17.2-1 input source: population having hair
calling units: customers wanting haircuts
queue: customers waiting for a barber
Service discipline: usually first in, first out.
Service mechanism: barbers and equipment.

17.2-2

a) $L = 0\left(\frac{1}{16}\right) + 1\left(\frac{4}{16}\right) + 2\left(\frac{6}{16}\right) + 3\left(\frac{4}{16}\right) + 4\left(\frac{1}{16}\right) = 2$ which represents the average number of customers in the shop, including those getting their hair cut.

b)

n	# in queue	probability	product
0	0		
1	0		
2	0		
3	1	0.25	0.25
4	2	0.0625	0.125

Lq = 0.375 which represents the average number of customers in the shop waiting to get a haircut.

c) E(# customers being served) $= 1 \cdot P_1 + 2\,(P_2 + P_3 + P_4) =$
$= \frac{4}{16} + 2\left(\frac{6}{16} + \frac{4}{16} + \frac{1}{16}\right) = \frac{13}{8}$

d)

$$W = \frac{L}{\lambda} = \frac{2}{4} = 0.5 = 30 \text{ minutes}$$

$$W_q = \frac{L_q}{\lambda} = \frac{0.375}{4} = 0.094 = 5.625 \text{ minutes}$$

These quantities mean that customers will be in the shop an average of half an hour, including the time to get a haircut, and will have to wait an average of 5.625 minutes before their haircut will begin.

e) W - Wq = 0.5 – 0.094 = 0.406 hours = 24.36 minutes

17.2-3

a) A parking lot is a queueing system for providing parking with cars as the customers, and parking spaces as the servers. The service time is the amount of time a car spends in a space. The queue capacity is 0.

b)

$$L = 0(P_0) + 1(P_1) + 2(P_2) + 3(P_3) = 0(0.2) + 1(0.3) + 2(0.3) + 3(0.2) = 1.5 \text{ cars}$$

$$L_q = 0 \text{ cars}$$

$$W = \left(\frac{L}{\lambda}\right) = \left(\frac{1.5}{2}\right) = 0.75 \text{ hours}$$

$$W_q = \left(\frac{L_q}{\lambda}\right) = \left(\frac{0}{2}\right) = 0 \text{ hours}$$

c) A car spends an average of 45 minutes in a parking space.

17.2-4

a) False. The queue is where customers wait before being served.

b) False. Queueing models conventionally assume that the queue is an infinite queue.

c) True. The most common is first come first served.

17.2-5

a) A bank is a queueing system with people as the customers, and tellers as the servers.

b)

$$W_q = 1 \text{ minute}$$

$$W = W_q + \frac{1}{\mu} = 1 + 2 = 3 \text{ minutes}$$

$$L_q = \lambda W_q = \frac{40}{60}(1) = 0.667 \text{ customers}$$

$$L = \lambda W = \frac{40}{60}(3) = 2 \text{ customers}$$

17.2-6

The utilization factor ρ represents the fraction of time that the server is busy. The server is busy except when there are zero people in the system. P_0 is the probability of having 0 customers in the system. Hence, $\rho = 1 - P_0$.

17.2-7

$$\lambda_2 = 2\lambda_1, \quad \mu_2 = 2\mu_1, \quad L_2 = 2L_1$$

$$\frac{W_2}{W_1} = \frac{L_2/\lambda_2}{L_1/\lambda_1} = 1$$

17.2-8

(a) $L = \begin{cases} L_q & \text{when no one is in the system} \\ L_q+1 & \text{otherwise} \end{cases}$

So $L = P_0 L_q + (1-P_0)(L_q+1) = L_q + (1-P_0)$

(b) $L = \lambda W = \lambda(W_q + 1/\mu) = \lambda W_q + \lambda/\mu = L_q + \rho$

(c) $L = L_q + \rho = L_q + (1-P_0)$ from (a) and (b). So $\rho = 1-P_0$.

17.2-9

$$L = \sum_{n=0}^{\infty} nP_n = \sum_{n=0}^{s-1} nP_n + \sum_{n=s}^{\infty} nP_n = \sum_{n=0}^{s-1} nP_n + \sum_{n=s}^{\infty}(n-s)P_n + \sum_{n=s}^{\infty} sP_n =$$
$$= \sum_{n=0}^{s-1} nP_n + L_q + s\sum_{n=s}^{\infty} P_n =$$
$$= \sum_{n=0}^{s-1} nP_n + L_q + s\left(1 - \sum_{n=0}^{s-1} P_n\right)$$

17.3-1

Part	Customers	Servers
(a)	customers waiting checkout	checkers
(b)	fires	fire fighting units
(c)	cars	toll collectors
(d)	broken bicycles	bicycle repairpersons
(e)	ships to be loaded or unloaded	Longshoremen + equipment
(f)	machines needing operator	operator
(g)	materials to be handled	handling equipment
(h)	calls for plumbers	plumbers
(i)	custom orders	customized process
(j)	typing requests	typists

17.4-1

$\lambda_n = 1/2$ for $n>0$ and $\mu_n = \begin{cases} 1/2 & \text{for } n=1 \\ 1 & \text{for } n \geq 2 \end{cases}$

(a) $P\{\text{next arrival before 1:00}\} = 1 - e^{-1/2} = .393$

$P\{\text{next arrival between 1:00 and 2:00}\} = (1-e^{-1/2 \cdot 2}) - (1 - e^{-1/2}) = .239$

$P\{\text{next arrival after 2:00}\} = e^{-2\cdot 1/2} = .368$

(b) $P\left\{\begin{matrix}\text{next arrival between} \\ \text{1:00 and 2:00}\end{matrix} \middle| \begin{matrix}\text{no arrivals between} \\ \text{12:00 and 1:00}\end{matrix}\right\} = 1 - e^{-1/2} = .393$

(c) $P\{\text{no arrivals between 1:00 and 2:00}\} = \dfrac{(\lambda t)^0 e^{-\lambda t}}{0!} = e^{-1/2} = .607$

$P\{\text{one arrival between 1:00 and 2:00}\} = \dfrac{(\lambda t)^1 e^{-\lambda t}}{1!} = \tfrac{1}{2}\cdot e^{-1/2} = .303$

$P\{\text{two or more arrivals between 1:00 and 2:00}\} = 1 - e^{-1/2} - \tfrac{1}{2}e^{-1/2}$

17.4-1

(d) $P\{\text{none served by 2:00}\} = e^{-1} = .368$

$P\{\text{none served by 1:10}\} = e^{-1(1/6)} = .846$

$P\{\text{none served by 1:01}\} = e^{-1(1/60)} = .983$

17.4-2

$\lambda_n = 2$ for $n \geq 0 \Rightarrow P\{n \text{ arrivals in an hour}\} = \frac{2^n e^{-2}}{n!}$

(a) $P\{0 \text{ arrivals in an hour}\} = e^{-2} = .135$

b) $P\{2 \text{ arrivals in an hour}\} = \frac{2^2 e^{-2}}{2!} = 2e^{-2} = .270$

c) $P\{5 \text{ or more arrivals in an hour}\} = 1 - \sum_{n=0}^{4} P\{n \text{ arrivals in an hour}\}$

$= 1 - e^{-2} - 2e^{-2} - 2e^{-2} - (4/3) \cdot e^{-2} - (2/3) e^{-2}$

$= 1 - 7e^{-2} = .0527$

17.4-3

$Pay = 100 \cdot P\{T<2\} + 80 \cdot P\{T>2\} = 100 - 20 \cdot P\{T>2\}$

$P\{T_{old} > 2\} = e^{-\frac{1}{4} \cdot 2} = e^{-\frac{1}{2}} = 0.607$

$P\{T_{special} > 2\} = e^{-\frac{1}{2} \cdot 2} = e^{-1}$

$\text{Increase} = Pay_{special} - Pay_{old} = 20(P\{T_{old} > 2\} - P\{T_{special} > 2\})$

$= 20(e^{-\frac{1}{2}} - e^{-1})$

17.4-4

Given the memoryless property, the system turns into a two-server queue after first completion occurs.

T = amount of time after 1 and before next service completion

$P\{T < t\} = P\{\min(T_2, T_3) < t\}$

So, T satisfies exponential distribution with mean $0.5/2 = 0.25$ (property 3)

17.4-5 By memoryless property, $U = \min(T_1, T_2, T_3)$

$T_1 \sim \exp(1/20)$, $T_2 \sim \exp(1/15)$, $T_3 \sim \exp(1/10)$

$U \sim \exp(\frac{1}{20} + \frac{1}{15} + \frac{1}{10}) = \exp(\frac{13}{60})$

So, expected waiting time $= \frac{60}{13} = 4\frac{8}{13}$ minutes

17.4-6 a) From aggregation property of poisson process, the arrival process is still poisson with mean rate 10 per hour. So, distribution of time between consecutive arrivals is exponential with mean of 6 minutes.

17.4-6

b) The probability distribution is the minimum of two exponential random variables. By property 3, it is exponential with mean of 5 minutes.

17.4-7 a) Exponential with mean of 5 minutes.

b) $W = W_q + T_s$, W_q and T_s are independent

$EW = EW_q + ET_s = 5 + 10 = 15$ minutes $= \frac{1}{4}$ hour

$\text{Var}W = \text{var}W_q + \text{var}T_s = (\frac{1}{12})^2 + (\frac{1}{6})^2 = \frac{5}{144} = 0.0347$

d) $\tilde{W} = 5 + W_s$

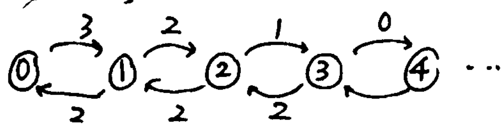
$\tilde{W}$ has mean of 20 minutes, but holds the same variance as W_s

17.4-8 a) False b) False c) False

17.4-9 Let $U = \min\{T_1, \dots, T_n\}$. So $P(T_j = U) = \int_0^\infty P\{T_i > T_j \text{ for all } i \neq j \mid T_j = t\}\, \alpha_j e^{-\alpha_j t} dt =$

$$= \int_0^\infty e^{-t\sum_{i=1}^{n}\alpha_i}\, \cancel{e^{\alpha_j t}} \cdot \alpha_j \cdot \cancel{e^{-\alpha_j t}}\, dt = \alpha_j \int_0^\infty e^{-t\sum_{i=1}^{n}\alpha_i} dt = \alpha_j \Big/ \sum_{i=1}^{n} \alpha_i . \quad \#$$

17.5-1 a)



b) $p_1 = \frac{\lambda_0}{\mu_1} p_0 = \frac{3}{2} p_0$, $p_2 = C_2 p_0 = \frac{\lambda_0 \lambda_1}{\mu_1 \mu_2} \cdot p_0 = \frac{3}{2} p_0$

$p_3 = \frac{\lambda_0 \lambda_1 \lambda_2}{\mu_1 \mu_2 \mu_3} \cdot p_0 = \frac{3}{4} p_0$, $p_4 = p_5 = \cdots = 0$

$p_0 + p_1 + p_2 + p_3 = (1 + \frac{3}{2} + \frac{3}{2} + \frac{3}{4}) p_0 = 1$, $p_0 = \frac{4}{19}$

$p_1 = \frac{12}{38}$, $p_2 = \frac{12}{38}$, $p_3 = \frac{6}{38}$

c) $L = \sum_{n=0}^{\infty} n p_n = p_1 + 2p_2 + 3p_3 = \frac{12}{38} + \frac{24}{38} + \frac{9}{19} = \frac{27}{19} = 1.421$

