MAP 2220 – FUNDAMENTOS DE ANÁLISE NUMÉRICA

2º Semestre - 2017

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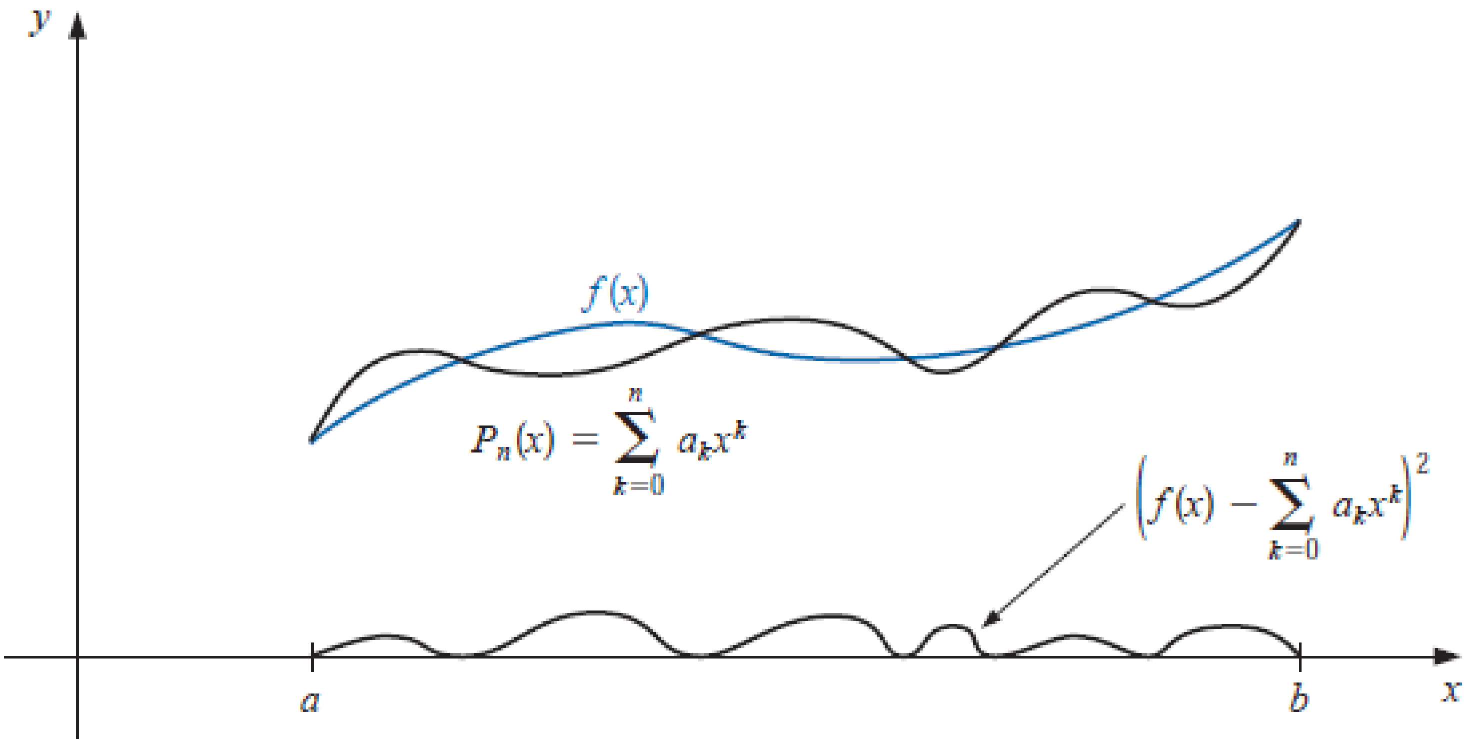
8 Approximation Theory 497

$$E \equiv E_2(a_0, a_1, \ldots, a_n) = \int_a^b \left(f(x) - \sum_{k=0}^{n} a_k x^k \right)^2 dx$$



linear **normal equations**

$$\sum_{k=0}^{n} a_k \int_a^b x^{j+k}\,dx = \int_a^b x^j f(x)\,dx, \quad \text{for each } j = 0, 1, \ldots, n, \tag{8.6}$$

$$\int_a^b x^{j+k}\,dx = \frac{b^{j+k+1} - a^{j+k+1}}{j+k+1},$$

Hilbert matrix, which is a classic example for demonstrating round-off error difficulties.

Theorem 8.6

If $\{\phi_0, \ldots, \phi_n\}$ is an orthogonal set of functions on an interval $[a, b]$ with respect to the weight function w, then the least squares approximation to f on $[a, b]$ with respect to w is

$$P(x) = \sum_{j=0}^{n} a_j \phi_j(x),$$

where, for each $j = 0, 1, \ldots, n$,

$$a_j = \frac{\int_a^b w(x)\phi_j(x) f(x)\, dx}{\int_a^b w(x)[\phi_j(x)]^2\, dx} = \frac{1}{\alpha_j} \int_a^b w(x)\phi_j(x) f(x)\, dx. \quad \blacksquare$$

Definition 8.5

$\{\phi_0, \phi_1, \ldots, \phi_n\}$ is said to be an **orthogonal set of functions** for the interval $[a, b]$ with respect to the weight function w if

$$\int_a^b w(x)\phi_k(x)\phi_j(x)\, dx = \begin{cases} 0, & \text{when } j \neq k, \\ \alpha_j > 0, & \text{when } j = k. \end{cases}$$

Theorem 8.7 Gram-Schmidt process

The set of polynomial functions $\{\phi_0, \phi_1, \ldots, \phi_n\}$ defined in the following way is orthogonal on $[a, b]$ with respect to the weight function w.

$$\phi_0(x) \equiv 1, \quad \phi_1(x) = x - B_1, \quad \text{for each } x \text{ in } [a, b],$$

where

$$B_1 = \frac{\int_a^b x w(x)[\phi_0(x)]^2\, dx}{\int_a^b w(x)[\phi_0(x)]^2\, dx},$$

and when $k \geq 2$,

$$\phi_k(x) = (x - B_k)\phi_{k-1}(x) - C_k\phi_{k-2}(x), \quad \text{for each } x \text{ in } [a, b],$$

where

$$B_k = \frac{\int_a^b x w(x)[\phi_{k-1}(x)]^2\, dx}{\int_a^b w(x)[\phi_{k-1}(x)]^2\, dx}$$

and

$$C_k = \frac{\int_a^b x w(x)\phi_{k-1}(x)\phi_{k-2}(x)\, dx}{\int_a^b w(x)[\phi_{k-2}(x)]^2\, dx}.$$

■

The set of **Legendre polynomials**, $\{P_n(x)\}$, is orthogonal on $[-1, 1]$ with respect to the weight function $w(x) \equiv 1$.

