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Differential forms as a basis for vector analysis—with applications to electrodynamics

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(Received 26 March 1982; accepted for publication 25 August 1982)

I give a concise and self-contained presentation of the theory of differential forms and how it subsumes all of classical vector analysis when applied to a three-dimensional space. The differential form analog of all vector operators, identities, and theorems are given (many of which are proved), all in the context of classical electrodynamics.

I. INTRODUCTION

Vector analysis is usually introduced to physicists in the context of classical electrodynamics. The theorems of Stokes and Gauss allow one to convert integral equations involving the **E** and **B** fields into differential equations—the Maxwell equations. The similarity between the two theorems is striking, in that each of them states that the integral of some type of derivative of a vector field over a region of certain dimension is equal to the integral of the vector field over the boundary of that region. And yet there does not seem to be a deeper underlying structure connecting the two theorems. Search as one may, classical vector analysis does not supply any answers.

Though this conceptual coincidence may be the most compelling one, it is by no means unique. Take, for example, the fact that curl·grad = 0 and div·curl = 0. The former equation applies of course to a scalar field and the latter to a vector field. Could there possibly exist a more general type of "derivative" operator that would give us *both* these equations? Moreover, could not this new type of derivative shed light on the connection between Gauss's and Stokes's theorems?

A deeper and unifying structure does indeed exist, and is provided by the theory of differential forms. Mathematicians invented forms at the beginning of the century and physicists working in general relativity have been using them for some twenty years. Forms give one a certain ease in calculations, elegance of notation and, usually, a more generalized viewpoint.

The theory of differential forms is partially motivated by the consideration of how integrands of multiple integrals transform under coordinate transformations. Consider the integral of the scalar function $f(x,y)$ over a two-dimensional region R: $\int_R f(x,y)dx\,dy$. Under the transformation to polar coordinates, $x = r\cos\theta$, $y = r\sin\theta$, the integral becomes $\int_R f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$. That is, the function $f(x,y)$ transforms as a scalar field and the term $dx\,dy \rightarrow r\,dr\,d\theta$ (or the Jacobian multiplied by $dr\,d\theta$). One does *not* calculate dx and dy as functions of dr and $d\theta$ and then multiply them. The point is that the integral and $dx\,dy$ exist as a whole and hence must be transformed as such. It is conceivable though that one could construct an algebra of quantities like dx and dy that, among other things, would transform appropriately under coordinate transformations. The theory of differential forms accomplishes this.

In the course of studying differential forms, I came across the Hodge decomposition theorem which states: Every p-form can be decomposed into the sum of the (exterior) derivative of a $(p-1)$-form, the coderivative of a $(p+1)$-form and a harmonic term, and, moreover, the decomposition is unique. I realized this to be the differential form analog of the well-known Helmholtz theorem, used in electrodynamics, which states: A vector field can be uniquely decomposed into the gradient of some scalar field, the curl of some vector field and a vector that satisfies Laplace's equation.

It became apparent that one could translate any statement about vectors into a statement about forms. In fact, as mathematicians well know, the *entire* classical vector analysis is subsumed by the theory of differential forms as applied to a three-dimensional space. One can derive all the theorems in vector analysis from theorems about forms in addition to obtaining all the standard vector identities—most of them quite effortlessly. I intend to use this latter fact as a vehicle to introduce the theory of differential forms, by first outlining the general theory, then by showing the operation on forms that corresponds to each operation on vectors, and then proving many of the standard vector identities.

Section II will be an unrigorous but rather intuitive account of the theory of differential forms. This will then be followed by (Sec. III) the application of forms to a three-dimensional space and the resulting connection to classical vector analysis will be made. The standard vector identities will be proved and theorems about vectors will be derived from theorems about forms (not all of which will be proved). I will then conclude with a brief section (Sec. IV) on how Maxwell's equations can be phrased in the language of forms.

II. THEORY OF DIFFERENTIAL FORMS

A. Exterior algebra: The algebra of forms[1]

I will be working in an n-dimensional Euclidean space with rectangular coordinates $x^1, x^2, \ldots, x^n$, a point being described by $x = (x^1, x^2, \ldots, x^n)$.

At each point x we consider objects dx^i, $i = 1,2,\ldots,n$, called our *basis one-forms*. An arbitrary one-form ω, will be a linear combination of basis one-forms: $\omega = \Sigma_{i=1}^{n} a_i\, dx^i$. That is, the set of one-forms at a point form a vector space.[2] The sum of two one-forms $\omega_1 = \Sigma a_i\, dx^i$ and $\omega_2 = \Sigma b_i\, dx^i$ is given by $\omega_1 + \omega_2 = \Sigma(a_i + b_i)dx^i$.

One can construct new objects called *2-forms* by defining a product of basis one-forms, called the *exterior* or wedge product. Thus a typical basis 2-form will be $dx^i \wedge dx^j$ with $\wedge$ denoting the wedge product. Once again, an arbitrary 2-form will be given by a linear combination of basis 2-forms. The wedge product is to satisfy the following rules: (a) $dx^i \wedge dx^j = -dx^j \wedge dx^i$, i.e., the product is *anti*commutative; (b) $(\Sigma a_i\, dx^i) \wedge dx^j = \Sigma a_i (dx^i \wedge dx^j)$, i.e., the product is distributive. Note that in particular, (a) says that $dx^i \wedge dx^j = 0$ for $i = j$. Thus, since $dx^i \wedge dx^j = -dx^j \wedge dx^i$, the number of independent 2-forms at a point is given by $n(n-1)/2$.

One can extend the above product to the multiplication of an arbitrary number of one-forms to get a basis *p-form*: $dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_p}$. Note, however, that if the dimensionality of the space is n and if $p > n$, then at least one of the dx^i's is duplicated in this product. By anticommuting the necessary number of times we will obtain in the product a term of the form $dx^i \wedge dx^i$—which is zero. Hence a p-form is zero for $p > n$, and we need only consider p-forms for $p \leqslant n$.

Note again that an arbitrary p-form ω will be given by $\omega = \Sigma_{i_1 \ldots i_p} a_{i_1 \ldots i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p}$. By using the same argument as I did for 2-forms, it is obvious that the number of independent basis p-forms is $n!/p!(n-p)!$.

