftrightarrow$	circle,
κ const > 0 and τ const > 0	$\Leftrightarrow$	circular helix,
τ/κ const $\neq 0$	$\Leftrightarrow$	cylindrical helix.

Exercises

Computer commands that produce the Frenet apparatus, κ, τ, T, N, B, of a curve are given in the Appendix. Their use is optional in the following exercises.

1. For the curve $\alpha(t) = (2t, t^2, t^3/3)$,

(a) Compute the Frenet apparatus.

(b) Sketch the curve for $-4 \leqq t \leqq 4$, showing T, N, B at $t = 2$.

(c) Find the limiting values of T, N, and B as $t \to -\infty$ and $t \to \infty$.

2. Express the curvature and torsion of the curve $\alpha(t) = (\cosh t, \sinh t, t)$ in terms of arc length s measured from $t = 0$.

3. The curve $\alpha(t) = (t\cos t, t\sin t, t)$ lies on a double cone and passes through the vertex at $t = 0$.
(a) Find the Frenet apparatus of α at $t = 0$.
(b) Sketch the curve for $-2\pi \leqq t \leqq 2\pi$, showing T, N, B at $t = 0$.

4. Show that the curvature of a regular curve in $\mathbf{R}^3$ is given by

$$\kappa^2 v^4 = \| \alpha'' \|^2 - (dv/dt)^2.$$

5. If α is a curve with constant speed $c > 0$, show that

$$T = \alpha'/c, \quad N = \alpha''/\| \alpha'' \|, \quad B = \alpha' \times \alpha''/(c\| \alpha'' \|),$$

$$\kappa = \frac{\| \alpha'' \|}{c^2}, \quad \tau = \frac{\alpha' \times \alpha'' \bullet \alpha'''}{c^2 \| \alpha'' \|^2},$$

where for N, B, τ, we assume α'' never zero, that is, $\kappa > 0$.

6. (a) If α is a cylindrical helix, prove that its unit vector $\mathbf{u}$ (Thm. 4.5) is

$$\mathbf{u} = \frac{\tau}{\sqrt{\kappa^2 + \tau^2}} T + \frac{\kappa}{\sqrt{\kappa^2 + \tau^2}} B,$$

and the coefficients here are $\cos\vartheta$ and $\sin\vartheta$ (for ϑ as in Def. 4.5).
(b) Check (a) for the cylindrical helix in Example 4.2 of Chapter 1.

7. Let $\alpha: I \to \mathbf{R}^3$ be a cylindrical helix with unit vector $\mathbf{u}$. For $t_0 \in I$, the curve

$$\gamma(t) = \alpha(t) - ((\alpha(t) - \alpha(t_0)) \bullet \mathbf{u})\,\mathbf{u}$$

is called a *cross-sectional curve* of the cylinder on which α lies. Prove:
(a) γ lies in the plane through $\alpha(t_0)$ orthogonal to $\mathbf{u}$.
(b) The curvature of γ is $\kappa/\sin^2\vartheta$, where κ is the curvature of α.

8. Verify that the following curves are cylindrical helices and, for each, find the unit vector $\mathbf{u}$, angle ϑ, and cross-sectional curve σ.
(a) The curve in Exercise 1. (b) The curve in Example 4.4.
(c) The curve in Exercise 2.

9. If α is a curve with $\kappa > 0$ and τ both constant, show that α is a circular helix.

10. (a) Prove that a curve is a cylindrical helix if and only if its spherical image is part of a circle.
(b) Sketch the spherical image of the cylindrical helix in Exercise 1. Is it a complete circle? Find its center.

11. If α is a curve with $\kappa > 0$, its *central curve* $\alpha^* = \alpha + (1/\kappa)N$ consists of all centers of curvature of α (Ex. 6 of Sec. 3). For nonzero numbers a and b, let β_{ab} be the helix in Example 3.3.

(a) Show that the central curve of β_{ab} is the helix $\beta_{\hat{a}b}$, where $\hat{a} = -b^2/a$.

(b) Deduce that the central curve of $\beta_{\hat{a}b}$ is the original helix β_{ab}.

(c) (*Computer graphics.*) Plot three complete turns of the mutually central helices $\beta_{2,1}$ and $\beta_{-1/2,1}$ in the same figure.

12. If $\alpha(t) = (x(t), y(t))$ is a regular curve in $\mathbf{R}^2$, show that its plane curvature (Ex. 8 of Sec. 3) is given by

$$\tilde{\kappa} = \frac{\alpha'' \bullet J(\alpha')}{v^3} = \frac{x'y'' - x''y'}{(x'^2 + y'^2)^{3/2}},$$

where J is the rotation operator $J(a, b) = (-b, a)$.

13. (*Continuation.*) For a plane curve α with $\tilde{\kappa} \neq 0$, the central curve $\alpha^* = \alpha + (1/\tilde{\kappa})N$ is called the *evolute* of α. Thus α^* gives a direct pointwise description of the turning of α.

(a) Show that

$$\alpha^* = \alpha + \frac{\alpha' \bullet \alpha'}{\alpha'' \bullet J(\alpha')} J(\alpha').$$

(b) Find a formula for the line segment λ_t from $\alpha(t)$ to $\alpha^*(t)$. This segment is the radius (line) of the approximating circle to α near $\alpha(t)$ (Ex. 6 of Sec. 3)

(c) Prove that λ_t is normal to α at $\alpha(t)$ and tangent to α^* at $\alpha^*(t)$. (*Hint:* It can be assumed that α has unit speed.)

14. (*Continuation, Computer graphics.*) In each case, plot the given plane curve and its evolute on the same figure, showing some of the construction lines λ_t.

(a) The ellipse $a(t) = (2\cos t, \sin t)$.

(b) The cycloid $\alpha(t) = (t + \sin t, 1 + \cos t)$ for $-2\pi \leq t \leq 2\pi$. (Here the evolute bears an unexpected relation to the original curve.)

15. (*Computer continuation of Ex. 9 of Sec. 3.*)

(a) Write the commands that, given a regular curve α with $\kappa(0) > 0$, plot—on a small interval $-\varepsilon \leqq t \leqq \varepsilon$—the orthogonal projection of α into the osculating, rectifying, and normal planes at $\alpha(0)$. Show the projections as curves in $\mathbf{R}^2$.

(b) Test (a) on the curves (3), (4), (5) in Example 4.2 of Chapter 1 and those in Example 4.3 of Chapter 3. Compare results.

The following exercise shows that the condition $\kappa > 0$ cannot be avoided in a detailed study of the geometry of curves in $\mathbf{R}^3$ for even if κ is zero at only a single point, the geometric character of the curve can change radically at that point. (This difficulty does not arise for curves in the plane.)

16. It is shown in advanced calculus that the function

$$f(t) = \begin{cases} 0 & \text{if } t \le 0, \\ e^{-1/t^2} & \text{if } t > 0. \end{cases}$$

is infinitely differentiable (has continuous derivatives of all orders). Thus

$$\alpha(t) = (t, f(t), f(-t))$$

is a well-defined differentiable curve.

