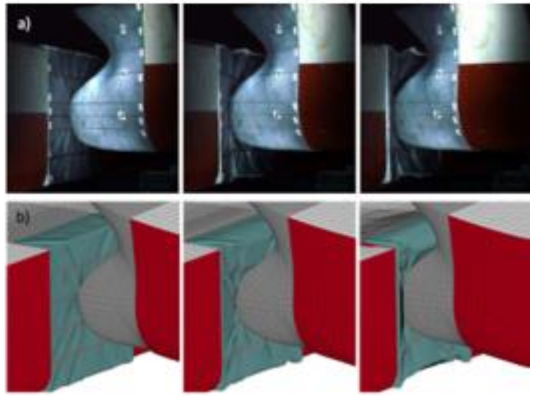
Vibração de Vigas

Sistemas Dinâmicos II

Profs. Ettore de Barros, Larissa Driemeier e Marcílio Alves



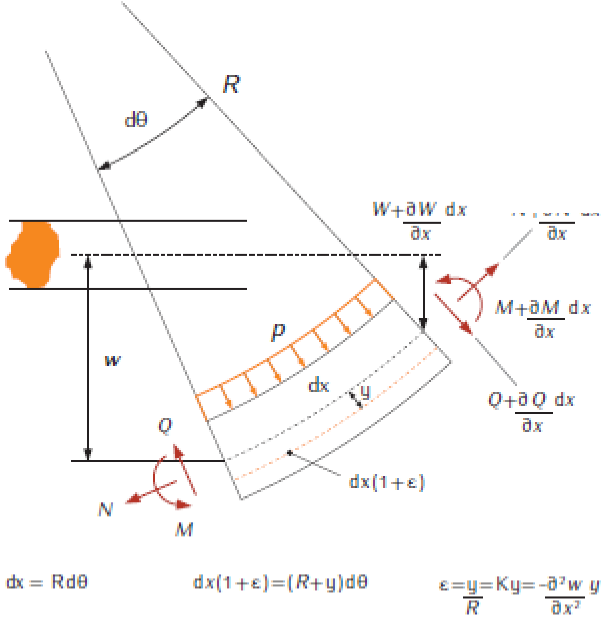
IMPACT ENGINEERING

Fundamentals

Experiments

Nonlinear Finite Elements

Marcílio Alves



A beam element subjected to moderate transverse displacements.

Moment equilibrium on the beam element yields

$$\frac{\partial^2 M}{\partial x^2} + N\frac{\partial^2 w}{\partial x^2} + p(x,t) = m\frac{\partial^2 w}{\partial t^2},$$

$$Q = \frac{\partial M}{\partial x},$$

$$\kappa = -\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}$$

$$\varepsilon^z = z\kappa,$$

for small slope and moderate displacement

Adopt now the linear elastic material law

$$\sigma = E\varepsilon$$

and calculate the bending moment across the beam section by integrating $\mathrm{d}M = z\mathrm{d}F = z\sigma\mathrm{d}A$, *ie*

$$M = EI\kappa = -EI\frac{\partial^2 w}{\partial x^2},$$

with $I = \int_A z^2 \mathrm{d}A$ being the inertia moment of the beam cross–section. Introducing this expression in the equilibrium equation and disregarding the membrane force we obtain the linear governing equation of a beam as

$$EI\frac{\partial^4 w}{\partial x^4} = f(x,t) - m\frac{\partial^2 w}{\partial t^2},$$

Let us start our study on the dynamics of beams by investigating the linear elastic beam response in the absence of load, *ie* $f = 0$. This is the case when the beam presents a free motion which, due to its repetitive nature, is called free vibration. The governing equation becomes

$$c^2 \frac{\partial^4 w}{\partial x^4} + \frac{\partial^2 w}{\partial t^2} = 0, \qquad \text{with} \qquad a = \sqrt{\frac{EI}{\rho A}},$$

which can be solved for some relatively simple, yet practical, cases.

Note that the beam starts its motion due to some external load or initial condition, like an initial displacement, and it is then left to vibrate freely. Observe that here a has units of m^2/s and therefore does not represent a wave velocity.