$P_0(x) = 1$

$P_1(x) = x$

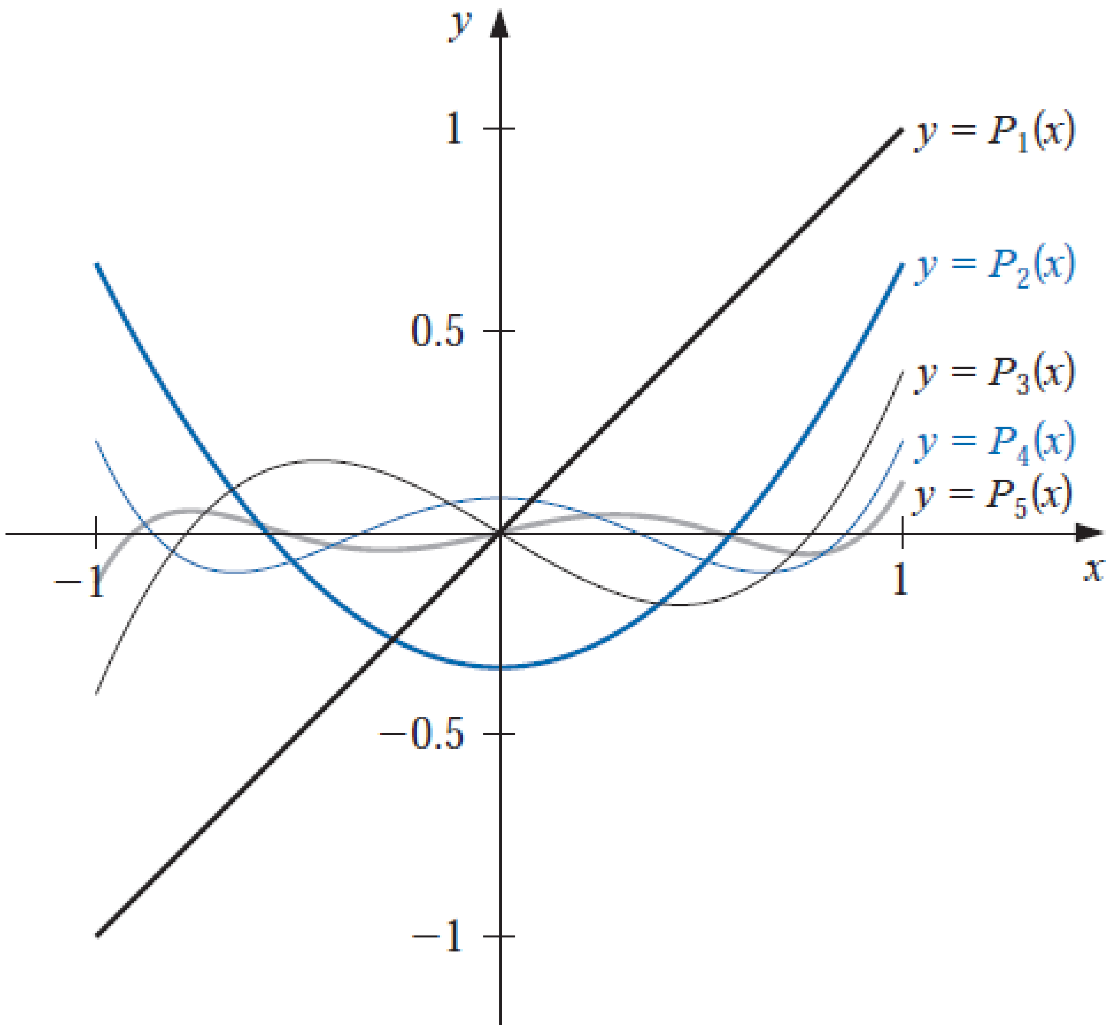
$P_2(x) = x^2 - \frac{1}{3}$

$P_3(x) = x^3 - \frac{3}{5}x$

$P_4(x) = x^4 - \frac{6}{7}x^2 + \frac{3}{35}$

$P_5(x) = x^5 - \frac{10}{9}x^3 + \frac{5}{21}x$

...



EXERCISE SET 8.2

11. Use the Gram-Schmidt procedure to calculate L_1, L_2, and L_3, where $\{L_0(x), L_1(x), L_2(x), L_3(x)\}$ is an orthogonal set of polynomials on $(0, \infty)$ with respect to the weight functions $w(x) = e^{-x}$ and $L_0(x) \equiv 1$. The polynomials obtained from this procedure are called the **Laguerre polynomials**.

12. Use the Laguerre polynomials calculated in Exercise 11 to compute the least squares polynomials of degree one, two, and three on the interval $(0, \infty)$ with respect to the weight function $w(x) = e^{-x}$ for the following functions:

a. $f(x) = x^2$ **b.** $f(x) = e^{-x}$ **c.** $f(x) = x^3$ **d.** $f(x) = e^{-2x}$

