MAP 2220 – FUNDAMENTOS DE ANÁLISE NUMÉRICA

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3.3 Divided Differences

Iterated interpolation was used in the previous section to generate successively higher-degree polynomial approximations at a specific point. Divided-difference methods introduced in this section are used to successively generate the polynomials themselves.

Suppose that $P_n(x)$ is the nth Lagrange polynomial that agrees with the function f at the distinct numbers $x_0, x_1, \ldots, x_n$. Although this polynomial is unique, there are alternate algebraic representations that are useful in certain situations. The divided differences of f with respect to $x_0, x_1, \ldots, x_n$ are used to express $P_n(x)$ in the form

$$P_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \cdots + a_n(x - x_0)\cdots(x - x_{n-1}), \quad (3.5)$$

for appropriate constants $a_0, a_1, \ldots, a_n$. To determine the first of these constants, a_0, note that if $P_n(x)$ is written in the form of Eq. (3.5), then evaluating $P_n(x)$ at x_0 leaves only the constant term a_0; that is,

$$a_0 = P_n(x_0) = f(x_0).$$

Similarly, when $P(x)$ is evaluated at x_1, the only nonzero terms in the evaluation of $P_n(x_1)$ are the constant and linear terms,

$$f(x_0) + a_1(x_1 - x_0) = P_n(x_1) = f(x_1);$$

so

$$a_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}. \tag{3.6}$$

We now introduce the divided-difference notation, which is related to Aitken's Δ^2 notation used in Section 2.5. The *zeroth divided difference* of the function f with respect to x_i, denoted $f[x_i]$, is simply the value of f at x_i:

$$f[x_i] = f(x_i). \tag{3.7}$$

The remaining divided differences are defined recursively; the *first divided difference* of f with respect to x_i and x_{i+1} is denoted $f[x_i, x_{i+1}]$ and defined as

$$f[x_i, x_{i+1}] = \frac{f[x_{i+1}] - f[x_i]}{x_{i+1} - x_i}. \tag{3.8}$$

The *second divided difference*, $f[x_i, x_{i+1}, x_{i+2}]$, is defined as

$$f[x_i, x_{i+1}, x_{i+2}] = \frac{f[x_{i+1}, x_{i+2}] - f[x_i, x_{i+1}]}{x_{i+2} - x_i}.$$

Similarly, after the $(k-1)$st divided differences,

$$f[x_i, x_{i+1}, x_{i+2}, \ldots, x_{i+k-1}] \quad \text{and} \quad f[x_{i+1}, x_{i+2}, \ldots, x_{i+k-1}, x_{i+k}],$$

have been determined, the **kth divided difference** relative to $x_i, x_{i+1}, x_{i+2}, \ldots, x_{i+k}$ is

$$f[x_i, x_{i+1}, \ldots, x_{i+k-1}, x_{i+k}] = \frac{f[x_{i+1}, x_{i+2}, \ldots, x_{i+k}] - f[x_i, x_{i+1}, \ldots, x_{i+k-1}]}{x_{i+k} - x_i}. \qquad (3.9)$$

The process ends with the single *nth divided difference*,

$$f[x_0, x_1, \ldots, x_n] = \frac{f[x_1, x_2, \ldots, x_n] - f[x_0, x_1, \ldots, x_{n-1}]}{x_n - x_0}.$$

Because of Eq. (3.6) we can write $a_1 = f[x_0, x_1]$, just as a_0 can be expressed as $a_0 = f(x_0) = f[x_0]$. Hence the interpolating polynomial in Eq. (3.5) is

$$P_n(x) = f[x_0] + f[x_0, x_1](x - x_0) + a_2(x - x_0)(x - x_1) + \cdots + a_n(x - x_0)(x - x_1)\cdots(x - x_{n-1}).$$

As might be expected from the evaluation of a_0 and a_1, the required constants are

$$a_k = f[x_0, x_1, x_2, \ldots, x_k],$$

for each $k = 0, 1, \ldots, n$. So $P_n(x)$ can be rewritten in a form called Newton's Divided-Difference:

$$P_n(x) = f[x_0] + \sum_{k=1}^{n} f[x_0, x_1, \ldots, x_k](x - x_0)\cdots(x - x_{k-1}). \tag{3.10}$$

The generation of the divided differences is outlined in Table 3.9. Two fourth and one fifth difference can also be determined from these data.

