Basic Structures: Sets, Functions, Sequences, and Sums

CSC-2259 Discrete Structures

Theorem: The set of rational numbers is countable

Proof:

We need to find a method to list

all rational numbers: $\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, K$

Naïve Approach

Start with nominator=1

Rational numbers: $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots$

One-to-one correspondence: $\updownarrow \quad \updownarrow \quad \updownarrow$

Positive integers: $1, 2, 3, \ldots$

Doesn't work:

we will never list numbers with nominator 2: $\frac{2}{1}, \frac{2}{2}, \frac{2}{3}, \ldots$

Better Approach: scan diagonals

Nomin.=1 $$\frac{1}{1} \quad \frac{1}{2} \quad \frac{1}{3} \quad \frac{1}{4} \quad \dots$$

Nomin.=2 $$\frac{2}{1} \quad \frac{2}{2} \quad \frac{2}{3} \quad \dots$$

Nomin.=3 $$\frac{3}{1} \quad \frac{3}{2} \quad \dots$$

Nomin.=4 $$\frac{4}{1} \quad \dots$$

first diagonal

$$\frac{1}{1} \qquad \frac{1}{2} \qquad \frac{1}{3} \qquad \frac{1}{4} \qquad ʌ$$

$$\frac{2}{1} \qquad \frac{2}{2} \qquad \frac{2}{3} \qquad ʌ$$

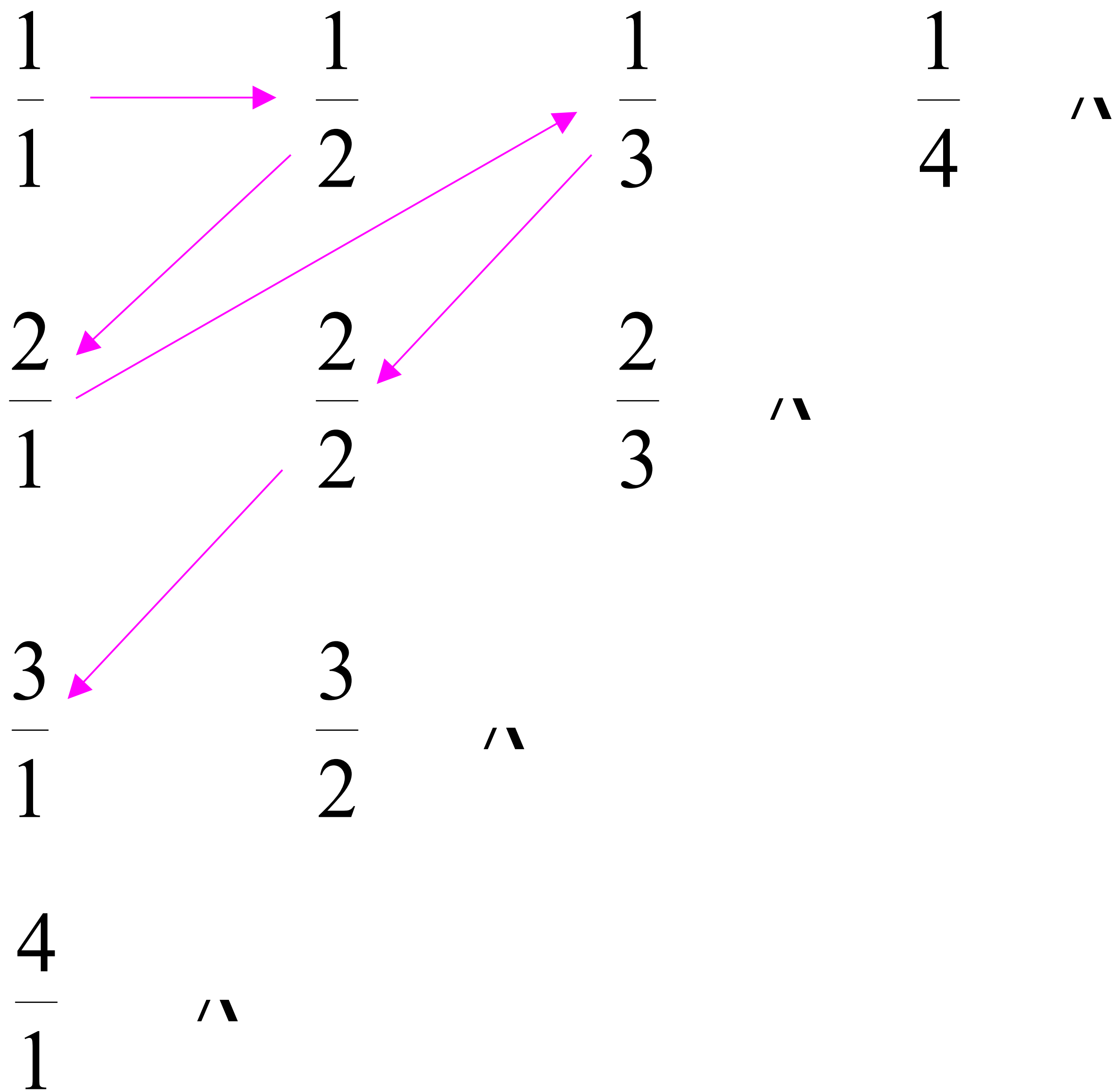
$$\frac{3}{1} \qquad \frac{3}{2} \qquad ʌ$$