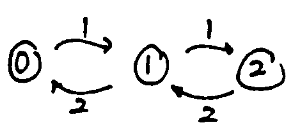
$L_q = p_2 + 2p_3 = \frac{12}{19} = 0.632$

$\bar{\lambda} = \sum_{n=0}^{\infty} \lambda_n p_n = 3p_0 + 2p_1 + p_2 = \frac{30}{19} = 1.579$

$W = L/\bar{\lambda} = \frac{27}{19} \times \frac{19}{30} = \frac{9}{10} = 0.9$

$W_q = L_q/\bar{\lambda} = \frac{12}{19} \cdot \frac{19}{30} = \frac{2}{5} = 0.4$

17.5-2 a)



b) $\begin{cases} 2p_1 = p_0 & p_0 + 2p_2 = 3p_1 \\ p_1 = 2p_2 & p_0 + p_1 + p_2 = 1 \end{cases}$

c) $p_0 = \frac{4}{7}$ $p_1 = \frac{2}{7}$ $p_2 = \frac{1}{7}$

17.5-2

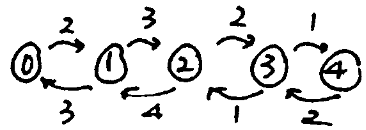
d) $p_1 = \frac{\lambda_0}{\mu_1} p_0 = \frac{1}{2} p_0 \qquad p_2 = \frac{\lambda_0 \lambda_1}{\mu_1 \mu_2} p_0 = \frac{1}{4} p_0$

$(1 + \frac{1}{2} + \frac{1}{4}) p_0 = p_1 + p_2 + p_0 = 1$, we have $p_0 = \frac{4}{7}$, $p_1 = \frac{2}{7}$, $p_2 = \frac{1}{7}$

$L = p_1 + 2p_2 = \frac{4}{7} \qquad L_q = p_2 = \frac{1}{7} \qquad \bar{\lambda} = \lambda_0 p_0 + \lambda_1 p_1 = \frac{6}{7}$

$W = L/\bar{\lambda} = \frac{2}{3} \qquad W_q = L_q/\lambda = \frac{1}{6}$

17.5-3 a)



b)
$$\begin{cases} 2p_0 = 3p_1 & ① \\ 2p_0 + 4p_2 = 6p_1 & ② \\ 3p_1 + p_3 = 6p_2 & ③ \\ 2p_2 + 2p_4 = 2p_3 & ④ \\ p_3 = 2p_4 & ⑤ \\ p_1 + p_2 + p_3 + p_4 = 1 & ⑥ \end{cases}$$

c) ① $\Rightarrow p_1 = \frac{2}{3} p_0$

② $\Rightarrow p_2 = (6 \cdot \frac{2}{3} p_0 - 2p_0)/4 = \frac{1}{2} p_0$

③ $\Rightarrow p_3 = \frac{1}{2} \cdot 6p_0 - 3 \cdot \frac{2}{3} p_0 = p_0$

④ $\Rightarrow p_4 = (2p_0 - 2 \cdot \frac{1}{2} p_0)/2 = \frac{1}{2} p_0$

⑤ $p_0 + \frac{2}{3} p_0 + \frac{1}{2} p_0 + p_0 + \frac{1}{2} p_0 = 1$

$\Rightarrow p_0 = \frac{3}{11} \qquad p_1 = \frac{2}{11} \qquad p_2 = \frac{3}{22} \qquad p_3 = \frac{3}{11} \qquad p_4 = \frac{3}{22}$

d) $p_1 = \frac{\lambda_0}{\mu_1} \cdot p_0 = \frac{2}{3} p_0$,

$p_2 = \frac{\lambda_0 \lambda_1}{\mu_1 \mu_2} \cdot p_0 = \frac{1}{2} p_0$

$p_3 = \frac{\lambda_0 \lambda_1 \lambda_2}{\mu_1 \mu_2 \mu_3} \cdot p_0 = p_0$

$p_4 = \frac{\lambda_0 \lambda_1 \lambda_2 \lambda_3}{\mu_1 \mu_2 \mu_3 \mu_4} \cdot p_0 = \frac{1}{2} p_0$

From $p_0 + p_1 + p_2 + p_3 + p_4 = 1 \Rightarrow p_0 = \frac{3}{11}$.

17.5-3 (CONT'D)

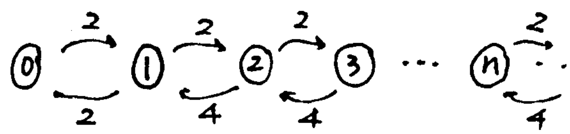
$L = p_1 + 2p_2 + 3p_3 = \frac{20}{11}$

$L_q = p_2 + 2p_3 = \frac{12}{11}$

$\bar{\lambda} = \lambda_0 p_0 + \lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 = \frac{18}{11}$

$W = L/\bar{\lambda} = \frac{10}{9} \qquad W_q = L_q/\bar{\lambda} = \frac{2}{3}$

17.5-4 a)



b) $p_1 = \frac{2}{2} p_0$

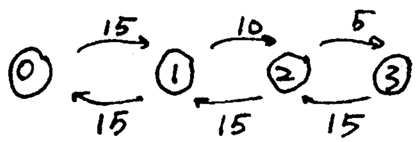
$p_2 = \frac{1}{2} p_0 \quad \cdots \quad p_n = (\frac{1}{2})^{n-1} p_0$

$\sum_{n=0}^{\infty} p_n = \sum_{n=0}^{\infty} (\frac{1}{2})^n p_0 + p_0 = 3p_0 = 1 \quad , \quad p_0 = \frac{1}{3}$

$p_n = (\frac{1}{2})^{n-1} \cdot \frac{1}{3}$

c) arrival rate 1,
service rate 2.

17.5-5 a)



b)
$$\begin{cases} 15p_0 = 15p_1 & ① \\ 15p_0 + 15p_2 = 25p_1 & ② \\ 10p_1 + 15p_3 = 20p_2 & ③ \\ 5p_2 = 15p_3 & ④ \end{cases}$$

c) $p_0 = p_1 \quad , \quad p_2 = \frac{5p_1 - 3p_0}{3} = \frac{2}{3}p_0 \quad , \quad p_3 = \frac{1}{3}p_2 = \frac{2}{9}p_0$

$p_0 + p_1 + p_2 + p_3 = 1 = \frac{26}{9} p_0 \Rightarrow p_0 = \frac{9}{26}$

$p_1 = \frac{9}{26} \qquad p_2 = \frac{3}{13} \qquad p_3 = \frac{1}{13}$

Or $p_1 = \frac{\lambda_0}{\mu_1} p_0 = p_0$

$p_2 = \frac{\lambda_0 \lambda_1}{\mu_1 \mu_2} p_0 = \frac{2}{3} p_0$

$p_3 = \frac{\lambda_0 \lambda_1 \lambda_2}{\mu_1 \mu_2 \mu_3} p_0 = \frac{2}{9} p_0$

17.5-5

d) $L = P_1 + 2P_2 + 3P_3 = \frac{27}{26} = 1.04$, $\bar{\lambda} = \lambda_0 P_0 + \lambda_1 P_1 + \lambda_2 P_2 = \frac{255}{26}$
$= 9.81$

$W = L/\bar{\lambda} = \frac{9}{85}$ (hour) $= 0.106$

17.5-6 a)

$$\begin{tikzcd} 0 \arrow[r, bend left, "\frac{1}{5}"] & 1 \arrow[l, bend left, "\frac{1}{8}"] \arrow[r, bend left, "\frac{1}{10}"] & 2 \arrow[l, bend left, "\frac{1}{8}"] \end{tikzcd}$$

state = # of machines in breakdown state.

b) $P_1 = \frac{8}{5} P_0$, $P_2 = \frac{8}{5} \cdot \frac{8}{10} P_0 = \frac{32}{25} P_0$,

$P_0 + P_1 + P_2 = 1 \Rightarrow P_0 = \frac{25}{97}$, $P_1 = \frac{40}{97}$, $P_2 = \frac{32}{97}$

c) $\bar{\lambda} = P_0 \cdot \lambda_0 + P_1 \lambda_1 = \frac{1}{5} \cdot \frac{25}{97} + \frac{1}{10} \cdot \frac{40}{97} = \frac{9}{97} = 0.093$

$L = P_1 + 2P_2 = \frac{104}{97} = 1.072$, $L_q = \frac{32}{97} = 0.330$

$W = L/\bar{\lambda} = \frac{104}{9}$, $W_q = \frac{32}{9} = 3.556$

d) $P_1 + P_2 = \frac{72}{97} = 0.742$

e) $P_0 + \frac{1}{2} P_1 = \frac{45}{97} = 0.464$

17.5-7 a)

$$\begin{tikzcd}[column sep=small] 0 \arrow[r, bend left, "\lambda"] & 1 \arrow[l, bend left, "\mu"] \arrow[r, bend left, "\lambda"] & 2 \arrow[l, bend left, "\mu+\theta"] \arrow[r, bend left, "\lambda"] & 3 \arrow[l, bend left, "\mu+2\theta"] & \cdots & n-2 \arrow[r, bend left, "\lambda"] & n-1 \arrow[l, bend left, "\mu(n-2)\theta"] \arrow[r, bend left, "\lambda"] & n \arrow[l, bend left, "\mu(n-1)\theta"] \arrow[r, bend left, "\lambda"] & n+1 \arrow[l, bend left, "\mu+n\theta"] & \cdots \end{tikzcd}$$

(b) $\mu P_1 = \lambda P_0$

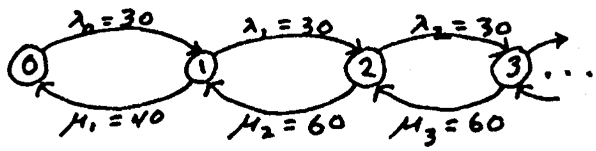
$\lambda P_0 + (\mu + \theta) P_2 = (\mu + \lambda) P_1$

$\vdots$

$\lambda P_{n-1} + (\mu + n\theta) P_{n+1} = (\lambda + \mu + (n-1)\theta) P_n$

$\vdots$

17.5-8 a)



(b) $P_0 = \left[1 + \sum_{n=1}^{\infty} \frac{\lambda^n}{\mu_1 \mu_2^{n-1}}\right]^{-1} = \left[1 + \frac{\lambda}{\mu_1} \sum_{n=1}^{\infty} \left(\frac{\lambda}{\mu_2}\right)^{n-1}\right]^{-1} = \left[1 + \frac{\lambda}{\mu_1}\left(\frac{1}{1-\frac{\lambda}{\mu_2}}\right)\right]^{-1}$

$= \left[1 + \frac{3}{4}\left(\frac{1}{1-1/2}\right)\right]^{-1} = 2/5 = .4$

$P_n = P_0 \frac{\lambda^n}{\mu_1 \mu_2^{n-1}} = \left(\frac{3}{5}\right)\left(\frac{1}{2}\right)^n$ for $n \geq 1$