For example, in a three-dimensional space, the basis one-forms at a point would be dx^1, dx^2, and dx^3. The basis 2-forms would be $dx^1 \wedge dx^2$, $dx^1 \wedge dx^3$, and $dx^2 \wedge dx^3$. There is only one basis 3-form: $dx^1 \wedge dx^2 \wedge dx^3$. Thus arbitrary one-, two-, and three-forms would be written, respectively, as

$$\phi = a_1\, dx^1 + a_2\, dx^2 + a_3\, dx^3 = \Sigma a_i\, dx^i,$$

$$\omega = a_{12}\, dx^1 \wedge dx^2 + a_{13}\, dx^1 \wedge dx^3 + a_{23}\, dx^2 \wedge dx^3$$

$$= \sum_{i<j} a_{ij}\, dx^i \wedge dx^j,$$

$$\eta = a\, dx^1 \wedge dx^2 \wedge dx^3.$$

Now all of our considerations were at a *point* of the underlying space $x^1, x^2, \ldots, x^n$. I can extend these ideas to a *field* of forms by defining forms at each point such that they are smoothly connected. Thus a field of one-forms would be given by $\omega = \Sigma a_i(x) dx^i$ where $a_i(x)$ is a set of differentiable functions rather than a set of numbers. This of course defines a one-form at any given point $x = x_0$ by $\omega(x_0) = \Sigma a_i(x_0) dx^i$.

This is generalized to a field of p-forms in the obvious way, and then one has for an arbitrary p-form,

$$\omega = \sum_{i_1 \ldots i_p} a_{i_1 \ldots i_p}(x) dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_p}.$$

A zero-form is defined as a scalar field.

I now introduce a key idea that turns out to be at the heart of the possibility of introducing a vector calculus. This is the Hodge star operator, or dual, which associates to every p-form ω an $(n-p)$-form $*\omega$ (this dual is not to be confused with the idea of dual vector space).

The dual of a basis p-form $dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_p}$ is defined to be the $(n-p)$-form $\epsilon_{i_1 \ldots i_p i_{p+1} \ldots i_n} dx^{i_{p+1}} \wedge \cdots \wedge dx^{i_n}$, where the set $\{i_1, \ldots, i_p, i_{p+1}, \ldots, i_n\} = \{\text{integers from 1 to } n\}$, i.e., the $i_{p+1}, \ldots, i_n$ "complete" the set $i_1, \ldots, i_p$.

Taking again a three-dimensional space as an example, one has

$$*dx^1 = dx^2 \wedge dx^3,\quad *dx^2 = dx^3 \wedge dx^1,\quad *dx^3 = dx^1 \wedge dx^2.$$

Also

$$*(dx^1 \wedge dx^2) = dx^3,\quad *(dx^1 \wedge dx^3) = -dx^2,$$

$$*(dx^2 \wedge dx^3) = dx^1,$$

$$*f = f\, dx^1 \wedge dx^2 \wedge dx^3,\quad *(dx^1 \wedge dx^2 \wedge dx^3) = 1.$$

The dual of an arbitrary p-form is then defined to be the sum of the duals of its component elements, i.e., the dual is a linear operator. Note that in 3-space one has $*(*\omega) = \omega$.

As we will see later, the dual will allow us (in 3-space) to identify a pseudovector with a vector, thus allowing for operations such as vector cross product and curl.

Before closing this section I would like to mention an important identity that forms satisfy: $\omega \wedge \eta = (-)^{pq} \eta \wedge \omega$, where ω is a p-form and η is a q-form. This is obvious for ω a basis p-form and η a basis q-form, since each of the q one-forms of η must be commuted through each of the p one-forms of ω thus introducing a factor of $(-)^{pq}$. The result for abitrary forms follows by distributivity.

The following is a summary of the algebraic properties of forms:

A(1): A general p-form can be written as

$$\omega = \sum_{i_1 \ldots i_p} a_{i_1 \ldots i_p}(x) dx^{i_1} \wedge \cdots \wedge dx^{i_p}.$$

A(2): If $\omega = \Sigma a_{i_1 \ldots i_p}\, dx^{i_1} \wedge \cdots \wedge dx^{i_p}$ and $\eta = \Sigma b_{i_1 \ldots i_p}\, dx^{i_1} \wedge \cdots \wedge dx^{i_p}$ then

$$\omega + \eta = \Sigma(a_{i_1 \ldots i_p} + b_{i_1 \ldots i_p}) dx^{i_1} \wedge \cdots \wedge dx^{i_p}.$$

If ω is a p-form and η a q-form, we have

A(3): $\omega \wedge \eta = \sum_i \sum_j a_{i_1 \ldots i_p} b_{j_1 \ldots j_q} dx^{i_1} \wedge \cdots \wedge dx^{i_p} \wedge dx^{j_1}$

$\wedge \cdots \wedge dx^{j_q}$,

i.e., associativity and distributivity, and
A(4): $\omega \wedge \eta = (-)^{pq} \eta \wedge \omega$.

B. Exterior derivative

The *exterior derivative* $d\omega$ of the p-form ω is defined to be the $(p+1)$-form,

$$d\omega = \sum_{j, i_1 \cdots i_p} \frac{\partial a_{i_1 \ldots i_p}}{\partial x^j} dx^j \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p}.$$

Note that if ω is an n-form then $d\omega = 0$.

We will see later that the d operator embodies in it (for $n = 3$) the gradient, curl, and divergence operators, depending on whether ω is a zero-, one-, or two-form.

To get a hint of this, let us examine what $d\omega$ looks like for various ω. If ω is a zero-form, i.e., a scalar function f, then

$$d\omega = df = \sum \frac{\partial f}{\partial x^i} dx^i,$$

which is reminiscent of the gradient operator. If ω is a one-form, $\omega = \phi = \Sigma a_i \, dx^i$ then

$$d\phi = \sum \frac{\partial a_i}{\partial x^j} dx^j \wedge dx^i,$$

and since $dx^i \wedge dx^j$ is skew-symmetric,

$$d\phi = \tfrac{1}{2} \sum_{i,j} \left(\frac{\partial a_i}{\partial x^j} - \frac{\partial a_j}{\partial x^i} \right) dx^j \wedge dx^i$$

$$= \sum_{j<i} \left(\frac{\partial a_i}{\partial x^j} - \frac{\partial a_j}{\partial x^i} \right) dx^j \wedge dx^i.$$

Thus if $n = 3$,

$$d\phi = \left(\frac{\partial a_2}{\partial x^1} - \frac{\partial a_1}{\partial x^2} \right) dx^1 \wedge dx^2 + \left(\frac{\partial a_3}{\partial x^1} - \frac{\partial a_1}{\partial x^3} \right) dx^1 \wedge dx^3$$

$$+ \left(\frac{\partial a_3}{\partial x^2} - \frac{\partial a_2}{\partial x^3} \right) dx^2 \wedge dx^3,$$

which is reminiscent of the curl operator. The precise connections will be made in Sec. III.