$$\frac{4}{1} \qquad ʌ$$

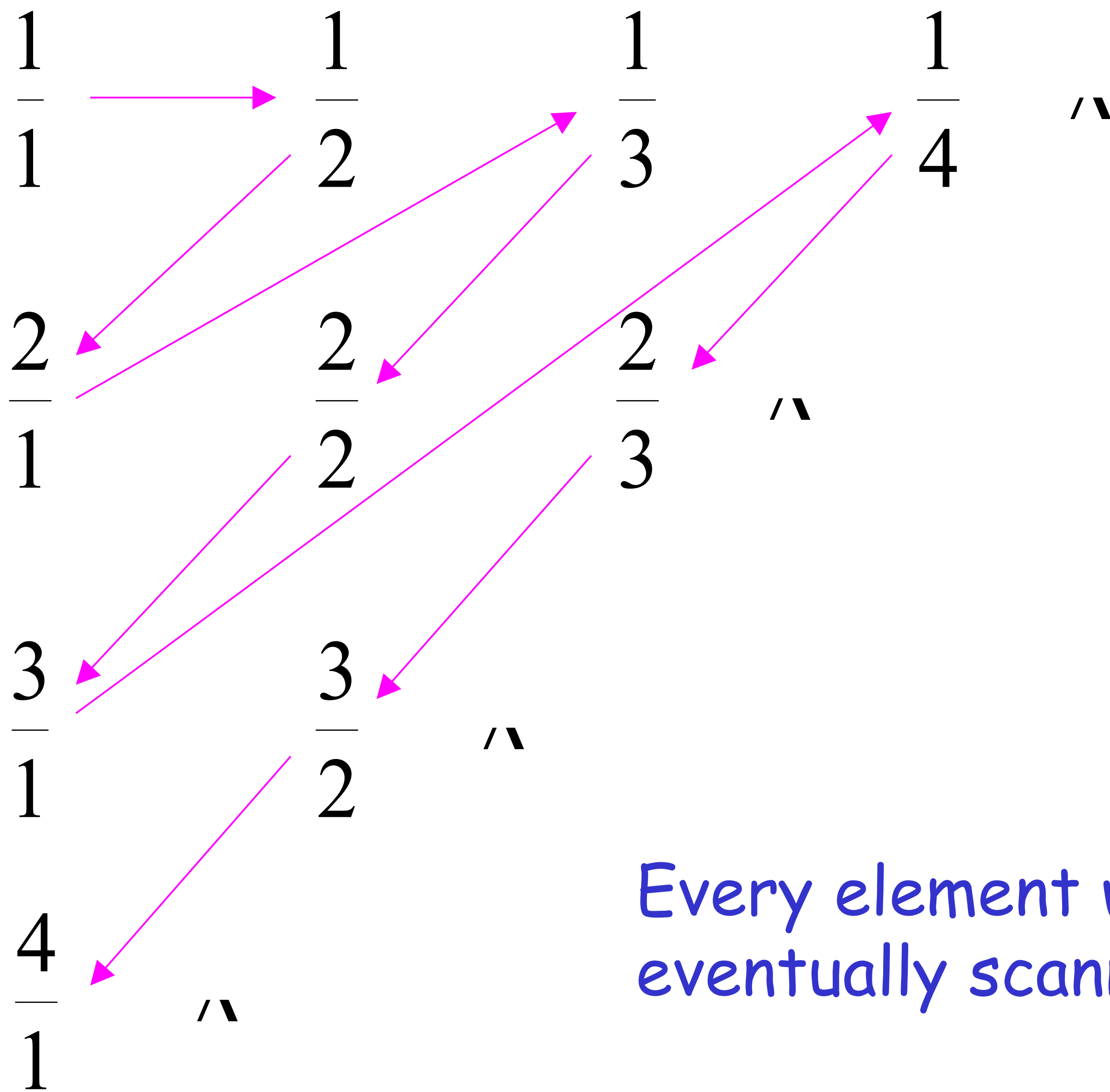
second diagonal

$\frac{1}{1}$ →	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	Λ
$\frac{2}{1}$ ↙	$\frac{2}{2}$	$\frac{2}{3}$	Λ	
$\frac{3}{1}$	$\frac{3}{2}$	Λ		
$\frac{4}{1}$	Λ			

third diagonal



fourth diagonal…



Every element will be eventually scanned

Diagonal listing

Rational Numbers: $\frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \frac{2}{2}, \ldots$

One-to-one correspondence: $\updownarrow \quad \updownarrow \quad \updownarrow \quad \updownarrow \quad \updownarrow$

Positive Integers: $1, 2, 3, 4, 5, \ldots$

End of Proof

Theorem: Set $S = (0,1) \subseteq R$ is uncountable

Proof: Assume that S is countable,

then we can list its elements

$$S = \{s_1, s_2, s_3, \ldots\}$$

↑

Elements of S

List the elements of $S = (0,1)$

$$s_1 = 0 \;.\; 0\;1\;4\;5\;2\;9\;4\;2\;1\;6\;\Lambda$$

$$s_2 = 0 \;.\; 1\;2\;1\;3\;2\;1\;5\;7\;3\;1\;\Lambda$$

$$s_3 = 0 \;.\; 1\;3\;0\;2\;0\;5\;3\;1\;8\;4\;\Lambda$$

$$s_4 = 0 \;.\; 3\;2\;1\;0\;0\;3\;2\;1\;1\;3\;\Lambda$$

$$s_5 = 0 \;.\; 4\;6\;1\;8\;4\;2\;1\;5\;2\;1\;\Lambda$$

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$$s_1 = 0 \;.\; 0\;1\;4\;5\;2\;9\;4\;2\;1\;6\;\Lambda$$