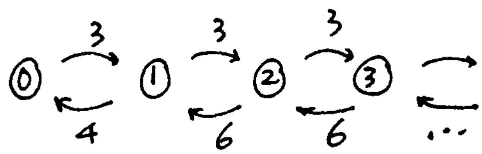
c) $L = \sum_{n=0}^{\infty} n P_n = \frac{3}{5} \sum_{n=1}^{\infty} n\left(\frac{1}{2}\right)^n = \frac{3}{5} \cdot \frac{1}{2} \sum_{n=1}^{\infty} n\left(\frac{1}{2}\right)^{n-1} = \frac{3}{5} \cdot \frac{1}{2} \cdot \frac{1}{(1-1/2)^2} = \frac{6}{5}$

$L_q = L - (1 - P_0) = 6/5 - (1 - 2/5) = 3/5$

$W = L/\lambda = 1/25$

$W_q = L_q/\lambda = 1/50$

17.5-9 a)



b) $p_1 = \frac{3}{4} p_0$, $p_2 = \frac{3}{4} \cdot \frac{1}{2} p_0$, ... , $p_n = \frac{3}{4} \cdot \left(\frac{1}{2}\right)^{n-1} p_0$

$\sum_{n=0}^{\infty} p_n = p_0 + \frac{3}{4} p_0 + \frac{3}{4} \cdot \frac{1}{2} p_0 + \cdots$

$= \frac{5}{2} p_0 = 1$

So, $p_0 = \frac{2}{5}$, $p_1 = \frac{3}{10}$, $p_n = \left(\frac{3}{4}\right)\left(\frac{1}{2}\right)^{n-1}\left(\frac{2}{5}\right) = \frac{3}{10} \cdot \left(\frac{1}{2}\right)^{n-1}$

Let $p_i' = P\{$in steady-state i documents have been received but not yet completed$\}$

17.5-9 (CONT'D)

Then $P_0' = P_0 + P_1 = \frac{7}{10}$

$P_n' = P_{n+1} = \frac{3}{10}\cdot\left(\frac{1}{2}\right)^n \qquad , n \geq 1$

c) $L = \sum_{n=1}^{\infty} n P_n = \frac{3}{10}\left(1 + 2\cdot\frac{1}{2} + 3\cdot\frac{1}{2^2} + \cdots\right)$

$= \frac{3}{10}\cdot 4\cdot\left(\frac{1}{2^2} + \frac{2}{2^3} + \frac{3}{2^4} + \cdots\right)$

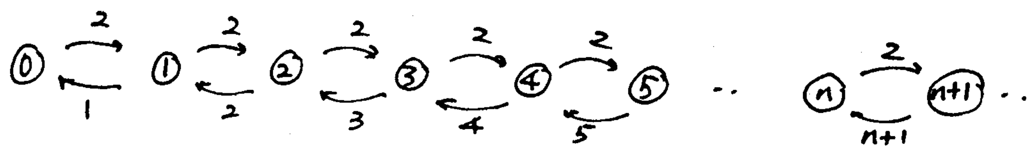
$= -\frac{6}{5}\ \frac{d}{d\rho}\left(\frac{1}{\rho-1}\right)\Big|_{\rho=2} = \frac{6}{5}$

$W = L/\lambda = \frac{2}{3}$

$L_q = \sum_{n=1}^{\infty}(n-1)P_n = L - (1-P_0) = \frac{3}{5}$

$W_q = L_q/\lambda = \frac{1}{5}$

17.5-10



Then, $P_1 = \frac{\lambda_0}{\mu_1}P_0 = 2P_0$,

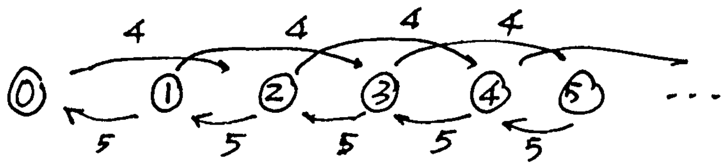
$P_2 = 2\cdot 1\cdot P_0$

$\vdots$

$P_n = \frac{2^n}{n!}\cdot P_0$

$\sum_{n=0}^{\infty} P_n = e^2\cdot P_0 = 1$, So $P_0 = e^{-2}$, $P_1 = 2e^{-2}$

17.5-11 a)



b)

$5p_1 = 4p_0$

$5p_2 = 9p_1$

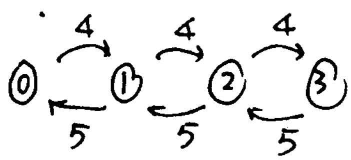
$5p_3 + 4p_0 = 9p_2$

$\vdots$

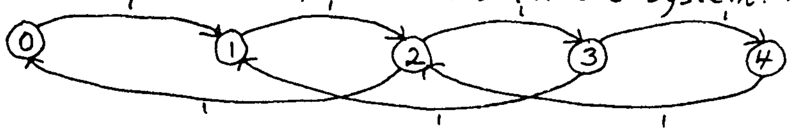
$5p_{n+1} + 4p_{n-2} = 9p_n$

$\vdots$

c)



17.5-12 (a) Let n = number of customers in the system. Then the rate diagram is:



17.5-12 (CONT'D)

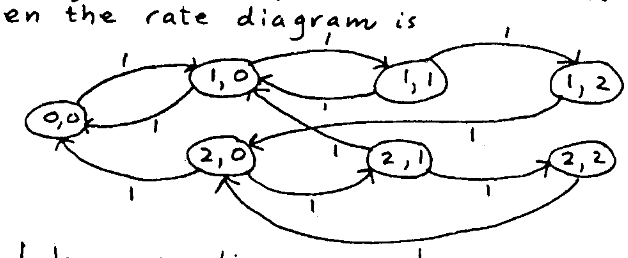
The balance equations are:

$P_0 = P_2$
$P_1 = P_0 + P_3$
$2P_2 = P_1 + P_4$
$2P_3 = P_2$
$P_4 = P_3$

(b) The state space has to be more complex in this case because you need to know how many customers are being worked on by the server.

Let the state be (s, q) where
s = number of customers being served
q = number of customers in the queue

Then the rate diagram is



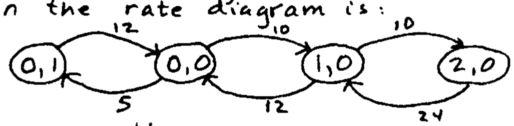
The balance equations are:

$P_{00} = P_{10} + P_{20}$
$2P_{10} = P_{00} + P_{11} + P_{21}$
$2P_{11} = P_{10}$
$P_{12} = P_{11}$
$2P_{20} = P_{12} + P_{22}$
$2P_{21} = P_{20}$
$P_{22} = P_{21}$

17.5-13

(a) Let the state be (n_1, n_2) where
n_1 = number of type 1 customers in the system
n_2 = number of type 2 customers in the system

Then the rate diagram is:



(b) The balance equations are:
(c)

$12P_{01} = 5P_{00}$
$15P_{00} = 12(P_{01} + P_{10})$
$22P_{10} = 10P_{00} + 24P_{20}$
$24P_{20} = 10P_{10}$
$(P_{00} + P_{10} + P_{01} + P_{20} = 1)$

$\Rightarrow P_{00} = \frac{72}{187}, P_{10} = \frac{60}{187}, P_{01} = \frac{30}{187}, P_{20} = \frac{25}{187}$

(d) Type 1 customers are blocked when the system is in state (2,0) or (0,1) which means that the fraction unable to enter the system is $P_{20} + P_{01} = 55/187$.
Type 2 customers are blocked when the system is in state (2,0),(1,0) or (0,1) which means that the fraction unable to enter the system is $P_{20} + P_{10} + P_{01} = 115/187$.

17.6-1 a) $\lambda=2$, $\mu=4$, $s=1$, $\rho=1/2$

For M/M/1 queue, $P_0 = 1-\lambda/\mu = 1/2$ and $P_n = (1-\rho)\rho^n = (1/2)^{n+1}$

desired proportion of time $= \sum_{i=0}^{4} P_i = 31/32$

b)

P_0 =	0.5
P_1 =	0.25
P_2 =	0.125
P_3 =	0.0625
P_4 =	0.03125

Total = 97%

17.6-2 $\lambda=10$, $\mu=15$, $p_0 = \left(1-\frac{\lambda}{\mu}\right) = \frac{1}{3}$ = propotion of time no one is waiting

17.6-3 (a) $\mathcal{W} \sim \exp(\mu-\lambda)$, $W = \frac{1}{\mu-\lambda}$

$$P\{\mathcal{W} > W\} = (\mu-\lambda)\, e^{-(\mu-\lambda)\frac{1}{\mu-\lambda}} = (\mu-\lambda)/e$$

(b) $W_q = \frac{\lambda}{\mu(\mu-\lambda)}$, $\mathcal{W}_q(t) = \begin{cases} 1-\rho & t=0 \\ 1-\rho e^{-\mu(1-\rho)t} & t>0 \end{cases}$

$$\text{So, } P\{\mathcal{W}_q > W_q\} = 1 - \mathcal{W}_q(W_q) = \rho e^{-\mu(1-\rho)\frac{\lambda}{\mu(\mu-\lambda)}}$$

$$= \frac{\lambda}{\mu}\cdot e^{-\frac{\lambda}{\mu}}$$

17.6-4 $P_0 = 1-\rho$, $W_q = \frac{\lambda}{\mu(\mu-\lambda)}$

$$\frac{(1-P_0)^2}{W_q P_0} = \frac{\rho^2}{\frac{\lambda}{\mu(\mu-\lambda)}\cdot\left(1-\frac{\lambda}{\mu}\right)} = \frac{\lambda^2/\mu}{\lambda/\mu^2} = \lambda$$

$$\frac{1-P_0}{W_q P_0} = \frac{\rho}{\frac{\lambda}{\mu(\mu-\lambda)}\cdot\frac{\mu-\lambda}{\mu}} = \frac{\lambda/\mu}{\lambda/\mu^2} = \mu$$

17.6-5 $\lambda=3$, $\mu=4$, $s=1$, $\rho=3/4$

The system without the storage restriction is a M/M/1 queue. If n square feet of floor space were available for waiting, the proportion of time this would be sufficient is $\sum_{i=0}^{n+1} P_i$. Thus we want to find n_ℓ such that $\sum_{i=0}^{n_\ell+1} P_i \geq q_\ell$ for $\ell=1,2,3$, where $q_1=.5$, $q_2=.9$, $q_3=.99$

Now $\sum_{i=0}^{n_\ell+1} P_i \geq q_\ell \Leftrightarrow \sum_{i=0}^{n_\ell+1}(1-\rho)\rho^i \geq q_\ell \Leftrightarrow (1-\rho)\frac{(1-\rho^{n_\ell+2})}{(1-\rho)} \geq q_\ell \Leftrightarrow$

$\Leftrightarrow 1-\rho^{n_\ell+2} \geq q_\ell \Leftrightarrow \rho^{n_\ell+2} \leq 1-q_\ell \Leftrightarrow (n_\ell+2)\ln\rho \leq \ln(1-q_\ell)$

$\Leftrightarrow (n_\ell+2) \geq \frac{\ln(1-q_\ell)}{\ln\rho} \Leftrightarrow n_\ell \geq \frac{\ln(1-q_\ell)}{\ln\rho} - 2$

part	q_ℓ	$\frac{\ln(1-q_\ell)}{\ln\rho} - 2$	floor space required
(a)	.50	.409	1
(b)	.90	6.004	7
(c)	.99	14.008	15

17.6-6. a) True

b) False. $L = \lambda W = \frac{\rho}{1-\rho}$

c) False. $L = \rho/(1-\rho)$, $L = 9$ when $\rho = 0.9$, but $L = 99$ when $\rho = 0.99$

