MAP 2220 – FUNDAMENTOS DE ANÁLISE NUMÉRICA

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Introduction

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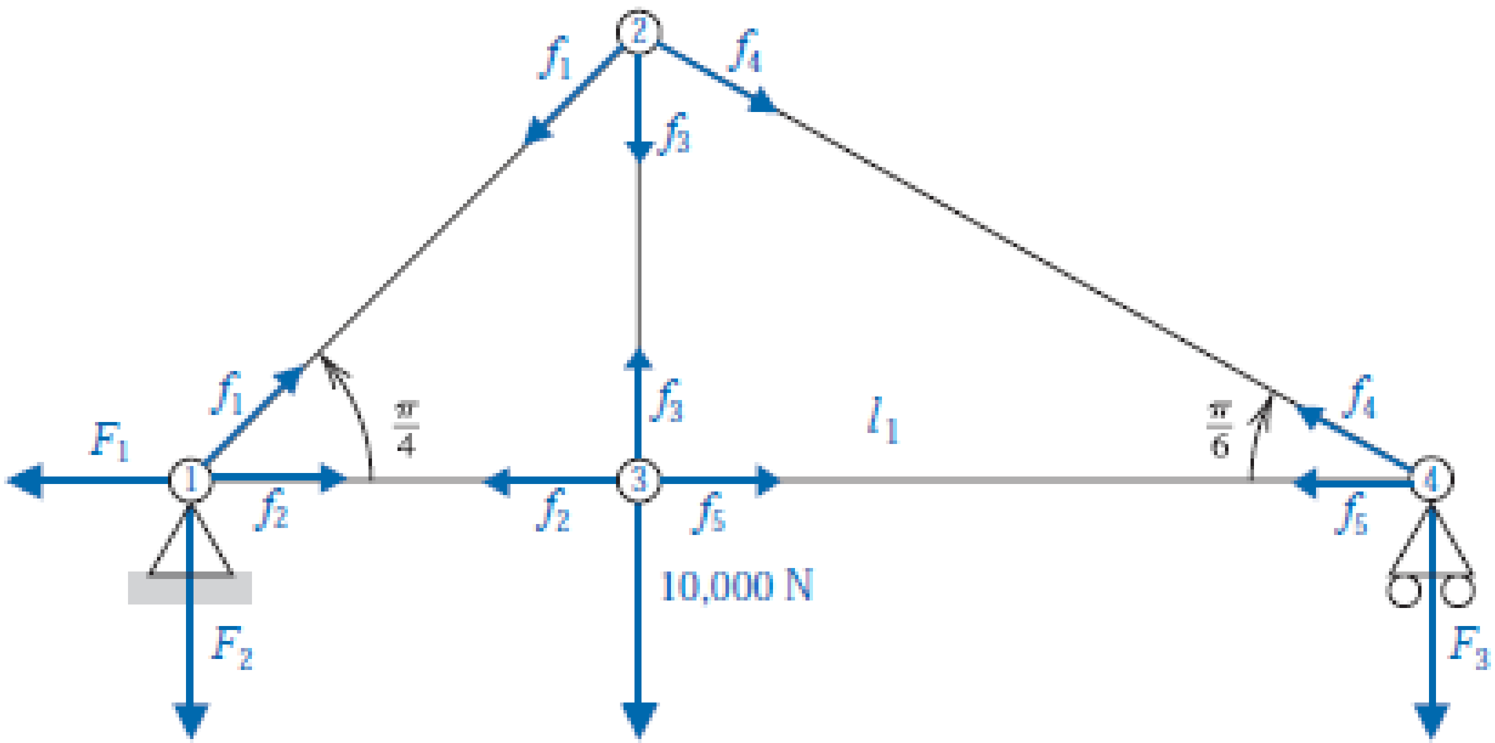


Joint	Horizontal Component	Vertical Component
①	$-F_1 + \frac{\sqrt{2}}{2} f_1 + f_2 = 0$	$\frac{\sqrt{2}}{2} f_1 - F_2 = 0$
②	$-\frac{\sqrt{2}}{2} f_1 + \frac{\sqrt{3}}{2} f_4 = 0$	$-\frac{\sqrt{2}}{2} f_1 - f_3 - \frac{1}{2} f_4 = 0$
③	$-f_2 + f_5 = 0$	$f_3 - 10{,}000 = 0$
④	$-\frac{\sqrt{3}}{2} f_4 - f_5 = 0$	$\frac{1}{2} f_4 - F_3 = 0$

If the truss is in static equilibrium, the forces at each joint must add to the zero vector, so the sum of the horizontal and vertical components at each joint must be 0. This produces the system of linear equations shown in the accompanying table. An 8×8 matrix describing this system has 47 zero entries and only 17 nonzero entries. Matrices with a high percentage of zero entries are called *sparse* and are often solved using iterative, rather than direct, techniques. The iterative solution to this system is considered in Exercise 18 of Section 7.3 and Exercise 10 in Section 7.4.

The methods presented in Chapter 6 used direct techniques to solve a system of $n \times n$ linear equations of the form $A\mathbf{x} = \mathbf{b}$. In this chapter, we present iterative methods to solve a system of this type.

This linear system can be placed in the matrix form

$$\begin{bmatrix} -1 & 0 & 0 & \frac{\sqrt{2}}{2} & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & \frac{\sqrt{2}}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{\sqrt{2}}{2} & 0 & -1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\frac{\sqrt{2}}{2} & 0 & 0 & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{\sqrt{3}}{2} & -1 \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 10{,}000 \\ 0 \\ 0 \end{bmatrix}.$$

7.1 Norms of Vectors and Matrices

Definition 7.1 A vector norm on $\mathbb{R}^n$ is a function, $\|\cdot\|$, from $\mathbb{R}^n$ into $\mathbb{R}$ with the following properties:

(i) $\|\mathbf{x}\| \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$,

(ii) $\|\mathbf{x}\| = 0$ if and only if $\mathbf{x} = \mathbf{0}$,

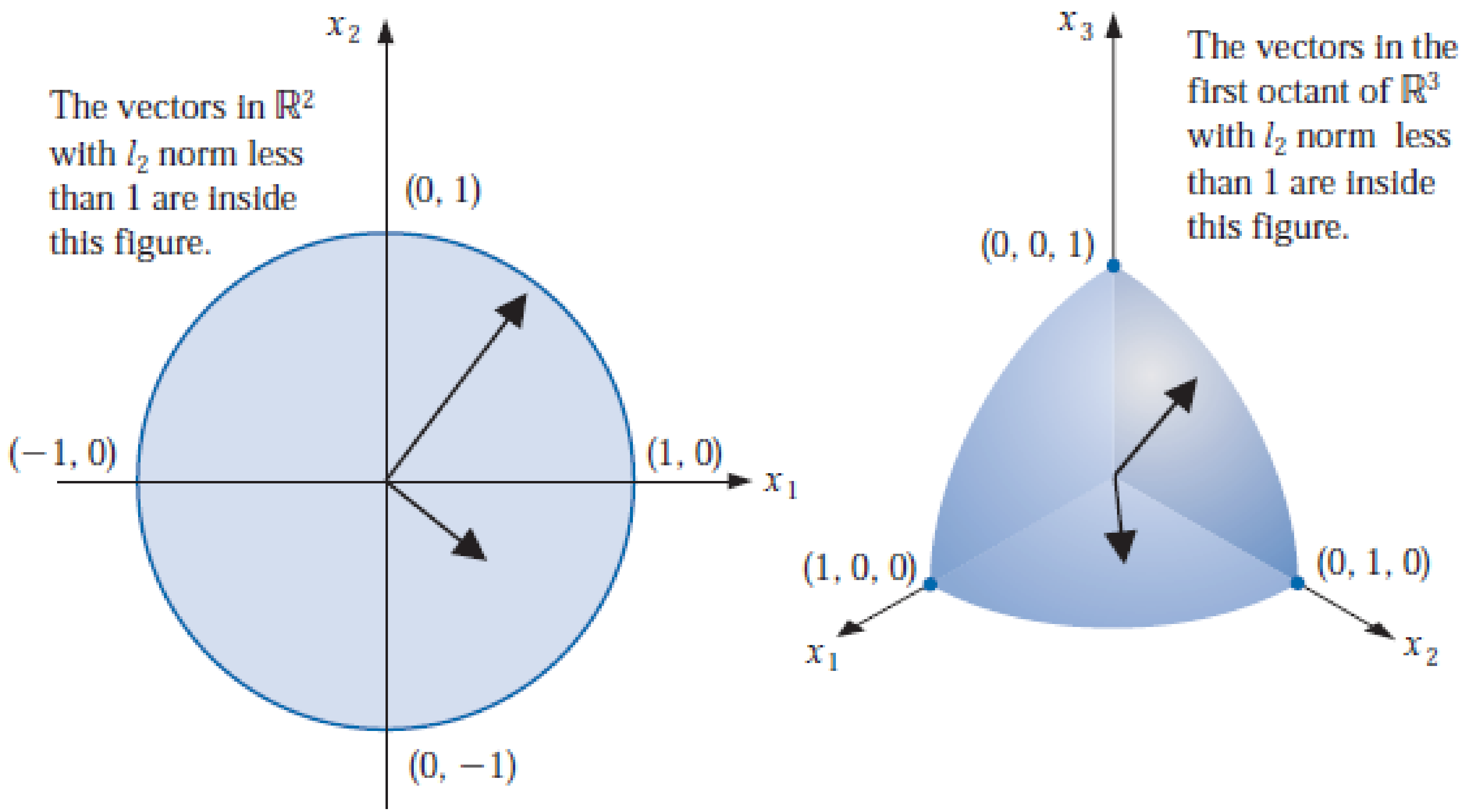
(iii) $\|\alpha\mathbf{x}\| = |\alpha|\|\mathbf{x}\|$ for all $\alpha \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$,

(iv) $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. ■

Definition 7.2 The l_2 and l_∞ norms for the vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)^t$ are defined by

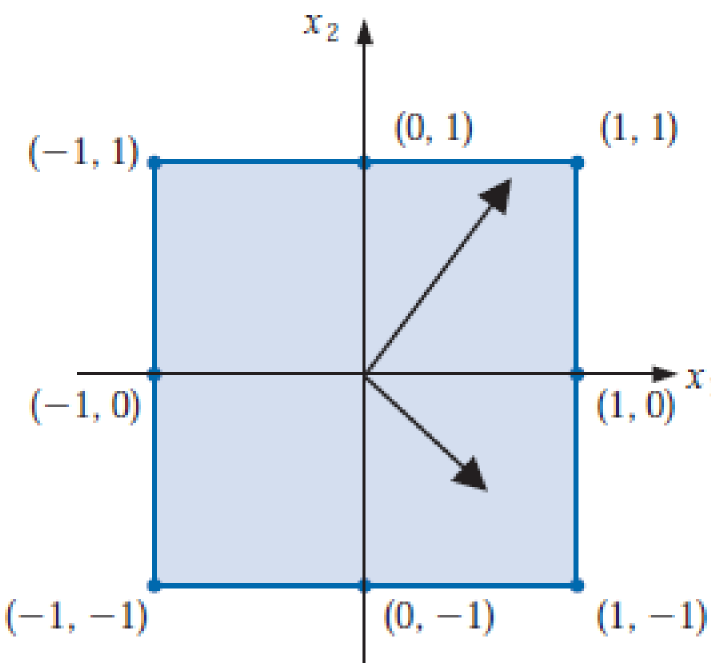
$$\|\mathbf{x}\|_2 = \left\{\sum_{i=1}^{n} x_i^2\right\}^{1/2} \quad \text{and} \quad \|\mathbf{x}\|_\infty = \max_{1\leq i\leq n} |x_i|. \qquad ■$$

Figure 7.1

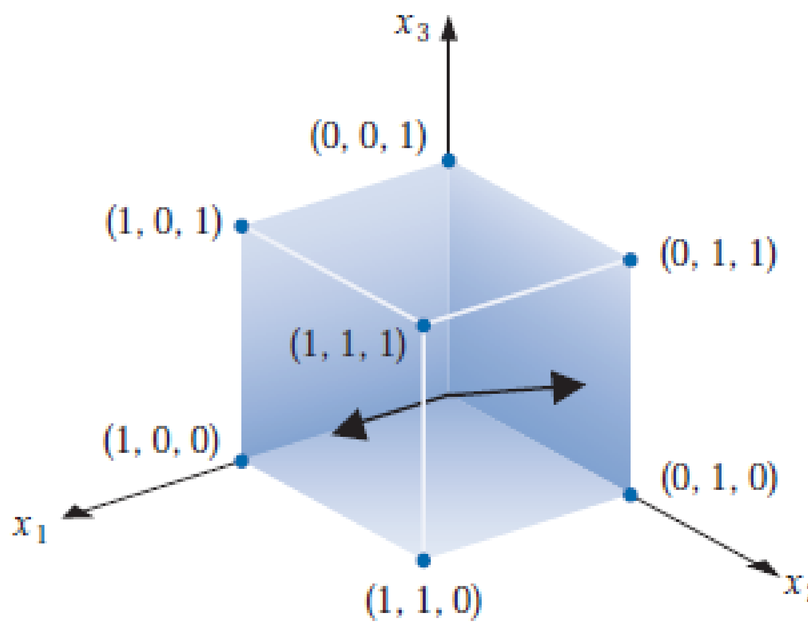


x_2
The vectors in $\mathbb{R}^2$ with l_2 norm less than 1 are inside this figure.
(0, 1)
(−1, 0)
(1, 0)
x_1
(0, −1)
x_3
The vectors in the first octant of $\mathbb{R}^3$ with l_2 norm less than 1 are inside this figure.
(0, 0, 1)
(1, 0, 0)
(0, 1, 0)
x_1
x_2

Figure 7.2



The vectors in $\mathbb{R}^2$ with l_∞ norm less than 1 are inside this figure.



The vectors in the first octant of $\mathbb{R}^3$ with l_∞ norm less than 1 are inside this figure.

Theorem 7.3 **(Cauchy-Bunyakovsky-Schwarz Inequality for Sums)**

For each $\mathbf{x} = (x_1, x_2, \ldots, x_n)^t$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)^t$ in $\mathbb{R}^n$,

$$\mathbf{x}^t\mathbf{y} = \sum_{i=1}^{n} x_i y_i \le \left\{\sum_{i=1}^{n} x_i^2\right\}^{1/2} \left\{\sum_{i=1}^{n} y_i^2\right\}^{1/2} = \|\mathbf{x}\|_2 \cdot \|\mathbf{y}\|_2. \qquad (7.1)$$

■

Definition 7.4 If $\mathbf{x} = (x_1, x_2, \ldots, x_n)^t$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)^t$ are vectors in $\mathbb{R}^n$, the l_2 and l_∞ distances between $\mathbf{x}$ and $\mathbf{y}$ are defined by

$$\|\mathbf{x} - \mathbf{y}\|_2 = \left\{\sum_{i=1}^{n} (x_i - y_i)^2\right\}^{1/2} \quad \text{and} \quad \|\mathbf{x} - \mathbf{y}\|_\infty = \max_{1 \le i \le n} |x_i - y_i|.$$