$$s_2 = 0 \;.\; 1\;2\;1\;3\;2\;1\;5\;7\;3\;1\;\Lambda$$

$$s_3 = 0 \;.\; 1\;3\;0\;2\;0\;5\;3\;1\;8\;4\;\Lambda$$

$$s_4 = 0 \;.\; 3\;2\;1\;0\;0\;3\;2\;1\;1\;3\;\Lambda$$

$$s_5 = 0 \;.\; 4\;6\;1\;8\;4\;2\;1\;5\;2\;1\;\Lambda$$

$$\mathrm{M}$$

Create new element based on diagonal

$$t = 0 \;.\; x_1\;x_2\;x_3\;x_4\;x_5\;x_6\;x_7\;x_8\;x_9\;x_{10}\;\Lambda$$

$$s_1 = 0 . 0\ 1\ 4\ 5\ 2\ 9\ 4\ 2\ 1\ 6\ \Lambda$$

$$s_2 = 0 . 1\ 2\ 1\ 3\ 2\ 1\ 5\ 7\ 3\ 1\ \Lambda$$

$$s_3 = 0 . 1\ 3\ 0\ 2\ 0\ 5\ 3\ 1\ 8\ 4\ \Lambda$$

$$s_4 = 0 . 3\ 2\ 1\ 0\ 0\ 3\ 2\ 1\ 1\ 3\ \Lambda$$

$$s_5 = 0 . 4\ 6\ 1\ 8\ 4\ 2\ 1\ 5\ 2\ 1\ \Lambda$$

$$\mathrm{M}$$

If diagonal element is 0 then set digit to 1

$$t = 0 . 1\ x_2\ x_3\ x_4\ x_5\ x_6\ x_7\ x_8\ x_9\ x_{10}\ \Lambda$$

$$
\begin{array}{lcccccccccccccc}
s_1 & = & 0 & . & 0 & 1 & 4 & 5 & 2 & 9 & 4 & 2 & 1 & 6 & \Lambda \\
s_2 & = & 0 & . & 1 & 2 & 1 & 3 & 2 & 1 & 5 & 7 & 3 & 1 & \Lambda \\
s_3 & = & 0 & . & 1 & 3 & 0 & 2 & 0 & 5 & 3 & 1 & 8 & 4 & \Lambda \\
s_4 & = & 0 & . & 3 & 2 & 1 & 0 & 0 & 3 & 2 & 1 & 1 & 3 & \Lambda \\
s_5 & = & 0 & . & 4 & 6 & 1 & 8 & 4 & 2 & 1 & 5 & 2 & 1 & \Lambda \\
\mathrm{M} & & & & & & & & & & & & & &
\end{array}
$$

If diagonal element is not 0 then set digit to 0

$$
t = 0 \; . \; 1 \; 0 \; x_3 \; x_4 \; x_5 \; x_6 \; x_7 \; x_8 \; x_9 \; x_{10} \; \Lambda
$$

$$s_1 = 0\ .\ 0\ 1\ 4\ 5\ 2\ 9\ 4\ 2\ 1\ 6\ \Lambda$$

$$s_2 = 0\ .\ 1\ 2\ 1\ 3\ 2\ 1\ 5\ 7\ 3\ 1\ \Lambda$$

$$s_3 = 0\ .\ 1\ 3\ 0\ 2\ 0\ 5\ 3\ 1\ 8\ 4\ \Lambda$$

$$s_4 = 0\ .\ 3\ 2\ 1\ 0\ 0\ 3\ 2\ 1\ 1\ 3\ \Lambda$$

$$s_5 = 0\ .\ 4\ 6\ 1\ 8\ 4\ 2\ 1\ 5\ 2\ 1\ \Lambda$$

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If diagonal element is 0 then set digit to 1