Table 3.9

x	$f(x)$	First divided differences	Second divided differences	Third divided differences
x_0	$f[x_0]$			
		$f[x_0,x_1] = \dfrac{f[x_1]-f[x_0]}{x_1-x_0}$		
x_1	$f[x_1]$		$f[x_0,x_1,x_2] = \dfrac{f[x_1,x_2]-f[x_0,x_1]}{x_2-x_0}$	
		$f[x_1,x_2] = \dfrac{f[x_2]-f[x_1]}{x_2-x_1}$		$f[x_0,x_1,x_2,x_3] = \dfrac{f[x_1,x_2,x_3]-f[x_0,x_1,x_2]}{x_3-x_0}$
x_2	$f[x_2]$		$f[x_1,x_2,x_3] = \dfrac{f[x_2,x_3]-f[x_1,x_2]}{x_3-x_1}$	
		$f[x_2,x_3] = \dfrac{f[x_3]-f[x_2]}{x_3-x_2}$		$f[x_1,x_2,x_3,x_4] = \dfrac{f[x_2,x_3,x_4]-f[x_1,x_2,x_3]}{x_4-x_1}$
x_3	$f[x_3]$		$f[x_2,x_3,x_4] = \dfrac{f[x_3,x_4]-f[x_2,x_3]}{x_4-x_2}$	
		$f[x_3,x_4] = \dfrac{f[x_4]-f[x_3]}{x_4-x_3}$		$f[x_2,x_3,x_4,x_5] = \dfrac{f[x_3,x_4,x_5]-f[x_2,x_3,x_4]}{x_5-x_2}$
x_4	$f[x_4]$		$f[x_3,x_4,x_5] = \dfrac{f[x_4,x_5]-f[x_3,x_4]}{x_5-x_3}$	
		$f[x_4,x_5] = \dfrac{f[x_5]-f[x_4]}{x_5-x_4}$		
x_5	$f[x_5]$			

Newton's divided-difference formula can be expressed in a simplified form when the nodes are arranged consecutively with equal spacing. In this case, we introduce the notation $h = x_{i+1} - x_i$, for each $i = 0, 1, \ldots, n-1$ and let $x = x_0 + sh$. Then the difference $x - x_i$ is $x - x_i = (s - i)h$. So Eq. (3.10) becomes

$$P_n(x) = P_n(x_0 + sh) = f[x_0] + shf[x_0, x_1] + s(s-1)h^2 f[x_0, x_1, x_2]$$
$$+ \cdots + s(s-1)\cdots(s-n+1)h^n f[x_0, x_1, \ldots, x_n]$$
$$= f[x_0] + \sum_{k=1}^{n} s(s-1)\cdots(s-k+1)h^k f[x_0, x_1, \ldots, x_k].$$

Using binomial-coefficient notation,

$$\binom{s}{k} = \frac{s(s-1)\cdots(s-k+1)}{k!},$$

we can express $P_n(x)$ compactly as

$$P_n(x) = P_n(x_0 + sh) = f[x_0] + \sum_{k=1}^{n} \binom{s}{k} k! h^k f[x_0, x_i, \ldots, x_k]. \qquad (3.11)$$

Example 1

Table 3.10

x	$f(x)$
1.0	0.7651977
1.3	0.6200860
1.6	0.4554022
1.9	0.2818186
2.2	0.1103623

Complete the divided difference table for the data used in Example 1 of Section 3.2, and reproduced in Table 3.10, and construct the interpolating polynomial that uses all this data.

Table 3.11

i	x_i	$f[x_i]$	$f[x_{i-1}, x_i]$	$f[x_{i-2}, x_{i-1}, x_i]$	$f[x_{i-3}, \ldots, x_i]$	$f[x_{i-4}, \ldots, x_i]$
0	1.0	0.7651977				
			−0.4837057			
1	1.3	0.6200860		−0.1087339		
			−0.5489460		0.0658784	
2	1.6	0.4554022		−0.0494433		0.0018251
			−0.5786120		0.0680685	
3	1.9	0.2818186		0.0118183		
			−0.5715210			
4	2.2	0.1103623				

The coefficients of the Newton forward divided-difference form of the interpolating polynomial are along the diagonal in the table. This polynomial is

$$P_4(x) = 0.7651977 - 0.4837057(x-1.0) - 0.1087339(x-1.0)(x-1.3)$$
$$+ 0.0658784(x-1.0)(x-1.3)(x-1.6)$$
$$+ 0.0018251(x-1.0)(x-1.3)(x-1.6)(x-1.9).$$

Notice that the value $P_4(1.5) = 0.5118200$ agrees with the result in Table 3.6 for Example 2 of Section 3.2, as it must because the polynomials are the same. ■

