MAP 2220 – FUNDAMENTOS DE ANÁLISE NUMÉRICA

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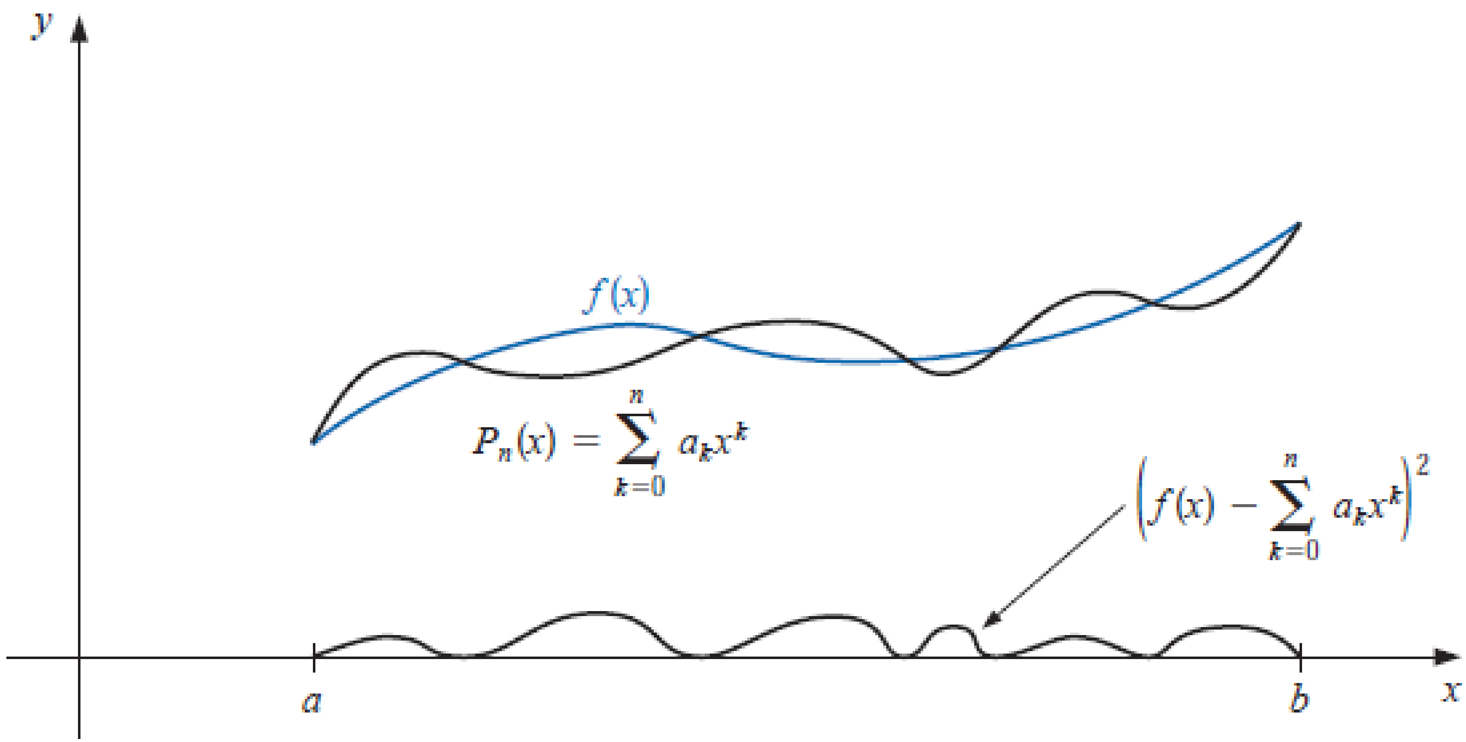
$$E \equiv E_2(a_0, a_1, \ldots, a_n) = \int_a^b \left(f(x) - \sum_{k=0}^{n} a_k x^k \right)^2 dx$$



Theorem 8.6

If $\{\phi_0, \ldots, \phi_n\}$ is an orthogonal set of functions on an interval $[a, b]$ with respect to the weight function w, then the least squares approximation to f on $[a, b]$ with respect to w is

$$P(x) = \sum_{j=0}^{n} a_j \phi_j(x),$$

where, for each $j = 0, 1, \ldots, n$,

$$a_j = \frac{\int_a^b w(x)\phi_j(x) f(x)\, dx}{\int_a^b w(x)[\phi_j(x)]^2\, dx} = \frac{1}{\alpha_j} \int_a^b w(x)\phi_j(x) f(x)\, dx.$$

■

Definition 8.5

$\{\phi_0, \phi_1, \ldots, \phi_n\}$ is said to be an **orthogonal set of functions** for the interval $[a, b]$ with respect to the weight function w if

$$\int_a^b w(x)\phi_k(x)\phi_j(x)\, dx = \begin{cases} 0, & \text{when } j \neq k, \\ \alpha_j > 0, & \text{when } j = k. \end{cases}$$

Theorem 8.7 Gram-Schmidt process

The set of polynomial functions $\{\phi_0, \phi_1, \ldots, \phi_n\}$ defined in the following way is orthogonal on $[a, b]$ with respect to the weight function w.