$$t = 0\ .\ 1\ 0\ 1\ x_4\ x_5\ x_6\ x_7\ x_8\ x_9\ x_{10}\ \Lambda$$

$$s_1 = 0\ .\ 0\ 1\ 4\ 5\ 2\ 9\ 4\ 2\ 1\ 6\ \Lambda$$

$$s_2 = 0\ .\ 1\ 2\ 1\ 3\ 2\ 1\ 5\ 7\ 3\ 1\ \Lambda$$

$$s_3 = 0\ .\ 1\ 3\ 0\ 2\ 0\ 5\ 3\ 1\ 8\ 4\ \Lambda$$

$$s_4 = 0\ .\ 3\ 2\ 1\ 0\ 0\ 3\ 2\ 1\ 1\ 3\ \Lambda$$

$$s_5 = 0\ .\ 4\ 6\ 1\ 8\ 4\ 2\ 1\ 5\ 2\ 1\ \Lambda$$

$$\mathrm{M}$$

If diagonal element is 0 then set digit to 1

$$t = 0\ .\ 1\ 0\ 1\ 1\ x_5\ x_6\ x_7\ x_8\ x_9\ x_{10}\ \Lambda$$

$$
\begin{array}{llllllllllllll}
s_1 & = & 0 & . & 0 & 1 & 4 & 5 & 2 & 9 & 4 & 2 & 1 & 6 & \Lambda \\
s_2 & = & 0 & . & 1 & 2 & 1 & 3 & 2 & 1 & 5 & 7 & 3 & 1 & \Lambda \\
s_3 & = & 0 & . & 1 & 3 & 0 & 2 & 0 & 5 & 3 & 1 & 8 & 4 & \Lambda \\
s_4 & = & 0 & . & 3 & 2 & 1 & 0 & 0 & 3 & 2 & 1 & 1 & 3 & \Lambda \\
s_5 & = & 0 & . & 4 & 6 & 1 & 8 & 4 & 2 & 1 & 5 & 2 & 1 & \Lambda \\
\mathrm{M} & & & & & & & & & & & & & &
\end{array}
$$

If diagonal element is not 0 then set digit to 0

$$
t = 0 \;.\; 1 \;\; 0 \;\; 1 \;\; 1 \;\; 0 \;\; x_6 \;\; x_7 \;\; x_8 \;\; x_9 \;\; x_{10} \;\; \Lambda
$$

$$s_1 = 0\;.\;0\;1\;4\;5\;2\;9\;4\;2\;1\;6\;\Lambda$$

$$s_2 = 0\;.\;1\;2\;1\;3\;2\;1\;5\;7\;3\;1\;\Lambda$$

$$s_3 = 0\;.\;1\;3\;0\;2\;0\;5\;3\;1\;8\;4\;\Lambda$$

$$s_4 = 0\;.\;3\;2\;1\;0\;0\;3\;2\;1\;1\;3\;\Lambda$$

$$s_5 = 0\;.\;4\;6\;1\;8\;4\;2\;1\;5\;2\;1\;\Lambda$$

$$\mathrm{M}$$

By repeating process we obtain new number

$$t = 0\;.\;1\;0\;1\;1\;0\;1\;\Lambda \quad \in (0,1)$$

$$s_1 = 0 \; . \; 0 \; 1 \; 4 \; 5 \; 2 \; 9 \; 4 \; 2 \; 1 \; 6 \; \Lambda$$

$$s_2 = 0 \; . \; 1 \; 2 \; 1 \; 3 \; 2 \; 1 \; 5 \; 7 \; 3 \; 1 \; \Lambda$$

$$s_3 = 0 \; . \; 1 \; 3 \; 0 \; 2 \; 0 \; 5 \; 3 \; 1 \; 8 \; 4 \; \Lambda$$

$$s_4 = 0 \; . \; 3 \; 2 \; 1 \; 0 \; 0 \; 3 \; 2 \; 1 \; 1 \; 3 \; \Lambda$$

$$s_5 = 0 \; . \; 4 \; 6 \; 1 \; 8 \; 4 \; 2 \; 1 \; 5 \; 2 \; 1 \; \Lambda$$

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Observation: $t \neq s_1$ (differ on first digit)

$$t = 0 \; . \; 1 \; 0 \; 1 \; 1 \; 0 \; 1 \; \Lambda$$

$$s_1 = 0 \; . \; 0 \; 1 \; 4 \; 5 \; 2 \; 9 \; 4 \; 2 \; 1 \; 6 \; \Lambda$$

$$s_2 = 0 \; . \; 1 \; 2 \; 1 \; 3 \; 2 \; 1 \; 5 \; 7 \; 3 \; 1 \; \Lambda$$

$$s_3 = 0 \; . \; 1 \; 3 \; 0 \; 2 \; 0 \; 5 \; 3 \; 1 \; 8 \; 4 \; \Lambda$$

$$s_4 = 0 \; . \; 3 \; 2 \; 1 \; 0 \; 0 \; 3 \; 2 \; 1 \; 1 \; 3 \; \Lambda$$

$$s_5 = 0 \; . \; 4 \; 6 \; 1 \; 8 \; 4 \; 2 \; 1 \; 5 \; 2 \; 1 \; \Lambda$$

$$\mathrm{M}$$

Observation: $t \neq s_2$ (differ on second digit)