17.6-7 a) False

b) True. when $\rho > 1$, $L \to \infty$

c) True

17.6-8 a) True

b) False

c) True

17.6-9

a)

$$L = \frac{\lambda}{\mu - \lambda} = \frac{30}{40-30} = 3 \text{ customers}$$

$$W = \frac{1}{\mu - \lambda} = \frac{1}{40-30} = 0.1 \text{ hours}$$

$$W_q = \frac{\lambda}{\mu(\mu - \lambda)} = \frac{30}{40(40-30)} = 0.075 \text{ hours}$$

$$L_q = \lambda W_q = 30(0.075) = 2.25 \text{ customers}$$

$$P_0 = 1 - \rho = 1 - 0.75 = 0.25$$

$$P_1 = (1-\rho)\rho = (1-0.75)0.75 = 0.188$$

$$P_2 = (1-\rho)\rho^2 = (1-0.75)0.75^2 = 0.141$$

There is a 42% chance of having more than 2 customers at the checkout stand.

b)

Data

λ =	30	(mean arrival rate)
μ =	40	(mean service rate)
s =	1	(# servers)

Pr(ω>t) =	3.975E-31
when t =	7

Prob(ω_q>t) =	1.447E-22
when t =	5

P0+P1+P2= 0.578125

Results

L =	3
L_q =	2.25
W =	0.1
W_q =	0.075
ρ =	0.75
P_0 =	0.25
P_1 =	0.1875
P_2 =	0.140625

17.6-9 c)

$$L = \frac{\lambda}{\mu - \lambda} = \frac{30}{60 - 30} = 1 \text{ customer}$$

$$W = \frac{1}{\mu - \lambda} = \frac{1}{60 - 30} = 0.033 \text{ hours}$$

$$W_q = \frac{\lambda}{\mu(\mu - \lambda)} = \frac{30}{60(60 - 30)} = 0.017 \text{ hours}$$

$$L_q = \lambda W_q = 30(0.075) = 0.5 \text{ customer}$$

$$P_0 = 1 - \rho = 1 - 0.5 = 0.5$$

$$P_1 = (1 - \rho)\rho = (1 - 0.5)0.5 = 0.25$$

$$P_2 = (1 - \rho)\rho^2 = (1 - 0.5)0.5^2 = 0.125$$

There is a 12.5% chance of having more than 2 customers at the checkout stand.

d)

Data

$\lambda =$	30	(mean arrival rate)
$\mu =$	60	(mean service rate)
s =	1	(# servers)

$Pr(\omega>t) =$	6.283E-92
when t =	7

$Prob(\omega_q>t) =$	3.588E-66
when t =	5

P0+P1+P2= 0.875

Results

L =	1
L_q =	0.5
W =	0.03333333
W_q =	0.01666667
ρ =	0.5
P_0 =	0.5
P_1 =	0.25
P_2 =	0.125

e) The manager should adopt the new approach of adding another person to bag the groceries.

17.6-10

a)

Data

$\lambda =$	10	(mean arrival rate)
$\mu =$	20	(mean service rate)
s =	1	(# servers)

$Pr(\omega>t) =$	0.0067379
when t =	0.5

$Prob(\omega_q>t) =$	0.003369
when t =	0.5

P0+P1+P2+P3+P4+P5= 0.984375

Results

L =	1
L_q =	0.5
W =	0.1
W_q =	0.05
ρ =	0.5
P_0 =	0.5
P_1 =	0.25
P_2 =	0.125
P_3 =	0.0625
P_4 =	0.03125
P_5 =	0.015625

All the criteria are currently being satisfied.

b)

Data

$\lambda =$	15	(mean arrival rate)
$\mu =$	20	(mean service rate)
s =	1	(# servers)

$Pr(\omega>t) =$	0.082085
when t =	0.5

$Prob(\omega_q>t) =$	0.0615637
when t =	0.5

P0+P1+P2+P3+P4+P5= 0.822021484

Results

L =	3
L_q =	2.25
W =	0.2
W_q =	0.15
ρ =	0.75
P_0 =	0.25
P_1 =	0.1875
P_2 =	0.140625
P_3 =	0.10546875
P_4 =	0.07910156
P_5 =	0.05932617

None of the criteria are now satisfied.

17.6-10

c)

Data

$\lambda =$	25	(mean arrival rate)
$\mu =$	20	(mean service rate)
s =	2	(# servers)

$Pr(\omega>t) =$	0.0010217
when t =	0.5

$Prob(\omega_q>t) =$	0.0002659
when t =	0.5

P0+P1+P2+P3+P4+P5= 0.926640437

Results

L =	2.05128205
L_q =	0.80128205
W =	0.08205128
W_q =	0.03205128
ρ =	0.625
P_0 =	0.23076923
P_1 =	0.28846154
P_2 =	0.18028846
P_3 =	0.11268029
P_4 =	0.07042518
P_5 =	0.04401574

In this case, the first and third criteria are satisfied but the second is not.

17.6-11

a)

Data

$\lambda =$	2	(mean arrival rate)
$\mu =$	1	(mean service rate)
s =	4	(# servers)

$Pr(\omega>t) =$	0.0079019
when t =	5

$Prob(\omega_q>t) =$	7.896E-06
when t =	5

P0+P1+P2+P3+P4+P5+P6+P7+P8+P9=

0.997282609

Results

L =	2.17391304
L_q =	0.17391304
W =	1.08695652
W_q =	0.08695652
ρ =	0.5
P_0 =	0.13043478
P_1 =	0.26086957
P_2 =	0.26086957
P_3 =	0.17391304
P_4 =	0.08695652
P_5 =	0.04347826
P_6 =	0.02173913
P_7 =	0.01086957
P_8 =	0.00543478
P_9 =	0.00271739

All the guidelines are currently being met.

17.6-11

b)

Data

$\lambda =$	3	(mean arrival rate)
$\mu =$	1	(mean service rate)
s =	4	(# servers)

$Pr(\omega>t)$ =	0.0239006
when t =	5

$Prob(\omega_q>t)$ =	0.0034325
when t =	5

P0+P1+P2+P3+P4+P5+P6+P7+P8+P9=

0.9093317

Results

$L =$	4.52830189
$L_q =$	1.52830189
$W =$	1.50943396
$W_q =$	0.50943396
$\rho =$	0.75
$P_0 =$	0.03773585
$P_1 =$	0.11320755
$P_2 =$	0.16981132
$P_3 =$	0.16981132
$P_4 =$	0.12735849
$P_5 =$	0.09551887
$P_6 =$	0.07163915
$P_7 =$	0.05372936
$P_8 =$	0.04029702
$P_9 =$	0.03022277

The first two guidelines will not be satisfied in one year but the third will be.

c) Five tellers will be needed in a year,

17.6-12

(a)

λ	L	L_q	W	W_q	$P\{\mathcal{W} > 5\}$
.5	1	.50	2	1	.082
.9	9	8.10	10	9	.607
.99	99	98.01	100	99	.951

(b)

λ	λ/μ	ρ	P_0	L	L_q	W	W_q	$P\{\mathcal{W}>5\}$
.5	1	.5	.3333	1.333	.333	2.667	.667	.150
.9	1.8	.9	.0526	9.474	7.674	10.526	8.526	.641
.99	1.98	.99	.0050	99.497	97.517	100.503	98.503	.956

17.6-13

Data

λ =	10	(mean arrival rate)
μ =	12	(mean service rate)
s =	1	(# servers)

$Pr(\omega>t)$ =	2.061E-09
when t =	10

$Prob(\omega_q>t)$ =	0.8333333
when t =	0

P0+P1= 0.305555556

Results

L =	5
L_q =	4.16666667
W =	0.5
W_q =	0.41666667
ρ =	0.83333333
P_0 =	0.16666667
P_1 =	0.13888889
P_2 =	0.11574074
P_3 =	0.09645062

Data

λ =	10	(mean arrival rate)
μ =	12	(mean service rate)
s =	2	(# servers)

$Pr(\omega>t)$ =	1.894E-52
when t =	10

$Prob(\omega_q>t)$ =	0.245098
when t =	0

P0+P1+P2= 0.897875817

Results

L =	1.00840336
L_q =	0.17507003
W =	0.10084034
W_q =	0.017507
ρ =	0.41666667
P_0 =	0.41176471
P_1 =	0.34313725
P_2 =	0.14297386
P_3 =	0.05957244

Data

λ =	10	(mean arrival rate)
μ =	12	(mean service rate)
s =	3	(# servers)

$Pr(\omega>t)$ =	8.047E-53
when t =	10

$Prob(\omega_q>t)$ =	0.0577101
when t =	0

P0+P1+P2+P3= 0.983969426

Results

L =	0.85552951
L_q =	0.02219618
W =	0.08555295
W_q =	0.00221962
ρ =	0.27777778
P_0 =	0.43213296
P_1 =	0.3601108
P_2 =	0.15004617
P_3 =	0.04167949

Data

λ =	10	(mean arrival rate)
μ =	12	(mean service rate)
s =	4	(# servers)

$Pr(\omega>t)$ =	7.707E-53
when t =	10

$Prob(\omega_q>t)$ =	0.0110241
when t =	0

P0+P1+P2+P3+P4= 0.997703314

Results

L =	0.83623441
L_q =	0.00290108
W =	0.08362344
W_q =	0.00029011
ρ =	0.20833333
P_0 =	0.43433168
P_1 =	0.36194306
P_2 =	0.15080961
P_3 =	0.04189156
P_4 =	0.00872741

(CONT'D)

17.6-13
(CONT'D)

Data

λ =	10	(mean arrival rate)
μ =	12	(mean service rate)
s =	5	(# servers)

Pr(ω>t) =	7.672E-53
when t =	10

Prob(ω_q>t) =	0.0017464
when t =	0

P0+P1+P2+P3+P4+P5= 0.999708926

Results

L =	0.83368262
L_q =	0.00034929
W =	0.08336826
W_q =	3.4929E-05
ρ =	0.16666667
P_0 =	0.43457121
P_1 =	0.36214268
P_2 =	0.15089278
P_3 =	0.04191466
P_4 =	0.00873222
P_5 =	0.00145537

a) 2 servers
b) 3 servers
c) 2 servers
d) 1 server
e) 5 servers
f) 1 server
g) 3 servers

17.6-14 $\lambda = 15$, $\mu = 20$, this is a M/M/1 Queue

$P\{\text{customer does not have to wait}\} = P_0 = 1 - \frac{\lambda}{\mu} = \frac{1}{3}$

$\text{price/per gallon} = 1.2 \times \frac{1}{3} + 1 \times \frac{2}{3} = 1.067$

17.6-15 $\text{Expected cost} = \sum_{n=1}^{\infty} n \cdot P_n = \sum_{n=1}^{\infty} n \cdot \rho^n (1-\rho) = \frac{\rho}{1-\rho} = \frac{\lambda}{\mu - \lambda}$

17.6-16
Let $P\{\mathcal{W} \le t\} = G(t)$ and let $\frac{d\,G(t)}{dt} = g(t)$

Then $P\{\mathcal{W} > t\} = 1 - P(t)$

So $[1 - G(t)] = \sum_{n=0}^{\infty} P_n \, P\{S_{n+1} > t\} =$

$$= \sum_{n=0}^{\infty} (1-\rho)\rho^n \left[\int_t^{\infty} \frac{\mu^{n+1} x^n e^{-\mu x}}{n!} dx \right] =$$

$$= \sum_{n=0}^{\infty} (1-\rho)\rho^n \left[1 - \int_0^{t} \frac{\mu^{n+1} x^n e^{-\mu x}}{n!} dx \right]$$