$$\phi_0(x) \equiv 1, \quad \phi_1(x) = x - B_1, \quad \text{for each } x \text{ in } [a, b],$$

where

$$B_1 = \frac{\int_a^b x w(x)[\phi_0(x)]^2\, dx}{\int_a^b w(x)[\phi_0(x)]^2\, dx},$$

and when $k \geq 2$,

$$\phi_k(x) = (x - B_k)\phi_{k-1}(x) - C_k \phi_{k-2}(x), \quad \text{for each } x \text{ in } [a, b],$$

where

$$B_k = \frac{\int_a^b x w(x)[\phi_{k-1}(x)]^2\, dx}{\int_a^b w(x)[\phi_{k-1}(x)]^2\, dx}$$

and

$$C_k = \frac{\int_a^b x w(x)\phi_{k-1}(x)\phi_{k-2}(x)\, dx}{\int_a^b w(x)[\phi_{k-2}(x)]^2\, dx}.$$

■

The set of **Legendre polynomials**, $\{P_n(x)\}$, is orthogonal on $[-1, 1]$ with respect to the weight function $w(x) \equiv 1$.

$P_0(x) = 1$

$P_1(x) = x$

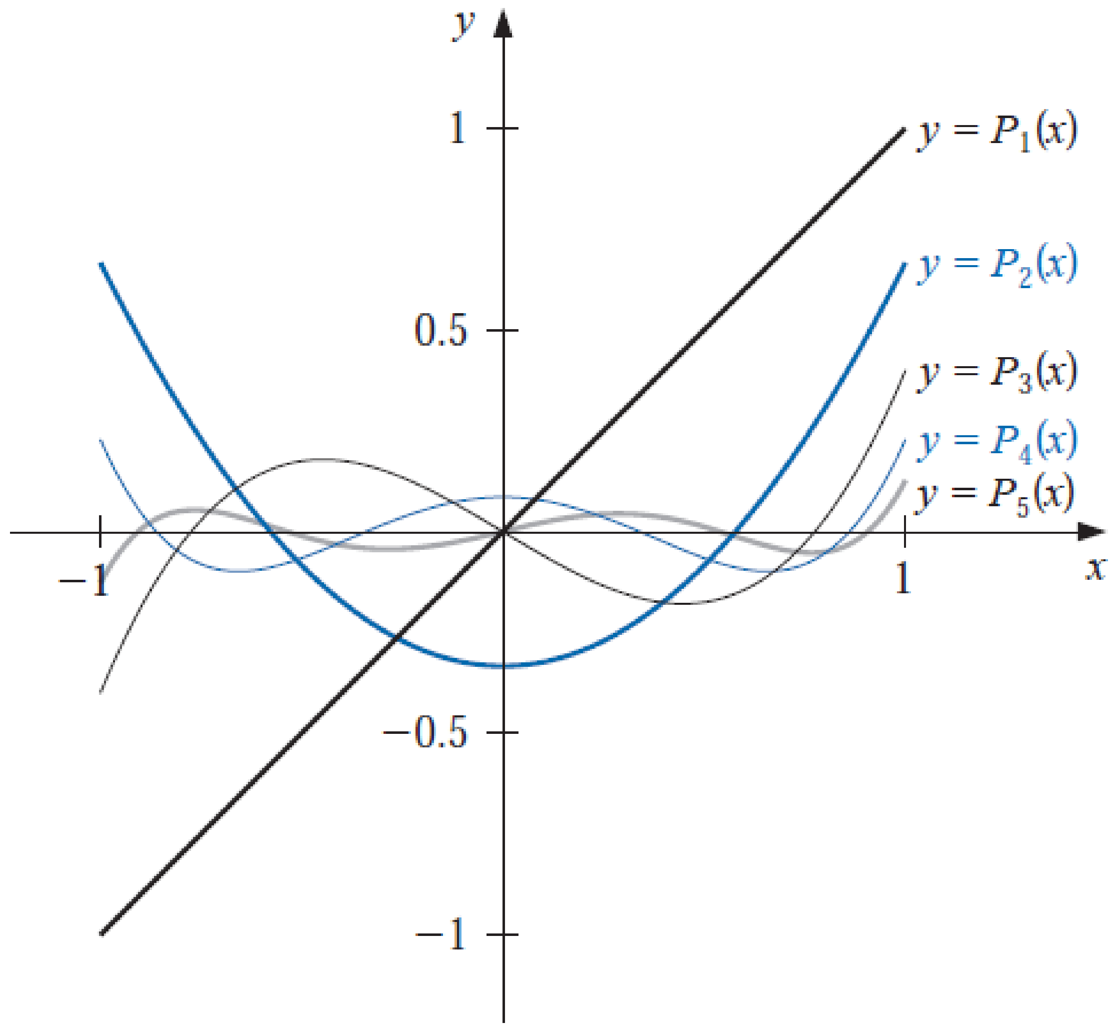
$P_2(x) = x^2 - \frac{1}{3}$

$P_3(x) = x^3 - \frac{3}{5}x$

$P_4(x) = x^4 - \frac{6}{7}x^2 + \frac{3}{35}$

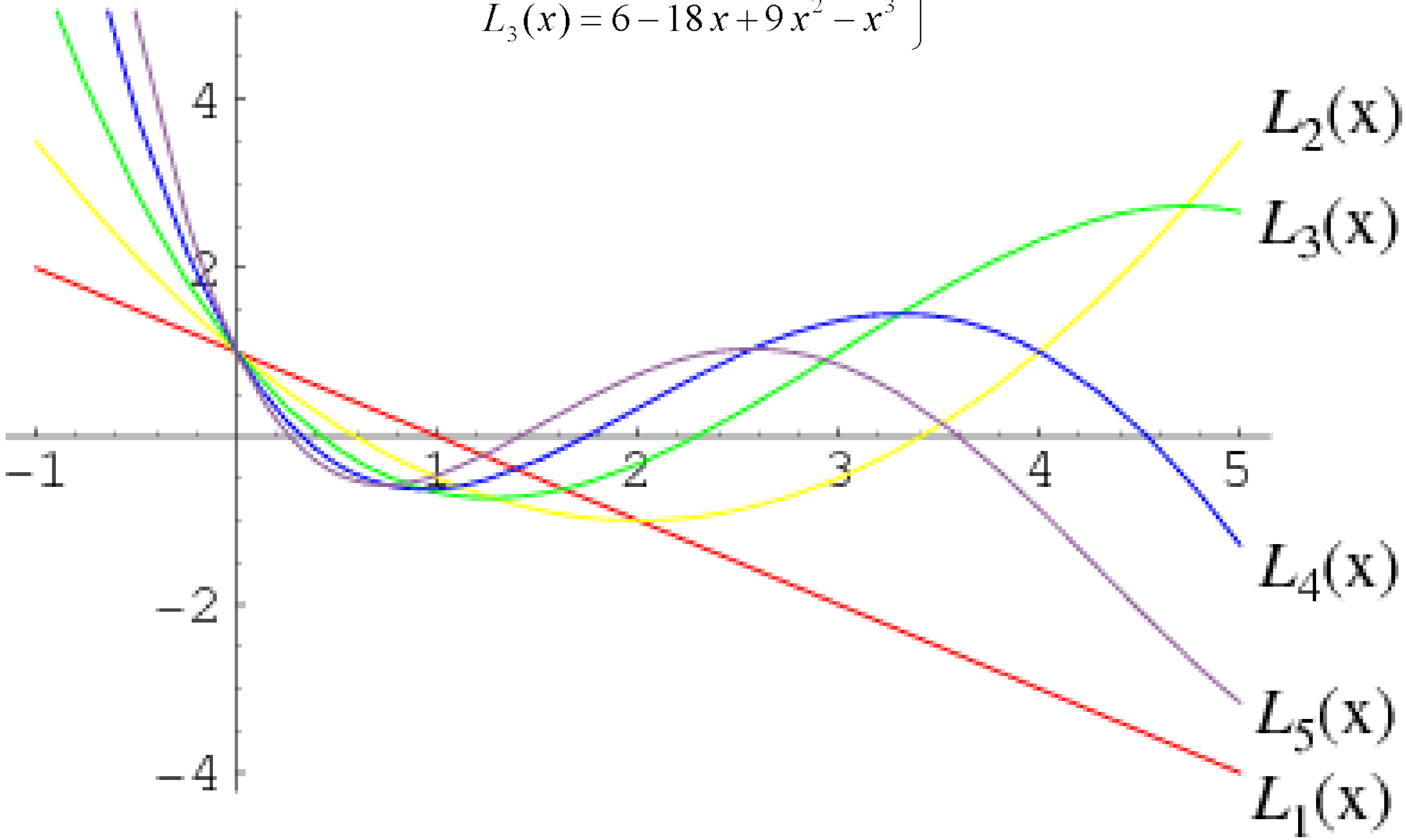
$P_5(x) = x^5 - \frac{10}{9}x^3 + \frac{5}{21}x$

...