Perceba que o espaçamento da tabela é constante, logo poderíamos usar a forma modificada do polinômio de diferenças divididas reduzindo o esforço

$$P_n(x) = P_n(x_0 + sh) = f[x_0] + \sum_{k=1}^{n} \binom{s}{k} k! h^k f[x_0, x_i, \ldots, x_k].$$

x	1,5	s	1,666666667	h	0,3	
x	f[xi]	f[xi-1,xi]	f[xi-2,xi-1,xi]	f[xi-3,xi-2,xi-1,xi]	f[xi-4,xi-3,xi-2,xi-1,xi]	
1	0,7651977					
		-0,4837057				
1,3	0,6200860		-0,1087339			
		-0,5489460		0,0658784		
1,6	0,4554022		-0,0494433		0,0018251	
		-0,5786120		0,0680685		
1,9	0,2818186		0,0118183			
		-0,5715210				
2,2	0,1103623					
P(s)	0,7651977	-0,24185283	-0,01087339	-0,000658784	7,30041E-06	0,5118200

Forward Differences

The **Newton forward-difference formula**, is constructed by making use of the forward difference notation Δ introduced in Aitken's Δ^2 method. With this notation,

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{1}{h}(f(x_1) - f(x_0)) = \frac{1}{h}\Delta f(x_0)$$

$$f[x_0, x_1, x_2] = \frac{1}{2h}\left[\frac{\Delta f(x_1) - \Delta f(x_0)}{h}\right] = \frac{1}{2h^2}\Delta^2 f(x_0),$$

and, in general,

$$f[x_0, x_1, \ldots, x_k] = \frac{1}{k!h^k}\Delta^k f(x_0).$$

Since $f[x_0] = f(x_0)$, Eq. (3.11) has the following form.

Newton Forward-Difference Formula

$$P_n(x) = f(x_0) + \sum_{k=1}^{n} \binom{s}{k} \Delta^k f(x_0) \tag{3.12}$$

Backward Differences

If the interpolating nodes are reordered from last to first as $x_n, x_{n-1}, \ldots, x_0$, we can write the interpolatory formula as

$$P_n(x) = f[x_n] + f[x_n, x_{n-1}](x - x_n) + f[x_n, x_{n-1}, x_{n-2}](x - x_n)(x - x_{n-1})$$
$$+ \cdots + f[x_n, \ldots, x_0](x - x_n)(x - x_{n-1}) \cdots (x - x_1).$$

If, in addition, the nodes are equally spaced with $x = x_n + sh$ and $x = x_i + (s + n - i)h$, then

$$\begin{aligned} P_n(x) &= P_n(x_n + sh) \\ &= f[x_n] + shf[x_n, x_{n-1}] + s(s+1)h^2 f[x_n, x_{n-1}, x_{n-2}] + \cdots \\ &\quad + s(s+1)\cdots(s+n-1)h^n f[x_n, \ldots, x_0]. \end{aligned}$$

This is used to derive a commonly applied formula known as the **Newton backward-difference formula**. To discuss this formula, we need the following definition.

Definition 3.7

Given the sequence $\{p_n\}_{n=0}^{\infty}$, define the backward difference ∇p_n (read *nabla* p_n) by

$$\nabla p_n = p_n - p_{n-1}, \quad \text{for } n \geq 1.$$

Higher powers are defined recursively by

$$\nabla^k p_n = \nabla(\nabla^{k-1} p_n), \quad \text{for } k \geq 2.$$

■

Definition 3.7 implies that

$$f[x_n, x_{n-1}] = \frac{1}{h} \nabla f(x_n), \quad f[x_n, x_{n-1}, x_{n-2}] = \frac{1}{2h^2} \nabla^2 f(x_n),$$

and, in general,

$$f[x_n, x_{n-1}, \ldots, x_{n-k}] = \frac{1}{k!h^k} \nabla^k f(x_n).$$

Consequently,

$$P_n(x) = f[x_n] + s\nabla f(x_n) + \frac{s(s+1)}{2} \nabla^2 f(x_n) + \cdots + \frac{s(s+1)\cdots(s+n-1)}{n!} \nabla^n f(x_n).$$

Consequently,

$$P_n(x) = f[x_n] + s\nabla f(x_n) + \frac{s(s+1)}{2}\nabla^2 f(x_n) + \cdots + \frac{s(s+1)\cdots(s+n-1)}{n!}\nabla^n f(x_n).$$

If we extend the binomial coefficient notation to include all real values of s by letting

$$\binom{-s}{k} = \frac{-s(-s-1)\cdots(-s-k+1)}{k!} = (-1)^k \frac{s(s+1)\cdots(s+k-1)}{k!},$$

then

$$P_n(x) = f[x_n] + (-1)^1\binom{-s}{1}\nabla f(x_n) + (-1)^2\binom{-s}{2}\nabla^2 f(x_n) + \cdots + (-1)^n\binom{-s}{n}\nabla^n f(x_n).$$

This gives the following result.