(a) Sketch α on an interval $-a \leqq t \leqq a$.

(b) Show that the curvature of α is zero only at $t = 0$.

(c) What are the osculating planes of α for $t < 0$ and $t > 0$?

In the following exercise, a global geometric invariant of curves is gotten by integrating a local invariant.

17. The *total curvature* of a unit-speed curve $\alpha: I \to \mathbf{R}^3$ is $\int_I \kappa(s)ds$. If α is merely regular, the formula becomes $\int_I \kappa(t)v(t)dt$. Find the total curvature of the following curves:

(a) The curve in Example 4.4.

(b) The helix in Example 3.3.

(c) The curve in Exercise 2.

(d) The ellipse $\alpha(t) = (a\cos t, b\sin t)$ on $0 \leqq t \leqq 2\pi$.

18. One definition of convexity for a smoothly closed plane curve is that its curvature κ is positive (hence its plane curvature $\tilde{\kappa}$ is either always positive or always negative). Prove that a convex closed plane curve has total curvature 2π. (*Hint:* Consider its spherical image.)

A theorem of Fenchel asserts that every regular closed curve α in $\mathbf{R}^3$ has total curvature $\geqq 2\pi$. Surprisingly, this has an easy proof in terms of surface theory (see Sec. 8 of Ch. 6).

19. (*Computer*.)

(a) Plot the curve

$$\tau(t) = (4 \cos 2t + 2 \cos t, 4 \sin 2t - 2 \sin t, \sin 3t) \quad \text{on } 0 \leqq t \leqq 2\pi.$$

Even looking at this curve from different viewpoints may not make its crossing pattern clear, but Exercise 21 of Section 5.4 will show that τ is a *trefoil knot.*

(Intuitively, a simple closed curve in $\mathbf{R}^3$ is a *knot* provided it cannot be continuously deformed—always remaining simply closed—until it becomes a circle.)

The Fary-Milnor theorem asserts that every knot has total curvature strictly greater than 4π. Show:

(b) The plane curve obtained from τ by removing the z-component $\sin 3t$ has total curvature exactly 4π. (This curve is not simply closed, and hence is not a knot.)

(c) τ can be deformed to a knot that has (numerically estimated) total curvature less than 4.01π.

20. (*Computer*.)

(a) Write a command that, given an arbitrary regular curve, returns the test function in Exercise 10 of Section 3 whose constancy implies that the curve lies on a sphere. (Plotting this function provides a good test for constancy and does not require simplifying it.) (*Hint:* To allow for arbitrary parametrization, replace derivatives $f'(s)$ by $f'(t)v(t)$, where $v(t) = ds/dt$.)

(b) In each case, decide whether the curve lies on a sphere, and if so, find its radius and center:

(i) $\alpha(t) = (2\sin t, \sin 2t, 2\sin^2 t)$;

(ii) $\beta(t) = (\cos^2 t, \sin 2t, 2\sin t)$;

(iii) $\gamma(t) = (\cos t, 1 + \sin t, 2\sin \frac{t}{2})$.

21. Prove that the cubic curve $\gamma(t) = (at, bt^2, ct^3)$, $abc \neq 0$, is a cylindrical helix if and only if $3ac = \pm 2b^2$. (Computer optional.)

2.5 Covariant Derivatives

In Chapter 1 the definition of a new object (curve, differential form, mapping, . . .) was usually followed by an appropriate notion of *derivative* of that object. To see how to define the derivative of a vector field on a Euclidean space, we mimic the definition of the derivative $\mathbf{v}[f]$ of a function f relative to a tangent vector $\mathbf{v}$ at a point $\mathbf{p}$ (Definition 3.1 of Chapter 1). In fact, replacing f by a vector field W on $\mathbf{R}^3$ gives a vector field $t \to W(\mathbf{p} + t\mathbf{v})$ on the curve $t \to \mathbf{p} + t\mathbf{v}$. The derivative of such a vector field was defined in Section 2. Then the derivative of W with respect to $\mathbf{v}$ will be the derivative of $t \to W(\mathbf{p} + t\mathbf{v})$ at $t = 0$.

5.1 Definition Let W be a vector field on $\mathbf{R}^3$, and let $\mathbf{v}$ be a tangent vector field to $\mathbf{R}^3$ at the point $\mathbf{p}$. Then the *covariant derivative* of W with respect to $\mathbf{v}$ is the tangent vector

$$\nabla_{\mathbf{v}} W = W(\mathbf{p} + t\mathbf{v})'(0)$$

at the point $\mathbf{p}$.

Evidently $\nabla_v W$ *measures the initial rate of change of* $W(\mathbf{p})$ *as* $\mathbf{p}$ *moves in the* $\mathbf{v}$ *direction.* (The term "covariant" derives from the generalization of this notion discussed in Chapter 7.)

For example, suppose $W = x^2 U_1 + yzU_3$, and

$$\mathbf{v} = (-1, 0, 2) \quad \text{at} \quad \mathbf{p} = (2, 1, 0).$$

Then

$$\mathbf{p} + t\mathbf{v} = (2 - t, 1, 2t),$$

so

$$W(\mathbf{p} + t\mathbf{v}) = (2 - t)^2 U_1 + 2tU_3,$$

where strictly speaking U_1 and U_3 are also evaluated at $\mathbf{p} + t\mathbf{v}$. Thus,

$$\nabla_v W = W(\mathbf{p} + t\mathbf{v})'(0) = -4U_1(\mathbf{p}) + 2U_3(\mathbf{p}).$$

5.2 Lemma If $W = \sum w_i U_i$ is a vector field on $\mathbf{R}^3$, and $\mathbf{v}$ is a tangent vector at $\mathbf{p}$, then

$$\nabla_v W = \sum \mathbf{v}[w_i] U_i(\mathbf{p}).$$

Proof. We have

$$W(\mathbf{p} + t\mathbf{v}) = \sum w_i(\mathbf{p} + t\mathbf{v}) U_i(\mathbf{p} + t\mathbf{v})$$

for the restriction of W to the curve $t \to \mathbf{p} + t\mathbf{v}$. To differentiate such a vector field (at $t = 0$), one simply differentiates its Euclidean coordinates (at $t = 0$). But by the definition of directional derivative (Definition 3.1 of Chapter 1), the derivative of $w_i(\mathbf{p} + t\mathbf{v})$ at $t = 0$ is precisely $\mathbf{v}[w_i]$. Thus

$$\nabla_v W = W(\mathbf{p} + t\mathbf{v})'(0) = \sum \mathbf{v}[w_i] U_i(\mathbf{p}).$$ ◆

In short, *to apply* ∇_v *to a vector field, apply* $\mathbf{v}$ *to its Euclidean coordinates.* Thus the following linearity and Leibnizian properties of covariant derivative follow easily from the corresponding properties (Theorem 3.3 of Chapter 1) of directional derivatives.