Differentiating both sides we have:

$$g(t) = \sum_{n=0}^{\infty} (1-\rho)\rho^n \left[\frac{\mu^{n+1} t^n e^{-\mu t}}{n!} \right] = (1-\rho)\mu e^{-\mu t} \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!}$$

$$= (1-\rho)\mu e^{-\mu t} e^{\lambda t} = \mu(1-\rho) e^{-\mu(1-\rho)t}$$

Hence, by integration, $P(W > t) = 1 - \int_0^t g(x)dx = e^{-\mu(1-\rho)t}$

17.6-17 (a) Let $P\{\mathcal{W}_q \le t\} = G(t)$ and let $\frac{dG(t)}{dt} = g(t)$

Then $P\{\mathcal{W}_q > t\} = 1 - G(t)$

So $[1-G(t)] = \sum_{n=1}^{\infty} P_n P\{S_n > t\} = \sum_{n=1}^{\infty} (1-\rho)\rho^n \left[1 - \int_0^t \frac{\mu^n \, x^{n-1} e^{-\mu x}}{(n-1)!} dx\right]$

Differentiating both sides gives:

$$g(t) = \sum_{n=1}^{\infty} (1-\rho)\rho^n \left[\frac{\mu^n \; t^{n-1} e^{-\mu t}}{(n-1)!}\right]$$

$$= (1-\rho)\lambda e^{-\mu t} \sum_{n=1}^{\infty} \frac{(\lambda t)^{n-1}}{(n-1)!}$$

$$= (1-\rho)\lambda e^{-\mu t} \; e^{\lambda t}$$

$$= \left(\frac{\lambda}{\mu}\right)(\mu - \lambda) e^{-(\mu-\lambda)t}$$

Then $W_q = \left(\frac{\lambda}{\mu}\right)\int_0^{\infty} t(\mu-\lambda)e^{-(\mu-\lambda)t} dt = \frac{\lambda}{\mu(\mu-\lambda)}$

(b) Let $P\{\mathcal{W}_q \le t\} = G(t)$ and let $\frac{dG(t)}{dt} = g(t)$

Then $P\{\mathcal{W}_q > t\} = 1 - G(t)$

So $[1-G(t)] = \sum_{n=s}^{\infty} P_n P\{S_{n-s+1} > t\}$

$$= \sum_{n=s}^{\infty} P_n \left[1 - \int_0^t \frac{(s\mu)^{n-s+1}}{(n-s)!} x^{n-s} e^{-(s\mu)x} dx\right]$$

Now $P_n = \frac{(\lambda/\mu)^n}{s!\, s^{n-s}} P_0$ for $n \ge s$

So differentiating both sides gives

$$g(t) = \sum_{n=s}^{\infty}\left[\frac{(\lambda/\mu)^n P_0}{s!\, s^{n-s}}\right]\left[\frac{(s\mu)^{n-s+1} t^{n-s} e^{-(s\mu)t}}{(n-s)!}\right]$$

$$= \frac{P_0 (s\mu)(\lambda/\mu)^s}{s!} e^{-s\mu t} \sum_{n=s}^{\infty} \frac{(\lambda t)^{n-s}}{(n-s)!}$$

$$= \frac{P_0 (s\mu)(\lambda/\mu)^s}{s!} e^{-s\mu t} \, e^{\lambda t}$$

$$= \frac{P_0 (s\mu)(\lambda/\mu)^s}{s!} e^{-(s\mu)(1-\rho)t}$$

So $W_q = \frac{P_0(\lambda/\mu)^s}{s!} \int_0^{\infty} t(s\mu) e^{-(s\mu)(1-\rho)t} dt$

$$= \frac{P_0(\lambda/\mu)^s}{s!(1-\rho)} \int_0^{\infty} t(s\mu)(1-\rho) e^{-(s\mu)(1-\rho)t} dt$$

$$= \frac{P_0 (\lambda/\mu)^s}{s!(1-\rho)^2 (s\mu)}$$

$$= \frac{P_0 (\lambda/\mu)^s \rho}{s!(1-\rho)^2 \lambda}$$

$$= L_q/\lambda$$

17.6-18

$\lambda = 4 \quad \mu = 3 \quad s = 2$

we have: $P_0 = 0.2$; $P_1 = 0.267$; $P_2 = 0.178$

the mean rate is:

$$\frac{\mu_0 P_0 + \mu_1 P_1 + \mu_2 P_2}{P_0 + P_1 + P_2} = \frac{0 \cdot P_0 + 3P_1 + 6P_2}{P_0 + P_1 + P_2} = 2.90$$

17.6-19

$\lambda = 4$

$\mu = 6$

$s = 2$

$L = 0.75$

$L_q = 0.083$

$W = 0.188$

$W_q = 0.021$

$P(\mathcal{W} > t) = 1$, where $t = 0$

$P(\mathcal{W}_q > t) = 0.003$, where $t = 0.5$

P_0	= 0.5
P_1	= 0.33333
P_2	= 0.11111
P_3	= 0.03704
P_4	= 0.01235
P_5	= 0.00412
P_6	= 0.00137
P_7	= 0.00046
P_8	= 0.00015
P_9	= 0.00005
P_{10}	= 0.00002
P_{11}	= 0.00001
P_{12}	= 0
P_{13}	= 0
P_{14}	= 0
P_{15}	= 0
P_{16}	= 0
P_{17}	= 0
P_{18}	= 0
P_{19}	= 0
P_{20}	= 0

$$P\{W_q > 0.5 \mid \#\text{customers} \geq 2\} = \frac{P\{W_q > 0.5, \#\text{customers} \geq 2\}}{P\{\#\text{customers} \geq 2\}}$$

$$= \frac{P\{W_q > 0.5\}}{1 - P_0 - P_1} = \frac{0.003}{1-(0.5+0.3333)} = 0.018$$

17.6-20 (a) $W = \frac{1}{\mu - \lambda}$

$W_{clara} = \frac{1}{20-16} = \frac{1}{4}$ hour ≈ 15 minutes

$W_{clarence} = \frac{1}{20-14} = \frac{1}{6}$ hour $= 10$ minutes

$W_{total} = P\{\text{clara}\} \cdot W_{clara} + P\{\text{clarence}\} \cdot W_{clarence}$

$= \frac{16}{30} \cdot 15 + \frac{14}{30} \cdot 10 = 12.67$ minutes $\doteq 0.211$ hour

(b) It is M/M/2 queue, $\lambda = 16+14 = 30$, $\mu = 20$, $S = 2$

Run OR courseware, $W = 0.114$ hour

(c) $\mu = 60/3.5$, $W = 0.249$

$\mu = 60/3.4$, $W = 0.204$

$\mu = 60/3.45$, $W = 0.225$

$\mu = 60/3.425$, $W = 0.214$

$\mu = 60/3.419$, $W = 0.212$

$\mu = 60/3.4185$, $W = 0.211$

$\vdots$

So, if the expected processing time is 3.485 minutes, it causes the same expected waiting time.

17.6-21 a) When $\lambda = 10$, $\mu = 7.5$, $S = 2$, (current system)

$L = 2.4$, $L_q = 1.067$, $W = 0.24$, $W_q = 0.107$

When $\lambda = 5$, $\mu = 7.5$, $s = 1$, (next year)

$L = 2$, $L_q = 1.333$, $W = 0.4$, $W_q = 0.267$

So, next year yields smaller L but larger L_q, W and W_q

b) $W = \frac{1}{\mu - \lambda} \Rightarrow \mu = \frac{1}{W} + \lambda = \frac{1}{0.24} + 5 = 9.17$

c) $W_q = \frac{\lambda}{\mu(\mu-\lambda)} \Rightarrow \mu = \frac{\lambda W_q \pm \sqrt{\lambda^2 W_q^2 + 4\lambda W_q}}{2 W_q} \Rightarrow \mu = 9.78$

17.6-22

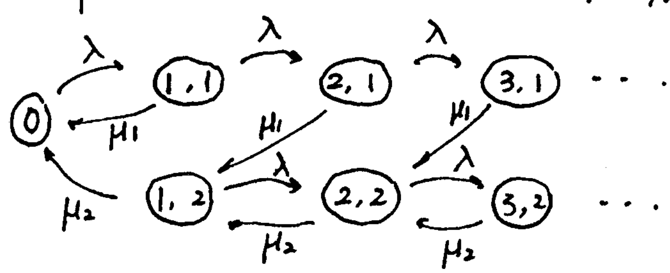
(a) The future evolution of the queueing system is affected by whether the parameter of the service time distribution for the customer currently in service is μ_1 or μ_2. So the current state of the system needs to include this information from the history of the process.

The states are (n, s) where

n = number of customers in the system ($n = 1, 2, \cdots$)

$$s = \begin{cases} 1 & \text{if the current parameter is } \mu_1 \\ 2 & \text{if the current parameter is } \mu_2 \end{cases}$$

except that $n=0$ does not need s. Then we have:



(b) $\lambda p_0 = \mu_1 \cdot P_{1,1} + \mu_2 \cdot P_{1,2}$

$(\lambda + \mu_1) P_{1,1} = \lambda p_0$

$\vdots$

$(\lambda + \mu_1) P_{n,1} = \lambda P_{n-1,1}$

$\vdots$

$(\lambda + \mu_2) P_{1,2} = \mu_1 \cdot P_{2,1} + \mu_2 P_{2,2}$

$\vdots$

$(\lambda + \mu_2) P_{n,2} = \lambda P_{n-1,2} + \mu_1 P_{n+1,1} + \mu_2 P_{n+1,2} \quad (n \geq 2)$

$\vdots$

17.6-22

(c) Truncate (cut off) the balance equations at a very large n and then solve the resulting finite system of equations numerically. The resulting approximation of the stationary distribution should be essentially exact if the probability of exceeding the truncating value of n (in the exact model) is negligible.