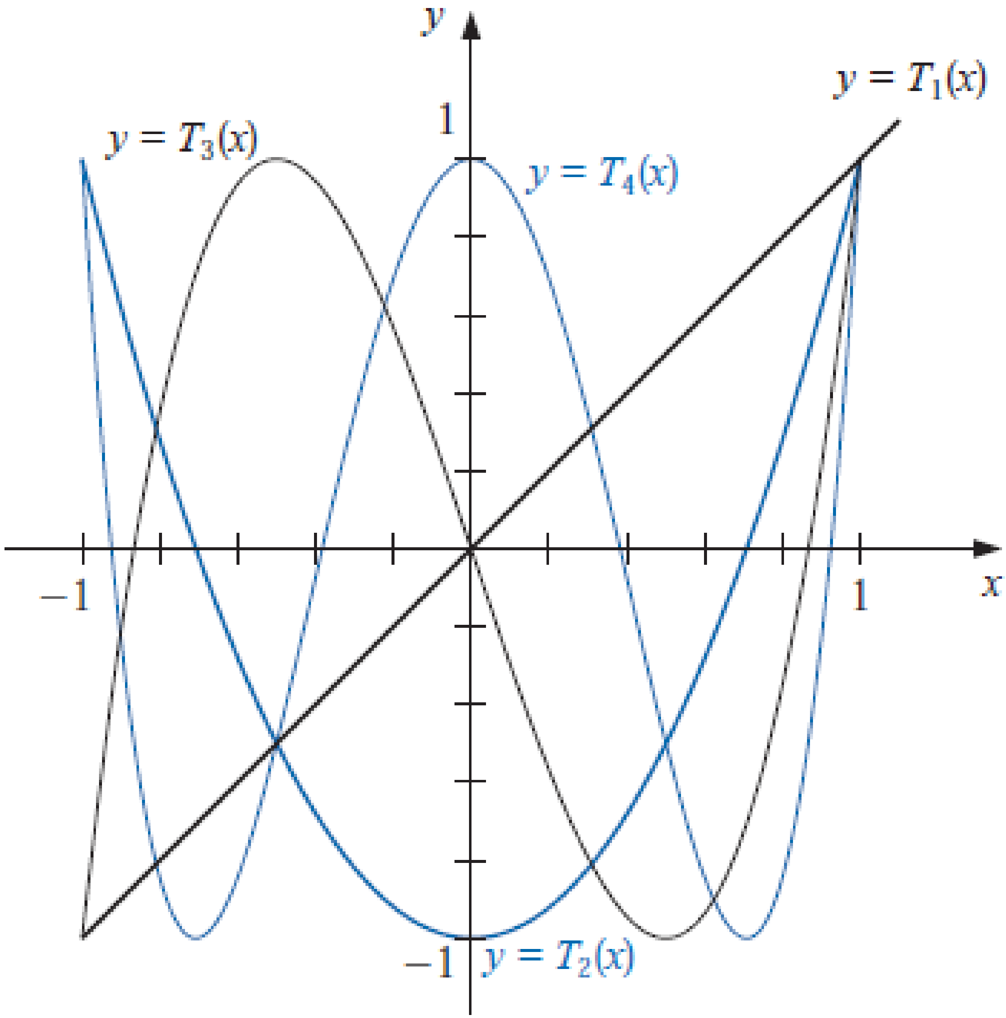
Laguerre Polynomials

$$L_o(x) = 1$$
$$L_1(x) = 1 - x$$
$$L_2(x) = 2 - 4x + x^2$$
$$L_3(x) = 6 - 18x + 9x^2 - x^3$$



Chebyshev Polynomials

$$T_0(x) = 1$$
$$T_1(x) = x$$
$$T_2(x) = 2x^2 - 1$$
$$T_3(x) = 4x^3 - 3x$$
$$T_4(x) = 8x^4 - 8x^2 + 1$$
$$T_5(x) = 16x^5 - 20x^3 + 5x$$
$$T_6(x) = 32x^6 - 48x^4 + 18x^2 - 1$$
$$T_7(x) = 64x^7 - 112x^5 + 56x^3 - 7x$$



8.4 Rational Function Approximation

The class of algebraic polynomials has some distinct advantages for use in approximation:

- There are a sufficient number of polynomials to approximate any continuous function on a closed interval to within an arbitrary tolerance;
- Polynomials are easily evaluated at arbitrary values; and
- The derivatives and integrals of polynomials exist and are easily determined.

The disadvantage of using polynomials for approximation is their tendency for oscillation. This often causes error bounds in polynomial approximation to significantly exceed the average approximation error, because error bounds are determined by the maximum approximation error. We now consider methods that spread the approximation error more evenly over the approximation interval. These techniques involve rational functions.

A **rational function** r of degree N has the form

$$r(x) = \frac{p(x)}{q(x)},$$

where $p(x)$ and $q(x)$ are polynomials whose degrees sum to N.

Every polynomial is a rational function (simply let $q(x) \equiv 1$), so approximation by rational functions gives results that are no worse than approximation by polynomials. However, rational functions whose numerator and denominator have the same or nearly the same degree often produce approximation results superior to polynomial methods for the same amount of computation effort. (This statement is based on the assumption that the amount of computation effort required for division is approximately the same as for multiplication.)

Rational functions have the added advantage of permitting efficient approximation of functions with infinite discontinuities near, but outside, the interval of approximation. Polynomial approximation is generally unacceptable in this situation.

Padé Approximation

Suppose r is a rational function of degree $N = n + m$ of the form

$$r(x) = \frac{p(x)}{q(x)} = \frac{p_0 + p_1 x + \cdots + p_n x^n}{q_0 + q_1 x + \cdots + q_m x^m},$$

that is used to approximate a function f on a closed interval I containing zero. For r to be defined at zero requires that $q_0 \neq 0$. In fact, we can assume that $q_0 = 1$, for if this is not the case we simply replace $p(x)$ by $p(x)/q_0$ and $q(x)$ by $q(x)/q_0$. Consequently, there are $N + 1$ parameters $q_1, q_2, \ldots, q_m, p_0, p_1, \ldots, p_n$ available for the approximation of f by r.

