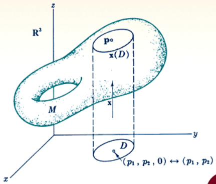
ELEMENTARY DIFFERENTIAL GEOMETRY

BARRETT O'NEILL

REVISED SECOND EDITION



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Preface to the Revised Second Edition

This book is an elementary account of the geometry of curves and surfaces. It is written for students who have completed standard courses in calculus and linear algebra, and its aim is to introduce some of the main ideas of differential geometry.

The language of the book is established in Chapter 1 by a review of the core content of differential calculus, emphasizing linearity. Chapter 2 describes the method of *moving frames*, which is introduced, as in elementary calculus, to study curves in space. (This method turns out to apply with equal efficiency to surfaces.) Chapter 3 investigates the rigid motions of space, in terms of which congruence of curves and surfaces is defined in the same way as congruence of triangles in the plane.

Chapter 4 requires special comment. One weakness of classical differential geometry is its lack of any adequate definition of *surface*. In this chapter we decide just what a surface is, and show that every surface has a differential and integral calculus of its own, strictly analogous to the familiar calculus of the plane. This exposition provides an introduction to the notion of *differentiable manifold*, which is the foundation for those branches of mathematics and its applications that are based on the calculus.

The next two chapters are devoted to the geometry of surfaces in 3-space. Chapter 5 measures the *shape* of a surface and derives basic geometric invariants, notably Gaussian curvature. Intuitive and computational aspects are stressed to give geometrical meaning to the theory in Chapter 6.

In the final two chapters, although our methods are unchanged, there is a radical shift of viewpoint. Roughly speaking, we study the geometry of a surface *as seen by its inhabitants*, with no assumption that the surface can be found in ordinary three-dimensional space. Chapter 7 is dominated by *curvature* and culminates in the Gauss-Bonnet theorem and its geometric and topological consequences. In particular, we use the Gauss-Bonnet theorem to prove the Poincaré-Hopf theorem, which relates the singularities of a vector field on M to the topology of M.

Chapter 8 studies the local and global properties of geodesics. Full development of the global properties requires the notion of covering surface. With it, we can give a comprehensive survey of the surfaces of constant Gaussian curvature and prove the theorems of Bonnet and Hadamard on, respectively, positive and nonnegative curvature.

No branch of mathematics makes a more direct appeal to the intuition than geometry. I have sought to emphasize this by a large number of illustrations that form an integral part of the text.

Each chapter of the book is divided into sections, and in each section a single sequence of numbers designates collectively the theorems, lemmas, examples, and so on. Each section ends with a set of exercises; these range from routine checks of comprehension to moderately challenging problems.

In this revision, the structure of the text, including the numbering of its contents, remains the same, but there are many changes around this framework. The most significant are, first, correction of all known errors; second, a better way of referencing exercises (the most common reference); third, general improvement of the exercises. These improvements include deletion of a few unreasonably difficult exercises, simplification of others, and fuller answers to odd-numbered ones.

In teaching from earlier versions of this book, I have usually covered the background material in Chapter 1 rather rapidly and not devoted any classroom time to Chapter 3. A short course in the geometry of curves and surfaces in 3-space might consist of Chapter 2 (omit Sec. 8), Chapter 4 (omit Sec. 8), Chapter 5, Chapter 6 (covering Secs. 6–9 lightly), and a leap to Section 6 of Chapter 7: the Gauss-Bonnet theorem. This is essentially the content of a traditional undergraduate course in differential geometry, with clarification of the notions of surface and mapping.

Such a course, however, neglects the shift of viewpoint mentioned earlier, in which the geometric concept of surface evolved from a *shape* in 3-space to an independent entity—a two-dimensional *Riemannian manifold*.

This development is important from a practical viewpoint since it makes surface theory applicable throughout the range of scientific applications where 2-parameter objects appear that meet the requisite conditions—for example, in the four-dimensional manifolds of general relativity.

Such a surface is logically simpler than a surface in 3-space since it is constructed (at the start of Chapter 7) by discarding effects of Euclidean space. However, readers can neglect this transition and—as suggested for the Gauss-Bonnet theorem—proceed directly to most of the topics considered in the final two chapters, for example, properties of geodesics (length-minimization and completeness), singularities of vector fields, and the theorems of Bonnet and Hadamard.

For readers with access to a computer containing either the *Mathematica* or *Maple* computation system, I have included some forty computer exercises. These offer an opportunity to amplify the text in various ways.

Previous computer experience is not required. The Appendix contains a summary of the syntaxes of the most recent versions of *Mathematica* and *Maple*, together with a list of explicit computer commands covering the basic geometry of curves and surfaces. Further commands appear in the answers to exercises.