$$t = 0 \; . \; 1 \; 0 \; 1 \; 1 \; 0 \; 1 \; \Lambda$$

$$s_1 = 0 \,.\, 0\ 1\ 4\ 5\ 2\ 9\ 4\ 2\ 1\ 6\ \Lambda$$

$$s_2 = 0 \,.\, 1\ 2\ 1\ 3\ 2\ 1\ 5\ 7\ 3\ 1\ \Lambda$$

$$s_3 = 0 \,.\, 1\ 3\ 0\ 2\ 0\ 5\ 3\ 1\ 8\ 4\ \Lambda$$

$$s_4 = 0 \,.\, 3\ 2\ 1\ 0\ 0\ 3\ 2\ 1\ 1\ 3\ \Lambda$$

$$s_5 = 0 \,.\, 4\ 6\ 1\ 8\ 4\ 2\ 1\ 5\ 2\ 1\ \Lambda$$

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Observation: $t \neq s_3$ (differ on third digit)

$$t = 0 \,.\, 1\ 0\ 1\ 1\ 0\ 1\ \Lambda$$

Observation: $t \neq s_i$ (differ on i digit)

for every i

$$t \notin S = \{s_1, s_2, \mathrm{K}\} = (0,1)$$

Contradiction!

$$t = 0 \; . \; 1 \; 0 \; 1 \; 1 \; 0 \; 1 \; \Lambda \quad \in (0,1)$$

End of Proof

We have proven: $(0,1) \subseteq R$ is uncountable

It can be proven: Every subset of a countable set is countable

It follows that the set of real numbers R is uncountable

The previous proof technique is known as:

Cantor diagonalization argument

The same technique can be used in other proofs

Theorem: If S is an infinite countable set, then the power set $P(S)$ is uncountable

Proof:

Since S is countable, we can list its elements

$$S = \{s_1, s_2, s_3, \ldots\}$$

Elements of S

Elements of the power set $P(S)$ have the form:

$$\emptyset$$

$$\{s_1\}$$

$$\{s_1, s_3\}$$

$$\{s_1, s_3, s_4\}$$

$$\{s_5, s_7, s_9, s_{10}\}$$

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We encode each element of the powerset with a binary string of 0's and 1's

Powerset elements $P(S)$ (in arbitrary order)	Binary encoding				
	s_1	s_2	s_3	s_4	$\cdots$
$\{s_1\}$	1	0	0	0	$\cdots$
$\{s_2, s_3\}$	0	1	1	0	$\cdots$
$\{s_1, s_3, s_4\}$	1	0	1	1	$\cdots$

Observation:

Every infinite binary string corresponds to an element of the power set

Example: $1\,0\,0\,1\,1\,1\,0\,\ldots$

Corresponds to: $\{s_1, s_4, s_5, s_6, \ldots\} \in P(S)$

Let's assume (for contradiction) that the power set $P(S)$ is countable

Then: we can enumerate the elements of the powerset

$$P(S) = \{t_1, t_2, t_3, \ldots\}$$

suppose that this is the respective

Power set element $P(S)$	Binary encoding					
t_1	1	0	0	0	0	$\cdots$
t_2	1	1	0	0	0	$\cdots$
t_3	1	1	0	1	0	$\cdots$
t_4	1	1	0	0	1	$\cdots$
$\vdots$	$\vdots$					

Take the binary string whose bits are the complement of the diagonal

t_1	**1**	0	0	0	0	Λ
t_2	1	**1**	0	0	0	Λ
t_3	1	1	**0**	1	0	Λ
t_4	1	1	0	**0**	1	Λ
Complement of diagonal	0	0	1	1		Λ

Binary string: $t = 0011\Lambda$

The binary string $t = 0011\ldots$

corresponds to an element of the power set $P(S)$: $t = \{s_3, s_4, \ldots\} \in P(S)$

Thus, t must be equal to some t_i: $t = t_i$

$$t \in P(S)$$

However,

the i-th bit in the binary string of t is different than the i-th bit of t_i, thus: $t \neq t_i$

$$t \notin P(S) = \{t_1, t_2, \ldots, t_n\}$$

Contradiction!!!

End of Proof