5.3 Theorem Let $\mathbf{v}$ and $\mathbf{w}$ be tangent vectors to $\mathbf{R}^3$ at $\mathbf{p}$, and let Y and Z be vector fields on $\mathbf{R}^3$. Then for numbers a, b and functions f,

(1) $\nabla_{av+bw} Y = a\nabla_v Y + b\nabla_w Y$.
(2) $\nabla_v(aY + bZ) = a\nabla_v Y + b\nabla_v Z$.
(3) $\nabla_v(fY) = \mathbf{v}[f]Y(\mathbf{p}) + f(\mathbf{p})\nabla_v Y$.
(4) $\mathbf{v}[Y \bullet Z] = \nabla_v Y \bullet Z(\mathbf{p}) + Y(\mathbf{p}) \bullet \nabla_v Z$.

Proof. For example, let us prove (4). If

$$Y = \sum y_i U_i \quad \text{and} \quad Z = \sum z_i U_i ,$$

then

$$Y \bullet Z = \sum y_i z_i .$$

Hence by Theorem 3.3 of Chapter 1,

$$\mathbf{v}[Y \bullet Z] = \mathbf{v}\left[\sum y_i z_i\right] = \sum \mathbf{v}[y_i] z_i(\mathbf{p}) + \sum y_i(\mathbf{p})\mathbf{v}[z_i].$$

But by the preceding lemma,

$$\nabla_{\mathbf{v}} Y = \sum \mathbf{v}[y_i] U_i(\mathbf{p}) \quad \text{and} \quad \nabla_{\mathbf{v}} Z = \sum \mathbf{v}[z_i] U_i(\mathbf{p}).$$

Thus the two sums displayed above are precisely $\nabla_{\mathbf{v}} Y \bullet Z(\mathbf{p})$ and $Y(\mathbf{p}) \bullet \nabla_{\mathbf{v}} Z$. ◆

Using the pointwise principle (Chapter 1, Section 2), we can take the covariant derivative of a vector field W with respect to a *vector field* V, rather than a single tangent vector $\mathbf{v}$. The result is the *vector field* $\nabla_V W$ whose value at each point $\mathbf{p}$ is $\nabla_{V(p)} W$. Thus $\nabla_V W$ consists of all the covariant derivatives of W with respect to the vectors of V. It follows immediately from the lemma above that if $W = \sum w_i U_i$, then

$$\nabla_V W = \sum V[w_i] U_i .$$

Coordinate computations are easy using the basic identity $U_i[f] = \partial f/\partial x_i$. For example, suppose $V = (y - x)U_1 + xyU_3$ and (as in the example above) $W = x^2 U_1 + yzU_3$. Then

$$V[x^2] = (y - x)U_1[x^2] = 2x(y - x),$$
$$V[yz] = xyU_3[yz] = xy^2.$$

Hence

$$\nabla_V W = 2x(y - x)U_1 + xy^2 U_3.$$

For the covariant derivative $\nabla_V W$ as expressed entirely in terms of vector fields, the properties in the preceding theorem take the following form.

5.4 Corollary Let V, W, Y, and Z be vector fields on $\mathbf{R}^3$. Then

(1) $\nabla_{fV+gW} Y = f\nabla_V Y + g\nabla_W Y$, for all functions f and g.

(2) $\nabla_V(aY + bZ) = a\nabla_V Y + b\nabla_V Z$, for all numbers a and b.

(3) $\nabla_V(fY) = V[f]Y + f\nabla_V Y$, for all functions f.
(4) $V[Y \bullet Z] = \nabla_V Y \bullet Z + Y \bullet \nabla_V Z$.

We shall omit the proof, which is an exercise in the use of parentheses based on the (pointwise principle) definition $(\nabla_V Y)(\mathbf{p}) = \nabla_{V(p)} Y$.

Note that $\nabla_V Y$ does not behave symmetrically with respect to V and Y. This is to be expected, since it is Y that is being differentiated, while the role of V is merely algebraic. In particular, $\nabla_{fV} Y$ is $f\nabla_V Y$, but $\nabla_V(fY)$ is not $f\nabla_V Y$: There is an extra term arising from the differentiation of f by V.

Exercises

1. Consider the tangent vector $\mathbf{v} = (1, -1, 2)$ at the point $\mathbf{p} = (1, 3, -1)$. Compute $\nabla_{\mathbf{v}} W$ directly from the definition, where

(a) $W = x^2 U_1 + yU_2$. (b) $W = xU_1 + x^2 U_2 - z^2 U_3$.

2. Let $V = -yU_1 + xU_3$ and $W = \cos x U_1 + \sin x U_2$. Express the following covariant derivatives in terms of U_1, U_2, U_3:

(a) $\nabla_V W$. (b) $\nabla_V V$.
(c) $\nabla_V(z^2 W)$. (d) $\nabla_W(V)$.
(e) $\nabla_V(\nabla_{\mathbf{v}} W)$. (f) $\nabla_V(xV - zW)$.

3. If W is a vector field with constant length $\|W\|$, prove that for any vector field V, the covariant derivative $\nabla_V W$ is everywhere orthogonal to W.

4. Let X be the special vector field $\sum x_i U_i$, where x_1, x_2, x_3 are the natural coordinate functions of $\mathbf{R}^3$. Prove that $\nabla_V X = V$ for every vector field V.

5. Let W be a vector field defined on a region containing a regular curve α. Then $t \to W(\alpha(t))$ is a vector field on α called the *restriction* of W to α and denoted by W_α.

(a) Prove that $\nabla_{\alpha'(t)} W = (W_\alpha)'(t)$.

(b) Deduce that the straight line in Definition 5.1 may be replaced by *any* curve with initial velocity $\mathbf{v}$. Thus the derivative Y' of a vector field Y on a curve α is (almost) $\nabla_{\alpha'} Y$.

2.6 Frame Fields

When the Frenet formulas were discovered (by Frenet in 1847, and independently by Serret in 1851), the theory of *surfaces* in $\mathbf{R}^3$ was already a richly developed branch of geometry. The success of the Frenet approach to curves led Darboux (around 1880) to adapt this "method of moving frames" to the study of surfaces. Then, as we mentioned earlier, it was Cartan who brought the method to full generality. His essential idea was very simple: To each point of the object under study (a curve, a surface, Euclidean space itself, . . .) assign a frame; then using orthonormal expansion express the rate of change of the frame in terms of the frame itself. This, of course, is just what the Frenet formulas do in the case of a curve.

In the next three sections we shall carry out this scheme for the Euclidean space $\mathbf{R}^3$. We shall see that geometry of curves and surfaces in $\mathbf{R}^3$ is not merely an analogue, but actually a *corollary*, of these basic results. Since the main application (to surface theory) comes only in Chapter 6, these sections may be postponed, and read later as a preliminary to that chapter.

By means of the pointwise principle (Chapter 1, Section 2) we can automatically extend operations on individual tangent vectors to operations on vector fields. For example, if V and W are vector fields on $\mathbf{R}^3$, then the *dot product* $V \bullet W$ of V and W is the (differentiable) real-valued function on $\mathbf{R}$ whose value at each point $\mathbf{p}$ is $V(\mathbf{p}) \bullet W(\mathbf{p})$. The *norm* $\|\mathbf{V}\|$ of V is the real-valued function on $\mathbf{R}^3$ whose value at $\mathbf{p}$ is $\|V(\mathbf{p})\|$. Thus $\|V\| = (V \bullet V)^{1/2}$. By contrast with $V \bullet W$, the norm function $\|V\|$ need not be differentiable at points for which $V(\mathbf{p}) = 0$, since the square-root function is badly behaved at 0.