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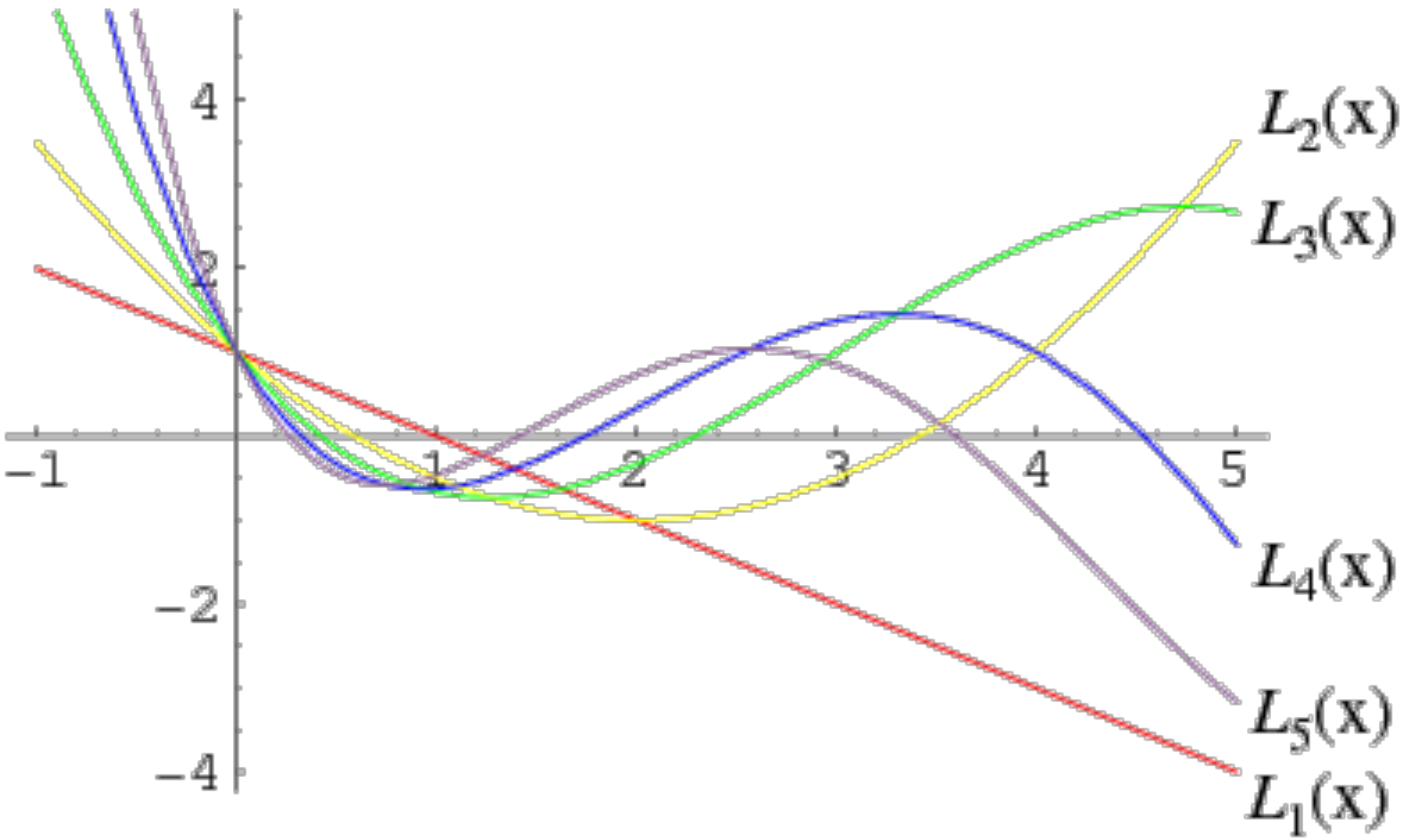
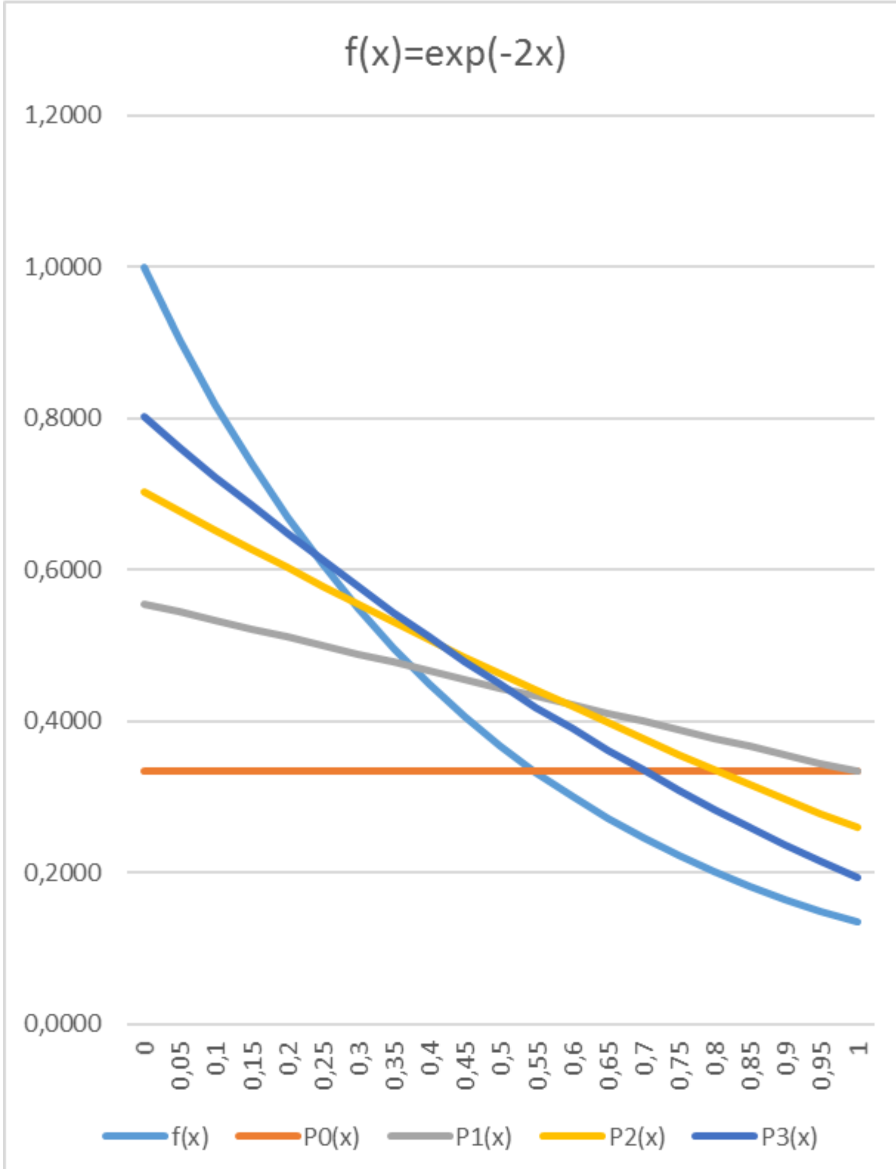