The following useful lemmas can easily be proved[3]:

L(1): $d(p \wedge q) = dp \wedge q + (-)^p p \wedge dq$. This applies to arbitrary p-forms and q-forms. In particular, if f and g are functions, and ϕ and ψ are one-forms, we have

(a) $d(fg) = df\, g + f\, dg$,

(b) $d(f\phi) = df \wedge \phi + f\, d\phi$,

(c) $d(\phi \wedge \psi) = d\phi \wedge \psi - \phi \wedge d\psi$.

L(2): If

$$\phi_i = \sum_{j=1}^{3} f_{ij}\, dx^j \quad 1 \leqslant i \leqslant 3,$$

then

$$\phi_1 \wedge \phi_2 \wedge \phi_3 = \begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{vmatrix} dx^1 \wedge dx^2 \wedge dx^3.$$

L(3): $*(*\omega) = \omega$ (for $n = 3$).

L(4): $\alpha \wedge *\beta = \beta \wedge *\alpha$, where α and β are both p-forms.

The definition of the exterior derivative d gives us an alternate interpretation of the one-form dx^i. By considering the coordinate x^i as a *function* (on whichever point of the space it is), we can take its exterior derivative:

$$dx^i = \sum_j \frac{\partial x^i}{\partial x^j} dx^j = \sum_j \delta_{ij}\, dx^j = dx^i.$$

Thus the one-form dx^i can be thought of as the exterior derivative of the coordinate function x^i. With this interpretation we can see how forms will transform if we undertake a coordinate transformation on the underlying space. Particularly if $(x^1,\ldots,x^n) \rightarrow (\bar{x}^1,\ldots,\bar{x}^n)$, or $\bar{x}^i = f_i(x)$, i.e., we can think of the new coordinates as functions defined on the space, then

$$d\bar{x}^i = \sum_j \frac{\partial f_i}{\partial x^j} dx^j.$$

Applying L(2), we then have $d\bar{x}^1 \wedge d\bar{x}^2 \wedge d\bar{x}^3 =$ (Jacobian) $dx^1 \wedge dx^2 \wedge dx^3$. This explains the transformation properties of integrands. I return to the example given in Sec. I of $\int f(x,y) dx\, dy$. When integration is properly formulated in terms of forms, this would actually read $\int f(x,y) dx \wedge dy$. Then under the coordinate transformation

$x = r \cos\theta$, $y = r \sin\theta$:

$f(x,y) dx \wedge dy = f(r\cos\theta, r\sin\theta) J(x,y;r,\theta) dr \wedge d\theta$,

by L(2). Thus if one wants to treat dx and dy as algebraic objects one can do so provided the algebra of forms is used.

In addition to the exterior derivative d, one can define the coderivative δ, by $\delta\omega = *(d\, *\omega)$. The coderivative thus maps a p-form into a $(p-1)$-form. The combination $d\delta + \delta d$ is defined as the generalized Laplacian operator, since for $n = 3$ it is precisely that classical operator (see Appendix A).[4]

C. Important theorems

The three fundamental theorems of differential forms:

T(1): Poincare lemma. If ω is an arbitrary p-form, then $d^2\omega = 0$. The proof is essentially trivial:

$$\omega = \sum_{i\text{'s}} a_{i_1 \ldots i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p},$$

$$d\omega = \sum_{j, i\text{'s}} \frac{\partial a_{i_1 \ldots i_p}}{\partial x^j} dx^j \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p},$$

$$d^2\omega = \sum_{k,j,i\text{'s}} \frac{\partial^2 a_{i_1 \ldots i_p}}{\partial x^k\, \partial x^j} dx^k \wedge dx^j \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p}$$

$$= \sum_{j,k,i\text{'s}} \frac{\partial^2 a_{i_1 \ldots i_p}}{\partial x^j \partial x^k} dx^j \wedge dx^k \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p}$$

by interchanging dummy indices j and k, and this is equal to

$$-\sum_{j,k,i\text{'s}} \frac{\partial^2 a_{i_1 \ldots i_p}}{\partial x^k\, \partial x^j} dx^k \wedge dx^j \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p} = -d^2\omega = 0.$$

by the skew-symmetry of $dx^j \wedge dx^k$, and the equality of mixed partial derivatives.

This will prove to be a very powerful theorem.

T(2): Converse of Poincare lemma. If $d\lambda = 0$, where λ is a p-form, then there exists a $(p-1)$-form ω such that $\lambda = d\omega$. This has certain restrictions on the topological nature of the domain of its validitiy, and I will not state its proof here.

T(3): Hodge decomposition theorem. Any p-form can be written as $\omega = d\alpha + \delta\beta + \gamma$, where α is a $(p-1)$-form, β a $(p+1)$-form, and γ a harmonic p-form, i.e., satisfying Laplace's equation $\Delta\gamma = 0$. The forms $d\alpha$, $\delta\beta$, and γ are unique.

Finally, one can prove, by defining integration appropriately (which we do not do in this paper—see, however, the comment on integration in Sec. II B, and also Appendix D):

T(4): Generalized Stokes theorem. If ω is a p-form and D a $(p+1)$-dimensional domain, then $\int_{\partial D}\omega = \int_D d\omega$. that is, the integral of ω over the boundary of the region D is equal to the integral of the exterior derivative over the region D. In three-space, when ω is, respectively, a one- and two-form, we will see that this theorem encompasses the duo of theorems, Stokes and Gauss. Moreover, for ω a zero-form we get the fundamental theorem of calculus.