(d) $L = \sum_{n=1}^{\infty} n(P_{n,1} + P_{n,2})$, $W = \frac{L}{\lambda}$

$L_q = \sum_{n=1}^{\infty} (n-1)(P_{n,1} + P_{n,2})$, $W_q = \frac{L_q}{\lambda}$

(e) Because the input is Poisson, the distribution of the state of the system is the same just before an arrival and at an arbitrary point in time

$$P\{W \le t\} = P\{W \le t \mid \text{arrival finds state } 0\} P_0 + \sum_{n=1}^{\infty} P\{W \le t \mid \text{arrival finds state } (n,1)\} P_{n,1} + \sum_{n=1}^{\infty} P\{W \le t \mid \text{arrival finds state } (n,2)\} P_{n,2}$$

These 3 conditional distributions of W are, respectively (1) $\exp(\mu_1)$ (2) a convolution of $\exp(\mu_1)$ and Erlang$(n/\mu_2, n)$ and (3) Erlang$((n+1)\mu_2, n+1)$

Then, $$P\{W \le t\} = (1 - e^{-\mu_1 t}) P_0 + \sum_{n=1}^{\infty} \left[\int_0^t [1 - e^{-\mu_1 (t - t_1)}] \cdot \frac{\mu_2^n t_1^{n-1} e^{-\mu_2 t}}{(n-1)!} dt_1 \right] P_{n,1} + \sum_{n=1}^{\infty} \left[\int_0^t \frac{\mu_2^{n+1} x^n e^{-\mu_2 x}}{n!} dx \right] P_{n,2}$$

17.6-23

(a) $\lambda P_0 = \mu P_1$ . . . . (0)

$\lambda P_0 + \mu P_2 = (\lambda + \mu) P_n$ . . . (1)

$\vdots$

$\lambda P_{n-1} + \mu P_{n+1} = (\lambda + \mu) P_n$ . . . (n)

The solution given in Sec. 16.6 is:

$P_n = (1-\rho)\rho^n$ for $n = 0, 1, 2 \ldots$

Verifying that the above satisfy the balance equations:

equation (0) : $\lambda \cdot (1-\rho) = \mu(1-\rho) \cdot \rho \Leftrightarrow \lambda = \mu \cdot \rho = \mu \frac{\lambda}{\mu}$ OK

equation (n) : $\lambda(1-\rho)\rho^{n-1} + \mu(1-\rho)\rho^{n+1} = (\lambda+\mu)(1-\rho)\rho^n$

$$\Leftrightarrow \underbrace{\lambda + \mu\rho^2}_{\lambda + \frac{\lambda^2}{\mu}} = \underbrace{(\lambda + \mu) \cdot \rho}_{\frac{\lambda^2}{\mu} + \lambda}$$ OK

17.6-23 (b) $\lambda P_0 = \mu P_1$

$\lambda P_0 + \mu P_2 = (\lambda+\mu) P_1$

$\lambda P_1 = \mu P_2$

The solution given in Sec. 16.6 is:

$$P_n = \left(\frac{1-\rho}{1-\rho^3}\right)\rho^n \quad \text{for } n = 0,1,2$$

Verifying:

- $\lambda \cdot \frac{1-\rho}{1-\rho^3} = \mu \cdot \frac{1-\rho}{1-\rho^3} \cdot \rho \Leftrightarrow \lambda = \mu \cdot \rho = \mu \cdot \frac{\lambda}{\mu}$ . . . $\underline{\text{ok}}$

- $\lambda \cdot \frac{1-\rho}{1-\rho^3} + \mu \frac{1-\rho}{1-\rho^3} \cdot \rho^2 = (\lambda+\mu) \frac{1-\rho}{1-\rho^3}\rho \Leftrightarrow \underbrace{\lambda + \mu \cdot \rho^2}_{= \lambda + \frac{\lambda^2}{\mu}} = \underbrace{(\lambda+\mu)\cdot\rho}_{= \frac{\lambda^2}{\mu} + \lambda}$

- $\lambda \cdot \frac{1-\rho}{1-\rho^3} \cdot \rho = \mu \frac{1-\rho}{1-\rho^3} \cdot \rho^2 \Leftrightarrow \underbrace{\lambda \cdot \rho}_{= \frac{\lambda^2}{\mu}} = \underbrace{\mu \rho^2}_{= \frac{\lambda^2}{\mu}}$ . . . $\underline{\text{ok}}$

(c) $2\lambda P_0 = \mu P_1$

$2\lambda P_0 + \mu P_2 = (\lambda+\mu) P_1$

$2 P_1 = \mu P_2$

The solution given in Sec. 16.6 is:

$$P_0 = \left[\sum_{n=0}^{2} \frac{2!}{(2-n)!}\left(\frac{\lambda}{\mu}\right)^n\right]^{-1} = \left[1 + 2\left(\frac{\lambda}{\mu}\right) + 2\left(\frac{\lambda}{\mu}\right)^2\right]^{-1}$$

$$P_n = \frac{2!}{(2-n)!}\left(\frac{\lambda}{\mu}\right)^n P_0 \quad \text{for } n = 1,2$$

Verifying:

- $2\lambda/(1 + 2(\frac{\lambda}{\mu}) + 2(\frac{\lambda}{\mu})^2) = \mu \cdot 2\left(\frac{\lambda}{\mu}\right)/(1 + 2(\frac{\lambda}{\mu}) + 2(\frac{\lambda}{\mu})^2)$

  $\Leftrightarrow 2\lambda = \mu \cdot 2 \cdot \frac{\lambda}{\mu}$ . . . $\underline{\text{ok}}$

- $2\lambda/(1 + 2(\frac{\lambda}{\mu}) + 2(\frac{\lambda}{\mu})^2) + \mu \cdot 2(\frac{\lambda}{\mu})^2/(1 + 2(\frac{\lambda}{\mu}) + 2(\frac{\lambda}{\mu})^2) =$

  $= (\lambda+\mu) 2 (\lambda/\mu)/(1 + 2(\lambda/\mu) + 2(\lambda/\mu)^2) \Leftrightarrow \underbrace{2\lambda + 2\mu\left(\frac{\lambda}{\mu}\right)^2}_{= 2\left(\lambda + \frac{\lambda^2}{\mu}\right)} = \underbrace{2(\lambda+\mu)\left(\frac{\lambda}{\mu}\right)}_{= 2\left(\frac{\lambda^2}{\mu} + \lambda\right)}$ ...

- $\lambda \cdot 2\left(\frac{\lambda}{\mu}\right)/(1 + 2(\lambda/\mu) + 2(\lambda/\mu)^2) = \mu \cdot 2 \cdot (\lambda/\mu)^2/(1 + 2(\lambda/\mu) + 2(\lambda/\mu)^2)$

  $\Leftrightarrow 2\frac{\lambda^2}{\mu} = 2\frac{\lambda^2}{\mu}$ . . . $\underline{\text{ok}}$

17.6-24 (a) The M/M/s Model:

$\lambda = 6$

$\mu = 4$

$s = 3$

$L = 1.737$

$L_q = 0.237$

$W = 0.289$

$W_q = 0.039$

$P(\mathcal{W} > t) = 0.026$, where $t = 1$

$P(\mathcal{W}_q > t) = 0.237$, where $t = 0$

P_0	= 0.21053
P_1	= 0.31579
P_2	= 0.23684
P_3	= 0.11842
P_4	= 0.05921
P_5	= 0.02961
P_6	= 0.0148
P_7	= 0.0074
P_8	= 0.0037
P_9	= 0.00185
P_{10}	= 0.00093
P_{11}	= 0.00046
P_{12}	= 0.00023
P_{13}	= 0.00012
P_{14}	= 0.00006
P_{15}	= 0.00003
P_{16}	= 0.00001
P_{17}	= 0.00001
P_{18}	= 0
P_{19}	= 0
P_{20}	= 0

(b) $P\{$a phone is answered immediately$\} = 1 - P\{\mathcal{W}_q > 0\} = 0.763$

Or, $= P\{$at least one server is free$\} = P_0 + P_1 + P_2$

$= 0.21053 + 0.31579 + 0.23684 = 0.763$

(c) $P\{n$ calls on hold$\} = P_n' = P_{n+3}$ $(n \geq 1)$

$P_0' = P_0 + P_1 + P_2 + P_3 = 0.88158$

(d) The printed measures are in the next page.

$P\{$arriving call is lost$\}$

$= P\{$all three servers are busy$\}$

$= P_3 = 0.13433$

17.6-24 *(continued)*

Finite Queue Variation of the M/M/s Model:

$\lambda = 6$		P_0	$= 0.23881$
		P_1	$= 0.35821$
$\mu = 4$		P_2	$= 0.26866$
		P_3	$= 0.13433$
$s = 3$		P_4	$= 0$
		P_5	$= 0$
$K = 3$		P_6	$= 0$
		P_7	$= 0$
		P_8	$= 0$
		P_9	$= 0$
$L = 1.299$		P_{10}	$= 0$
		P_{11}	$= 0$
$L_q = 0$		P_{12}	$= 0$
		P_{13}	$= 0$
$W = 0.25$		P_{14}	$= 0$
		P_{15}	$= 0$
$W_q = 0$		P_{16}	$= 0$
		P_{17}	$= 0$
		P_{18}	$= 0$
		P_{19}	$= 0$
		P_{20}	$= 0$

17.6-25 This is a M/M/1/K queue with $K=1, 3$ and 5, respectively. Also, $\lambda = 1/4$ and $\mu = 1/3$ so that $\rho = 3/4$.
The fraction of customers lost $= P_K = \frac{(1-\rho)}{(1-\rho^{K+1})} \cdot \rho^K$

(a) zero spaces : $P_1 = \frac{(1-3/4)}{(1-(3/4)^2)} \cdot (3/4) = \frac{3}{7} = .429$

(b) two spaces : $P_3 = \frac{(1-3/4)}{(1-(3/4)^4)} \cdot (3/4)^3 = \frac{27}{175} = .154$

(c) four spaces : $P_5 = \frac{(1-3/4)}{(1-(3/4)^6)} \cdot (3/4)^5 = \frac{243}{3367} = .072$

17.6-26 M/M/s/K model

$$L_q = \sum_{n=s}^{\infty} (n-s) P_n =$$

$$= \sum_{n=s}^{K} (n-s) \frac{(\lambda/\mu)^n}{s!\, s^{n-s}} P_0$$

$$= \frac{P_0 (\lambda/\mu)^{s+1}}{s!\; s} \sum_{n=s}^{K} (n-s) \left(\frac{\lambda}{s\mu}\right)^{n-s-1} =$$

$$= \frac{P_0 (\lambda/\mu)^s \rho}{s!} \sum_{j=0}^{K-s} j \rho^{j-1} =$$

$$= \frac{P_0 (\lambda/\mu)^s \rho}{s!} \sum_{j=0}^{K-s} \frac{d(\rho^j)}{d\rho} =$$

17.6-26 (continued)

$$= \frac{P_0 (\lambda/\mu)^s \rho}{s!} \frac{d}{d\rho}\left[\sum_{j=0}^{K-s} \rho^j\right] =$$

$$= \frac{P_0 (\lambda/\mu)^s \rho}{s!} \frac{d}{d\rho}\left(\frac{1-\rho^{K-s+1}}{1-\rho}\right) =$$

$$= \frac{P_0 (\lambda/\mu)^s \rho}{s!}\left[\frac{1-\rho^{K-s}-(K-s)\rho^{K-s}(1-\rho)}{(1-\rho)^2}\right]$$

17.6-27

$\mathcal{W}$ and $\mathcal{W}_q$ represent the waiting times of arriving customers who enter the system. The probability that such a customer finds n customers already there is:

$$P\{n \text{ customers in system} \mid \text{system not full}\} = \begin{cases} \frac{P_n}{1-P_K} & 0 \le n \le K-1 \\ 0 & n = K \end{cases}$$

And so:

(a) $P\{\mathcal{W} > t\} = \frac{1}{1-P_K} \sum_{n=0}^{K-1} P_n P\{S_{n+1} > t\}$

(b) $P\{\mathcal{W}_q > t\} = \frac{1}{1-P_K} \sum_{n=0}^{K-1} P_n P\{S_n > t\}$

17.6-28

a) & b)

Data

λ =	20	(mean arrival rate)
μ =	30	(mean service rate)
s =	1	(# servers)
K =	2	(max customers)

Results

L =	0.73684211
L_q =	0.21052632
W =	0.04666667
W_q =	0.01333333
ρ =	0.66666667
P_0 =	0.47368421
P_1 =	0.31578947
P_2 =	0.21052632

CONT'D)

17.6-28

a) & b) (CONT'D)

Data

$\lambda =$	20	(mean arrival rate)
$\mu =$	30	(mean service rate)
s =	1	(# servers)
K =	3	(max customers)