4
2
−2
−4
−1
1
2
3
4
5
$L_2(x)$
$L_3(x)$
$L_4(x)$
$L_5(x)$
$L_1(x)$

x	f(x)	P0(x)	P1(x)	P2(x)	P3(x)	df1	df2	df3
0	1,0000	0,3333	0,5556	0,7037	0,8025	0,1975	0,0878	0,0390
0,05	0,9048	0,3333	0,5444	0,6780	0,7623	0,1365	0,0541	0,0214
0,1	0,8187	0,3333	0,5333	0,6526	0,7232	0,0900	0,0305	0,0101
0,15	0,7408	0,3333	0,5222	0,6276	0,6852	0,0555	0,0149	0,0036
0,2	0,6703	0,3333	0,5111	0,6030	0,6483	0,0310	0,0055	0,0006
0,25	0,6065	0,3333	0,5000	0,5787	0,6124	0,0146	0,0010	0,0000
0,3	0,5488	0,3333	0,4889	0,5548	0,5776	0,0048	0,0000	0,0011
0,35	0,4966	0,3333	0,4778	0,5313	0,5438	0,0005	0,0017	0,0032
0,4	0,4493	0,3333	0,4667	0,5081	0,5110	0,0004	0,0052	0,0057
0,45	0,4066	0,3333	0,4556	0,4854	0,4793	0,0038	0,0097	0,0083
0,5	0,3679	0,3333	0,4444	0,4630	0,4486	0,0097	0,0149	0,0107
0,55	0,3329	0,3333	0,4333	0,4409	0,4188	0,0175	0,0202	0,0128
0,6	0,3012	0,3333	0,4222	0,4193	0,3900	0,0267	0,0254	0,0144
0,65	0,2725	0,3333	0,4111	0,3980	0,3622	0,0368	0,0301	0,0154
0,7	0,2466	0,3333	0,4000	0,3770	0,3353	0,0474	0,0343	0,0159
0,75	0,2231	0,3333	0,3889	0,3565	0,3094	0,0582	0,0376	0,0158
0,8	0,2019	0,3333	0,3778	0,3363	0,2844	0,0688	0,0402	0,0152
0,85	0,1827	0,3333	0,3667	0,3165	0,2603	0,0792	0,0419	0,0141
0,9	0,1653	0,3333	0,3556	0,2970	0,2371	0,0890	0,0427	0,0127
0,95	0,1496	0,3333	0,3444	0,2780	0,2148	0,0982	0,0426	0,0110
1	0,1353	0,3333	0,3333	0,2593	0,1934	0,1066	0,0417	0,0092
					Erro	0,1021	0,0517	0,0216



f(x)=exp(-2x)
1,2000
1,0000
0,8000
0,6000
0,4000
0,2000
0,0000
0
0,05
0,1
0,15
0,2
0,25
0,3
0,35
0,4
0,45
0,5
0,55
0,6
0,65
0,7
0,75
0,8
0,85
0,9
0,95
1
f(x)
P0(x)
P1(x)
P2(x)
P3(x)

8.3 Chebyshev Polynomials and Economization of Power Series

The Chebyshev polynomials $\{T_n(x)\}$ are orthogonal on $(-1, 1)$ with respect to the weight function $w(x) = (1 - x^2)^{-1/2}$. Although they can be derived by the method in the previous section, it is easier to give their definition and then show that they satisfy the required orthogonality properties.

For $x \in [-1, 1]$, define

$$T_n(x) = \cos[n \arccos x], \quad \text{for each } n \geq 0. \tag{8.8}$$

It might not be obvious from this definition that for each n, $T_n(x)$ is a polynomial in x, but we will now show this. First note that

$$T_0(x) = \cos 0 = 1 \quad \text{and} \quad T_1(x) = \cos(\arccos x) = x.$$

For $n \geq 1$, we introduce the substitution $\theta = \arccos x$ to change this equation to

$$T_n(\theta(x)) \equiv T_n(\theta) = \cos(n\theta), \quad \text{where } \theta \in [0, \pi].$$

A recurrence relation is derived by noting that

$$T_{n+1}(\theta) = \cos(n+1)\theta = \cos\theta\ \cos(n\theta) - \sin\theta\ \sin(n\theta)$$

and

$$T_{n-1}(\theta) = \cos(n-1)\theta = \cos\theta\ \cos(n\theta) + \sin\theta\ \sin(n\theta)$$

Adding these equations gives

$$T_{n+1}(\theta) = 2\cos\theta\ \cos(n\theta) - T_{n-1}(\theta).$$

Returning to the variable $x = \cos\theta$, we have, for $n \geq 1$,

$$T_{n+1}(x) = 2x\cos(n\arccos x) - T_{n-1}(x),$$

that is,

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x). \tag{8.9}$$

Because $T_0(x) = 1$ and $T_1(x) = x$, the recurrence relation implies that the next three Chebyshev polynomials are

$$T_2(x) = 2xT_1(x) - T_0(x) = 2x^2 - 1,$$

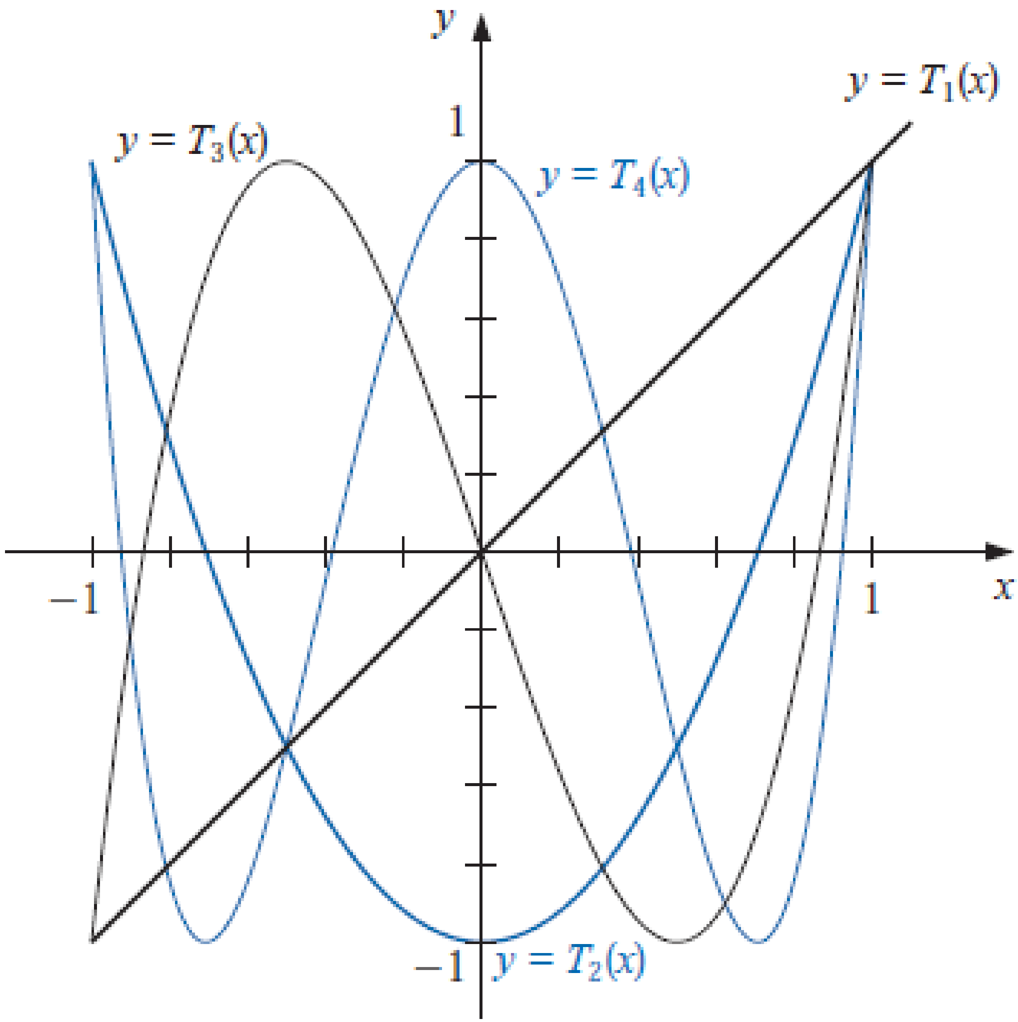
$$T_3(x) = 2xT_2(x) - T_1(x) = 4x^3 - 3x,$$