The **Padé approximation technique**, is the extension of Taylor polynomial approximation to rational functions. It chooses the $N+1$ parameters so that $f^{(k)}(0) = r^{(k)}(0)$, for each $k = 0, 1, \ldots, N$. When $n = N$ and $m = 0$, the Padé approximation is simply the Nth Maclaurin polynomial.

Consider the difference

$$f(x) - r(x) = f(x) - \frac{p(x)}{q(x)} = \frac{f(x)q(x) - p(x)}{q(x)} = \frac{f(x)\sum_{i=0}^{m} q_i x^i - \sum_{i=0}^{n} p_i x^i}{q(x)},$$

and suppose f has the Maclaurin series expansion $f(x) = \sum_{i=0}^{\infty} a_i x^i$. Then

$$f(x) - r(x) = \frac{\sum_{i=0}^{\infty} a_i x^i \sum_{i=0}^{m} q_i x^i - \sum_{i=0}^{n} p_i x^i}{q(x)}. \tag{8.14}$$

The object is to choose the constants $q_1, q_2, \ldots, q_m$ and $p_0, p_1, \ldots, p_n$ so that

$$f^{(k)}(0) - r^{(k)}(0) = 0, \quad \text{for each } k = 0, 1, \ldots, N.$$

In Section 2.4 (see, in particular, Exercise 10 on page 86) we found that this is equivalent to $f - r$ having a zero of multiplicity $N + 1$ at $x = 0$. As a consequence, we choose $q_1, q_2, \ldots, q_m$ and $p_0, p_1, \ldots, p_n$ so that the numerator on the right side of Eq. (8.14),

$$(a_0 + a_1 x + \cdots)(1 + q_1 x + \cdots + q_m x^m) - (p_0 + p_1 x + \cdots + p_n x^n), \tag{8.15}$$

has no terms of degree less than or equal to N.

To simplify notation, we define $p_{n+1} = p_{n+2} = \cdots = p_N = 0$ and $q_{m+1} = q_{m+2} = \cdots = q_N = 0$. We can then express the coefficient of x^k in expression (8.15) more compactly as

$$\left(\sum_{i=0}^{k} a_i q_{k-i} \right) - p_k.$$

The rational function for Padé approximation results from the solution of the $N + 1$ linear equations

$$\sum_{i=0}^{k} a_i q_{k-i} = p_k, \quad k = 0, 1, \ldots, N$$

in the $N + 1$ unknowns $q_1, q_2, \ldots, q_m, p_0, p_1, \ldots, p_n$.

Example 1 The Maclaurin series expansion for e^{-x} is

$$\sum_{i=0}^{\infty} \frac{(-1)^i}{i!} x^i.$$

Find the Padé approximation to e^{-x} of degree 5 with $n = 3$ and $m = 2$.

Solution To find the Padé approximation we need to choose p_0, p_1, p_2, p_3, q_1, and q_2 so that the coefficients of x^k for $k = 0, 1, \ldots, 5$ are 0 in the expression

$$\left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \cdots\right)(1 + q_1x + q_2x^2) - (p_0 + p_1x + p_2x^2 + p_3x^3).$$

Expanding and collecting terms produces

$$\begin{aligned}
x^5:&\quad -\frac{1}{120} + \frac{1}{24}q_1 - \frac{1}{6}q_2 = 0; &\qquad x^2:&\quad \frac{1}{2} - q_1 + q_2 = p_2;\\
x^4:&\quad \frac{1}{24} - \frac{1}{6}q_1 + \frac{1}{2}q_2 = 0; & x^1:&\quad -1 + q_1 = p_1;\\
x^3:&\quad -\frac{1}{6} + \frac{1}{2}q_1 - q_2 = p_3; & x^0:&\quad 1 = p_0.
\end{aligned}$$

This gives

$$\left\{p_1 = -\frac{3}{5},\ p_2 = \frac{3}{20},\ p_3 = -\frac{1}{60},\ q_1 = \frac{2}{5},\ q_2 = \frac{1}{20}\right\}$$

So the Padé approximation is

$$r(x) = \frac{1 - \frac{3}{5}x + \frac{3}{20}x^2 - \frac{1}{60}x^3}{1 + \frac{2}{5}x + \frac{1}{20}x^2}.$$

Table 8.10 lists values of $r(x)$ and $P_5(x)$, the fifth Maclaurin polynomial. The Padé approximation is clearly superior in this example. ■

x	e^{-x}	$P_5(x)$	$\lvert e^{-x} - P_5(x)\rvert$	$r(x)$	$\lvert e^{-x} - r(x)\rvert$
0.2	0.81873075	0.81873067	8.64×10^{-8}	0.81873075	7.55×10^{-9}
0.4	0.67032005	0.67031467	5.38×10^{-6}	0.67031963	4.11×10^{-7}
0.6	0.54881164	0.54875200	5.96×10^{-5}	0.54880763	4.00×10^{-6}
0.8	0.44932896	0.44900267	3.26×10^{-4}	0.44930966	1.93×10^{-5}
1.0	0.36787944	0.36666667	1.21×10^{-3}	0.36781609	6.33×10^{-5}