Let us assume that the transverse displacement of the beam can be represented by the product of two functions,

$$w(x,t) = W(x)T(t),$$

which, when substituted in the governing equations gives

$$a^2 \frac{\partial^4 W(x)/\partial x^4}{W(x)} = -\frac{\partial^2 T(t)/\partial t^2}{T(t)}.$$

One side of this equation depends only on x and the other only on t, which is only possible when they are a constant, say ω^2, allowing us to write

$$\frac{\partial^4 W(x)}{\partial x^4} - \left(\frac{\omega}{a}\right)^2 W(x) = 0 \qquad \text{and} \qquad \frac{\partial^2 T(t)}{\partial t^2} + \omega^2 T(t) = 0.$$

These two ordinary differential equations have the solutions

$$W(x) = a_1 \sin \beta x + a_2 \cos \beta x + a_3 \sinh \beta x + a_4 \cosh \beta x$$

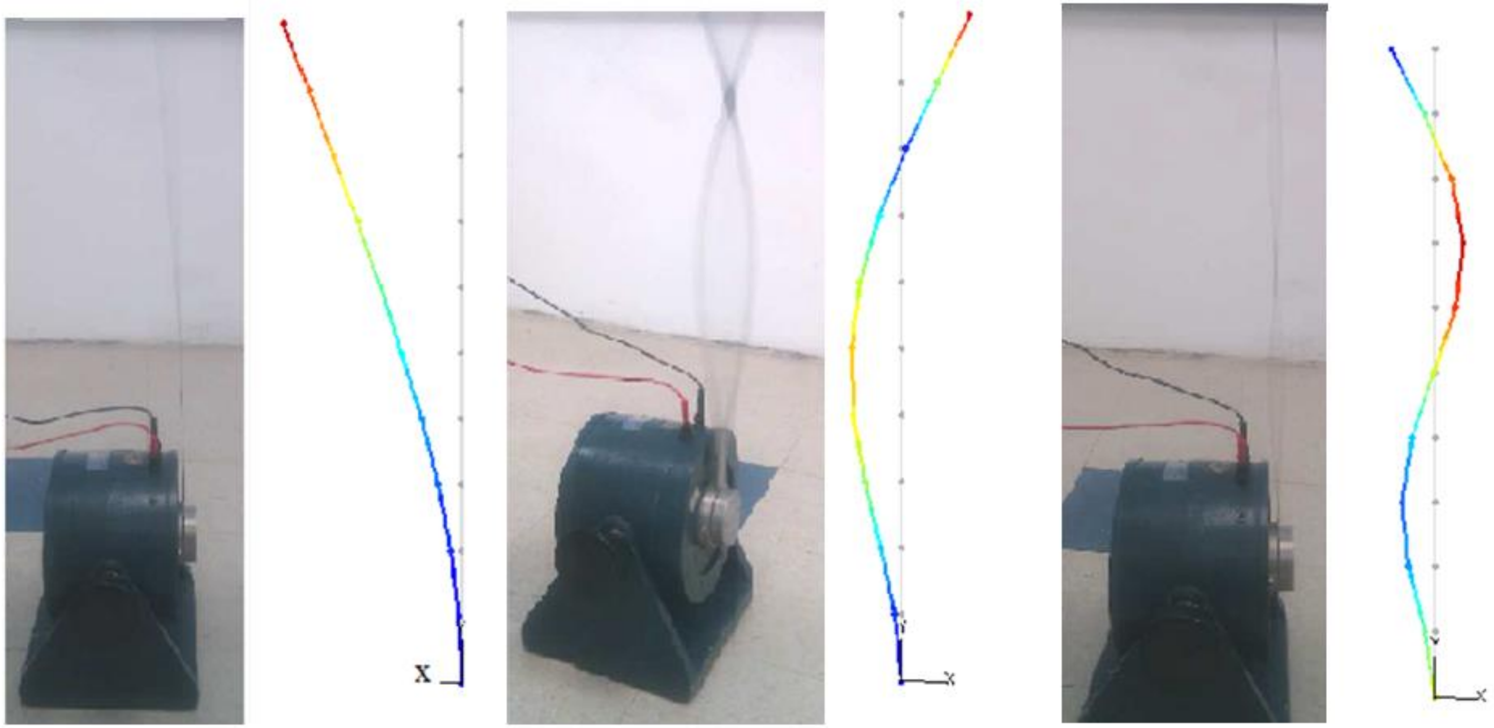
and

$$T(t) = A_1 \sin \omega t + A_2 \cos \beta t,$$

with $\beta^4 = \omega^2/a^2$. A_n and a_n are constants to be determined from the boundary and initial conditions, as the next example shows.

The next figure shows a shaker to which a beam is connected on its base via a clamped fixture. Suggest suitable boundary and initial conditions for the beam and obtain its natural frequencies.

A cantilever beam connected to a shaker and undergoing free vibration in the first, second and third vibration mode. By each photo there is the FE result. The respective measured natural resonant frequencies are 3.2 Hz, 20.1 Hz and 54.0 Hz, which are in error no larger than 3.42% in comparison with the FE analysis.