■

Definition 7.5 A sequence $\{\mathbf{x}^{(k)}\}_{k=1}^{\infty}$ of vectors in $\mathbb{R}^n$ is said to **converge** to $\mathbf{x}$ with respect to the norm $\|\cdot\|$ if, given any $\varepsilon > 0$, there exists an integer $N(\varepsilon)$ such that

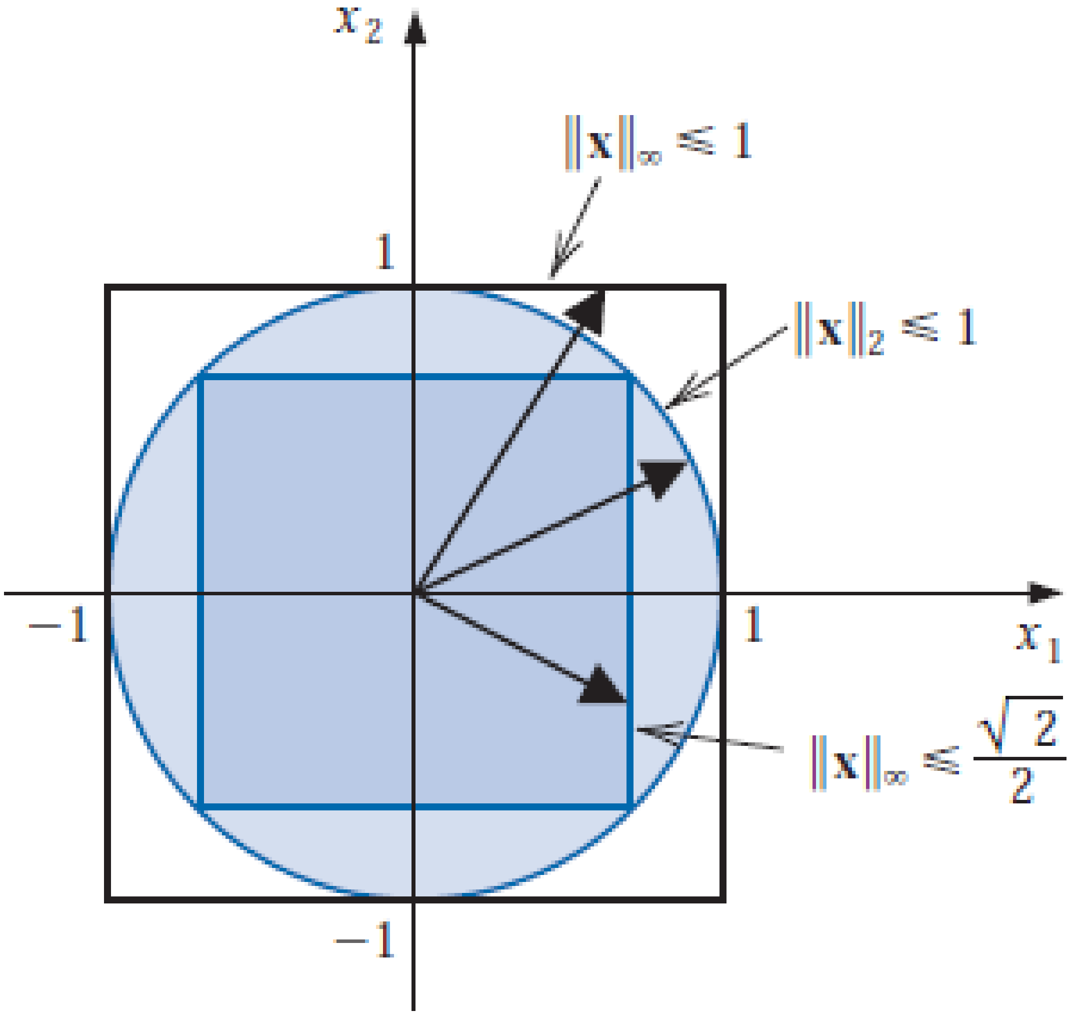
$$\|\mathbf{x}^{(k)} - \mathbf{x}\| < \varepsilon, \quad \text{for all } k \geq N(\varepsilon). \blacksquare$$

Theorem 7.6 The sequence of vectors $\{\mathbf{x}^{(k)}\}$ converges to $\mathbf{x}$ in $\mathbb{R}^n$ with respect to the l_∞ norm if and only if $\lim_{k\to\infty} x_i^{(k)} = x_i$, for each $i = 1, 2, \ldots, n$. $\blacksquare$

Theorem 7.7 For each $\mathbf{x} \in \mathbb{R}^n$,

$$\|\mathbf{x}\|_\infty \leq \|\mathbf{x}\|_2 \leq \sqrt{n}\|\mathbf{x}\|_\infty. \blacksquare$$

Figure 7.3


x_2
$\|\mathbf{x}\|_\infty \leqslant 1$
1
$\|\mathbf{x}\|_2 \leqslant 1$
-1
1
x_1
$\|\mathbf{x}\|_\infty \leqslant \frac{\sqrt{2}}{2}$
-1

Matrix Norms and Distances

Definition 7.8 A **matrix norm** on the set of all $n \times n$ matrices is a real-valued function, $\|\cdot\|$, defined on this set, satisfying for all $n \times n$ matrices A and B and all real numbers α:

(i) $\|A\| \geq 0$;

(ii) $\|A\| = 0$, if and only if A is O, the matrix with all 0 entries;

(iii) $\|\alpha A\| = |\alpha|\|A\|$;

(iv) $\|A + B\| \leq \|A\| + \|B\|$;

(v) $\|AB\| \leq \|A\|\|B\|$. ■

Theorem 7.9 If $||\cdot||$ is a vector norm on $\mathbb{R}^n$, then

$$\|A\| = \max_{\|\mathbf{x}\|=1} \|A\mathbf{x}\| \tag{7.2}$$

is a matrix norm. ■

Corollary 7.10 For any vector $\mathbf{z} \neq \mathbf{0}$, matrix A, and any natural norm $\|\cdot\|$, we have

$$\|A\mathbf{z}\| \leq \|A\| \cdot \|\mathbf{z}\|.$$ ■

The measure given to a matrix under a natural norm describes how the matrix stretches unit vectors relative to that norm. The maximum stretch is the norm of the matrix. The matrix norms we will consider have the forms

$$\|A\|_\infty = \max_{\|\mathbf{x}\|_\infty=1} \|A\mathbf{x}\|_\infty, \quad \text{the } l_\infty \text{ norm,}$$

and

$$\|A\|_2 = \max_{\|\mathbf{x}\|_2=1} \|A\mathbf{x}\|_2, \quad \text{the } l_2 \text{ norm.}$$

An illustration of these norms when $n = 2$ is shown in Figures 7.4 and 7.5 for the matrix