Newton Backward–Difference Formula

$$P_n(x) = f[x_n] + \sum_{k=1}^{n}(-1)^k\binom{-s}{k}\nabla^k f(x_n) \tag{3.13}$$

The divided-difference Table 3.12 corresponds to the data in Example 1.

Table 3.12

		First divided differences	Second divided differences	Third divided differences	Fourth divided differences
1.0	0.7651977				
		−0.4837057			
1.3	0.6200860		−0.1087339		
		−0.5489460		0.0658784	
1.6	0.4554022		−0.0494433		0.0018251
		−0.5786120		0.0680685	
1.9	0.2818186		0.0118183		
		−0.5715210			
2.2	0.1103623				

If an approximation to $f(1.1)$ is required, the reasonable choice for the nodes would be $x_0 = 1.0$, $x_1 = 1.3$, $x_2 = 1.6$, $x_3 = 1.9$, and $x_4 = 2.2$ since this choice makes the earliest possible use of the data points closest to $x = 1.1$, and also makes use of the fourth divided difference. This implies that $h = 0.3$ and $s = \frac{1}{3}$, so the Newton forward divided-difference formula is used with the divided differences that have a *solid* underline (___) in Table 3.12:

$$
\begin{aligned}
P_4(1.1) &= P_4(1.0 + \frac{1}{3}(0.3)) \\
&= 0.7651977 + \frac{1}{3}(0.3)(-0.4837057) + \frac{1}{3}\left(-\frac{2}{3}\right)(0.3)^2(-0.1087339) \\
&\quad + \frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)(0.3)^3(0.0658784) \\
&\quad + \frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)\left(-\frac{8}{3}\right)(0.3)^4(0.0018251) \\
&= 0.7196460.
\end{aligned}
$$

To approximate a value when x is close to the end of the tabulated values, say, $x = 2.0$, we would again like to make the earliest use of the data points closest to x. This requires using the Newton backward divided-difference formula with $s = -\frac{2}{3}$ and the divided differences in Table 3.12 that have a *wavy* underline ($\underset{\sim\sim\sim}{\quad}$). Notice that the fourth divided difference is used in both formulas.

$$
\begin{aligned}
P_4(2.0) &= P_4\left(2.2 - \frac{2}{3}(0.3)\right) \\
&= 0.1103623 - \frac{2}{3}(0.3)(-0.5715210) - \frac{2}{3}\left(\frac{1}{3}\right)(0.3)^2(0.0118183) \\
&\quad - \frac{2}{3}\left(\frac{1}{3}\right)\left(\frac{4}{3}\right)(0.3)^3(0.0680685) - \frac{2}{3}\left(\frac{1}{3}\right)\left(\frac{4}{3}\right)\left(\frac{7}{3}\right)(0.3)^4(0.0018251) \\
&= 0.2238754. \qquad \square
\end{aligned}
$$

Centered Differences

The Newton forward- and backward-difference formulas are not appropriate for approximating $f(x)$ when x lies near the center of the table because neither will permit the highest-order difference to have x_0 close to x. A number of divided-difference formulas are available for this case, each of which has situations when it can be used to maximum advantage. These methods are known as **centered-difference formulas**. We will consider only one centered-difference formula, Stirling's method.

For the centered-difference formulas, we choose x_0 near the point being approximated and label the nodes directly below x_0 as $x_1, x_2, \ldots$ and those directly above as $x_{-1}, x_{-2}, \ldots$. With this convention, **Stirling's formula** is given by

$$\begin{aligned} P_n(x) = P_{2m+1}(x) &= f[x_0] + \frac{sh}{2}(f[x_{-1}, x_0] + f[x_0, x_1]) + s^2h^2 f[x_{-1}, x_0, x_1] \\ &+ \frac{s(s^2-1)h^3}{2}(f[x_{-2}, x_{-1}, x_0, x_1] + f[x_{-1}, x_0, x_1, x_2]) \\ &+ \cdots + s^2(s^2-1)(s^2-4)\cdots(s^2-(m-1)^2)h^{2m} f[x_{-m}, \ldots, x_m] \\ &+ \frac{s(s^2-1)\cdots(s^2-m^2)h^{2m+1}}{2}(f[x_{-m-1}, \ldots, x_m] + f[x_{-m}, \ldots, x_{m+1}]), \end{aligned} \tag{3.14}$$

if $n = 2m+1$ is odd. If $n = 2m$ is even, we use the same formula but delete the last line. The entries used for this formula are underlined in Table 3.13.