To solve this problem we need to determine the constants $A_{,}$ and a_n in the above solution. We know that at the support o this cantilever beam, the displacement and curvature are zero, i $W(0) = 0$ and $\mathrm{d}W(0)/\mathrm{d}x = 0$. Also, the bending moment anc the transverse shear force at $x = L$ are zero, *ie* $\mathrm{d}^2W(L)/\mathrm{d}x^2 =$ and $\mathrm{d}^3W(L)/\mathrm{d}x^3 = 0$. Introducing these conditions in $W(x$ gives

erivatives of hyperbolic sine nd co–sine do not change gnal. Also, $\sinh^2 x + \cosh^2 x = 1$.

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ -\sin\beta L & -\cos\beta L & \sinh\beta L & \cosh\beta L \\ -\cos\beta L & \sin\beta L & \cosh\beta L & \sinh\beta L \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

To avoid a trivial solution for this system, its determinan must be zero, which gives

$$\cos\beta L \cosh\beta L = -1.$$

A sketch of the functions $\cos x$ and $-1/\cosh x$, together with a more refined calculation, will show that they intercept at

$$\beta_1 L = 1.875, \qquad \beta_n L \approx n\pi/2, \quad n = 3, 5, 7 \ldots$$

Knowing β_n, the frequencies, called natural, $\omega = \beta^2 c$ are obtained for this beam configuration. They are compared to experimental values in the previous figure, indicating that the model performs quite well for the first modes of vibration.

We explore further the problem examined above by determing an expression for $W(x)$. We need to know the constants a_n, for $n = 1, 2, \ldots,$ which can be obtained by expressing the constants a_2, a_3, a_4 in the above system of equation as a function of a_1, giving

$$W_n(x) = a_1 \left[\cos \beta_n x - \cosh \beta_n x - \frac{\cos \beta_n L + \cosh \beta_n L}{\sin \beta_n L + \sinh \beta_n L} (\sin \beta_n x - \sinh \beta_n x) \right].$$

These are the so called natural modes of vibration. Note that the constant a_1 can be obtained only when one impose some condition to the beam. For instance, at the beam free end we can set it to $W(L) = 5$mm and let the beam to vibrate. Using the above expression, it is then possible to obtain a_1 and the actual displacement, which can yield the stresses and strains in the beam.

The next table list the natural frequencies and modes of vibration for various supporting cases of single span beams. Note that the natural frequencies are

$$\omega_n = \beta_n^2 \sqrt{EI/m} \quad \text{(rad/s)}$$

and for $n \geq 5$, one can use $\lambda_n = \lambda_{n-1} + \pi$. The modes of vibration are given by

$$\phi(x)_n = C_{1n} \sin \beta_n x + C_{2n} \cos \beta_n x + C_{3n} \sinh \beta_n x + C_{4n} \cosh \beta_n x,$$

with $\beta_n = \lambda_n / L$ and

$$c_n = [(\sinh \lambda_n + A \sinh \lambda_n)/(\cosh \lambda_n + B \cos \lambda_n)]^{\alpha}.$$

beam configuration	 simply sup.	 clamped	 free	 clamped–s.sup.	 cantilever
frequency equation	$\sin\lambda = 0$	$\cos\lambda\cosh\lambda = 1$	$\cos\lambda\cosh\lambda = 1$	$\tan\lambda - \tanh\lambda = 0$	$\cos\lambda\cosh\lambda = -1$
λ_1	π	4.7300	4.7300	3.9266	1.8751
λ_2	2π	7.8532	7.8532	7.0686	4.6941
λ_3	3π	10.9956	10.9956	10.2102	7.8548
λ_4	4π	14.1372	14.1372	13.3518	10.9955
modes of vibration					
α	–	-1	-1	-1	1
A	–	-1	-1	-1	-1
B	–	-1	-1	-1	1
c_1	–	1.0008	1.0008	1.0008	0.7341
c_2	–	1.0008	1.0008	1.0000	1.0185
c_3	–	0.9999	0.9999	1.0000	0.9992
c_4	–	1.0000	1.0000	1.0000	1.0000
C_{1n}	1	c_n	$-c_n$	c_n	c_n
C_{2n}	0	-1	1	-1	-1
C_{3n}	0	$-c_n$	$-c_n$	$-c_n$	$-c_n$
C_{4n}	0	1	1	1	1

Natural frequencies and vibration modes equations for various beam configurations. Adapted from lecture notes by C.A. Nunes Dias, GMSIE—USP.

Demonstração