In Chapter 1 we called the three vector fields U_1, U_2, U_3 the natural frame field on $\mathbf{R}^3$. Here is a simple but crucial generalization.

6.1 Definition Vector fields E_1, E_2, E_3 on $\mathbf{R}^3$ constitute a *frame field* on $\mathbf{R}^3$ provided

$$E_i \bullet E_j = \delta_{ij} \quad (1 \le i, j \le 3),$$

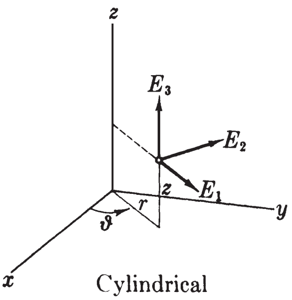
where δ_{ij} is the Kronecker delta.

Thus at each point $\mathbf{p}$ the vectors $E_1(\mathbf{p})$, $E_2(\mathbf{p})$, $E_3(\mathbf{p})$ do in fact form a frame (Definition 1.4) since they have unit length and are mutually orthogonal.

In elementary calculus, frame fields are usually derived from coordinate systems, as in the following cases.

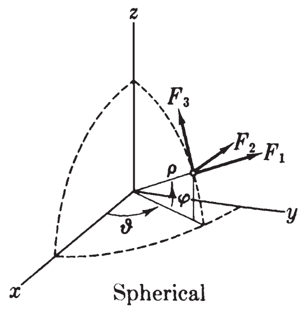
6.2 Example (1) *The cylindrical frame field* (Fig. 2.19). Let r, ϑ, z be the usual cylindrical coordinate functions on $\mathbf{R}^3$. We shall pick a unit vector field in the direction in which each coordinate increases (when the other two are held constant). For r, this is evidently

$$E_1 = \cos\vartheta\, U_1 + \sin\vartheta\, U_2,$$



Cylindrical

FIG. 2.19



Spherical

FIG. 2.20

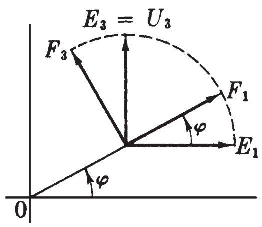


FIG. 2.21

pointing straight out from the z axis. Then

$$E_2 = -\sin\vartheta\, U_1 + \cos\vartheta\, U_2$$

points in the direction of increasing ϑ as in Fig. 2.19. Finally, the direction of increase of z is, of course, straight up, so

$$E_3 = U_3.$$

It is easy to check that $E_i \bullet E_j = \delta_{ij}$, so this is a frame field (defined on all of $\mathbf{R}^3$ except the z axis). We call it the *cylindrical frame field* on $\mathbf{R}^3$.

(2) *The spherical frame field on* $\mathbf{R}^3$ (Fig. 2.20). In a similar way, a frame field F_1, F_2, F_3 can be derived from the spherical coordinate functions ρ, ϑ, φ on $\mathbf{R}^3$. As indicated in the figure, we shall measure φ up from the xy plane rather than (as is usually done) down from the z axis.

Let E_1, E_2, E_3 be the cylindrical frame field. For spherical coordinates, the unit vector field F_2 in the direction of increasing ϑ is the same as above, so $F_2 = E_2$. The unit vector field F_1, in the direction of increasing ρ, points straight out from the origin; hence it can be expressed as

$$F_1 = \cos\varphi\ E_1 + \sin\varphi\ E_3$$

(Fig. 2.21). Similarly, the vector field for increasing φ is

$$F_3 = -\sin\varphi\ E_1 + \cos\varphi\ E_3.$$

Thus the formulas for E_1, E_2, E_3 in (1) yield

$$F_1 = \cos\varphi(\cos\vartheta\,U_1 + \sin\vartheta\,U_2) + \sin\varphi\,U_3,$$

$$F_2 = -\sin\vartheta\,U_1 + \cos\vartheta\,U_2,$$

$$F_3 = -\sin\varphi(\cos\vartheta\,U_1 + \sin\vartheta\,U_2) + \cos\varphi\,U_3.$$

By repeated use of the identity $\sin^2 + \cos^2 = 1$, we check that F_1, F_2, F_3 is a frame field—the *spherical frame field* on $\mathbf{R}^3$. (Its actual domain of definition is $\mathbf{R}^3$ minus the z axis, as in the cylindrical case.)

The following useful result is an immediate consequence of orthonormal expansion.

6.3 Lemma Let E_1, E_2, E_3 be a frame field on $\mathbf{R}^3$.

(1) If V is a vector field on $\mathbf{R}^3$, then $V = \sum f_i E_i$, where the functions $f_i = V \bullet E_i$ are called the *coordinate functions* of V with respect to E_1, E_2, E_3.

(2) If $V = \sum f_i E_i$ and $W = \sum g_i E_i$, then $V \bullet W = \sum f_i g_i$. In particular, $\| V \| = (\sum f_i^2)^{1/2}$.

Thus a given vector field V has a different set of coordinate functions with respect to each choice of a frame field E_1, E_2, E_3. The *Euclidean* coordinate functions (Lemma 2.5 of Chapter 1), of course, come from the natural frame field U_1, U_2, U_3. In Chapter 1, we used this natural frame field exclusively, but now we shall gradually shift to arbitrary frame fields. The reason is clear: In studying curves and surfaces in $\mathbf{R}^3$, we shall then be able to choose a frame field *specifically adapted to the problem at hand.* Not only does this simplify computations, but it gives a clearer understanding of geometry than if we had insisted on using the same frame field in every situation.

Exercises

1. If V and W are vector fields on $\mathbf{R}^3$ that are linearly independent at each point, show that

$$E_1 = \frac{V}{\| V \|}, \quad E_2 = \frac{\tilde{W}}{\| \tilde{W} \|}, \quad E_3 = E_1 \times E_2$$

is a frame field, where $\tilde{W} = W - (W \bullet E_1)E_1$.

2. Express each of the following vector fields (i) in terms of the cylindrical frame field (with coefficients in terms of r, ϑ, z) and (ii) in terms of the spherical frame field (with coefficients in terms of ρ, ϑ, φ):

(a) U_1. (b) $\cos\vartheta U_1 + \sin\vartheta U_2 + U_3$.

(c) $xU_1 + yU_2 + zU_3$.