Table 3.13

x	$f(x)$	First divided differences	Second divided differences	Third divided differences	Fourth divided differences
x_{-2}	$f[x_{-2}]$				
		$f[x_{-2}, x_{-1}]$			
x_{-1}	$f[x_{-1}]$		$f[x_{-2}, x_{-1}, x_0]$		
		$\underline{f[x_{-1}, x_0]}$		$\underline{f[x_{-2}, x_{-1}, x_0, x_1]}$	
x_0	$\underline{f[x_0]}$		$\underline{f[x_{-1}, x_0, x_1]}$		$\underline{f[x_{-2}, x_{-1}, x_0, x_1, x_2]}$
		$\underline{f[x_0, x_1]}$		$\underline{f[x_{-1}, x_0, x_1, x_2]}$	
x_1	$f[x_1]$		$f[x_0, x_1, x_2]$		
		$f[x_1, x_2]$			
x_2	$f[x_2]$				

Example 2

Consider the table of data given in the previous examples. Use Stirling's formula to approximate $f(1.5)$ with $x_0 = 1.6$.

Solution To apply Stirling's formula we use the *underlined* entries in the difference Table 3.14.

Table 3.14

x	$f(x)$	First divided differences	Second divided differences	Third divided differences	Fourth divided differences
1.0	0.7651977				
		−0.4837057			
1.3	0.6200860		−0.1087339		
		−0.5489460		0.0658784	
1.6	0.4554022		−0.0494433		0.0018251
		−0.5786120		0.0680685	
1.9	0.2818186		0.0118183		
		−0.5715210			
2.2	0.1103623				

The formula, with $h = 0.3$, $x_0 = 1.6$, and $s = -\frac{1}{3}$, becomes

$$f(1.5) \approx P_4\left(1.6 + \left(-\frac{1}{3}\right)(0.3)\right)$$

$$= 0.4554022 + \left(-\frac{1}{3}\right)\left(\frac{0.3}{2}\right)((-0.5489460) + (-0.5786120))$$

$$+ \left(-\frac{1}{3}\right)^2 (0.3)^2(-0.0494433)$$

$$+ \frac{1}{2}\left(-\frac{1}{3}\right)\left(\left(-\frac{1}{3}\right)^2 - 1\right)(0.3)^3(0.0658784 + 0.0680685)$$

$$+ \left(-\frac{1}{3}\right)^2\left(\left(-\frac{1}{3}\right)^2 - 1\right)(0.3)^4(0.0018251) = 0.5118200.$$

■

EXERCISE SET 3.3

9. a. Approximate $f(0.05)$ using the following data and the Newton forward-difference formula:

x	0.0	0.2	0.4	0.6	0.8
$f(x)$	1.00000	1.22140	1.49182	1.82212	2.22554

b. Use the Newton backward-difference formula to approximate $f(0.65)$.

c. Use Stirling's formula to approximate $f(0.43)$.

Para simplificar o trabalho vamos usar somente Polinômios de grau 2

Forward Differences

$$P_n(x) = P_n(x_0 + sh) = f[x_0] + sh\,f[x_0, x_1] + s(s-1)h^2 f[x_0, x_1, x_2]$$

Backward Differences

$$\begin{aligned} P_n(x) &= P_n(x_n + sh) \\ &= f[x_n] + sh\,f[x_n, x_{n-1}] + s(s+1)h^2 f[x_n, x_{n-1}, x_{n-2}] \end{aligned}$$

Centered Differences

$$P_n(x) = P_{2m+1}(x) = f[x_0] + \frac{sh}{2}(f[x_{-1}, x_0] + f[x_0, x_1]) + s^2h^2 f[x_{-1}, x_0, x_1]$$