ALGORITHM 8.1

Padé Rational Approximation

To obtain the rational approximation

$$r(x) = \frac{p(x)}{q(x)} = \frac{\sum_{i=0}^{n} p_i x^i}{\sum_{j=0}^{m} q_j x^j}$$

for a given function $f(x)$:

Continued Fraction Approximation

It is interesting to compare the number of arithmetic operations required for calculations of $P_5(x)$ and $r(x)$ in Example 1. Using nested multiplication, $P_5(x)$ can be expressed as

$$P_5(x) = \left(\left(\left(\left(-\frac{1}{120}x + \frac{1}{24}\right)x - \frac{1}{6}\right)x + \frac{1}{2}\right)x - 1\right)x + 1.$$

Assuming that the coefficients of $1, x, x^2, x^3, x^4$, and x^5 are represented as decimals, a single calculation of $P_5(x)$ in nested form requires five multiplications and five additions/subtractions.

Using nested multiplication, $r(x)$ is expressed as

$$r(x) = \frac{\left(\left(-\frac{1}{60}x + \frac{3}{20}\right)x - \frac{3}{5}\right)x + 1}{\left(\frac{1}{20}x + \frac{2}{5}\right)x + 1},$$

so a single calculation of $r(x)$ requires five multiplications, five additions/subtractions, and one division. Hence, computational effort appears to favor the polynomial approximation.

However, by reexpressing $r(x)$ by continued division, we can write

$$r(x) = \frac{1 - \frac{3}{5}x + \frac{3}{20}x^2 - \frac{1}{60}x^3}{1 + \frac{2}{5}x + \frac{1}{20}x^2}$$

$$= \frac{-\frac{1}{3}x^3 + 3x^2 - 12x + 20}{x^2 + 8x + 20}$$

$$= -\frac{1}{3}x + \frac{17}{3} + \frac{(-\frac{152}{3}x - \frac{280}{3})}{x^2 + 8x + 20}$$

$$= -\frac{1}{3}x + \frac{17}{3} + \frac{-\frac{152}{3}}{\left(\frac{x^2+8x+20}{x+(35/19)}\right)}$$

or

$$r(x) = -\frac{1}{3}x + \frac{17}{3} + \frac{-\frac{152}{3}}{\left(x + \frac{117}{19} + \frac{3125/361}{(x+(35/19))}\right)}. \tag{8.16}$$

Written in this form, a single calculation of $r(x)$ requires one multiplication, five additions/subtractions, and two divisions. If the amount of computation required for division is approximately the same as for multiplication, the computational effort required for an evaluation of the polynomial $P_5(x)$ significantly exceeds that required for an evaluation of the rational function $r(x)$.

https://www.khanacademy.org/math/algebra2/arithmetic-with-polynomials/long-division-of-polynomials/v/polynomial-division

Expressing a rational function approximation in a form such as Eq. (8.16) is called **continued-fraction** approximation. This is a classical approximation technique of current interest because of the computational efficiency of this representation. It is, however, a specialized technique that we will not discuss further. A rather extensive treatment of this subject and of rational approximation in general can be found in [RR], pp. 285–322.

Although the rational-function approximation in Example 1 gave results superior to the polynomial approximation of the same degree, note that the approximation has a wide variation in accuracy. The approximation at 0.2 is accurate to within 8×10^{-9}, but at 1.0 the approximation and the function agree only to within 7×10^{-5}. This accuracy variation is expected because the Padé approximation is based on a Taylor polynomial representation of e^{-x}, and the Taylor representation has a wide variation of accuracy in [0.2, 1.0].

Chebyshev Rational Function Approximation

To obtain more uniformly accurate rational-function approximations we use Chebyshev polynomials, a class that exhibits more uniform behavior. The general Chebyshev rational-function approximation method proceeds in the same manner as Padé approximation, except that each x^k term in the Padé approximation is replaced by the kth-degree Chebyshev polynomial $T_k(x)$.

Suppose we want to approximate the function f by an Nth-degree rational function r written in the form

$$r(x) = \frac{\sum_{k=0}^{n} p_k T_k(x)}{\sum_{k=0}^{m} q_k T_k(x)}, \quad \text{where } N = n + m \text{ and } q_0 = 1.$$

Writing $f(x)$ in a series involving Chebyshev polynomials as

$$f(x) = \sum_{k=0}^{\infty} a_k T_k(x),$$

gives

$$f(x) - r(x) = \sum_{k=0}^{\infty} a_k T_k(x) - \frac{\sum_{k=0}^{n} p_k T_k(x)}{\sum_{k=0}^{m} q_k T_k(x)}$$

or

$$f(x) - r(x) = \frac{\sum_{k=0}^{\infty} a_k T_k(x) \sum_{k=0}^{m} q_k T_k(x) - \sum_{k=0}^{n} p_k T_k(x)}{\sum_{k=0}^{m} q_k T_k(x)}. \tag{8.17}$$

The coefficients $q_1, q_2, \ldots, q_m$ and $p_0, p_1, \ldots, p_n$ are chosen so that the numerator on the right-hand side of this equation has zero coefficients for $T_k(x)$ when $k = 0, 1, \ldots, N$. This implies that the series

$$(a_0T_0(x) + a_1T_1(x) + \cdots)(T_0(x) + q_1T_1(x) + \cdots + q_mT_m(x))$$
$$- (p_0T_0(x) + p_1T_1(x) + \cdots + p_nT_n(x))$$

has no terms of degree less than or equal to N.