III. VECTOR ANALYSIS

A. Vector-form correspondences

The key idea is that one can associate to every vector a one-form and to every operation on vectors an operation on forms.

Since I will be doing vector analysis in a three-dimensional space, I restrict the space on which I have defined form to be $n = 3$.

The basic correspondence that one makes is that to every covariant vector field $\mathbf{A} = A_1\hat{x}_1 + A_2\hat{x}_2 + A_3\hat{x}_3$, one associates the one-form $\phi_A = A_1, dx^1 + A_2\,dx^2 + A_3\,dx^3$. Symbolically, I write

$$\mathbf{A} = A_1\hat{x}_1 + A_2\hat{x}_2 + A_3\hat{x}_3 \leftrightarrow$$
$$\phi_A = A_1\,dx^1 + A_2\,dx^2 + A_3\,dx^3.$$

ϕ_A will always represent the one-form associated with the vector **A**.[5] The symbol $\leftrightarrow$ denotes the association of a vector to an appropriate form.

To the function f one associates the 0-form f.

I now show how to obtain the correspondences between *operations* on vectors and operation on forms.

1. $\mathbf{A}\times\mathbf{B} = (A_2B_3 - A_3B_2)\hat{x}_1 + (A_3B_1 - A_1B_3)\hat{x}_2 + (A_1B_2 - A_2B_1)\hat{x}_3$

I would like to know what operation on ϕ_A and ϕ_B will yield a form $\phi_{\mathbf{A}\times\mathbf{B}}$, i.e., the form which is associated to the vector $\mathbf{A}\times\mathbf{B}$. Note that

$$\begin{aligned}\phi_A \wedge \phi_B &= (A_1\,dx^1 + A_2\,dx^2 + A_3\,dx^3)\\ &\quad\wedge(B_1\,dx^1 + B_2\,dx^2 + B_3\,dx^3)\\ &= (A_1B_2 - A_2B_1)dx^1\wedge dx^2 + (A_2B_3 - A_3B_2)dx^2\wedge dx^3\\ &\quad + (A_3B_1 - A_1B_3)dx^3\wedge dx^1.\end{aligned}$$

The coefficients of the basis forms look right, but this is a *two*-form, so how do I associate a vector to it? Simple, I first take its dual—which gives me a one-form—and then associate a vector to this one-form. So I claim that

$$\mathbf{A}\times\mathbf{B} = \mathbf{C} \leftrightarrow *(\phi_A\wedge\phi_B) = \phi_C.$$

Pictorially, I write

$$\begin{matrix}\mathbf{A}\times\mathbf{B} = \mathbf{C}\\ \updownarrow \quad \updownarrow \quad \updownarrow\\ *(\phi_A\wedge\phi_B) = \phi_C.\end{matrix}$$

To repeat, I can mimic the cross product between vectors, by taking the dual of the exterior product of the corresponding forms. Thus to find the vector given by the cross product of the vectors **A** and **B**, I write down the associated one-forms ϕ_A and ϕ_B, take the dual of their exterior product, which turns out to be a one-form, and then find the vector associated to this one-form.

2. $\mathbf{A}\cdot\mathbf{B} = A_1B_1 + A_2B_2 + A_3B_3$

Since $\mathbf{A}\cdot\mathbf{B}$ is a scalar, something must be done with ϕ_A and ϕ_B as to end up with a 0-form. It must be an algebraic operation since there are no derivatives involved. The only[6] possibility is $*(\phi_A\wedge *\phi_B)$. Note that ϕ_A is a one-form, ϕ_B a one-form, so $*\phi_B$ is a 2-form, $\phi_A\wedge *\phi_B$ a 3-form, and $*(\phi_A\wedge *\phi_B)$ a 0-form. So we guess: $\mathbf{A}\cdot\mathbf{B}\leftrightarrow *(\phi_A\wedge *\phi_B)$. It turns out that by using this type of argument, one can get *all* the vector operations. To check our conjecture:

$$\begin{aligned}*(\phi_A\wedge *\phi_B) &= *[(A_1\,dx^1 + A_2\,dx^2 + A_3\,dx^3)\\ &\quad\wedge *(B_1\,dx^1 + B_2\,dx^2 + B_3\,dx^3)]\\ &= *[(A_1\,dx^1 + A_2\,dx^2 + A_3\,dx^3)\\ &\quad\wedge(B_1\,dx^2\wedge dx^3 + B_2\,dx^3\wedge dx^1 + B_3\,dx^1\wedge dx^2)]\\ &= *[(A_1B_1 + A_2B_2 + A_3B_3)\,dx^1\wedge dx^2\wedge dx^3]\\ &= A_1B_1 + A_2B_2 + A_3B_3.\end{aligned}$$

One can check that $*(\phi_B\wedge *\phi_A) = *(\phi_A\wedge *\phi_B)$, as it indeed should be since $\mathbf{B}\cdot\mathbf{A} = \mathbf{A}\cdot\mathbf{B}$.

I now demonstrate this procedure for a vector operation involving *differentiation*, and then merely list all the other correspondences.

3. $$\nabla\times\mathbf{B} = \left(\frac{\partial B_3}{\partial x^2} - \frac{\partial B_2}{\partial x^3}\right)\hat{x}_1 + \left(\frac{\partial B_1}{\partial x^3} - \frac{\partial B_3}{\partial x^1}\right)\hat{x}_2 + \left(\frac{\partial B_2}{\partial x^1} - \frac{\partial B_1}{\partial x^2}\right)\hat{x}_3.$$

This clearly looks like $d\phi_B$—which is a 2-form (see Sec. II B)—so we take its dual and guess that $\nabla\times\mathbf{B}\leftrightarrow *(d\phi_B)$. To check it:

$$\begin{aligned}*(d\phi_B) &= *[d(B_1\,dx^1 + B_2\,dx^2 + B_3\,dx^3)]\\ &= *\left[\left(\frac{\partial B_1}{\partial x^2}dx^2\wedge dx^1 + \frac{\partial B_1}{\partial x^3}dx^3\wedge dx^1\right)\right.\\ &\quad + \left(\frac{\partial B_2}{\partial x^1}dx^1\wedge dx^2 + \frac{\partial B_2}{\partial x^3}dx^3\wedge dx^2\right)\\ &\quad\left. + \left(\frac{\partial B_3}{\partial x^1}dx^1\wedge dx^3 + \frac{\partial B_3}{\partial x^2}dx^2\wedge dx^3\right)\right]\\ &= \left(\frac{\partial B_3}{\partial x^2} - \frac{\partial B_2}{\partial x^3}\right)dx^1 + \left(\frac{\partial B_1}{\partial x^3} - \frac{\partial B_3}{\partial x^1}\right)dx^2\\ &\quad + \left(\frac{\partial B_2}{\partial x^1} - \frac{\partial B_1}{\partial x^2}\right)dx^3,\end{aligned}$$

which clearly $\leftrightarrow\nabla\times\mathbf{B}$.