MAP2220

x	0,05	s	0,25	h	0,2	
x	f[xi]	f[xi-1,xi]	f[xi-2,xi-1,xi]	f[xi-3,xi-2,xi-1,xi]	f[xi-4,xi-3,xi-2,xi-1,xi]	
0,0	1,00000					
		1,10700				
0,2	1,22140		0,61275			
		1,35210		0,2262500		
0,4	1,49182		0,74850		0,0619792	
		1,65150		0,2758333		
0,6	1,82212		0,91400			
		2,01710				
0,8	2,22554					
						P4
P(s)	1,0000000	0,05535	-0,00459562	0,000593906	-8,94824E-05	1,05126
		P2	1,05075			

x	0,65	s	-0,75	h	0,2	
x	f[xi]	f[xi-1,xi]	f[xi-2,xi-1,xi]	f[xi-3,xi-2,xi-1,xi]	f[xi-4,xi-3,xi-2,xi-1,xi]	
0,0	1,00000					
		1,10700				
0,2	1,22140		0,61275			
		1,35210		0,2262500		
0,4	1,49182		0,74850		0,0619792	
		1,65150		0,2758333		
0,6	1,82212		0,91400			
		2,01710				
0,8	2,22554					
						P4
P(s)	2,2255400	-0,302565	-0,00459562	-0,000517187	-5,22949E-05	1,91781
		P2	1,91838			

x	0,43	s	0,15	h	0,2	
x	f[xi]	f[xi-1,xi]	f[xi-2,xi-1,xi]	f[xi-3,xi-2,xi-1,xi]	f[xi-4,xi-3,xi-2,xi-1,xi]	
0,0	1,00000					
		1,10700				
0,2	1,22140		0,61275			
		1,35210		0,2262500		
0,4	1,49182		0,74850		0,0619792	
		1,65150		0,2758333		
0,6	1,82212		0,91400			
		2,01710				
0,8	2,22554					
						P4
P(s)	1,4918200	0,045054	0,00067365	-0,000294472	-2,18105E-06	1,53725
		P2	1,53755			

3.4 Hermite Interpolation

Hermite Polynomials

Theorem 3.9

If $f \in C^1[a,b]$ and $x_0, \dots, x_n \in [a,b]$ are distinct, the unique polynomial of least degree agreeing with f and f' at $x_0, \dots, x_n$ is the Hermite polynomial of degree at most $2n+1$ given by

$$H_{2n+1}(x) = \sum_{j=0}^{n} f(x_j)H_{n,j}(x) + \sum_{j=0}^{n} f'(x_j)\hat{H}_{n,j}(x),$$

where, for $L_{n,j}(x)$ denoting the jth Lagrange coefficient polynomial of degree n, we have

$$H_{n,j}(x) = [1 - 2(x - x_j)L'_{n,j}(x_j)]L^2_{n,j}(x) \quad \text{and} \quad \hat{H}_{n,j}(x) = (x - x_j)L^2_{n,j}(x).$$

Moreover, if $f \in C^{2n+2}[a,b]$, then

$$f(x) = H_{2n+1}(x) + \frac{(x - x_0)^2 \dots (x - x_n)^2}{(2n+2)!} f^{(2n+2)}(\xi(x)),$$

for some (generally unknown) $\xi(x)$ in the interval (a,b). ■

Hermite Polynomials Using Divided Differences

There is an alternative method for generating Hermite approximations that has as its basis the Newton interpolatory divided-difference formula (3.10) at $x_0, x_1, \ldots, x_n$, that is,

$$P_n(x) = f[x_0] + \sum_{k=1}^{n} f[x_0, x_1, \ldots, x_k](x - x_0) \cdots (x - x_{k-1}).$$

The alternative method uses the connection between the nth divided difference and the nth derivative of f, as outlined in Theorem 3.6 in Section 3.3.

Suppose that the distinct numbers $x_0, x_1, \ldots, x_n$ are given together with the values of f and f' at these numbers. Define a new sequence $z_0, z_1, \ldots, z_{2n+1}$ by

$$z_{2i} = z_{2i+1} = x_i, \quad \text{for each } i = 0, 1, \ldots, n,$$

and construct the divided difference table in the form of Table 3.9 that uses $z_0, z_1, \ldots, z_{2n+1}$.

Since $z_{2i} = z_{2i+1} = x_i$ for each i, we cannot define $f[z_{2i}, z_{2i+1}]$ by the divided difference formula. However, if we assume, based on Theorem 3.6, that the reasonable substitution in this situation is $f[z_{2i}, z_{2i+1}] = f'(z_{2i}) = f'(x_i)$, we can use the entries

$$f'(x_0), f'(x_1), \ldots, f'(x_n)$$

in place of the undefined first divided differences

$$f[z_0, z_1], f[z_2, z_3], \ldots, f[z_{2n}, z_{2n+1}].$$

The remaining divided differences are produced as usual, and the appropriate divided differences are employed in Newton's interpolatory divided-difference formula. Table 3.16 shows the entries that are used for the first three divided-difference columns when determining the Hermite polynomial $H_5(x)$ for x_0, x_1, and x_2. The remaining entries are generated in the same manner as in Table 3.9. The Hermite polynomial is then given by

$$H_{2n+1}(x) = f[z_0] + \sum_{k=1}^{2n+1} f[z_0, \ldots, z_k](x - z_0)(x - z_1) \cdots (x - z_{k-1}).$$