$$A = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$$

Figure 7.4

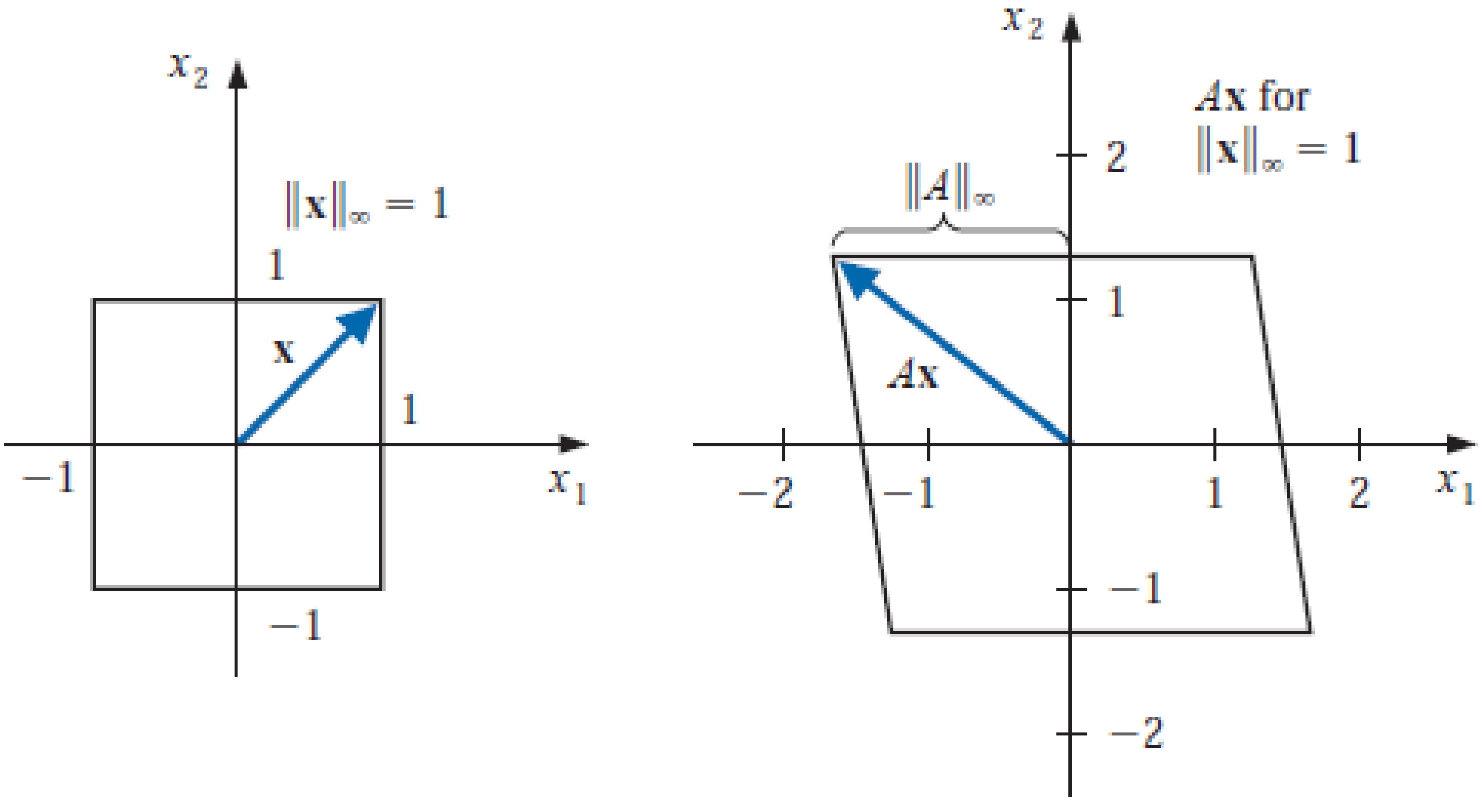
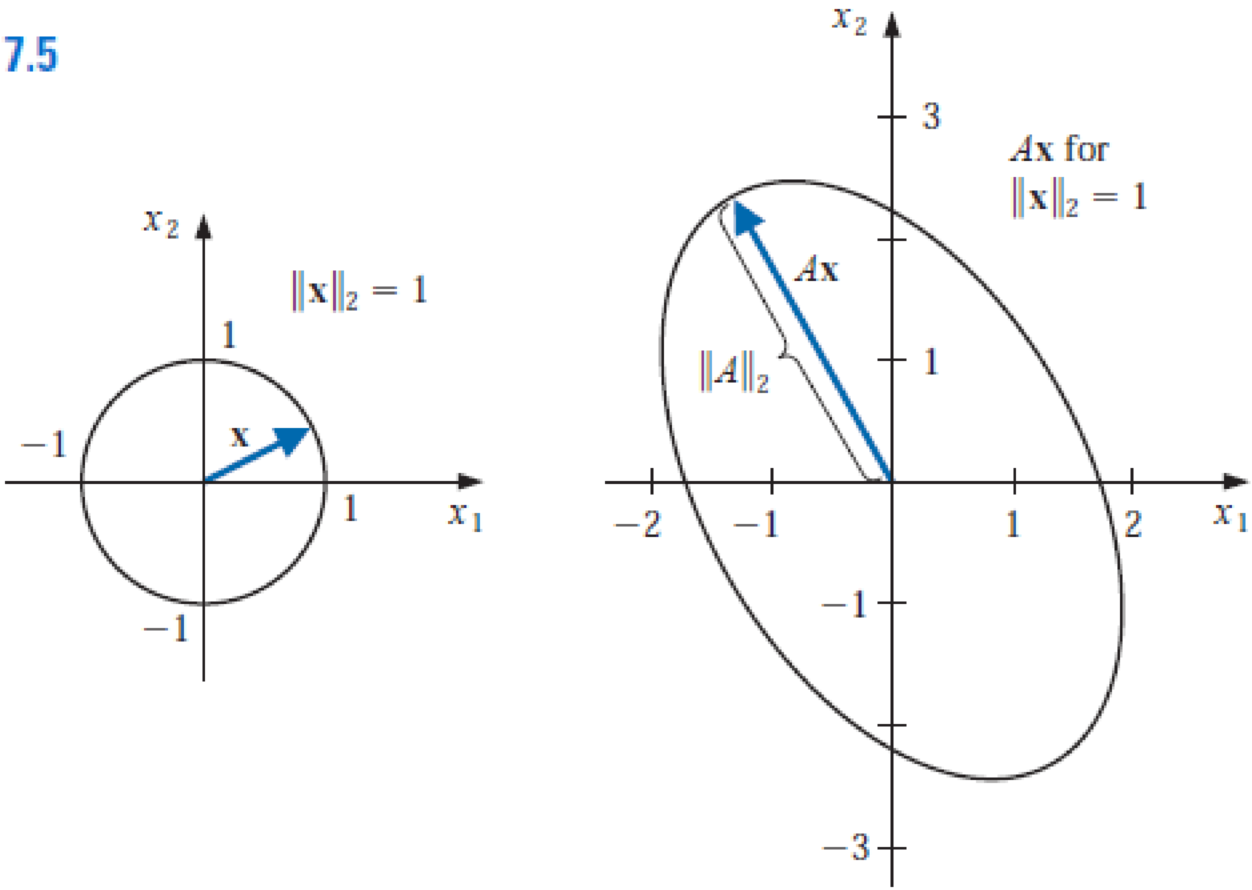

x_2
$\|\mathbf{x}\|_\infty = 1$
1
$\mathbf{x}$
1
-1
x_1
-1
x_2
$A\mathbf{x}$ for
$\|\mathbf{x}\|_\infty = 1$
2
$\|A\|_\infty$
1
$A\mathbf{x}$
-2
-1
1
2
x_1
-1
-2

Figure 7.5



Theorem 7.11 If $A = (a_{ij})$ is an $n \times n$ matrix, then

$$\|A\|_\infty = \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}|.$$

■

7.2 Eigenvalues and Eigenvectors

Definition 7.12 If A is a square matrix, the **characteristic polynomial** of A is defined by