Two problems arise with the Chebyshev procedure that make it more difficult to implement than the Padé method. One occurs because the product of the polynomial $q(x)$ and the series for $f(x)$ involves products of Chebyshev polynomials. This problem is resolved by making use of the relationship

$$T_i(x)T_j(x) = \frac{1}{2}\left[T_{i+j}(x) + T_{|i-j|}(x)\right]. \tag{8.18}$$

(See Exercise 8 of Section 8.3.) The other problem is more difficult to resolve and involves the computation of the Chebyshev series for $f(x)$. In theory, this is not difficult for if

$$f(x) = \sum_{k=0}^{\infty} a_k T_k(x),$$

then the orthogonality of the Chebyshev polynomials implies that

$$a_0 = \frac{1}{\pi} \int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}\, dx \quad \text{and} \quad a_k = \frac{2}{\pi} \int_{-1}^{1} \frac{f(x)T_k(x)}{\sqrt{1-x^2}}\, dx, \quad \text{where } k \geq 1.$$

Practically, however, these integrals can seldom be evaluated in closed form, and a numerical integration technique is required for each evaluation.

Example 2 The first five terms of the Chebyshev expansion for e^{-x} are

$$\tilde{P}_5(x) = 1.266066T_0(x) - 1.130318T_1(x) + 0.271495T_2(x) - 0.044337T_3(x) + 0.005474T_4(x) - 0.000543T_5(x).$$

Determine the Chebyshev rational approximation of degree 5 with $n = 3$ and $m = 2$.

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Solution Finding this approximation requires choosing p_0, p_1, p_2, p_3, q_1, and q_2 so that for $k = 0, 1, 2, 3, 4$, and 5, the coefficients of $T_k(x)$ are 0 in the expansion

$$\tilde{P}_5(x)[T_0(x) + q_1T_1(x) + q_2T_2(x)] - [p_0T_0(x) + p_1T_1(x) + p_2T_2(x) + p_3T_3(x)].$$

Using the relation (8.18) and collecting terms gives the equations

$$\begin{aligned}
T_0: &\quad 1.266066 - 0.565159q_1 + 0.1357485q_2 = p_0,\\
T_1: &\quad -1.130318 + 1.401814q_1 - 0.587328q_2 = p_1,\\
T_2: &\quad 0.271495 - 0.587328q_1 + 1.268803q_2 = p_2,\\
T_3: &\quad -0.044337 + 0.138485q_1 - 0.565431q_2 = p_3,\\
T_4: &\quad 0.005474 - 0.022440q_1 + 0.135748q_2 = 0,\\
T_5: &\quad -0.000543 + 0.002737q_1 - 0.022169q_2 = 0.
\end{aligned}$$

The solution to this system produces the rational function

$$r_T(x) = \frac{1.055265T_0(x) - 0.613016T_1(x) + 0.077478T_2(x) - 0.004506T_3(x)}{T_0(x) + 0.378331T_1(x) + 0.022216T_2(x)}.$$

We found at the beginning of Section 8.3 that

$$T_0(x) = 1,\ T_1(x) = x,\ T_2(x) = 2x^2 - 1,\ T_3(x) = 4x^3 - 3x.$$

Using these to convert to an expression involving powers of x gives

$$r_T(x) = \frac{0.977787 - 0.599499x + 0.154956x^2 - 0.018022x^3}{0.977784 + 0.378331x + 0.044432x^2}.$$

Table 8.11 lists values of $r_T(x)$ and, for comparison purposes, the values of $r(x)$ obtained in Example 1. Note that the approximation given by $r(x)$ is superior to that of $r_T(x)$ for $x = 0.2$ and 0.4, but that the maximum error for $r(x)$ is 6.33×10^{-5} compared to 9.13×10^{-6} for $r_T(x)$. ■

Table 8.11

x	e^{-x}	$r(x)$	$\|e^{-x} - r(x)\|$	$r_T(x)$	$\|e^{-x} - r_T(x)\|$
0.2	0.81873075	0.81873075	7.55×10^{-9}	0.81872510	5.66×10^{-6}
0.4	0.67032005	0.67031963	4.11×10^{-7}	0.67031310	6.95×10^{-6}
0.6	0.54881164	0.54880763	4.00×10^{-6}	0.54881292	1.28×10^{-6}
0.8	0.44932896	0.44930966	1.93×10^{-5}	0.44933809	9.13×10^{-6}
1.0	0.36787944	0.36781609	6.33×10^{-5}	0.36787155	7.89×10^{-6}

The Chebyshev approximation can be generated using Algorithm 8.2.