Table 3.16

z	$f(z)$	First divided differences	Second divided differences
$z_0 = x_0$	$f[z_0] = f(x_0)$		
		$f[z_0, z_1] = f'(x_0)$	
$z_1 = x_0$	$f[z_1] = f(x_0)$		$f[z_0, z_1, z_2] = \dfrac{f[z_1, z_2] - f[z_0, z_1]}{z_2 - z_0}$
		$f[z_1, z_2] = \dfrac{f[z_2] - f[z_1]}{z_2 - z_1}$	
$z_2 = x_1$	$f[z_2] = f(x_1)$		$f[z_1, z_2, z_3] = \dfrac{f[z_2, z_3] - f[z_1, z_2]}{z_3 - z_1}$
		$f[z_2, z_3] = f'(x_1)$	
$z_3 = x_1$	$f[z_3] = f(x_1)$		$f[z_2, z_3, z_4] = \dfrac{f[z_3, z_4] - f[z_2, z_3]}{z_4 - z_2}$
		$f[z_3, z_4] = \dfrac{f[z_4] - f[z_3]}{z_4 - z_3}$	
$z_4 = x_2$	$f[z_4] = f(x_2)$		$f[z_3, z_4, z_5] = \dfrac{f[z_4, z_5] - f[z_3, z_4]}{z_5 - z_3}$
		$f[z_4, z_5] = f'(x_2)$	
$z_5 = x_2$	$f[z_5] = f(x_2)$		

Example 2

Use the data given in Example 1 and the divided difference method to determine the Hermite polynomial approximation at $x = 1.5$.

Use the Hermite polynomial that agrees with the data listed in Table 3.15 to find an approximation of $f(1.5)$.

k	x_k	$f(x_k)$	$f'(x_k)$
0	1.3	0.6200860	−0.5220232
1	1.6	0.4554022	−0.5698959
2	1.9	0.2818186	−0.5811571

MAP2220

MAP2220

x	1,5						
z	f[zi]	f´[zi]	2nd	3rd	4rd	5th	
1,3	0,6200860						
		-0,5220232					
1,3	0,6200860		-0,0897427				
		-0,5489460		0,0663656			
1,6	0,4554022		-0,0698330		0,0026667		
		-0,5698959		0,0679656		-0,00277469	
1,6	0,4554022		-0,0290537		0,0010019		
		-0,5786120		0,0685667			
1,9	0,2818186		-0,0084837				
		-0,5811571					
1,9	0,2818186						
							H5
P(x)	0,6200860	-0,10440464	-0,00358971	-0,000265462	1,06667E-06	0,0000004	0,5118277

ALGORITHM 3.3

Hermite Interpolation

To obtain the coefficients of the Hermite interpolating polynomial $H(x)$ on the $(n+1)$ distinct numbers $x_0, \ldots, x_n$ for the function f:

INPUT numbers $x_0, x_1, \ldots, x_n$; values $f(x_0), \ldots, f(x_n)$ and $f'(x_0), \ldots, f'(x_n)$.

OUTPUT the numbers $Q_{0,0}, Q_{1,1}, \ldots, Q_{2n+1,2n+1}$ where

$$\begin{aligned} H(x) = Q_{0,0} &+ Q_{1,1}(x - x_0) + Q_{2,2}(x - x_0)^2 + Q_{3,3}(x - x_0)^2(x - x_1) \\ &+ Q_{4,4}(x - x_0)^2(x - x_1)^2 + \cdots \\ &+ Q_{2n+1,2n+1}(x - x_0)^2(x - x_1)^2 \cdots (x - x_{n-1})^2(x - x_n). \end{aligned}$$

Step 1 For $i = 0, 1, \ldots, n$ do Steps 2 and 3.

Step 2 Set $z_{2i} = x_i$;
$z_{2i+1} = x_i$;
$Q_{2i,0} = f(x_i)$;
$Q_{2i+1,0} = f(x_i)$;
$Q_{2i+1,1} = f'(x_i)$.