and

$$T_4(x) = 2xT_3(x) - T_2(x) = 8x^4 - 8x^2 + 1.$$

The recurrence relation also implies that when $n \geq 1$, $T_n(x)$ is a polynomial of degree n with leading coefficient 2^{n-1}. The graphs of T_1, T_2, T_3, and T_4 are shown in Figure 8.10.

Figure 8.10



To show the orthogonality of the Chebyshev polynomials with respect to the weight function $w(x) = (1 - x^2)^{-1/2}$, consider

$$\int_{-1}^{1} \frac{T_n(x)T_m(x)}{\sqrt{1-x^2}}\,dx = \int_{-1}^{1} \frac{\cos(n\arccos x)\cos(m\arccos x)}{\sqrt{1-x^2}}\,dx.$$

Reintroducing the substitution $\theta = \arccos x$ gives

$$d\theta = -\frac{1}{\sqrt{1-x^2}}\,dx$$

and

$$\int_{-1}^{1} \frac{T_n(x)T_m(x)}{\sqrt{1-x^2}}\,dx = -\int_{\pi}^{0} \cos(n\theta)\cos(m\theta)\,d\theta = \int_{0}^{\pi} \cos(n\theta)\cos(m\theta)\,d\theta.$$

Suppose $n \neq m$. Since

$$\cos(n\theta)\cos(m\theta) = \frac{1}{2}[\cos(n+m)\theta + \cos(n-m)\theta],$$

we have

$$\begin{aligned}\int_{-1}^{1} \frac{T_n(x)T_m(x)}{\sqrt{1-x^2}}\,dx &= \frac{1}{2}\int_{0}^{\pi} \cos((n+m)\theta)\,d\theta + \frac{1}{2}\int_{0}^{\pi} \cos((n-m)\theta)\,d\theta \\ &= \left[\frac{1}{2(n+m)}\sin((n+m)\theta) + \frac{1}{2(n-m)}\sin((n-m)\theta)\right]_{0}^{\pi} = 0.\end{aligned}$$

By a similar technique (see Exercise 9), we also have

$$\int_{-1}^{1} \frac{[T_n(x)]^2}{\sqrt{1-x^2}}\,dx = \frac{\pi}{2}, \quad \text{for each } n \geq 1. \tag{8.10}$$

The Chebyshev polynomials are used to minimize approximation error. We will see how they are used to solve two problems of this type:

- an optimal placing of interpolating points to minimize the error in Lagrange interpolation;
- a means of reducing the degree of an approximating polynomial with minimal loss of accuracy.

EXERCISE SET 8.2

9. Obtain the least squares approximation polynomial of degree 3 for the functions in Exercise 1 using the results of Exercise 7.

d. $f(x) = e^x$

$$P(x) = \sum_{j=0}^{n} a_j \phi_j(x),$$

$$a_j = \frac{\int_a^b w(x)\phi_j(x) f(x)\, dx}{\int_a^b w(x)[\phi_j(x)]^2\, dx} = \frac{1}{\alpha_j} \int_a^b w(x)\phi_j(x) f(x)\, dx.$$

$$T_0(x) = 1$$

$$T_1(x) = x$$

$$T_2(x) = 2x^2 - 1$$

$$T_3(x) = 4x^3 - 3x$$

$$T_4(x) = 8x^4 - 8x^2 + 1$$

$$T_5(x) = 16x^5 - 20x^3 + 5x$$

$$T_6(x) = 32x^6 - 48x^4 + 18x^2 - 1$$

$$T_7(x) = 64x^7 - 112x^5 + 56x^3 - 7x$$

Definite integral:

More digits

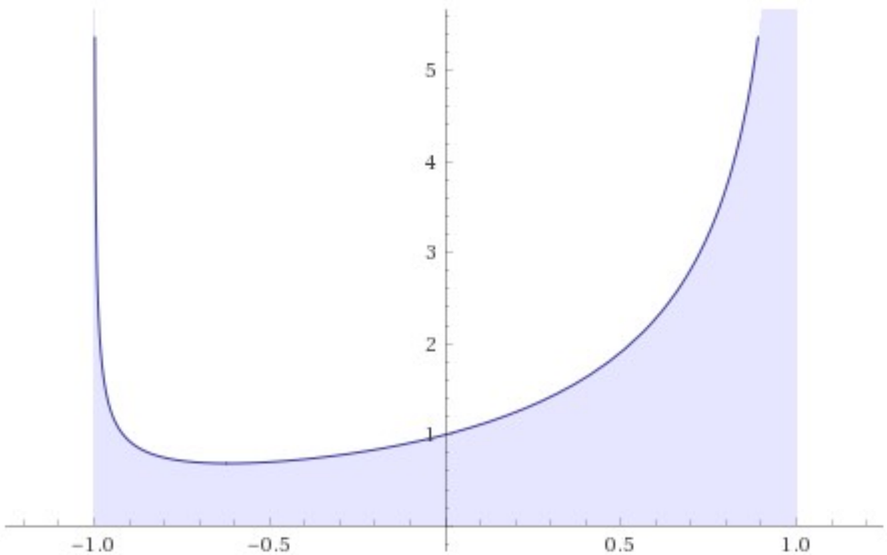
$$\int_{-1}^{1} \frac{\exp(x)}{\sqrt{1-x^2}}\,dx = \pi\, I_0(1) \approx 3.97746$$