3. Find a frame field E_1, E_2, E_3 such that

$$E_1 = \cos x\, U_1 + \sin x \cos z\, U_2 + \sin x \sin z\, U_3.$$

2.7 Connection Forms

Once more we state the essential point: The power of the Frenet formulas stems not from the fact that they tell what the derivatives T', N', B' are, but from the fact that *they express these derivatives in terms of* T, N, B—and thereby define curvature and torsion. We shall now do the same thing with an arbitrary frame field E_1, E_2, E_3 on $\mathbf{R}^3$; namely, *express the covariant derivatives of these vector fields in terms of the vector fields themselves.* We begin with the covariant derivative with respect to an arbitrary tangent vector $\mathbf{v}$ at a point $\mathbf{p}$. Then

$$\begin{aligned}
\nabla_v E_1 &= c_{11}E_1(\mathbf{p}) + c_{12}E_2(\mathbf{p}) + c_{13}E_3(\mathbf{p}),\\
\nabla_v E_2 &= c_{21}E_1(\mathbf{p}) + c_{22}E_2(\mathbf{p}) + c_{23}E_3(\mathbf{p}),\\
\nabla_v E_3 &= c_{31}E_1(\mathbf{p}) + c_{32}E_2(\mathbf{p}) + c_{33}E_3(\mathbf{p}),
\end{aligned}$$

where by orthonormal expansion the coefficients of these equations are

$$c_{ij} = \nabla_v E_i \bullet E_j(\mathbf{p}) \quad \text{for } 1 \leqq i,\ j \leqq 3.$$

These coefficients c_{ij}, depend on the particular tangent vector $\mathbf{v}$, so a better notation for them is

$$\omega_{ij}(\mathbf{v}) = \nabla_v E_i \bullet E_j(\mathbf{p}), \quad (1 \leqq i,\ j \leqq 3).$$

Thus for each choice of i and j, ω_{ij} is a real-valued function defined on all tangent vectors. But we have met that kind of function before.

7.1 Lemma Let E_1, E_2, E_3 be a frame field on $\mathbf{R}^3$. For each tangent vector $\mathbf{v}$ to $\mathbf{R}^3$ at the point $\mathbf{p}$, let

$$\omega_{ij}(\mathbf{v}) = \nabla_v E_i \bullet E_j(\mathbf{p}), \quad (1 \leqq i,\ j \leqq 3).$$

Then each ω_{ij} is a 1-form, and $\omega_{ij} = -\omega_{ji}$. These 1-forms are called the *connection forms* of the frame field E_1, E_2, E_3.

Proof. By definition, ω_{ij} is a real-valued function on tangent vectors, so to verify that ω_{ij} is a 1-form (Def. 5.1 of Ch. 1), it suffices to check the linearity condition. Using Theorem 5.3, we get

$$\begin{aligned}\omega_{ij}(a\mathbf{v} + b\mathbf{w}) &= \nabla_{av+bw}E_i \bullet E_j(\mathbf{p}) \\ &= (a\nabla_v E_i + b\nabla_w E_i) \bullet E_j(\mathbf{p}) \\ &= a\nabla_v E_i \bullet E_j(\mathbf{p}) + b\nabla_w E_i \bullet E_j(\mathbf{p}) \\ &= a\omega_{ij}(\mathbf{v}) + b\omega_{ij}(\mathbf{w}).\end{aligned}$$

To prove that $\omega_{ij} = -\omega_{ji}$ we must show that $\omega_{ij}(\mathbf{v}) = -\omega_{ji}(\mathbf{v})$ for every tangent vector $\mathbf{v}$. By definition of frame field, $E_i \bullet E_j = \delta_{ij}$, and since each Kronecker delta has constant value 0 or 1, the Leibnizian formula (4) of Theorem 5.3 yields

$$0 = \mathbf{v}[E_i \bullet E_j] = \nabla_v E_i \bullet E_j(\mathbf{p}) + E_i(\mathbf{p}) \bullet \nabla_v E_j.$$

By the symmetry of the dot product, the two vectors in this last term may be reversed, so we have found that $0 = \omega_{ij}(\mathbf{v}) + \omega_{ji}(\mathbf{v})$. ◆

The geometric significance of the connection forms is no mystery. The definition $\omega_{ij}(\mathbf{v}) = \nabla_v E_i \bullet E_j(\mathbf{p})$ shows that $\omega_{ij}(\mathbf{v})$ *is the initial rate at which* E_i *rotates toward* E_j *as* $\mathbf{p}$ *moves in the* $\mathbf{v}$ *direction.* Thus the 1-forms ω_{ij} contain this information for *all* tangent vectors to $\mathbf{R}^3$.

The following basic result is little more than a rephrasing of the definition of connection forms.

7.2 Theorem Let ω_{ij} $(1 \leqq i, j \leqq 3)$ be the connection forms of a frame field E_1, E_2, E_3 on $\mathbf{R}^3$. Then for any vector field V on $\mathbf{R}^3$,

$$\nabla_V E_i = \sum_j \omega_{ij}(V)E_j, \quad (1 \leqq i \leqq 3).$$

We call these the *connection equations* of the frame field E_1, E_2, E_3.

Proof. For fixed i, both sides of this equation are vector fields. Thus we must show that at each point $\mathbf{p}$,

$$\nabla_{V(p)}E_i = \sum_j \omega_{ij}(V(\mathbf{p}))E_j(\mathbf{p}).$$

But as we have already seen, the very definition of connection form makes this equation a consequence of orthonormal expansion. ◆

When $i = j$, the skew-symmetry condition $\omega_{ij} = -\omega_{ji}$ becomes $\omega_{ii} = -\omega_{ii}$; thus

$$\omega_{11} = \omega_{22} = \omega_{33} = 0.$$

Hence this condition has the effect of reducing the nine 1-forms ω_{ij} for $1 \leqq i, j \leqq 3$ to essentially only three, say ω_{12}, ω_{13}, ω_{23}. It is perhaps best to regard the connection forms ω_{ij} as the entries of a skew-symmetric matrix of 1-forms,

$$\omega = \begin{pmatrix} \omega_{11} & \omega_{12} & \omega_{13} \\ \omega_{21} & \omega_{22} & \omega_{23} \\ \omega_{31} & \omega_{32} & \omega_{33} \end{pmatrix} = \begin{pmatrix} 0 & \omega_{12} & \omega_{13} \\ -\omega_{12} & 0 & \omega_{23} \\ -\omega_{13} & -\omega_{23} & 0 \end{pmatrix}.$$

Thus in expanded form, the connection equations (Theorem 7.2) become

$$\begin{aligned} \nabla_V E_1 &= \qquad\qquad\quad \omega_{12}(V)E_2 + \omega_{13}(V)E_3, \\ \nabla_V E_2 &= -\omega_{12}(V)E_1 \qquad\qquad\quad + \omega_{23}(V)E_3, \\ \nabla_V E_3 &= -\omega_{13}(V)E_1 - \omega_{23}(V)E_2. \end{aligned} \tag{$*$}$$

showing an obvious relation to the Frenet formulas

$$\begin{aligned} T' &= \qquad\quad \kappa N, \\ N' &= -\kappa N \qquad\quad + \ \tau B, \\ B' &= \qquad -\tau N. \end{aligned}$$

The absence from the Frenet formulas of terms corresponding to $\omega_{13}(V)E_3$ and $-\omega_{13}(V)E_1$ is a consequence of the special way the Frenet frame field is fitted to its curve. Having gotten $T(\sim E_1)$, we chose $N(\sim E_2)$ so that the derivative T' would be a scalar multiple of N alone and not involve $B(\sim E_3)$.