Step 3 If $i \neq 0$ then set

$$Q_{2i,1} = \frac{Q_{2i,0} - Q_{2i-1,0}}{z_{2i} - z_{2i-1}}.$$

Step 4 For $i = 2, 3, \ldots, 2n + 1$

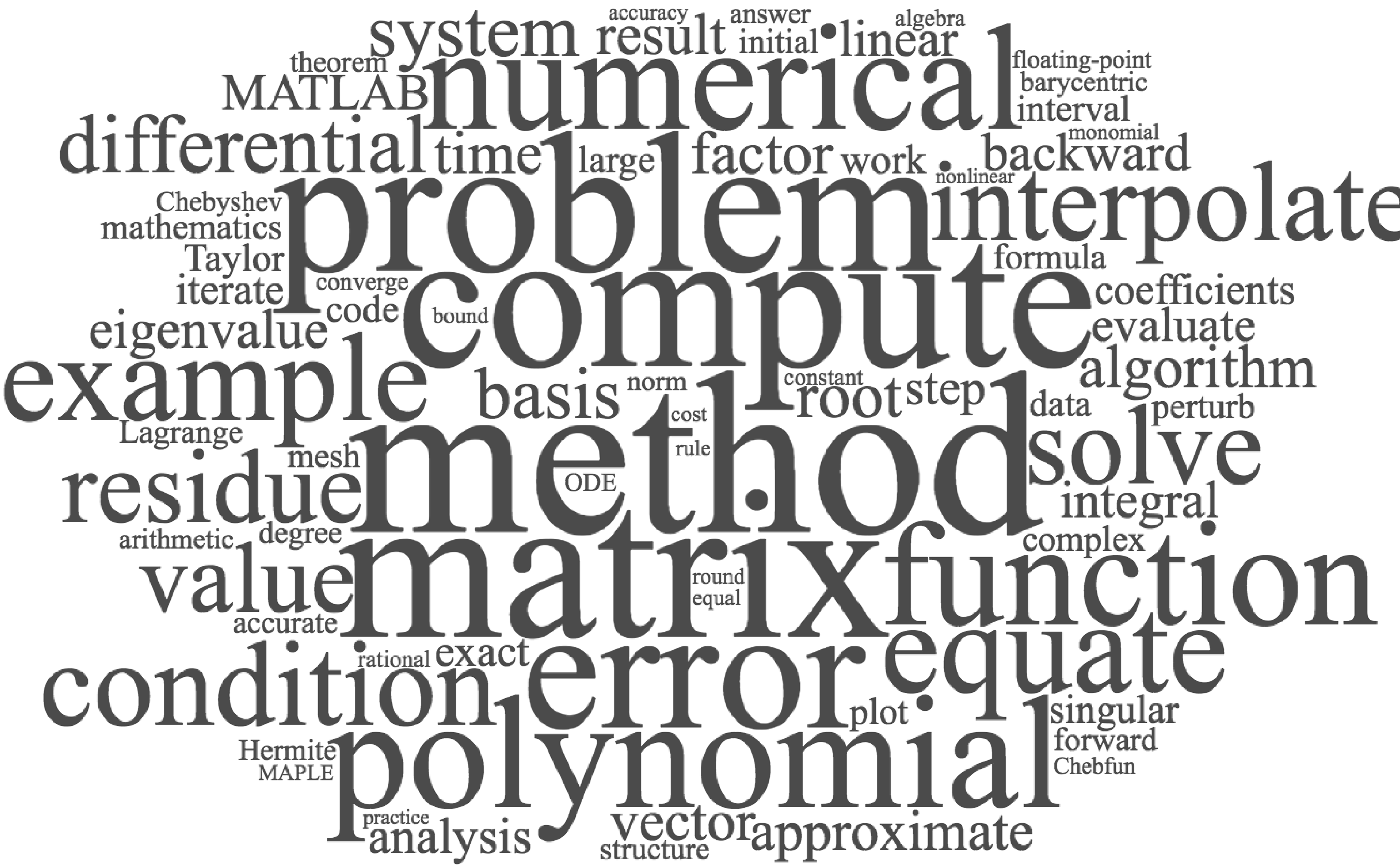
for $j = 2, 3, \ldots, i$ set $Q_{i,j} = \dfrac{Q_{i,j-1} - Q_{i-1,j-1}}{z_i - z_{i-j}}$.

Step 5 OUTPUT $(Q_{0,0}, Q_{1,1}, \ldots, Q_{2n+1,2n+1})$;
STOP

MAP2220

3 Interpolation and Polynomial Approximation 105

MAP2220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Hermite
polynomial
MAPLE
Chebfun
practice
vector
approximate
analysis
structure