Results

$L =$	1.01538462
$L_q =$	0.43076923
$W =$	0.05789474
$W_q =$	0.0245614
$\rho =$	0.66666667
$P_0 =$	0.41538462
$P_1 =$	0.27692308
$P_2 =$	0.18461538
$P_3 =$	0.12307692

Data

$\lambda =$	20	(mean arrival rate)
$\mu =$	30	(mean service rate)
s =	1	(# servers)
K =	4	(max customers)

Results

$L =$	1.24170616
$L_q =$	0.62559242
$W =$	0.06717949
$W_q =$	0.03384615
$\rho =$	0.66666667
$P_0 =$	0.38388626
$P_1 =$	0.25592417
$P_2 =$	0.17061611
$P_3 =$	0.11374408
$P_4 =$	0.07582938

Data

$\lambda =$	20	(mean arrival rate)
$\mu =$	30	(mean service rate)
s =	1	(# servers)
K =	5	(max customers)

Results

$L =$	1.42255639
$L_q =$	0.78796992
$W =$	0.07472354
$W_q =$	0.04139021
$\rho =$	0.66666667
$P_0 =$	0.36541353
$P_1 =$	0.24360902
$P_2 =$	0.16240602
$P_3 =$	0.10827068
$P_4 =$	0.07218045
$P_5 =$	0.0481203

c)

spaces	rate customers are lost (P_k)	change in P_k	profit/hour ($4)($\lambda$)($1-P_k$)	change in profit/hour
2	0.21		$63.20	
3	0.12	0.09	$70.40	$7.20
4	0.08	0.04	$73.60	$3.20
5	0.05	0.03	$76.00	$2.40

d) Since it cost $200 per month per car length rented, each additional space must bring at least $200 per month (or $1 per hour) in additional profit. Five spaces still bring more than that so 5 should be provided.

17.6-29

a) The M/M/s model with a finite calling population fits this queueing system.

b)

Data

λ =	0.333333	(max arrival rate)
μ =	0.5	(mean service rate)
s =	1	(# servers)
N =	3	(size of population)

Results

L =	0.71805274
L_q =	0.21095335
W =	2.832
W_q =	0.832
ρ =	0.66666667
λ-bar =	0.2535497
P_0 =	0.49290061
P_1 =	0.32860041
P_2 =	0.14604462
P_3 =	0.03245436

The probabilities that there are 0, 1, 2, or 3 machines not running are P_0, P_1, P_2, and P_3 respectively as shown in the spreadsheet above. The mean of this distribution is L=0.718 as shown above.

c) $W = \frac{L}{\bar{\lambda}} = \frac{0.718}{0.253} = 2.832$ hours.

d) The expected fraction of time that the repair technician will be busy is the system utilization, which is 0.667.

e) M/M/s model:

Data		
$\lambda =$	0.3333333	(mean arrival rate)
$\mu =$	0.5	(mean service rate)
s =	1	(# servers)

Results	
L =	2
L_q =	1.33333333

Finite queue variation of the M/M/s model with K=3:

Data		
$\lambda =$	0.3333333	(mean arrival rate)
$\mu =$	0.5	(mean service rate)
s =	1	(# servers)
K =	3	(max customers)

Results	
L =	1.01538462
L_q =	0.43076923
W =	3.47368421

f)

Data		
$\lambda =$	**0.333333**	(max arrival rate)
$\mu =$	**0.5**	(mean service rate)
s =	**2**	(# servers)
N =	**3**	(size of population)

Results	
L =	0.55280899
L_q =	0.00898876
W =	2.03305785
W_q =	0.03305785
ρ =	0.33333333
λ-bar =	0.27191011
P_0 =	0.54606742
P_1 =	0.36404494
P_2 =	0.08089888
P_3 =	0.00898876

The probabilities that there are 0, 1, 2, or 3 machines not running are P_0, P_1, P_2, and P_3 respectively as shown in the spreadsheet above. The mean of this distribution is L=0.553 as shown above.

The expected fraction of time that the repair technician will be busy is the system utilization, which is 0.333.

17.6-30 (a) This is a finite Calling Population of M/M/s Model.
Here $\lambda = 1$, $\mu = 2$, $S = 1$, $N = 3$

(b) Finite Calling Population of the M/M/s Model:

$\lambda = 1$

$\mu = 2$

$s = 1$

$N = 3$

$L = 1.421$

$L_q = 0.632$

$W = 0.9$

$W_q = 0.4$

$P_0 = 0.21053$
$P_1 = 0.31579$
$P_2 = 0.31579$
$P_3 = 0.15789$
$P_4 = 0$
$P_5 = 0$
$P_6 = 0$
$P_7 = 0$
$P_8 = 0$
$P_9 = 0$
$P_{10} = 0$
$P_{11} = 0$
$P_{12} = 0$
$P_{13} = 0$
$P_{14} = 0$
$P_{15} = 0$
$P_{16} = 0$
$P_{17} = 0$
$P_{18} = 0$
$P_{19} = 0$
$P_{20} = 0$

17.6-31

a) Alternative 1:

Data

$\lambda =$	1.2	(max arrival rate)
$\mu =$	4	(mean service rate)
s =	1	(# servers)
N =	3	(size of population)

Results

L =	0.32064422
L_q =	0.05270864
W =	0.29918033
W_q =	0.04918033

Three machines are the maximum that can be assigned to an operator while still achieving the required production rate. The average number not running is L=0.32. Thus, 1 – (0.32/3) = 89.7% of machines are running on the average.

$$\text{Utilization of servers} = \frac{\bar{\lambda}}{s\mu} = \frac{1.072}{(1)(4)} = 0.268.$$

17.6-31

b) Alternative 2:

Data

λ =	4.8	(max arrival rate)
μ =	4	(mean service rate)
s =	3	(# servers)
N =	12	(size of population)

Results

L =	1.12461693
L_q =	0.03708214
W =	0.25852352
W_q =	0.00852433

Three operators are required to achieve the required production rate. The average number not running is L=1.125. Thus, 1 – (1.125/12) = 90.6% of machines are running on the average.

$$\text{Utilization of servers} = \frac{\bar{\lambda}}{s\mu} = \frac{4.350}{(3)(4)} = 0.363.$$

c) Alternative 3:

Data

λ =	4.8	(max arrival rate)
μ =	8	(mean service rate)
s =	1	(# servers)
N =	12	(size of population)

Results

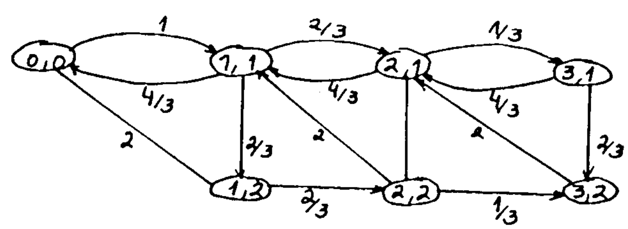
L =	1.03519555
L_q =	0.48698409
W =	0.23602691
W_q =	0.11103346

Two operators are required to achieve the required production rate. The average number not running is L=1.035. Thus, 1 – (1.035/12) = 91.4% of machines are running on the average.

$$\text{Utilization of servers} = \frac{\bar{\lambda}}{s\mu} = \frac{4.386}{(1)(8)} = 0.548.$$

17.6-32 (a) state is (n,i) where n is the number of failed machines $(n=0,1,2,3)$ and i is the stage of service (which operation) for the machine under repair ($i = 0$ (if no machines are failed), 1, 2).

(b)



(c)

State	Rate in = Rate out
(0,0)	$\frac{4}{3} P_{1,1} + 2 P_{1,2} = P_{0,0}$
(1,1)	$P_{0,0} + \frac{4}{3} P_{2,1} + 2 P_{2,2} = (\frac{4}{3} + \frac{2}{3} + \frac{2}{3}) P_{1,1}$
(2,1)	$\frac{2}{3} P_{1,1} + \frac{4}{3} P_{3,1} + 2 P_{3,2} = (\frac{4}{3} + \frac{2}{3} + \frac{1}{3}) P_{2,1}$
(3,1)	$\frac{1}{3} P_{2,1} = (\frac{4}{3} + \frac{2}{3}) P_{3,1}$
(1,2)	$\frac{2}{3} P_{1,1} = (2 + \frac{2}{3}) P_{1,2}$
(2,2)	$\frac{2}{3} (P_{1,2} + P_{2,1}) = (2 + \frac{1}{3}) P_{2,2}$
(3,2)	$\frac{1}{3} P_{2,2} + \frac{2}{3} P_{3,1} = 2 P_{3,2}$

17.7-1 (a) (i) exponential : $W_q = \frac{\lambda}{\mu(\mu-\lambda)}$

(ii) constant : $W_q = \frac{1}{2} \cdot \frac{\lambda}{\mu(\mu-\lambda)}$

(iii) Erlang : $\sigma = \frac{1}{2}(0 + \frac{1}{\mu}) = \frac{1}{2\mu} \Rightarrow \sigma^2 = \frac{1}{4\mu^2} \Rightarrow k = 4$

$W_q = \frac{1+4}{8} \cdot \frac{\lambda}{\mu(\mu-\lambda)} = \frac{5}{8} \frac{\lambda}{\mu(\mu-\lambda)}$

So $W_q^{exp} = 2 W_q^{c} = (8/5) W_q^{Erlang}$

(b) Let $B = 1, (1/2)$ and $(5/8)$ when the distribution is exponential, constant or Erlang, respectively.
Now $\lambda^{(2)} = 2\lambda^{(1)}$ and $\mu^{(2)} = 2\mu^{(1)}$

$W_q^{(2)} = B\left[\frac{2\lambda^{(1)}}{2\mu^{(1)}(2\mu^{(1)} - 2\lambda^{(1)})}\right] = \frac{W_q^{(1)}}{2}$

$L_q^{(2)} = \lambda^{(2)} W_q^{(2)} = 2\lambda^{(1)} W_q^{(1)}/2 = \lambda^{(1)} W_q^{(1)} = L_q^{(1)}$

So the waiting time is cut in half while the queue lenght is unchanged.

17.7-2

Data

$\lambda =$	0.2	(mean arrival rate)
$1/\mu =$	4	(expected service time)
$\sigma =$	4	(standard deviation)
s =	1	(# servers)

Results

$L =$	4
$L_q =$	3.2
$W =$	20
$W_q =$	16

Data

$\lambda =$	0.2	(mean arrival rate)
$1/\mu =$	4	(expected service time)
$\sigma =$	3	(standard deviation)
s =	1	(# servers)

Results

$L =$	3.3
$L_q =$	2.5
$W =$	16.5
$W_q =$	12.5

17.7-2 (CONT'D)

Data

$\lambda =$	0.2	(mean arrival rate)
$1/\mu =$	4	(expected service time)
$\sigma =$	2	(standard deviation)
s =	1	(# servers)

Results

L =	2.8
L_q =	2
W =	14
W_q =	10

Data

$\lambda =$	0.2	(mean arrival rate)
$1/\mu =$	4	(expected service time)
$\sigma =$	1	(standard deviation)
s =	1	(# servers)

Results

L =	2.5
L_q =	1.7
W =	12.5
W_q =	8.5

Data

$\lambda =$	0.2	(mean arrival rate)
$1/\mu =$	4	(expected service time)
$\sigma =$	0	(standard deviation)
s =	1	(# servers)

Results

L =	2.4
L_q =	1.6
W =	12
W_q =	8

b) L_q is half with $\sigma = 0$ therefore it is quite important to reduce the variability of the service times.

c)

σ	L_q	Change	
4	3.2		
3	2.5	0.7	largest reduction
2	2	0.5	
1	1.7	0.3	
0	1.6	0.1	smallest reduction

d) μ needs to be increased 0.05 to achieve the same L_q.