I now give a complete list of the various correspondences.

C(1): $\mathbf{A} = A_1\hat{x}_1 + A_2\hat{x}_2 + A_3\hat{x}_3 \leftrightarrow A_1\,dx^1 + A_2\,dx^2 + A_3\,dx^3 = \phi_A$.

C(2): f (as a scalar function) $\leftrightarrow f$ (as a 0-form).

C(3): $\mathbf{A}\cdot\mathbf{B}\leftrightarrow(\phi_A \wedge {*}\phi_B)\equiv{*}(\phi_B \wedge {*}\phi_A)$.
C(4): $\nabla\cdot\mathbf{B}\leftrightarrow{*}[d({*}\phi_B)]$.
C(5): $\nabla f\leftrightarrow df$.
C(6): $\mathbf{A}\times\mathbf{B}\leftrightarrow{*}(\phi_A \wedge \phi_B)$.
C(7): $\nabla\times\mathbf{B}\leftrightarrow{*}(d\phi_B)$.

The ϕ 's are all *one*-forms, allowing one to unambiguously make the distinction between C(4) and C(7), and C(3) and C(6).

B. Vector identities

I list the various identities and the corresponding statements in the language of forms.

I(1): $\nabla(fg) = f\nabla g + g\nabla f\leftrightarrow d(fg) = f\,dg + g\,df$.

I(2): $\nabla\cdot(f\mathbf{A}) = (\nabla f\cdot\mathbf{A}) + f(\nabla\cdot\mathbf{A})\leftrightarrow{*}[d\,{*}(f\phi_A)]$

$= {*}(df\wedge{*}\phi_A) + f{*}(d\,{*}\phi_A)$.

I(3): $\nabla(\mathbf{A}\times\mathbf{B})$
$= \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})\leftrightarrow{*}[d\,{*}\,{*}(\phi_A\wedge\phi_B)]$
$= {*}[\phi_B\wedge d\phi_A] - {*}[\phi_A\wedge d\phi_B]$.

I(4): $\nabla\times(f\mathbf{A}) = (\nabla f)\times\mathbf{A} + f(\nabla\times\mathbf{A})\leftrightarrow{*}[d(f\phi_A)]$

$= {*}(df\wedge\phi_A) + f{*}(d\phi_A)$.

I(5): $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A})\leftrightarrow{*}(\phi_A\wedge\phi_B\wedge\phi_C)$

$= {*}(\phi_C\wedge\phi_A\wedge\phi_B) = {*}(\phi_B\wedge\phi_C\wedge\phi_A)$.

I(6): $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$
$= \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})\leftrightarrow{*}[\phi_A\wedge{*}(\phi_B\wedge\phi_C)]$
$= \phi_B{*}[\phi_A\wedge{*}\phi_C] - \phi_C{*}[\phi_A\wedge{*}\phi_B]$.

I(7): $(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D}) = (\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{D}) - (\mathbf{A}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{C})\leftrightarrow$
${*}[{*}(\phi_A\wedge\phi_B)\wedge(\phi_C\wedge\phi_D)]$
$= {*}(\phi_A\wedge\phi_C){*}(\phi_B\wedge{*}\phi_D)$
$- {*}(\phi_A\wedge{*}\phi_D){*}(\phi_B\wedge{*}\phi_C)$.

All of these correspondences of identities are obtained by using C(1)–C(7).

The proofs of the form identities are mostly straightforward.[7] The first four identities are *all* contained in L(1), which states that $d(p\wedge q) = dp\wedge q + (-)^p p\wedge dq$. I(5) follows from A(4). The proofs of I(6) and I(7) are contained in Appendices B and C (they are the only ones that are not totally trivial).

I(8): $\nabla\times(\nabla\varphi) = 0\leftrightarrow{*}(d^2 f) = 0$.
I(9): $\nabla\cdot(\nabla\times\mathbf{A}) = 0\leftrightarrow{*}(d^2\phi_A) = 0$.

These two form identities follow simply from the Poincare lemma T(1). The only distinction between the two is that of applying the lemma to a 0-form and to a one-form.

One thus sees that many vector identities follow from the *same* form identity, only applied to forms of different orders. This is in fact characteristic of most all vector theorems and identities—the distinction between various vector statements is one only of degree in terms of forms.

C. Vector theorems

The following two theorems follow from the converse of the Poincare lemma:

VT(1): $\nabla\times\mathbf{E} = 0\Rightarrow\exists$ scalar ψ such that $\mathbf{E} = \nabla\psi$ $\leftrightarrow{*}(d\omega) = d\omega = 0$, with ω a one-form $\Rightarrow\exists$ a 0-form α such that $\omega = d\alpha$.

VT(2): $\nabla\cdot\mathbf{B} = 0\Rightarrow\exists$ a vector $\mathbf{A}$ such that $\mathbf{B} = \nabla\times\mathbf{A}$ $\leftrightarrow{*}(d\,{*}\omega) = 0\Rightarrow\exists$ a one-form β such that ${*}\omega = d\beta$ or $\omega = {*}(d\beta)$.

The first is simply the converse as applied to a one-form and the second as applied to a two-form.

Similarly, using the Hodge decomposition theorem T(3), I get

VT(3): For any vector field $\mathbf{A}$, $\exists\, f,\mathbf{B}$, and $\mathbf{Q}$ such that arbitrary $\mathbf{A} = \nabla f + \nabla\times\mathbf{B} + \mathbf{Q}$, where $\nabla^2\mathbf{Q} = 0\leftrightarrow$ For any p-form ω, $\exists\,\alpha,\beta$, and γ such that $\omega = d\alpha + \delta\beta + \gamma$ (γ harmonic).