ALGORITHM 8.2

Chebyshev Rational Approximation

To obtain the rational approximation

$$r_T(x) = \frac{\sum_{k=0}^{n} p_k T_k(x)}{\sum_{k=0}^{m} q_k T_k(x)}$$

for a given function $f(x)$:

EXERCISE SET 8.4

11. Find the Chebyshev rational approximation of degree 4 with $n = m = 2$ for $f(x) = \sin x$. Compare the results at $x_i = 0.1i$, for $i = 0, 1, 2, 3, 4, 5$, from this approximation with those obtained in Exercise 5 using a sixth-degree Padé approximation.

$$f(x) = \sum_{k=0}^{\infty} a_k T_k(x),$$

$$a_0 = \frac{1}{\pi}\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}\,dx \quad \text{and} \quad a_k = \frac{2}{\pi}\int_{-1}^{1} \frac{f(x)T_k(x)}{\sqrt{1-x^2}}\,dx, \quad \text{where } k \geq 1.$$

$$f(x) - r(x) = \frac{\sum_{k=0}^{\infty} a_k T_k(x) \sum_{k=0}^{m} q_k T_k(x) - \sum_{k=0}^{n} p_k T_k(x)}{\sum_{k=0}^{m} q_k T_k(x)}.$$

$$T_i(x)T_j(x) = \frac{1}{2}\left[T_{i+j}(x) + T_{|i-j|}(x)\right].$$

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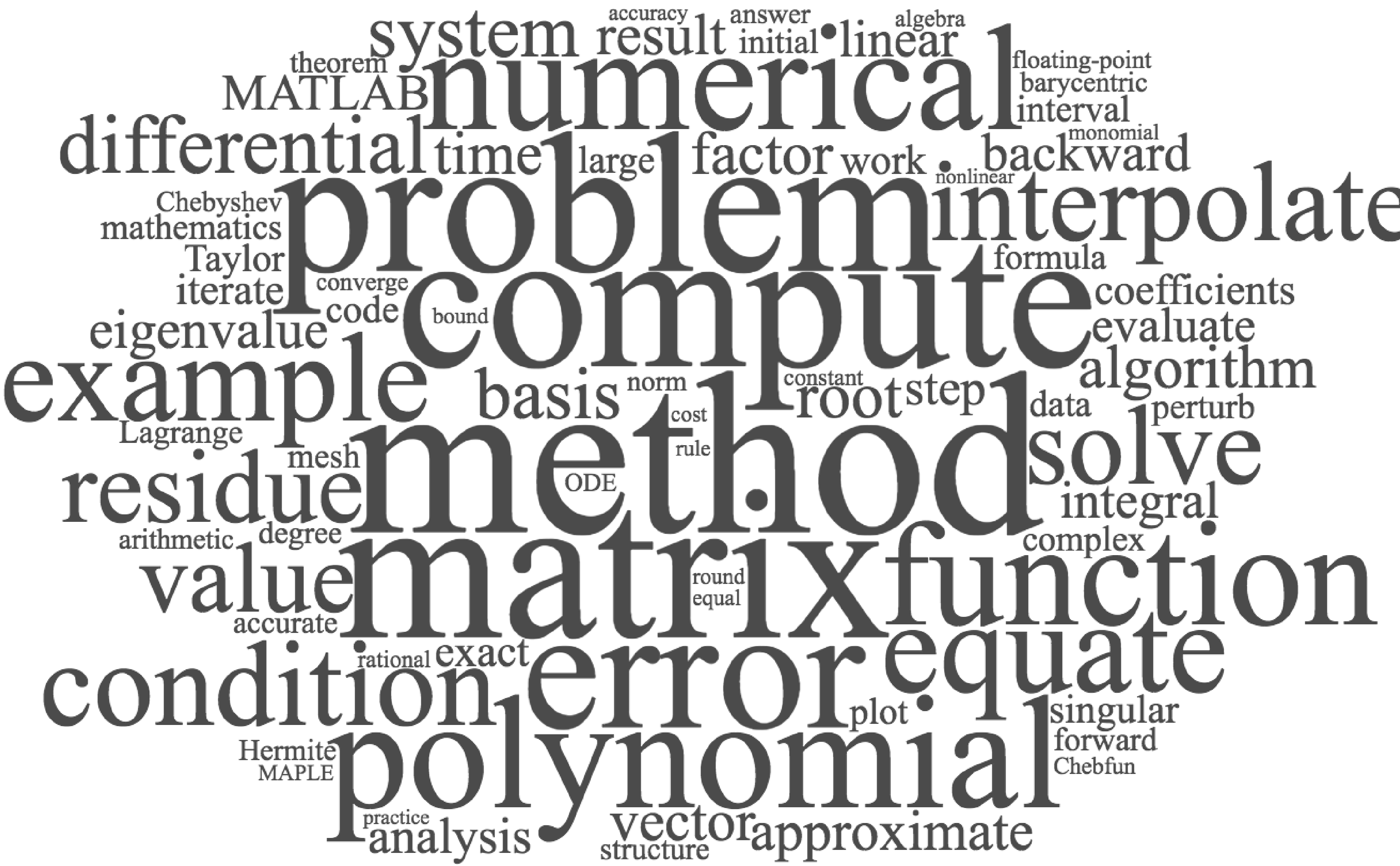
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