Another difference between the Frenet formulas and the equations above stems from the fact that $\mathbf{R}^3$ has three dimensions, while a curve has but one. The coefficients—curvature κ and torsion τ—in the Frenet formulas measure the rate of change of the frame field T, N, B only along its curve, that is, in the direction of T alone. But the coefficients in the connection equations must be able to make this measurement for E_1, E_2, E_3 with respect to *arbitrary* vector fields in $\mathbf{R}^3$. This is why the connection forms are 1-forms and not just functions.

These formal differences aside, a more fundamental distinction stands out. It is because a Frenet frame field is specially fitted to its curve that the Frenet formulas give information about that curve. Since the frame field E_1, E_2, E_3 used above is completely arbitrary, the connection equations give no direct information about $\mathbf{R}^3$, but only information about the "rate of rotation" of that particular frame field. This is not a weakness, but a strength, since as indicated earlier, if we can fit a frame field to a geometric problem arising in $\mathbf{R}^3$, then the connection equations will give direct information about that problem. Thus, these equations play a fundamental role in all the differential geometry of $\mathbf{R}^3$. For example, the Frenet formulas can be deduced from them (Exercise 8).

Given an arbitrary frame field E_1, E_2, E_3 on $\mathbf{R}^3$, it is fairly easy to find an explicit formula for its connection forms. First use orthonormal expansion to express the vector fields E_1, E_2, E_3 in terms of the natural frame field U_1, U_2, U_3 on $\mathbf{R}^3$:

$$E_1 = a_{11}U_1 + a_{12}U_2 + a_{13}U_3,$$

$$E_2 = a_{21}U_1 + a_{22}U_2 + a_{23}U_3,$$

$$E_3 = a_{31}U_1 + a_{32}U_2 + a_{33}U_3.$$

Here each $a_{ij} = E_i \bullet U_j$ is a real-valued function on $\mathbf{R}^3$. The matrix

$$A = (a_{ij}) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

with these functions as entries is called the *attitude matrix* of the frame field E_1, E_2, E_3. In fact, at each point $\mathbf{p}$, the numerical matrix

$$A(\mathbf{p}) = (a_{ij}(\mathbf{p}))$$

is exactly the attitude matrix of the frame $E_1(\mathbf{p})$, $E_2(\mathbf{p})$, $E_3(\mathbf{p})$ as in Definition 1.6. Since attitude matrices are orthogonal, the transpose ${}^tA$ of A is equal to its inverse A^{-1}.

Define the differential of $A = (a_{ij})$ to be $dA = (da_{ij})$, so dA is a matrix whose entries are 1-forms. We can now give a simple expression for the connection forms in terms of the attitude matrix.

7.3 Theorem If $A = (a_{ij})$ is the attitude matrix and $\omega = (\omega_{ij})$ the matrix of connection forms of a frame field E_1, E_2, E_3, then

$$\omega = dA\,{}^tA \quad \text{(matrix multiplication)},$$

or equivalently,

$$\omega_{ij} = \sum_k a_{jk}\, da_{ik} \quad \text{for} \quad 1 \leqq i,\, j \leqq 3.$$

Since the proof is routine, it may be more informative to illustrate the result by an example. For the cylindrical frame field in Example 6.2, we found the attitude matrix

$$A = \begin{pmatrix} \cos\vartheta & \sin\vartheta & 0 \\ -\sin\vartheta & \cos\vartheta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus

$$\omega = dA\,{}^tA = \begin{pmatrix} -\sin\vartheta\, d\vartheta & \cos\vartheta\, d\vartheta & 0 \\ -\cos\vartheta\, d\vartheta & -\sin\vartheta\, d\vartheta & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \cos\vartheta & -\sin\vartheta & 0 \\ \sin\vartheta & \cos\vartheta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & d\vartheta & 0 \\ -d\vartheta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Since $\omega_{12} = d\vartheta$ is the only nonzero connection form (except, of course, $\omega_{21} = -\omega_{12}$), the connection equations (∗) reduce to

$$\nabla_V E_1 = d\vartheta(V)E_2 = V[\vartheta]E_2,$$
$$\nabla_V E_2 = -d\vartheta(V)E_1 = -V[\vartheta]E_1,$$
$$\nabla_V E_3 = 0.$$

These equations have immediate geometrical significance. Because V is arbitrary, the third equation says that the vector field E_3 is parallel. We knew this already since in the cylindrical frame field, E_3 is just U_3.

The first two equations tell us that the covariant derivatives of E_1 and E_2 with respect to a vector field V depend only on the rate of change of the angle ϑ in the V direction.

For example, the definition of ϑ shows that $V[\vartheta] = 0$ whenever V is a vector field that at each point is tangent to a plane through the z axis. Thus for a vector field of this type the connection equations above predict that $\nabla_V E_1 = \nabla_V E_2 = 0$. In fact, it is clear from Fig. 2.19 that E_1 and E_2 do remain parallel on any plane through the z axis.

Exercises

1. For any function f, show that the vector fields

$$E_1 = (\sin f\, U_1 + U_2 - \cos f\, U_3)/\sqrt{2},$$
$$E_2 = (\sin f\, U_1 - U_2 - \cos f\, U_3)/\sqrt{2},$$
$$E_3 = \cos f\, U_1 + \sin f\, U_3$$

form a frame field, and find its connection forms.

2. Find the connection forms of the natural frame field U_1, U_2, U_3.

3. For any function f, show that

$$A = \begin{pmatrix} \cos^2 f & \cos f \sin f & \sin f \\ \sin f \cos f & \sin^2 f & -\cos f \\ -\sin f & \cos f & 0 \end{pmatrix}$$

is the attitude matrix of a frame field, and compute its connection forms.

4. Prove that the connection forms of the spherical frame field are

$$\omega_{12} = \cos\varphi\, d\vartheta, \quad \omega_{13} = d\varphi, \quad \omega_{23} = \sin\varphi\, d\vartheta.$$

5. If E_1, E_2, E_3 is a frame field and $W = \sum f_i E_i$, prove the *covariant derivative formula:*

$$\nabla_V W = \sum_j \left\{ V[f_j] + \sum_i f_i \omega_{ij}(V) \right\} E_j.$$

6. Let E_1, E_2, E_3 be the cylindrical frame field. If V is a vector field such that $V[r] = r$ and $V[\vartheta] = 1$, compute $\nabla_V (r \cos\vartheta E_1 + r \sin\vartheta E_3)$.

7. (*Computer*.) (a) Write a computer command that, given the attitude matrix A of a frame field on $\mathbf{R}^3$, returns the matrix $\omega = dA\, {}^tA$ of its connection forms. (*Hint:* For *Maple*, use the differential operator d from the package *difforms*. For *Mathematica*, use the total differential Dt.) (b) Test part (a) on the cylindrical frame field and on the spherical frame field (Ex. 4).