I thus obtain the Helmholtz theorem. The above was obtained with ω being a one-form. If I choose ω to be a 2-form, an identical theorem is obtained.

Finally, making use of the generalized Stokes theorem T(4):

VT(4): Gauss: $\int_V \nabla\cdot\mathbf{C}\,dV = \int_S \mathbf{C}\cdot d\,\mathbf{A}\leftrightarrow\int_D d\,{*}(\phi_C) = \int_{\partial D}{*}\phi_C$.

VT(5): Stokes: $\int_S(\nabla\times\mathbf{B})\cdot d\,\mathbf{A} = \oint\mathbf{B}\cdot d\,\mathbf{l}\leftrightarrow\int_D d\phi_B = \int_{\partial D}\phi_B$.

The first of the above is obtained by choosing in T(4), $\omega = {*}\phi_C$, a 2-form, and the second by choosing $\omega = \phi_B$, a one-form. See Appendix D for the exact statement as to how these relationships are obtained.

IV. ELECTRODYNAMICS

For completeness, I conclude with a brief description of how Maxwell's equations are expressed in the language of forms.[8]

Since the appropriate underlying space for electrodynamics is Minkowski space—which is *pseudo*-Euclidean rather than Euclidean—one needs to generalize the previously developed concept of duality to a space with arbitrary metric (the previous sections assumed a metric that was specifically Euclidean). In addition, one needs to extend the association between vectors and 1-forms to an association between higher-rank tensors and forms.

The more general definition of dual is given in terms of the inner product defined on the space.[9] The definition I use, however, will be more restrictive, though sufficient for our purposes.

The dual of a 1-form $J = J_\mu\,dx^\mu$ is defined as the 3-form ${*}J = \epsilon_{\mu\alpha\beta\gamma}J^\mu\,dx^\alpha\wedge dx^\beta\wedge dx^\gamma$, and the dual of the 2-form $F = F_{\mu\nu}\,dx^\mu\wedge dx^\nu$ is defined as the 2-form ${*}F = \frac{1}{2}\epsilon_{\alpha\beta\mu\nu}F^{\mu\nu}\,dx^\alpha\wedge dx^\beta$.

As previously, one associates to the current four-vector J_μ the 1-form $J_\mu\,dx^\mu$. To the covariant skew-symmetric electromagnetic field tensor $F_{\mu\nu}$ one associates the 2-form $\frac{1}{2}F_{\mu\nu}\,dx^\mu\wedge dx^\nu$, i.e., $F_{\mu\nu}$ (tensor) $\leftrightarrow\frac{1}{2}F_{\mu\nu}\,dx^\mu\wedge dx^\nu$ (2-form).

Maxwell's equations can then be shown to be given by

$$\left.\begin{array}{l}\nabla\cdot\mathbf{B} = 0\\ \nabla\times\mathbf{E} + \dfrac{1}{c}\dfrac{\partial\mathbf{B}}{\partial t} = 0\end{array}\right\}\Leftrightarrow F_{[\mu\nu,\lambda]} = 0\leftrightarrow dF = 0,$$

$$\left.\begin{array}{l}\nabla\cdot\mathbf{E} = 4\pi\rho\\ \nabla\times\mathbf{B} - \dfrac{1}{c}\dfrac{\partial E}{\partial t} = \dfrac{4\pi}{c}J\end{array}\right\}\Leftrightarrow F^{\mu\nu}{}_{,\nu} = \frac{4\pi}{c}J^\mu$$

$$\leftrightarrow d\,{*}F = \frac{4\pi}{c}{*}J,$$

where $[\mu\nu,\lambda]$ denotes skew-symmetry in all the indices, and a comma denotes differentiation with respect to the index following it.

The expressions above are, respectively, Maxwell's equations in vector, tensor, and form notation.

The four vector equations thus get reduced to two simple-looking differential-form equations

$$dF=0 \text{ and } d\,{*F}=(4\pi/c){*J}.$$

As usual, the simplicity of notation has been arrived at by constructing a deeper strata of definitions.

One can similarly express all the relevant electromagnetic quantities, e.g., the Lorentz force, Poynting vector, field energy, etc., in the language of forms[10] It is within this language that the question of the apparent nonexistence of magnetic monopoles is most compelling.[8]

V. CONCLUSION

I have shown how classical vector analysis is not only completely contained in the theory of differential forms, but also how the latter conceptually unifies many apparently disparate vector ideas.

From the point of view of differential forms, it is the conjunction of the concept of dual and the particular dimension of three that allows for a vector calculus. This can be seen in the following manner.

It is assumed that a vector is identified with a form of a given order. The dual of a form, providing an identification of an $(n-p)$-form with a p-form, allows for an introduction of differentiation which *changes* the order of the form, by identifying the higher-order form with a lower one, thus remaining in the space of forms identified with vectors.[11]

If the dimension of the space is larger than three the space will not be closed, in the sense that the operation of exterior product or differentiation may take one out of the space of forms that can be identified with vectors. Pictorially, one has

0-form	*1-form*	*2-form*	*3-form*
↓	↓	↓	↓
scalar	*vector*	*pseudovector*	*pseudoscalar*.

In any higher-dimensional space this would not be possible, as there would be forms that cannot be associated with vectors.

ACKNOWLEDGMENTS

My thanks are extended to Bronek Czarnocha for many inspiring discussion as well as a critical reading of the manuscript, and to Marlene Stulbach for her spiritual support.

APPENDIX A

Proof that $d\delta+\delta d$ agrees with classical Laplacian operator when applied to a function f:

$$\begin{aligned}(d\delta+\delta d)f &= (d{*d*}+{*d*d})f\\ &= {*d*df}={*d*}\left(\sum_i \frac{\partial f}{\partial x^i}\,dx^i\right)\\ &= {*d}\left(\sum_i \epsilon_{ijk}\frac{\partial f}{\partial x^i}dx^j\wedge dx^k\right)\\ &= {*}\left(\sum_{i,l}\epsilon_{ijk}\frac{\partial^2 f}{\partial x^l\,\partial x^i}dx^l\wedge dx^j\wedge dx^k\right)\\ &= \sum_{i,l}\epsilon_{ijk}\epsilon_{ljk}\frac{\partial^2 f}{\partial x^l\,\partial x^i}\\ &= \sum_{i,l}\delta_{il}\frac{\partial^2 f}{\partial x^l\,\partial x^i}=\sum_{i=1}^{3}\frac{\partial^2 f}{\partial x^{i2}},\end{aligned}$$

where the second equality arises from the fact that since ${*f}=f\,dx^1\wedge dx^2\wedge dx^3$ then $d*f=0$. q.e.d.