Open code

$I_n(z)$ is the modified Bessel function of the first kind

Enlarge | Data | Customize | Plaintext | Interactive

Visual representation of the integral:



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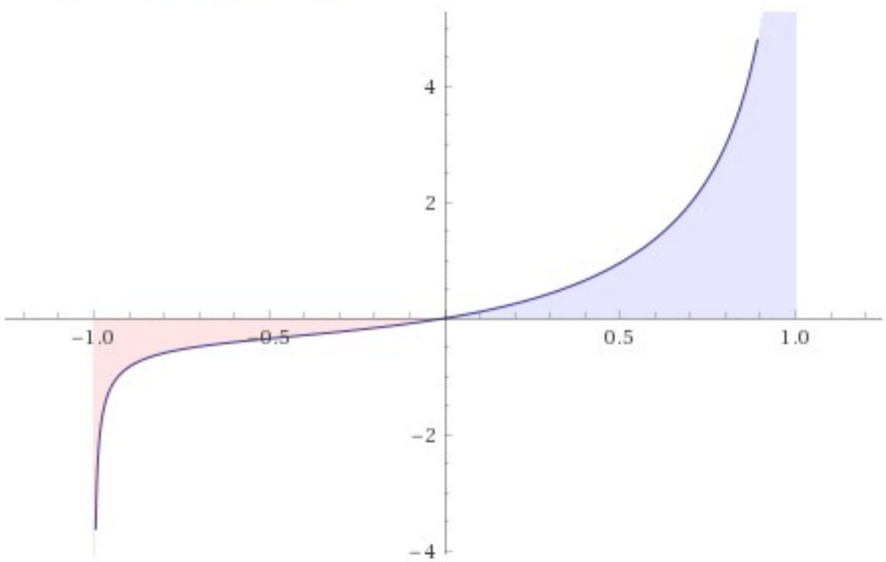
Definite integral:

More digits

$$\int_{-1}^{1} \frac{\exp(x)\, x}{\sqrt{1 - x^2}}\, dx = \pi\, I_1(1) \approx 1.7755$$

$I_n(z)$ is the modified Bessel function of the first kind

Visual representation of the integral:



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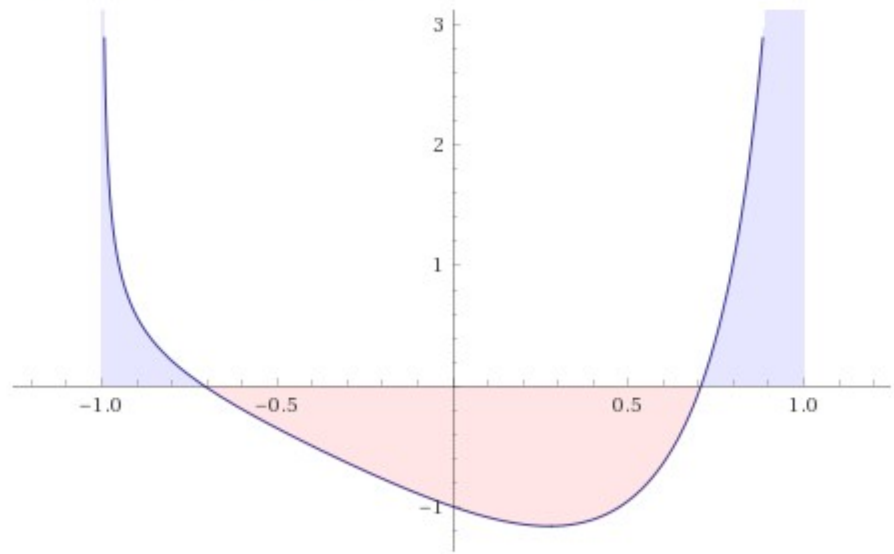
Definite integral:

More digits

$$\int_{-1}^{1} \frac{\exp(x)\,(2\,x^2 - 1)}{\sqrt{1 - x^2}}\,dx = \pi\, I_2(1) \approx 0.426464$$

$I_n(z)$ is the modified Bessel function of the first kind

Visual representation of the integral:



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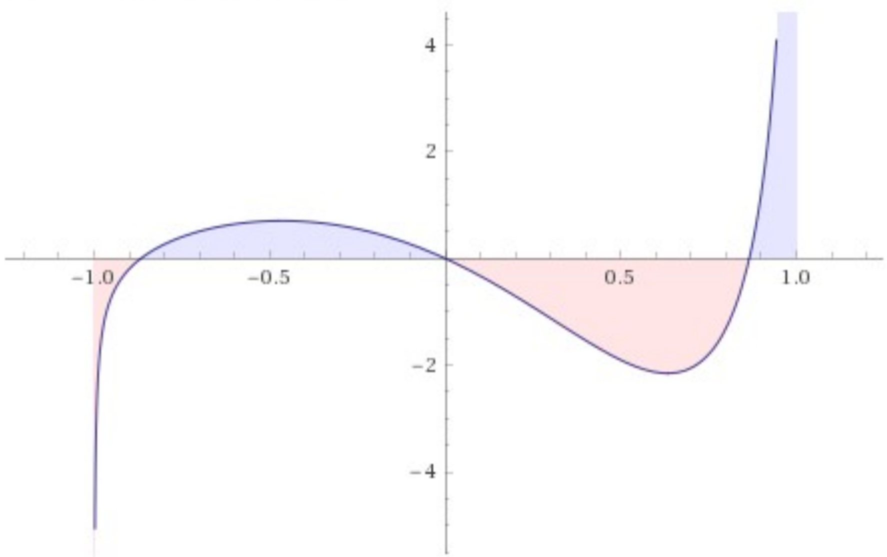
Definite integral:

More digits

$$\int_{-1}^{1} \frac{\exp(x)\,(4\,x^3 - 3\,x)}{\sqrt{1 - x^2}}\,dx = \pi\,I_3(1) \approx 0.0696442$$