It is important to go, step by step, through the hand calculation of the Gaussian curvature of a parametrized surface, but once this is understood, repetition becomes tedious. A surface in $\mathbf{R}^3$ given only by a formula is seldom easy to sketch. But using computer commands, a picture of a surface can be drawn and its curvature computed, often in no more than a few seconds. Analogous remarks hold for space curves.

Among other applications appearing in the exercises, the most valuable, since unreachable for humans, is the numerical solution of differential equations—and the plotting of these solutions.

This book would not have been possible without generous contributions by Allen B. Altman and Joseph E. Borzellino.

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Introduction

This book presupposes a reasonable knowledge of elementary calculus and linear algebra. It is a working knowledge of the fundamentals that is actually required. The reader will, for example, frequently be called upon to *use* the chain rule for differentiation, but its proof need not concern us.

Calculus deals mostly with real-valued functions of one or more variables, linear algebra with functions (linear transformations) from one vector space to another. We shall need functions of these and other types, so we give here general definitions that cover all types.

A *set* S is a collection of objects that are called the *elements* of S. A set A is a *subset* of S provided each element of A is also an element of S.

A *function* f from a set D to a set R is a rule that assigns to each element x of D a unique element $f(x)$ of R. The element $f(x)$ is called the *value* of f at x. The set D is called the *domain* of f; the set R is sometimes called the *range* of f. If we wish to emphasize the domain and range of a function f, the notation $f: D \to R$ is used. Note that the function is denoted by a single letter, say f, while $f(x)$ is merely a value of f.

Many different terms are used for functions—mappings, transformations, correspondences, operators, and so on. A function can be described in various ways, the simplest case being an explicit formula such as

$$f(x) = 3x^2 + 1,$$

which we may also write as $x \to 3x^2 + 1$.

If both f_1 and f_2 are functions from D to R, then $f_1 = f_2$ means that $f_1(x) = f_2(x)$ for all x in D. This is not a definition, but a logical consequence of the definition of *function*.

Let $f: D \to R$ and $g: E \to S$ be functions. In general, the *image* of f is the subset of R consisting of all elements of the form $f(x)$; it is usually denoted by $f(D)$. If this image happens to be a subset of the domain E of g, it is possible to combine these two functions to get the *composite function* $g(f)\colon D \to S$. By definition, $g(f)$ is the function whose value at each element x of D is the element $g(f(x))$ of S.

If $f\colon D \to R$ is a function and A is a subset of D, then the *restriction* of f to A is the function $f|A\colon A \to R$ defined by the same rule as f, but applied only to elements of A. This seems a rather minor change, but the function $f|A$ may have properties quite different from f itself.

Here are two vital properties that a function may possess. A function $f\colon D \to R$ is *one-to-one* provided that if x and y are any elements of D such that $x \neq y$, then $f(x) \neq f(y)$. A function $f\colon D \to R$ is *onto* (or *carries D onto R*) provided that for every element y of R there is at least one element x of D such that $f(x) = y$. In short, the image of f is the entire set R. For example, consider the following functions, each of which has the real numbers as both domain and range:

(1) The function $x \to x^3$ is both one-to-one and onto.
(2) The exponential function $x \to e^x$ is one-to-one, but not onto.
(3) The function $x \to x^3 + x^2$ is onto, but not one-to-one.
(4) The sine function $x \to \sin x$ is neither one-to-one nor onto.

If a function $f\colon D \to R$ is both one-to-one and onto, then for each element y of R there is one and only one element x such that $f(x) = y$. By defining $f^{-1}(y) = x$ for all x and y so related, we obtain a function $f^{-1}\colon R \to D$ called the *inverse* of f. Note that the function f^{-1} is also one-to-one and onto, and that *its* inverse function is the original function f.

Here is a short list of the main notations used throughout the book, in order of their appearance in Chapter 1:

$\mathbf{p}, \mathbf{q}$	points	(Section 1.1)
f, g	real-valued functions	(Section 1.1)
$\mathbf{v}, \mathbf{w}$	tangent vectors	(Section 1.2)
V, W	vector fields	(Section 1.2)
α, β	curves	(Section 1.4)
ϕ, ψ	differential forms	(Section 1.5)
F, G	mappings	(Section 1.7)

In Chapter 1 we define these concepts for Euclidean 3-space. (Extension to arbitrary dimensions is virtually automatic.) In Chapter 4 we show how these concepts can be adapted to a surface.

A few references are given to the brief bibliography at the end of the book; these are indicated by initials in square brackets.