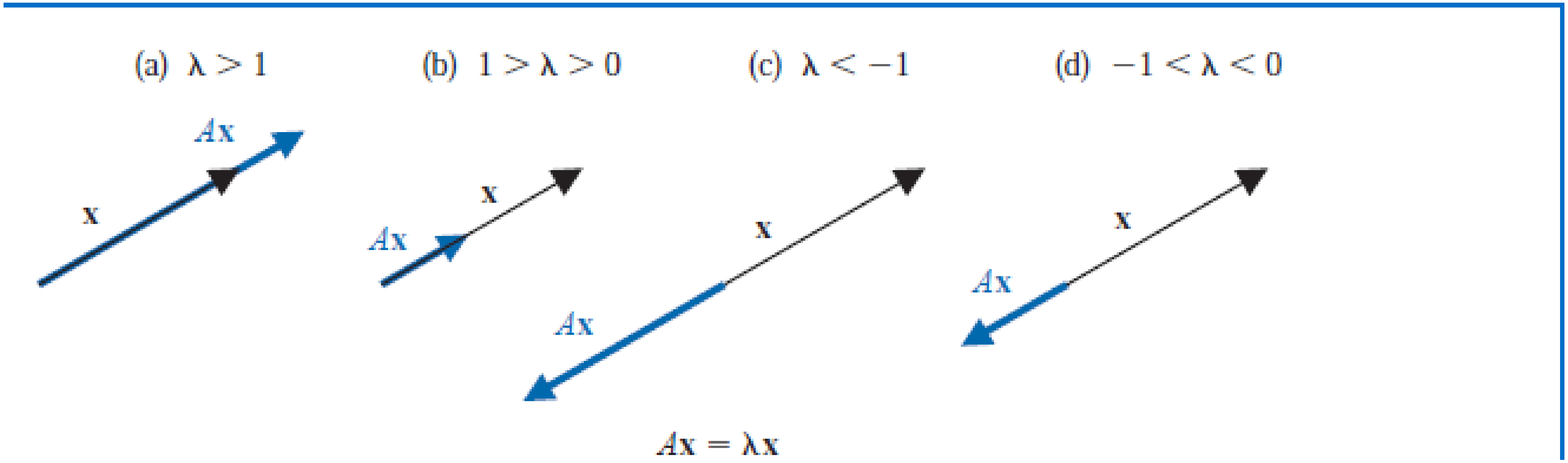
$$p(\lambda) = \det(A - \lambda I).$$

Definition 7.13 If p is the characteristic polynomial of the matrix A, the zeros of p are **eigenvalues**, or characteristic values, of the matrix A. If λ is an eigenvalue of A and $\mathbf{x} \neq \mathbf{0}$ satisfies $(A - \lambda I)\mathbf{x} = \mathbf{0}$, then $\mathbf{x}$ is an **eigenvector**, or characteristic vector, of A corresponding to the eigenvalue λ.

If $\mathbf{x}$ is an eigenvector associated with the real eigenvalue λ, then $A\mathbf{x} = \lambda\mathbf{x}$, so the matrix A takes the vector $\mathbf{x}$ into a scalar multiple of itself.

- If λ is real and $\lambda > 1$, then A has the effect of stretching $\mathbf{x}$ by a factor of λ, as illustrated in Figure 7.6(a).
- If $0 < \lambda < 1$, then A shrinks $\mathbf{x}$ by a factor of λ (see Figure 7.6(b)).
- If $\lambda < 0$, the effects are similar (see Figure 7.6(c) and (d)), although the direction of $A\mathbf{x}$ is reversed.

Figure 7.6



Spectral Radius

Definition 7.14 The **spectral radius** $\rho(A)$ of a matrix A is defined by

$$\rho(A) = \max |\lambda|, \quad \text{where } \lambda \text{ is an eigenvalue of } A.$$

(For complex $\lambda = \alpha + \beta i$, we define $|\lambda| = (\alpha^2 + \beta^2)^{1/2}$.) ■

Theorem 7.15 If A is an $n \times n$ matrix, then

(i) $\|A\|_2 = [\rho(A^t A)]^{1/2}$,

(ii) $\rho(A) \leq \|A\|$, for any natural norm $\|\cdot\|$. ■

Convergent Matrices

In studying iterative matrix techniques, it is of particular importance to know when powers of a matrix become small (that is, when all the entries approach zero). Matrices of this type are called *convergent*.

Definition 7.16 We call an $n \times n$ matrix A **convergent** if

$$\lim_{k\to\infty} (A^k)_{ij} = 0, \quad \text{for each } i = 1, 2, \ldots, n \text{ and } j = 1, 2, \ldots, n. \qquad \blacksquare$$

Theorem 7.17 The following statements are equivalent.

(i) A is a convergent matrix.

(ii) $\lim_{n\to\infty} \|A^n\| = 0$, for some natural norm.

(iii) $\lim_{n\to\infty} \|A^n\| = 0$, for all natural norms.

(iv) $\rho(A) < 1$.

(v) $\lim_{n\to\infty} A^n\mathbf{x} = \mathbf{0}$, for every $\mathbf{x}$. $\blacksquare$

Voltando ao rumo !

This linear system can be placed in the matrix form

$$
\begin{bmatrix}
-1 & 0 & 0 & \frac{\sqrt{2}}{2} & 1 & 0 & 0 & 0 \\
0 & -1 & 0 & \frac{\sqrt{2}}{2} & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & \frac{1}{2} & 0 \\
0 & 0 & 0 & -\frac{\sqrt{2}}{2} & 0 & -1 & -\frac{1}{2} & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & -\frac{\sqrt{2}}{2} & 0 & 0 & \frac{\sqrt{3}}{2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\frac{\sqrt{3}}{2} & -1
\end{bmatrix}
\begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{bmatrix}
=
\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 10{,}000 \\ 0 \\ 0 \end{bmatrix}.
$$

7.3 The Jacobi and Gauss-Siedel Iterative Techniques

In this section we describe the Jacobi and the Gauss-Seidel iterative methods, classic methods that date to the late eighteenth century. Iterative techniques are seldom used for solving linear systems of small dimension since the time required for sufficient accuracy exceeds that required for direct techniques such as Gaussian elimination. For large systems with a high percentage of 0 entries, however, these techniques are efficient in terms of both computer storage and computation. Systems of this type arise frequently in circuit analysis and in the numerical solution of boundary-value problems and partial-differential equations.

An iterative technique to solve the $n \times n$ linear system $A\mathbf{x} = \mathbf{b}$ starts with an initial approximation $\mathbf{x}^{(0)}$ to the solution $\mathbf{x}$ and generates a sequence of vectors $\{\mathbf{x}^{(k)}\}_{k=0}^{\infty}$ that converges to $\mathbf{x}$.

Jacobi's Method

The **Jacobi iterative method** is obtained by solving the ith equation in $A\mathbf{x} = \mathbf{b}$ for x_i to obtain (provided $a_{ii} \neq 0$)

$$x_i = \sum_{\substack{j=1 \\ j \neq i}}^{n} \left(-\frac{a_{ij}x_j}{a_{ii}} \right) + \frac{b_i}{a_{ii}}, \qquad \text{for } i = 1, 2, \ldots, n.$$

For each $k \geq 1$, generate the components $x_i^{(k)}$ of $\mathbf{x}^{(k)}$ from the components of $\mathbf{x}^{(k-1)}$ by

$$x_i^{(k)} = \frac{1}{a_{ii}} \left[\sum_{\substack{j=1 \\ j \neq i}}^{n} \left(-a_{ij}x_j^{(k-1)} \right) + b_i \right], \qquad \text{for } i = 1, 2, \ldots, n. \tag{7.5}$$

Example 1

The linear system $A\mathbf{x} = \mathbf{b}$ given by

$$
\begin{aligned}
E_1: &\quad 10x_1 - x_2 + 2x_3 &&= 6,\\
E_2: &\quad -x_1 + 11x_2 - x_3 + 3x_4 &&= 25,\\
E_3: &\quad 2x_1 - x_2 + 10x_3 - x_4 &&= -11,\\
E_4: &\quad 3x_2 - x_3 + 8x_4 &&= 15
\end{aligned}
$$

has the unique solution $\mathbf{x} = (1, 2, -1, 1)^t$. Use Jacobi's iterative technique to find approximations $\mathbf{x}^{(k)}$ to $\mathbf{x}$ starting with $\mathbf{x}^{(0)} = (0, 0, 0, 0)^t$ until

$$\frac{\|\mathbf{x}^{(k)} - \mathbf{x}^{(k-1)}\|_\infty}{\|\mathbf{x}^{(k)}\|_\infty} < 10^{-3}.$$