8. Let β be a unit-speed curve in $\mathbf{R}^3$ with $\kappa > 0$, and suppose that E_1, E_2, E_3 is a frame field on $\mathbf{R}^3$ such that the restriction of these vector fields to β gives the Frenet-frame field T, N, B of β. Prove that

$$\omega_{12}(T) = \kappa, \quad \omega_{13}(T) = 0, \quad \omega_{23}(T) = \tau.$$

Then deduce the Frenet formulas from the connection equations. (*Hint:* Ex. 5 of Sec. 5.)

2.8 The Structural Equations

We have seen that 1-forms—the connection forms—give the simplest description of the rate of rotation of a frame field. Furthermore, the frame field itself can be described in terms of 1-forms.

8.1 Definition If E_1, E_2, E_3 is a frame field on $\mathbf{R}^3$, then the *dual* 1-*forms* $\theta_1, \theta_2, \theta_3$ of the frame field are the 1-forms such that

$$\theta_i(\mathbf{v}) = \mathbf{v} \bullet E_i(\mathbf{p})$$

for each tangent vector $\mathbf{v}$ to $\mathbf{R}^3$ at $\mathbf{p}$.

Note that θ_i is linear on the tangent vectors at each point; hence it *is* a 1-form. In particular, $\theta_i(E_j) = \delta_{ij}$, so readers familiar with the notion of dual vector spaces will recognize that at each point, $\theta_1, \theta_2, \theta_3$ gives the dual basis of E_1, E_2, E_3.

In the case of the natural frame field U_1, U_2, U_3, the dual forms are just dx_1, dx_2, dx_3. In fact, from Example 5.3 of Chapter 1 we get

$$dx_i(\mathbf{v}) = v_i = \mathbf{v} \bullet U_i(\mathbf{p})$$

for each tangent vector $\mathbf{v}$; hence $dx_i = \theta_i$.

Using dual forms, the orthonormal expansion formula in Lemma 6.3 may be written $V = \sum \theta_i(V)E_i$. In the characteristic fashion of duality, this formula becomes the following lemma.

8.2 Lemma Let $\theta_1, \theta_2, \theta_3$ be the dual 1-forms of a frame field E_1, E_2, E_3. Then any 1-form ϕ on $\mathbf{R}^3$ has a unique expression

$$\phi = \sum \phi(E_i)\theta_i.$$

Proof. Two 1-forms are the same if they have the same value on any vector field V. But

$$\begin{aligned}\left(\sum \phi(E_i)\theta_i\right)(V) &= \sum \phi(E_i)\theta_i(V)\\ &= \phi\left(\sum \theta_i(V)E_i\right) = \phi(V).\end{aligned}$$

◆

Thus ϕ is expressed in terms of dual forms of E_1, E_2, E_3 by evaluating it on E_1, E_2, E_3. This useful fact is the generalization to arbitrary frame fields of Lemma 5.4 of Chapter 1.

We compared a frame field E_1, E_2, E_3 to the natural frame field by means of its attitude matrix $A = (a_{ij})$, for which

$$E_i = \sum a_{ij} U_j \quad (1 \leq i \leq 3).$$

The dual formulation is just

$$\theta_i = \sum a_{ij} dx_j$$

with *the same coefficients.* In fact, by the preceding lemma,

$$\theta_i = \sum \theta_i(U_j) dx_j.$$

But

$$\theta_i(U_j) = E_i \bullet U_j = \left(\sum a_{ik} U_k\right) \bullet U_j = \sum a_{ik} \delta_{kj} = a_{ij}.$$

These formulas for E_i and θ_i show plainly that θ_1, θ_2, θ_3 is merely the dual description of the frame field E_1, E_2, E_3.

In calculus, when a new function appears on the scene, it is natural to ask what its derivative is. Similarly with 1-forms—having associated with each frame field its dual forms and connection forms, it is reasonable to ask what their exterior derivatives are. The answer is given by two neat sets of equations discovered by Cartan.

8.3 Theorem (Cartan structural equations.) Let E_1, E_2, E_3 be a frame field on $\mathbf{R}^3$ with dual forms θ_1, θ_2, θ_3 and connection forms ω_{ij} $(1 \leqq i, j \leqq 3)$. The exterior derivatives of these forms satisfy

(1) the *first structural equations:*

$$d\theta_i = \sum_j \omega_{ij} \wedge \theta_j \quad (1 \leqq i \leqq 3);$$

(2) the s*econd structural equations:*

$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} \quad (1 \leqq i,\ j \leqq 3).$$

Because θ_i is the dual of E_i, the first structural equations may be easily recognized as the dual of the connection equations. Only later experience will show that the second structural equations mean that $\mathbf{R}^3$ is flat—roughly speaking, in the same sense that the plane $\mathbf{R}^2$ is flat.

The most efficient proof of the structural equations requires some preliminary remarks. In the Cartan approach, the fundamental objects are not individual forms, but rather *matrices whose entries are forms.* We have already seen that the simplest description of the connection forms ω_{ij} of a frame field is as a single skew-symmetric matrix ω with entries ω_{ij}. Then, for example, ω is expressed in terms of the attitude matrix A of the frame field by the matrix equation $\omega = dA\ {}^tA$. (Here, as always, to apply d to a matrix, apply it to each entry of the matrix.)

Similarly, the dual forms of a frame field can be described by a single $n \times 1$ matrix θ with entries θ_i. If ξ is the $n \times 1$ matrix whose entries are the natural coordinates x_i of $\mathbf{R}^3$, then

$$\theta = \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix} \quad \text{and} \quad d\xi = \begin{pmatrix} dx_1 \\ dx_2 \\ dx_3 \end{pmatrix},$$

so the formula $\theta_i = \sum a_{ij}\, dx_j$ above can be written as

$$\theta = A\, d\xi.$$

For such matrices of forms, matrix multiplication is defined as usual, but of course when *entries* are multiplied it is by the wedge product.

The proof of Theorem 8.3 is now quite simple. Recall that since the attitude matrix A is orthogonal, ${}^tAA$ is the identity matrix I, which can be inserted in any matrix formula without effect.

Proof of the First Structural Equation. Since $d^2 = 0$, we evidently have $d(d\xi) = 0$, so

$$d\theta = d(A\, d\xi) = dA \cdot d\xi = dA\ {}^tA \cdot A\, d\xi = \omega\theta.$$

Expressed in terms of entries, this is indeed the version in (1) of Theorem 8.3.