$I_n(z)$ is the modified Bessel function of the first kind

Visual representation of the integral:



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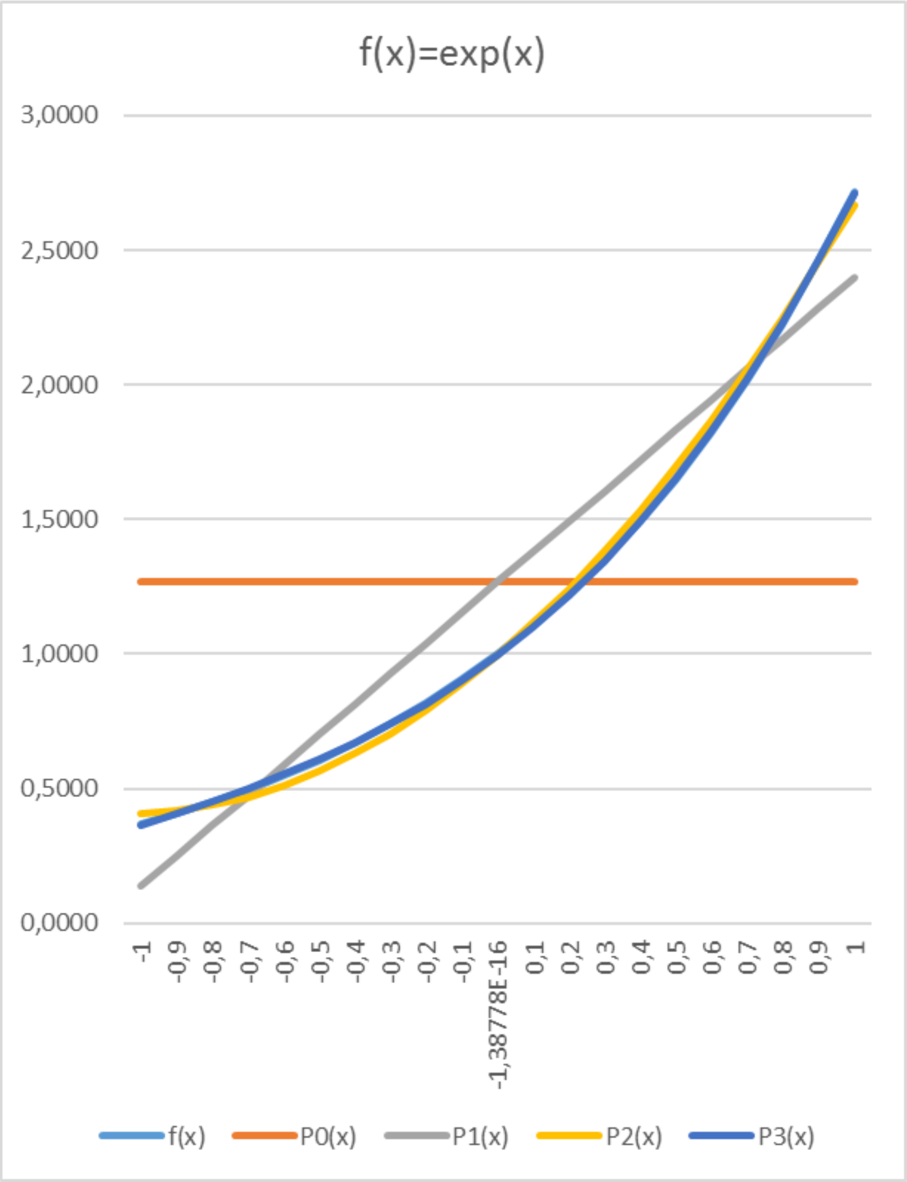

f(x)=exp(x)
3,0000
2,5000
2,0000
1,5000
1,0000
0,5000
0,0000
-1
-0,9
-0,8
-0,7
-0,6
-0,5
-0,4
-0,3
-0,2
-0,1
-1,38778E-16
0,1
0,2
0,3
0,4
0,5
0,6
0,7
0,8
0,9
1
f(x)
P0(x)
P1(x)
P2(x)
P3(x)

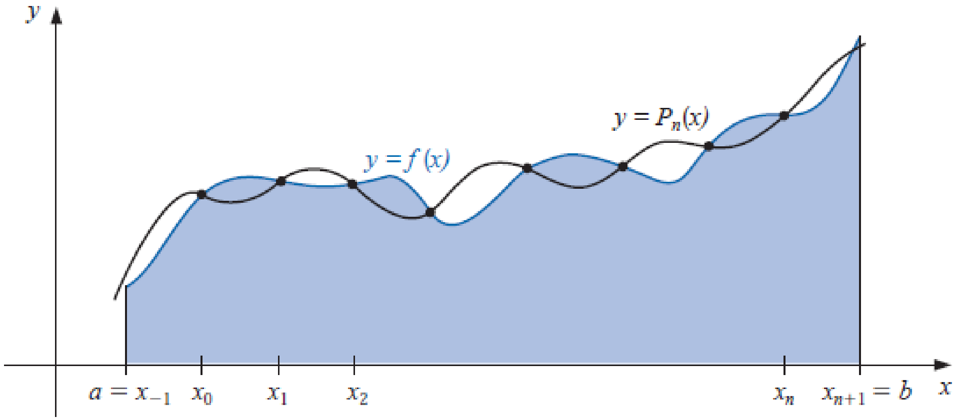
6. Compute the error E for the approximations in Exercise 4.