Solution We first solve equation E_i for x_i, for each $i = 1, 2, 3, 4$, to obtain

$$\begin{aligned}
x_1 &= \frac{1}{10}x_2 - \frac{1}{5}x_3 + \frac{3}{5},\\
x_2 &= \frac{1}{11}x_1 + \frac{1}{11}x_3 - \frac{3}{11}x_4 + \frac{25}{11},\\
x_3 &= -\frac{1}{5}x_1 + \frac{1}{10}x_2 + \frac{1}{10}x_4 - \frac{11}{10},\\
x_4 &= -\frac{3}{8}x_2 + \frac{1}{8}x_3 + \frac{15}{8}.
\end{aligned}$$

From the initial approximation $\mathbf{x}^{(0)} = (0, 0, 0, 0)^t$ we have $\mathbf{x}^{(1)}$ given by

$$\begin{aligned}
x_1^{(1)} &= \frac{1}{10}x_2^{(0)} - \frac{1}{5}x_3^{(0)} + \frac{3}{5} = 0.6000,\\
x_2^{(1)} &= \frac{1}{11}x_1^{(0)} + \frac{1}{11}x_3^{(0)} - \frac{3}{11}x_4^{(0)} + \frac{25}{11} = 2.2727,\\
x_3^{(1)} &= -\frac{1}{5}x_1^{(0)} + \frac{1}{10}x_2^{(0)} + \frac{1}{10}x_4^{(0)} - \frac{11}{10} = -1.1000,\\
x_4^{(1)} &= -\frac{3}{8}x_2^{(0)} + \frac{1}{8}x_3^{(0)} + \frac{15}{8} = 1.8750.
\end{aligned}$$

Additional iterates, $\mathbf{x}^{(k)} = (x_1^{(k)}, x_2^{(k)}, x_3^{(k)}, x_4^{(k)})^t$, are generated in a similar manner and are presented in Table 7.1.

Table 7.1

k	0	1	2	3	4	5	6	7	8	9	10
$x_1^{(k)}$	0.0000	0.6000	1.0473	0.9326	1.0152	0.9890	1.0032	0.9981	1.0006	0.9997	1.0001
$x_2^{(k)}$	0.0000	2.2727	1.7159	2.053	1.9537	2.0114	1.9922	2.0023	1.9987	2.0004	1.9998
$x_3^{(k)}$	0.0000	−1.1000	−0.8052	−1.0493	−0.9681	−1.0103	−0.9945	−1.0020	−0.9990	−1.0004	−0.9998
$x_4^{(k)}$	0.0000	1.8750	0.8852	1.1309	0.9739	1.0214	0.9944	1.0036	0.9989	1.0006	0.9998

We stopped after ten iterations because

$$\frac{\|\mathbf{x}^{(10)} - \mathbf{x}^{(9)}\|_\infty}{\|\mathbf{x}^{(10)}\|_\infty} = \frac{8.0 \times 10^{-4}}{1.9998} < 10^{-3}.$$

In fact, $\|\mathbf{x}^{(10)} - \mathbf{x}\|_\infty = 0.0002$. ■

In general, iterative techniques for solving linear systems involve a process that converts the system $A\mathbf{x} = \mathbf{b}$ into an equivalent system of the form $\mathbf{x} = T\mathbf{x} + \mathbf{c}$ for some fixed matrix T and vector $\mathbf{c}$. After the initial vector $\mathbf{x}^{(0)}$ is selected, the sequence of approximate solution vectors is generated by computing

$$\mathbf{x}^{(k)} = T\mathbf{x}^{(k-1)} + \mathbf{c},$$

for each $k = 1, 2, 3, \ldots$. This should be reminiscent of the fixed-point iteration studied in Chapter 2.

The Jacobi method can be written in the form $\mathbf{x}^{(k)} = T\mathbf{x}^{(k-1)} + \mathbf{c}$ by splitting A into its diagonal and off-diagonal parts. To see this, let D be the diagonal matrix whose diagonal entries are those of A, $-L$ be the strictly lower-triangular part of A, and $-U$ be the strictly upper-triangular part of A. With this notation,

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

is split into

$$A = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & a_{nn} \end{bmatrix} - \begin{bmatrix} 0 & \cdots & \cdots & 0 \\ -a_{21} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ -a_{n1} & \cdots & -a_{n,n-1} & 0 \end{bmatrix} - \begin{bmatrix} 0 & -a_{12} & \cdots & -a_{1n} \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & -a_{n-1,n} \\ 0 & \cdots & \cdots & 0 \end{bmatrix}$$

$$= D - L - U.$$

The equation $A\mathbf{x} = \mathbf{b}$, or $(D - L - U)\mathbf{x} = \mathbf{b}$, is then transformed into

$$D\mathbf{x} = (L + U)\mathbf{x} + \mathbf{b},$$

and, if D^{-1} exists, that is, if $a_{ii} \neq 0$ for each i, then

$$\mathbf{x} = D^{-1}(L + U)\mathbf{x} + D^{-1}\mathbf{b}.$$

This results in the matrix form of the Jacobi iterative technique:

$$\mathbf{x}^{(k)} = D^{-1}(L + U)\mathbf{x}^{(k-1)} + D^{-1}\mathbf{b}, \quad k = 1, 2, \ldots. \tag{7.6}$$

Introducing the notation $T_j = D^{-1}(L + U)$ and $\mathbf{c}_j = D^{-1}\mathbf{b}$ gives the Jacobi technique the form

$$\mathbf{x}^{(k)} = T_j\mathbf{x}^{(k-1)} + \mathbf{c}_j. \tag{7.7}$$

In practice, Eq. (7.5) is used in computation and Eq. (7.7) for theoretical purposes.

Example 2 Express the Jacobi iteration method for the linear system $A\mathbf{x} = \mathbf{b}$ given by

$$\begin{aligned} E_1: &\quad 10x_1 - x_2 + 2x_3 = 6, \\ E_2: &\quad -x_1 + 11x_2 - x_3 + 3x_4 = 25, \\ E_3: &\quad 2x_1 - x_2 + 10x_3 - x_4 = -11, \\ E_4: &\quad 3x_2 - x_3 + 8x_4 = 15 \end{aligned}$$

in the form $\mathbf{x}^{(k)} = T\mathbf{x}^{(k-1)} + \mathbf{c}$.

Solution We saw in Example 1 that the Jacobi method for this system has the form

$$\begin{aligned} x_1 &= \frac{1}{10}x_2 - \frac{1}{5}x_3 + \frac{3}{5}, \\ x_2 &= \frac{1}{11}x_1 + \frac{1}{11}x_3 - \frac{3}{11}x_4 + \frac{25}{11}, \\ x_3 &= -\frac{1}{5}x_1 + \frac{1}{10}x_2 + \frac{1}{10}x_4 - \frac{11}{10}, \\ x_4 &= -\frac{3}{8}x_2 + \frac{1}{8}x_3 + \frac{15}{8}. \end{aligned}$$

Hence we have

$$T = \begin{bmatrix} 0 & \frac{1}{10} & -\frac{1}{5} & 0 \\ \frac{1}{11} & 0 & \frac{1}{11} & -\frac{3}{11} \\ -\frac{1}{5} & \frac{1}{10} & 0 & \frac{1}{10} \\ 0 & -\frac{3}{8} & \frac{1}{8} & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{c} = \begin{bmatrix} \frac{3}{5} \\ \frac{25}{11} \\ -\frac{11}{10} \\ \frac{15}{8} \end{bmatrix}. \quad \blacksquare$$

ALGORITHM 7.1

Jacobi Iterative

To solve $A\mathbf{x} = \mathbf{b}$ given an initial approximation $\mathbf{x}^{(0)}$:

INPUT the number of equations and unknowns n; the entries a_{ij}, $1 \leq i, j \leq n$ of the matrix A; the entries b_i, $1 \leq i \leq n$ of $\mathbf{b}$; the entries XO_i, $1 \leq i \leq n$ of $\mathbf{XO} = \mathbf{x}^{(0)}$; tolerance TOL; maximum number of iterations N.