APPENDIX B

Proof that $*\,[\phi_A\wedge{*(\phi_B\wedge\phi_C)}]=\phi_B\,{*(\phi_A\wedge{*\phi_C})}-\phi_c{*(\phi_A\wedge{*\phi_B})}$:

$$\begin{aligned}{*(\phi_B\wedge\phi_C)} &= {*}\left[\left(\sum B_{i_1}\,dx^{i_1}\right)\wedge\left(\sum C_{i_2}dx^{i_2}\right)\right]\\ &= {*}\left(\sum_{i_1i_2}B_{i_1}C_{i_2}\,dx_1\wedge dx^{i_2}\right)\\ &= \sum_{i_1i_2}B_{i_1}C_{i_2}\epsilon_{i_1i_2i_3}\,dx^{i_3}.\end{aligned}$$

Then

$$\phi_A\wedge{*(\phi_B\wedge\phi_C)}=\left(\sum_{i_4}A_{i_4}\,dx^{i_4}\right)\wedge\sum_{i_1i_2}B_{i_1}C_{i_2}\epsilon_{i_1i_2i_3}\,dx^{i_3}.$$

Since i_4 cannot equal i_3, it must be either i_1 or i_2. Thus

$$\begin{aligned}\phi_A\wedge{*(\phi_B\wedge\phi_C)} &= \sum_{i_1i_2}\left[(A_{i_1}\,dx^{i_1}+A_{i_2}\,dx^{i_2})\right]\\ &\quad\wedge B_{i_1}C_{i_2}\epsilon_{i_1i_2i_3}\,dx^{i_3}\\ &= \sum_{i_1i_2}A_{i_1}B_{i_1}C_{i_2}\,\epsilon_{i_1i_2i_3}dx^{i_1}\wedge dx^{i_3}\\ &\quad+\sum_{i_1i_3}A_{i_2}B_{i_1}C_{i_2}\epsilon_{i_1i_2i_3}\,dx^{i_2}\wedge dx^{i_3}.\end{aligned}$$

Then

$$\begin{aligned}&{*}[\phi_A\wedge{*(\phi_B\wedge\phi_C)}]\\ &= \sum_{i_1i_2}A_{i_1}B_{i_1}C_{i_2}\epsilon_{i_1i_2i_3}\epsilon_{i_1i_3i_2}\,dx^{i_2}\\ &\quad+\sum_{i_1i_2}A_{i_2}B_{i_1}C_{i_2}\epsilon_{i_1i_2i_3}\epsilon_{i_2i_3i_1}\,dx^{i_1}\\ &= -\sum_{i_1i_2}A_{i_1}B_{i_1}C_{i_2}\,dx^{i_2}+\sum_{i_1i_2}A_{i_2}B_{i_1}C_{i_2}\,dx^{i_1}\\ &= -\phi_C{*(\phi_A\wedge{*\phi_B})}+\phi_B{*(\phi_A\wedge{*\phi_C})}\qquad\text{q.e.d.}\end{aligned}$$

This immediately translates into the vector theorem

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C})=\mathbf{B}(\mathbf{A}\cdot\mathbf{C})-\mathbf{C}(\mathbf{A}\cdot\mathbf{B}).$$

APPENDIX C

Proof of $(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D})=(\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{D})+(\mathbf{A}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{C})$.

In terms of forms we must prove that

$$*[*(\phi_A\wedge\phi_B)\wedge(\phi_C\wedge\phi_D)]$$
$$=*(\phi_A\wedge *\phi_C)*(\phi_B\wedge *\phi_D)-*(\phi_A\wedge *\phi_D)*(\phi_B\wedge *\phi_C).$$

From the previous proof, we have

$$*(\phi_A\wedge\phi_B)=\sum_{i_1i_2}\epsilon_{i_1i_2i}A_{i_1}B_{i_2}\,dx^i,$$

and

$$(\phi_C\wedge\phi_D)=\sum C_{i_3}D_{i_4}\,dx^{i_3}\wedge dx^{i_4}.$$

Then

$$*(\phi_A\wedge\phi_B)\wedge(\phi_C\wedge\phi_D)$$
$$=\sum_{i_1i_2i_3i_4}\epsilon_{i_1i_2i}A_{i_1}B_{i_2}C_{i_3}D_{i_4}\,dx^i\wedge dx^{i_3}\wedge dx^{i_4}.$$

Clearly $i=i_1\neq i_2$ and $i\neq i_3\neq i_4$, so we must have either $i_3=i_1$, $i_4=i_2$ *or* $i_4=i_1$, $i_3=i_2$. Then

$$*(\phi_A\wedge\phi_B)\wedge(\phi_C\wedge\phi_D)$$
$$=\sum_{i_1i_2}A_{i_1}B_{i_2}C_{i_1}D_{i_2}\epsilon_{i_1i_2i}\,dx^i\wedge dx^{i_1}\wedge dx^{i_2}$$
$$+\sum_{i_1i_2}A_{i_1}B_{i_2}C_{i_2}D_{i_1}\epsilon_{i_1i_2i}\,dx^i\wedge dx^{i_2}\wedge dx^{i_1}$$

and

$$*[*(\phi_A\wedge\phi_B)\wedge(\phi_C\wedge\phi_D)]=\sum_{i_1i_2}A_{i_1}B_{i_2}C_{i_1}D_{i_2}\epsilon_{i_1i_2i}\epsilon_{ii_1i_2}$$
$$+\sum_{i_1i_2}A_{i_1}B_{i_2}C_{i_2}D_{i_1}\epsilon_{i_1i_2i}\epsilon_{ii_2i_1}$$
$$=\sum_{i_1i_2}A_{i_1}B_{i_2}C_{i_1}D_{i_2}$$
$$-\sum_{i_1i_2}A_{i_1}B_{i_2}C_{i_2}D_{i_1}$$
$$=*(\phi_A\wedge *\phi_C)*(\phi_B\wedge *\phi_D)$$
$$-*(\phi_A\wedge *\phi_D)*(\phi_B\wedge *\phi_C)$$