$$E = E(a_0, \ldots, a_n) = \int_a^b w(x)\left[f(x) - \sum_{k=0}^{n} a_k \phi_k(x)\right]^2 dx.$$

$$w(x) = \frac{1}{\sqrt{1-x^2}}$$

4.3 Elements of Numerical Integration

Open Newton-Cotes Formulas



$n = 0$: Midpoint rule

$$\int_{x_{-1}}^{x_1} f(x)\,dx = 2hf(x_0) + \frac{h^3}{3} f''(\xi), \quad \text{where} \quad x_{-1} < \xi < x_1.$$

x	f(x)	P0(x)	P1(x)	P2(x)	P3(x)	df1	df2	df3
-1	0,3679	1,2661	0,1357	0,4072	0,3629			
-0,9	0,4066	1,2661	0,2488	0,4171	0,4075	1,6948	0,0571	0,0003
-0,8	0,4493	1,2661	0,3618	0,4378	0,4534			
-0,7	0,4966	1,2661	0,4748	0,4694	0,5017	0,8291	0,0007	0,0010
-0,6	0,5488	1,2661	0,5879	0,5119	0,5534			
-0,5	0,6065	1,2661	0,7009	0,5652	0,6095	0,5023	0,0103	0,0020
-0,4	0,6703	1,2661	0,8139	0,6293	0,6712			
-0,3	0,7408	1,2661	0,9270	0,7043	0,7395	0,2892	0,0363	0,0014
-0,2	0,8187	1,2661	1,0400	0,7902	0,8154			
-0,1	0,9048	1,2661	1,1530	0,8870	0,9001	0,1311	0,0619	0,0003
-1,4E-16	1,0000	1,2661	1,2661	0,9946	0,9946			
0,1	1,1052	1,2661	1,3791	1,1130	1,0999	0,0260	0,0754	0,0001
0,2	1,2214	1,2661	1,4921	1,2424	1,2172			
0,3	1,3499	1,2661	1,6052	1,3825	1,3474	0,0074	0,0683	0,0011
0,4	1,4918	1,2661	1,7182	1,5336	1,4917			
0,5	1,6487	1,2661	1,8312	1,6955	1,6511	0,1691	0,0385	0,0025
0,6	1,8221	1,2661	1,9443	1,8682	1,8267			
0,7	2,0138	1,2661	2,0573	2,0519	2,0196	0,7828	0,0027	0,0020
0,8	2,2255	1,2661	2,1703	2,2463	2,2307			
0,9	2,4596	1,2661	2,2834	2,4517	2,4613	3,2681	0,0713	0,0001
1	2,7183	1,2661	2,3964	2,6679	2,7122			
					Erro	1,5400	0,0845	0,0022

8 Approximation Theory 497

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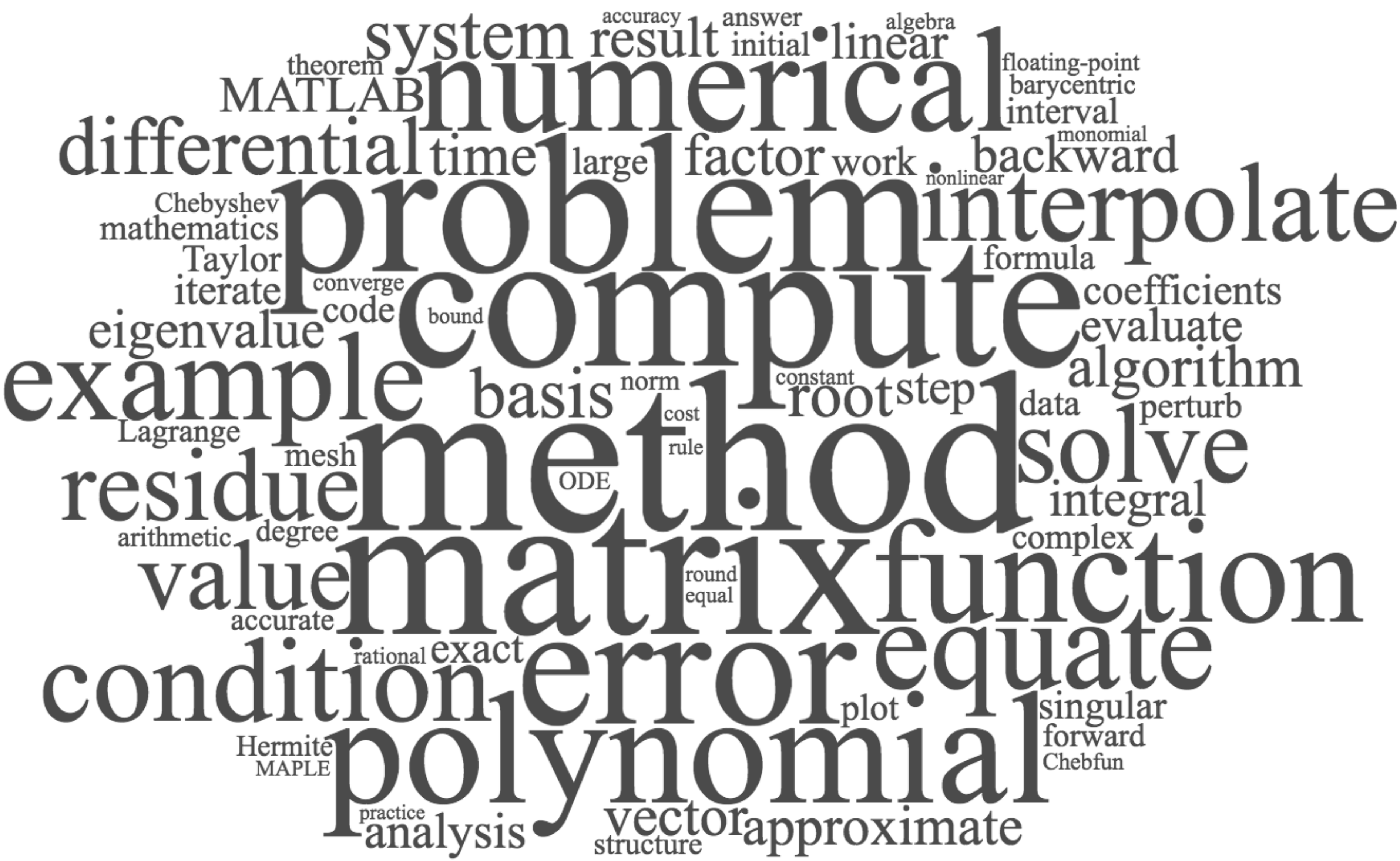
accuracy
answer
algebra
system
result
initial
linear
theorem
floating-point
numerical
MATLAB
barycentric
interval
differential
time
large
factor
work
monomial
backward
nonlinear
problem
interpolate
Chebyshev
mathematics
Taylor
formula
iterate
converge
compute
coefficients
code
bound
eigenvalue
evaluate
example
algorithm
basis
norm
constant
root
step
data
cost
perturb
Lagrange
method
solve
rule
mesh
ODE
residue
integral
arithmetic
degree
complex
matrix
function
value
round
equal
accurate
equate
exact
rational
error
condition
plot
singular
Hermite
forward
polynomial
MAPLE
Chebfun
practice
vector
analysis
approximate
structure