OUTPUT the approximate solution $x_1, \ldots, x_n$ or a message that the number of iterations was exceeded.

Step 1 Set $k = 1$.

Step 2 While ($k \leq N$) do Steps 3–6.

Step 3 For $i = 1, \ldots, n$

$$\text{set } x_i = \frac{1}{a_{ii}}\left[-\sum_{\substack{j=1 \\ j\neq i}}^{n} (a_{ij}XO_j) + b_i\right].$$

Step 4 If $||\mathbf{x} - \mathbf{XO}|| < TOL$ then OUTPUT $(x_1, \ldots, x_n)$;
(*The procedure was successful.*)
STOP.

Step 5 Set $k = k + 1$.

Step 6 For $i = 1, \ldots, n$ set $XO_i = x_i$.

Step 7 OUTPUT ('Maximum number of iterations exceeded');
(*The procedure was successful.*)
STOP.

Step 3 of the algorithm requires that $a_{ii} \neq 0$, for each $i = 1, 2, \ldots, n$. If one of the a_{ii} entries is 0 and the system is nonsingular, a reordering of the equations can be performed so that no $a_{ii} = 0$. To speed convergence, the equations should be arranged so that a_{ii} is as large as possible. This subject is discussed in more detail later in this chapter.

Another possible stopping criterion in Step 4 is to iterate until

$$\frac{\|\mathbf{x}^{(k)} - \mathbf{x}^{(k-1)}\|}{\|\mathbf{x}^{(k)}\|}$$

is smaller than some prescribed tolerance. For this purpose, any convenient norm can be used, the usual being the l_∞ norm.

The Gauss-Seidel Method

A possible improvement in Algorithm 7.1 can be seen by reconsidering Eq. (7.5). The components of $\mathbf{x}^{(k-1)}$ are used to compute all the components $x_i^{(k)}$ of $\mathbf{x}^{(k)}$. But, for $i > 1$, the components $x_1^{(k)}, \ldots, x_{i-1}^{(k)}$ of $\mathbf{x}^{(k)}$ have already been computed and are expected to be better approximations to the actual solutions $x_1, \ldots, x_{i-1}$ than are $x_1^{(k-1)}, \ldots, x_{i-1}^{(k-1)}$. It seems reasonable, then, to compute $x_i^{(k)}$ using these most recently calculated values. That is, to use

$$x_i^{(k)} = \frac{1}{a_{ii}}\left[-\sum_{j=1}^{i-1}(a_{ij}x_j^{(k)}) - \sum_{j=i+1}^{n}(a_{ij}x_j^{(k-1)}) + b_i\right], \tag{7.8}$$

for each $i = 1, 2, \ldots, n$, instead of Eq. (7.5). This modification is called the **Gauss-Seidel iterative technique** and is illustrated in the following example.

Example 3 Use the Gauss-Seidel iterative technique to find approximate solutions to

$$\begin{aligned} 10x_1 - x_2 + 2x_3 &= 6, \\ -x_1 + 11x_2 - x_3 + 3x_4 &= 25, \\ 2x_1 - x_2 + 10x_3 - x_4 &= -11, \\ 3x_2 - x_3 + 8x_4 &= 15 \end{aligned}$$

starting with $\mathbf{x} = (0, 0, 0, 0)^t$ and iterating until

$$\frac{\|\mathbf{x}^{(k)} - \mathbf{x}^{(k-1)}\|_\infty}{\|\mathbf{x}^{(k)}\|_\infty} < 10^{-3}.$$

Solution The solution $\mathbf{x} = (1, 2, -1, 1)^t$ was approximated by Jacobi's method in Example 1. For the Gauss-Seidel method we write the system, for each $k = 1, 2, \ldots$ as

$$\begin{aligned}
x_1^{(k)} &= \qquad\qquad \frac{1}{10}x_2^{(k-1)} - \frac{1}{5}x_3^{(k-1)} \qquad\qquad + \frac{3}{5},\\
x_2^{(k)} &= \frac{1}{11}x_1^{(k)} \qquad\qquad + \frac{1}{11}x_3^{(k-1)} - \frac{3}{11}x_4^{(k-1)} + \frac{25}{11},\\
x_3^{(k)} &= -\frac{1}{5}x_1^{(k)} + \frac{1}{10}x_2^{(k)} \qquad\qquad + \frac{1}{10}x_4^{(k-1)} - \frac{11}{10},\\
x_4^{(k)} &= \qquad - \frac{3}{8}x_2^{(k)} + \frac{1}{8}x_3^{(k)} \qquad\qquad + \frac{15}{8}.
\end{aligned}$$

When $\mathbf{x}^{(0)} = (0, 0, 0, 0)^t$, we have $\mathbf{x}^{(1)} = (0.6000, 2.3272, -0.9873, 0.8789)^t$. Subsequent iterations give the values in Table 7.2.

Table 7.2

k	0	1	2	3	4	5
$x_1^{(k)}$	0.0000	0.6000	1.030	1.0065	1.0009	1.0001
$x_2^{(k)}$	0.0000	2.3272	2.037	2.0036	2.0003	2.0000
$x_3^{(k)}$	0.0000	−0.9873	−1.014	−1.0025	−1.0003	−1.0000
$x_4^{(k)}$	0.0000	0.8789	0.9844	0.9983	0.9999	1.0000

Because

$$\frac{\|\mathbf{x}^{(5)} - \mathbf{x}^{(4)}\|_\infty}{\|\mathbf{x}^{(5)}\|_\infty} = \frac{0.0008}{2.000} = 4 \times 10^{-4},$$

$\mathbf{x}^{(5)}$ is accepted as a reasonable approximation to the solution. Note that Jacobi's method in Example 1 required twice as many iterations for the same accuracy. ■

To write the Gauss-Seidel method in matrix form, multiply both sides of Eq. (7.8) by a_{ii} and collect all kth iterate terms, to give

$$a_{i1}x_1^{(k)} + a_{i2}x_2^{(k)} + \cdots + a_{ii}x_i^{(k)} = -a_{i,i+1}x_{i+1}^{(k-1)} - \cdots - a_{in}x_n^{(k-1)} + b_i,$$

for each $i = 1, 2, \ldots, n$. Writing all n equations gives

$$\begin{aligned}
a_{11}x_1^{(k)} &= -a_{12}x_2^{(k-1)} - a_{13}x_3^{(k-1)} - \cdots - a_{1n}x_n^{(k-1)} + b_1,\\
a_{21}x_1^{(k)} + a_{22}x_2^{(k)} &= \phantom{-a_{12}x_2^{(k-1)}} - a_{23}x_3^{(k-1)} - \cdots - a_{2n}x_n^{(k-1)} + b_2,\\
&\vdots\\
a_{n1}x_1^{(k)} + a_{n2}x_2^{(k)} + \cdots + a_{nn}x_n^{(k)} &= b_n;
\end{aligned}$$

with the definitions of D, L, and U given previously, we have the Gauss-Seidel method represented by

$$(D - L)\mathbf{x}^{(k)} = U\mathbf{x}^{(k-1)} + \mathbf{b}$$

and

$$\mathbf{x}^{(k)} = (D - L)^{-1}U\mathbf{x}^{(k-1)} + (D - L)^{-1}\mathbf{b}, \quad \text{for each } k = 1, 2, \ldots. \tag{7.9}$$

Letting $T_g = (D - L)^{-1}U$ and $\mathbf{c}_g = (D - L)^{-1}\mathbf{b}$, gives the Gauss-Seidel technique the form

$$\mathbf{x}^{(k)} = T_g\mathbf{x}^{(k-1)} + \mathbf{c}_g. \tag{7.10}$$

For the lower-triangular matrix $D - L$ to be nonsingular, it is necessary and sufficient that $a_{ii} \neq 0$, for each $i = 1, 2, \ldots, n$.