APPENDIX D

Schematic derivation of Gauss's and Stokes's theorem from the generalized Stokes's theorem T(4):

(i) Gauss: $\int_V \nabla\cdot\mathbf{V}\,dV=\int_s \mathbf{V}\cdot d\,\mathbf{A}.$

We have the following correspondences of vectors to forms:

$$\nabla\cdot\mathbf{V}\,dV=\left(\frac{\partial V_1}{\partial x_1}+\frac{\partial V_2}{\partial x_2}+\frac{\partial V_3}{\partial x_3}\right)dx^1\,dx^2\,dx^3$$
$$\leftrightarrow d(*\phi_V)=\left(\frac{\partial V_1}{\partial x_1}+\frac{\partial V_2}{\partial x_2}+\frac{\partial V_3}{\partial x_3}\right)dx^1\wedge dx^2\wedge dx^3$$

and

$$\mathbf{V}\cdot d\,\mathbf{A}=V_3\,dx^1\,dx^2+V_2\,dx^3\,dx^1+V_1\,dx^2\,dx^3$$
$$\leftrightarrow *\phi_V=V_1\,dx^2\wedge dx^3+V_2\,dx^3\wedge dx^1+V_3\,dx^1\wedge dx^2.$$

Then, by choosing $\omega=*\phi_V$, a 2-form, we get

$$\int_V \nabla\cdot\mathbf{V}\,dV=\int_s \mathbf{V}\cdot d\,\mathbf{A}\leftrightarrow\int_D d(*\phi_V)=\int_{\partial D}*\phi_V.$$

(ii) Stokes: $\int_s \nabla\times\mathbf{A}\cdot d\,\mathbf{S}=\oint\mathbf{A}\cdot d\,\mathbf{l}.$

Let

$$d\,\mathbf{S}=dx^2\,dx^3\hat{x}_1+dx^3\,dx^1\,\hat{x}_2+dx^1\,dx^2\,\hat{x}_3,$$

then

$$\nabla\times\mathbf{A}\cdot d\,\mathbf{S}=\left(\frac{\partial A_3}{\partial x^2}-\frac{\partial A_2}{\partial x^3}\right)dx^2\,dx^3$$
$$+\left(\frac{\partial A_1}{\partial x^3}-\frac{\partial A_3}{\partial x^1}\right)dx^1\,dx^3$$
$$+\left(\frac{\partial A_2}{\partial x^1}-\frac{\partial A_1}{\partial x^2}\right)dx^1\,dx^2$$
$$\leftrightarrow d\phi_A=\left(\frac{\partial A_3}{\partial x^2}-\frac{\partial A_2}{\partial x^3}\right)dx^2\wedge dx^3$$
$$+\left(\frac{\partial A_1}{\partial x^3}-\frac{\partial A_3}{\partial x^1}\right)dx^3\wedge dx^1$$
$$+\left(\frac{\partial A_2}{\partial x^1}-\frac{\partial A_1}{\partial x^2}\right)dx^1\wedge dx^2.$$

Similarly, $\mathbf{A}\cdot d\,\mathbf{l}\leftrightarrow\phi_A$. Then, by choosing $\omega=\phi_A$, a one-form:

$$\int_s \nabla\times\mathbf{A}\cdot d\,\mathbf{S}=\oint\mathbf{A}\cdot d\,\mathbf{l}\leftrightarrow\int_D d\phi_A=\int_{\partial D}\phi_A.$$

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[1] A brief comment on notation: For referential convenience I label various sets of equations with alphabetic letters, e.g., T(3), that have the following meanings: A—Algebraic properties of forms, L—Lemmas pertaining to forms, T—Theorems pertaining to forms, C—Correspondences between vector operations and form operations, I—correspondences between form Identities and vector identities, and VT—Vector Theorem correspondences to form theorems. $J(x,y;r,\theta)$ is the Jacobian of the transformation from variables r,θ to variables x,y. The symbols ϕ and ψ generally represent *one*-forms, as distinct from ω whose order is arbitrary. The letters p and q, though usually denoting the order of a form, will occasionally represent a form itself of that order. The limits of a summation will be omitted when it is obvious.

[2] The space dual to the tangent space at the point.

[3] See, for example, H. Flanders, *Differential Forms* (Academic, New York, 1963).

[4] The relevance of this to fermion creation and annihilation operators will be presented in a future paper.

[5] This association is easily seen to be preserved under a coordinate transformation.

[6] As well as the symmetric product $*(\phi_B\wedge *\phi_A)$. See below.

[7] I would like to mention that the only type of vector identity I have not been able to prove is, e.g.,

$$\nabla\times(\mathbf{A}\times\mathbf{B})=(\mathbf{B}\cdot\nabla)\mathbf{A}-(\mathbf{A}\cdot\nabla)\mathbf{B}+(\nabla\cdot\mathbf{B})\mathbf{A}-(\nabla\cdot\mathbf{A})\mathbf{B}.$$

I believe that this is because it involves an abuse of vector notation, e.g., in a term like $(\mathbf{B}\cdot\nabla)\mathbf{A}$. It is conceivable, though, that a corresponding abuse of the form notation would yield the differential form analog of this type of vector identity.

[8] This topic, as well as the general theory of differential forms in a four-dimensional Minkowski space, will be discussed in detail in a forthcoming paper. Note that I adopt the Einstein (implied) summation convention in this section.

[9] An even more general definition can be given in terms of the "volume *n*-form." See B. Schutz, *Geometrical Methods of Mathematical Physics* (Cambridge University, Cambridge, 1980), Chap. 4. This is an excellent book on mathematical physics that is an elegant compromise between a physicist's desire for simplicity and a mathematician's for rigor.

[10] See C. Misner, K. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973), Chaps. 4 and 5.

[11] Which, in the language of vectors, is the identification of a pseudovector and a vector, allowing for the introduction of curl and cross product, while still remaining in the space of vectors.