Proof of the Second Structural Equation. For functions f and g.

$$d(df\ g) = d(g\ df) = dg \wedge df = -df \wedge dg.$$

Thus, using the transpose rule ${}^t(AB) = {}^tB\,{}^tA$, we get

$$d\omega = d(dA\,{}^tA) = -dA \cdot d({}^tA) = -dA\ {}^tA \cdot A^t(dA) = -\omega^t\omega = \omega\omega,$$

where the last step uses the skew-symmetry of ω. Again, in terms of entries, this is the version in (2) of Theorem 8.3. ◆

8.4 Example Structural equations for the spherical frame field (Example 6.2). The dual forms and connection forms are

$$\begin{aligned} \theta_1 &= d\rho, & \omega_{12} &= \cos\varphi\, d\vartheta, \\ \theta_2 &= \rho\cos\varphi\, d\vartheta, & \omega_{13} &= d\varphi, \\ \theta_3 &= \rho\, d\varphi, & \omega_{23} &= \sin\varphi\, d\vartheta. \end{aligned}$$

Let us check, say, the first structural equation

$$d\theta_3 = \sum \omega_{3j} \wedge \theta_j = \omega_{31} \wedge \theta_1 + \omega_{32} \wedge \theta_2.$$

Using the skew-symmetry $\omega_{ij} = -\omega_{ji}$ and the general properties of forms developed in Chapter 1, we get

$$\begin{aligned} \omega_{31} \wedge \theta_1 &= -d\varphi \wedge d\rho = d\rho \wedge d\varphi, \\ \omega_{32} \wedge \theta_2 &= (-\sin\varphi\, d\vartheta) \wedge (\rho\cos\varphi\, d\vartheta) = 0 \end{aligned}$$

(the latter since $d\vartheta \wedge d\vartheta = 0$). The sum of these terms is, correctly,

$$d\theta_3 = d(\rho\, d\varphi) = d\rho \wedge d\varphi.$$

Second structural equations involve only one wedge product. For example, since $\omega_{11} = \omega_{22} = 0$,

$$d\omega_{12} = \sum \omega_{1k} \wedge \omega_{k2} = \omega_{13} \wedge \omega_{32}.$$

In this case,

$$\omega_{13} \wedge \omega_{32} = d\varphi \wedge (-\sin\varphi\, d\vartheta) = -\sin\varphi\, d\varphi \wedge d\vartheta.$$

which is the same as

$$d\omega_{12} = d(\cos\varphi\, d\vartheta) = d(\cos\varphi) \wedge d\vartheta = -\sin\varphi\, d\varphi \wedge d\vartheta.$$

To derive the expressions given above for the dual 1-forms, first compute dx_1, dx_2, dx_3 by differentiating the well-known equations

$$\begin{aligned} x_1 &= \rho\cos\varphi\cos\vartheta, \\ x_2 &= \rho\cos\varphi\sin\vartheta, \\ x_3 &= \rho\sin\vartheta. \end{aligned}$$

Then substitute in the formula $\theta_i = \sum a_{ij}\, dx_j$, where $A = (a_{ij})$ is the attitude matrix from Example 6.2. This result, somewhat disguised, is derived in elementary calculus by a familiar plausibility argument: If at each point the spherical coordinates ρ, ϑ, φ are altered by increments $d\rho$, $d\vartheta$, $d\varphi$, then the sides of the resulting infinitesimal box (Fig. 2.22) are $d\rho$, $\rho\cos\varphi\, d\vartheta$, $\rho\, d\varphi$. These are exactly the formulas for θ_1, θ_2, θ_3.

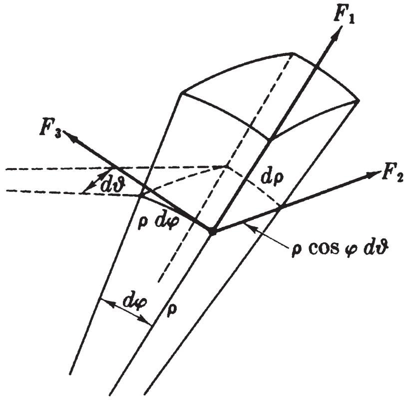


FIG. 2.22

The structural equations provide a powerful method for dealing with geometrical problems in $\mathbf{R}^3$: Select a frame field well adapted to the problem at hand; find its dual 1-forms and connection forms; apply the structural equations; interpret the results. We will use this method later to study the geometry of surfaces in $\mathbf{R}^3$.

Exercises

1. For a 1-form $\phi = \sum f_i \theta_i$, prove

$$d\phi = \sum_j \left\{ df_j + \sum_i f_i \omega_{ij} \right\} \wedge \theta_j .$$

(Compare Ex. 5 of Sec. 7.)

2. Check all the structural equations of the spherical frame field.

3. For the cylindrical frame field E_1, E_2, E_3.
(a) Starting from the basic cylindrical equations $x = r\cos\vartheta$, $y = r\sin\vartheta$, $z = z$, show that the dual 1-forms are

$$\theta_1 = dr, \quad \theta_2 = r\,d\vartheta, \quad \theta_3 = dz.$$

(b) Deduce that $E_1[r] = 1$, $E_2[\vartheta] = 1/r$, $E_3[z] = 1$ and that the other six possibilities $E_1[\vartheta]$, . . . are all zero.
(c) For a function $f(r, \vartheta, z)$, show that

$$E_1[f] = \frac{\partial f}{\partial r}, \quad E_2[f] = \frac{1}{r}\frac{\partial f}{\partial \vartheta}, \quad E_3[f] = \frac{\partial f}{\partial z}.$$

4. Frame fields on $\mathbf{R}^2$. Given a frame field E_1, E_2 on $\mathbf{R}^2$ there is an angle function ψ such that

$$E_1 = \cos\psi\, U_1 + \sin\psi\, U_2,$$
$$E_2 = -\sin\psi\, U_1 + \cos\psi\, U_2.$$

(a) Express the connection form and dual 1-forms in terms of ψ and the natural coordinates x, y.
(b) What are the structural equations in this case? Check that the results in part (a) satisfy these equations.

(*Hint:* Defining $E_3 = U_3$ gives a frame field on $\mathbf{R}^3$.)

2.9 Summary

We have accomplished the aims set at the beginning of this chapter. The idea of a moving frame has been expressed rigorously as a *frame field*—either on a curve in $\mathbf{R}^3$ or on an open set of $\mathbf{R}^3$ itself. In the case of a curve, we used only the Frenet frame field T, N, B of the curve. Expressing the derivatives of these vector fields in terms of the vector fields themselves, we discovered the *curvature* and *torsion* of the curve. It is already clear that curvature and torsion tell a lot about the geometry of a curve; we shall find in Chapter 3 that they tell everything. In the case of an open set of $\mathbf{R}^3$, we dealt with an arbitrary frame field E_1, E_2, E_3. Cartan's generalization (Theorem 7.2) of the Frenet formulas followed the same pattern of expressing the (covariant) derivatives of these vector fields in terms of the vector fields themselves. Omitting the vector field V from the notation in Theorem 7.2, we have

Cartan	*Frenet*
$\nabla E_1 = \omega_{12}E_2 + \omega_{13}E_3,$	$T' = \kappa N,$
$\nabla E_2 = -\omega_{12}E_1 + \omega_{23}E_3,$	$N' = -\kappa T + \tau B,$
$\nabla E_3 = -\omega_{13}E_1 - \omega_{23}E_2,$	$B' = -\tau N.$

Cartan's equations are not conspicuously more complicated than Frenet's, because the notion of 1-form is available for the coefficients ω_{ij}, the connection forms.