ALGORITHM 7.2

Gauss-Seidel Iterative

To solve $A\mathbf{x} = \mathbf{b}$ given an initial approximation $\mathbf{x}^{(0)}$:

INPUT the number of equations and unknowns n; the entries a_{ij}, $1 \le i, j \le n$ of the matrix A; the entries b_i, $1 \le i \le n$ of $\mathbf{b}$; the entries XO_i, $1 \le i \le n$ of $\mathbf{XO} = \mathbf{x}^{(0)}$; tolerance TOL; maximum number of iterations N.

OUTPUT the approximate solution $x_1, \ldots, x_n$ or a message that the number of iterations was exceeded.

Step 1 Set $k = 1$.

Step 2 While ($k \le N$) do Steps 3–6.

Step 3 For $i = 1, \ldots, n$

$$\text{set } x_i = \frac{1}{a_{ii}}\left[-\sum_{j=1}^{i-1} a_{ij}x_j - \sum_{j=i+1}^{n} a_{ij}XO_j + b_i\right].$$

Step 4 If $||\mathbf{x} - \mathbf{XO}|| < TOL$ then OUTPUT $(x_1, \ldots, x_n)$;
(*The procedure was successful.*)
STOP.

Step 5 Set $k = k + 1$.

Step 6 For $i = 1, \ldots, n$ set $XO_i = x_i$.

Step 7 OUTPUT ('Maximum number of iterations exceeded');
(*The procedure was successful.*)
STOP.

The comments following Algorithm 7.1 regarding reordering and stopping criteria also apply to the Gauss-Seidel Algorithm 7.2.

The results of Examples 1 and 2 appear to imply that the Gauss-Seidel method is superior to the Jacobi method. This is almost always true, but there are linear systems for which the Jacobi method converges and the Gauss-Seidel method does not (see Exercises 9 and 10).

EXERCISE SET 7.3

1. Find the first two iterations of the Jacobi method for the following linear systems, using $\mathbf{x}^{(0)} = \mathbf{0}$:

a.
$$\begin{aligned} 3x_1 - x_2 + x_3 &= 1, \\ 3x_1 + 6x_2 + 2x_3 &= 0, \\ 3x_1 + 3x_2 + 7x_3 &= 4. \end{aligned}$$

b.
$$\begin{aligned} 10x_1 - x_2 \phantom{{}- 2x_3} &= 9, \\ -x_1 + 10x_2 - 2x_3 &= 7, \\ - 2x_2 + 10x_3 &= 6. \end{aligned}$$

c.
$$\begin{aligned} 10x_1 + 5x_2 &= 6, \\ 5x_1 + 10x_2 - 4x_3 &= 25, \\ - 4x_2 + 8x_3 - x_4 &= -11, \\ - x_3 + 5x_4 &= -11. \end{aligned}$$

d.
$$\begin{aligned} 4x_1 + x_2 + x_3 + x_5 &= 6, \\ -x_1 - 3x_2 + x_3 + x_4 &= 6, \\ 2x_1 + x_2 + 5x_3 - x_4 - x_5 &= 6, \\ -x_1 - x_2 - x_3 + 4x_4 &= 6, \\ 2x_2 - x_3 + x_4 + 4x_5 &= 6. \end{aligned}$$

3. Repeat Exercise 1 using the Gauss-Seidel method.

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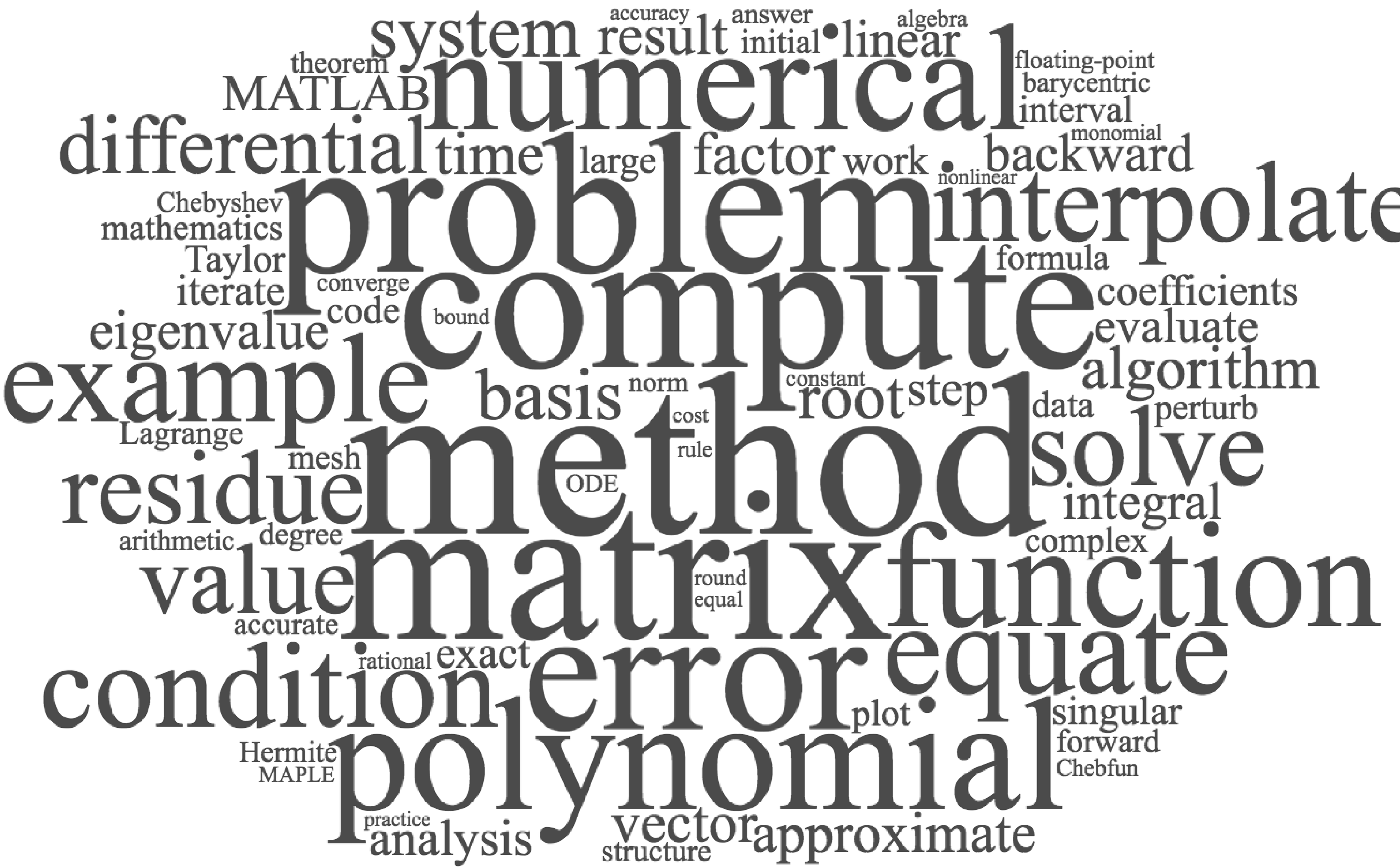
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