17.7-3 For M/G/1, $L = \rho + \frac{\rho^2 + \lambda^2 \sigma_s^2}{2(1-\rho)}$, $L_q = \frac{\rho^2 + \lambda^2 \sigma_s^2}{2(1-\rho)}$ $W = L/\lambda$, $W_q = \frac{L_q}{\lambda}$

(a) False. when L increases, W also increases.

(b) False. when μ and σ^2 are small, L_q is not necessarily small.

(c) True. For exponential service time, $L_q = \frac{2\rho^2}{2(1-\rho)}$ since $\sigma_s^2 = 1/\mu^2$

For constant service time, $L_q = \frac{\rho^2}{2(1-\rho)}$ since $\sigma_s^2 = 0$

(d) False. We can easily find distribution with $\sigma_s^2 > \frac{1}{\mu^2}$

17.7-4 a)

Data		
$\lambda =$	30	(mean arrival rate)
$1/\mu =$	0.0208333	(expected service time)
$\sigma =$	0.0208333	(standard deviation)
s =	1	(# servers)

Results	
L =	1.66666667
L_q =	1.04166667
W =	0.05555556
W_q =	0.03472222

b)

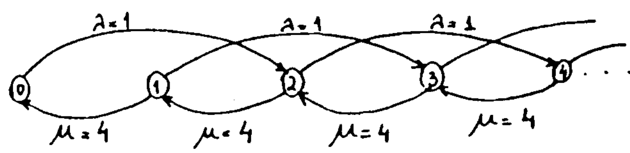
Data		
$\lambda =$	30	(mean arrival rate)
$1/\mu =$	0.0208333	(expected service time)
$\sigma =$	0	(standard deviation)
s =	1	(# servers)

Results	
L =	1.14583333
L_q =	0.52083333
W =	0.03819444
W_q =	0.01736111

c) L_q in part b) is half of L_q in part a).

d) Marsha needs to reduce her service time to approximately 61 seconds.

17.7-5 (a)



$\mu P_1 = \lambda P_0$

$\mu P_2 = (\lambda+\mu) P_1$

$\lambda P_0 + \mu P_3 = (\lambda+\mu) P_2$

$\vdots$

$\lambda P_{n-2} + \mu P_{n+1} = (\lambda+\mu) P_n$

(b) Poisson input with $\lambda = 1$ and Erlang service with $\mu = 4/2 = 2$ and $k = 2$.

c) $$L = L_q + \rho = \frac{\lambda^2\sigma^2 + \rho^2}{2(1-\rho)} + \rho = \frac{(1)^2(0.354)^2 + 0.5^2}{2(1-0.5)} + 0.5 = 0.875$$

d) $$W = W_q + \frac{1}{\mu} = \frac{L_q}{\lambda} + \frac{1}{\mu} = \frac{0.375}{1} + \frac{1}{2} = 0.875$$

e)

Data		
$\lambda =$	1	(mean arrival rate)
$\mu =$	2	(mean service rate)
k =	2	(shape parameter)
s =	1	(# servers)

Results	
L =	0.875
L_q =	0.375
W =	0.875
W_q =	0.375

17.7-6 a & b) Current policy:

Data

λ =	1	(mean arrival rate)
μ =	2	(mean service rate)
s =	1	(# servers)

Results

L =	1
L_q =	0.5
W =	1
W_q =	0.5

Proposal:

Data

λ =	0.25	(mean arrival rate)
μ =	0.5	(mean service rate)
k =	4	(shape parameter)
s =	1	(# servers)

Results

L =	0.8125
L_q =	0.3125
W =	3.25
W_q =	1.25

Under the current policy an airplane looses 1 day of flying time as opposed to 3.25 days under the proposed policy.

Under the current policy 1 airplane is loosing flying time per day as opposed to 0.8125 airplanes.

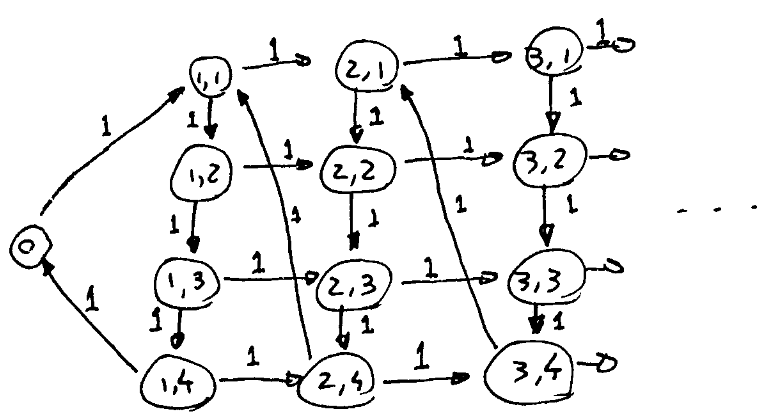
c) The comparison in part b) is the appropriate one for making the decision since it takes into account that airplanes will not have to come in for service as often.

17.7-7 a) Let the state be (n, s) where:

n = # of airplanes at the base

s = stage of service of the airplane being overhauled

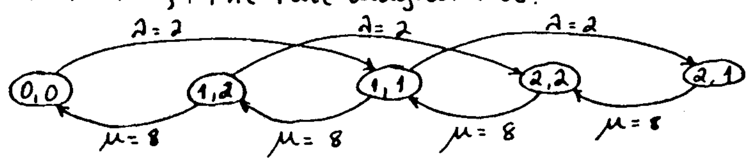
b) The rate diagram is:



17.7-8 For the current arrangement, $\lambda = 24$ and $\mu = 30 \Rightarrow \rho = .8$
For the proposal, $\lambda = 48$, $\mu = 30$ and $s = 2 \Rightarrow \rho = .8$

Model	Current			Proposal	
	L at each crib	Total L	$W = L/\lambda$	L	$W = L/\lambda$
Figure 17.7	4.0	8.0	0.167	4.444	0.093
Figure 17.11	2.4	4.8	0.098	3.1	0.064
Figure 17.13	3.2	6.4	0.133	3.7	0.078
Figure 17.14	2.2	4.4	0.091	2.8	0.058

17.7-9 (a) Let state (i,j) denote i calling units in the system, with the calling unit being served at the j^{th} stage of his service. Then the state space is: $\{(0,0), (1,2), (1,1), (2,2), (2,1)\}$. The rate diagram is:



Note this analysis is possible because an Erlang distribution with $1/\mu = 1/4$ and $k = 2$ is equivalent to the sum of two independent exponentials with parameter $1/\mu = 1/8$.

Hence, the steady state equations are:

$8 P_{1,2} = 2 P_{0,0}$

$8 P_{1,1} = 10 P_{1,2}$

$2 P_{0,0} + 8 P_{2,2} = 10 P_{1,1}$

$2 P_{1,2} + 8 P_{2,1} = 8 P_{2,2}$

$2 P_{1,1} = 8 P_{2,1}$

(b) The solution to these equations is:

$(P_{0,0}; P_{1,2}; P_{1,1}; P_{2,2}; P_{2,1}) = (64/114, 16/114, 20/114, 9/114, 5/114)$

Hence $P_0 = \frac{64}{114} = .561$

$P_1 = \frac{16+20}{114} = .316$

$P_2 = \frac{9+5}{114} = .123$

$L = \frac{18+14}{52} = .561$

(c) If the service time is exponential, then the system is an M/M/1 queue limited to $K = 2$ and with $\lambda = 2$ and $\mu = 4$. So,

$$P_0 = \frac{1-\rho}{1-\rho^{k+1}} = \frac{(1/2)}{(1-1/8)} = \frac{4}{7} = .571$$

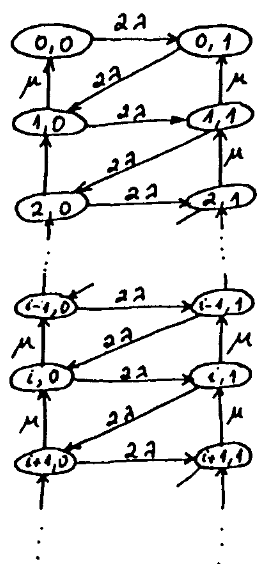
$$P_1 = \left(\frac{1}{2}\right) P_0 = \frac{2}{7} = .286$$

17.7-9 (CONT'D)

$$P_2 = \left(\tfrac{1}{2}\right)^2 P_0 = \tfrac{1}{7} = .143$$

$$L = \tfrac{2+2}{7} = \tfrac{4}{7} = .571$$

17.7-10 state is (n, i) where n is the number of customers in the system ($n \geq 1$) and i is the number of completed arrival stages for currently arriving customer ($i = 0, 1$).



17.7-11 (a) Let T be the repair time.

$E(T) = E(T \mid \text{minor repair needed}).(0.9) +$
$+ E(T \mid \text{major repair needed}).(0.1) = \frac{1}{2}(0.9) + 5(0.1) =$
$= .95$ hours

Now let X be a binary random variable with $P(X=1) = p = 0.9$ and $P(X=0) = q = 0.1$, Y_i be an exponential random variable with mean $1/\lambda_i$ $(i=1,2)$, with $\frac{1}{\lambda_1} = \frac{1}{2}$ and $\frac{1}{\lambda_2} = 5$. Then we may express T as follows:

$T = Y_1 . X + Y_2(1-X)$ where X, Y_1, Y_2 are independent

To calculate $\sigma^2 = Var(T)$ we use the formula:

$Var(T) = E(Var(T \mid X)) + Var(E(T \mid X))$

$Var(T \mid X) = Var(Y_1).X + Var(Y_2)(1-X) = (1/\lambda_1^2)X + (1/\lambda_2^2)(1-X)$

$\therefore E(Var(T \mid X)) = p/\lambda_1^2 + q/\lambda_2^2$

17.7-11 (CONT'D)

$$E(T|X) = E(Y_1).X + E(Y_2).(1-X) = \frac{1}{\lambda_1}.X + \frac{1}{\lambda_2}.(1-X) =$$

$$= \frac{1}{\lambda_2} + \left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right)X$$

$$\therefore Var(E(T|X)) = \left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right)^2 . Var X = \left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right)^2 pq$$

Therefore,

$$Var(T) = \frac{p}{\lambda^2} + \frac{q}{\lambda_2^2} + \left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right)^2 pq = 4.5475$$

Now we can see that T has a variance much bigger than that of an exponential random variable with same mean, which would be $(.95)^2 = .9025$

(b) $\left.\begin{array}{l}\mu = \frac{1}{.95} \\ \lambda = 1\end{array}\right\} \Rightarrow \rho = .95$

Since this is an M/G/1 queue we can apply the following formulas:

$P_0 = 1-\rho = 1.-.95 = .05$

$L_q = \frac{\lambda^2\sigma^2+\rho^2}{2(1-\rho)} = \frac{(4.5475)^2+(.95)^2}{2 \times .05} = 215.82$

$L = \rho + L_q = 216.77$

$W_q = \frac{L_q}{\lambda} = 215.82$

$W = W_q + \frac{1}{\mu} = 216.77$

(c) W | major repair needed $= W_q + 5 = 220.82$

W | minor repair needed $= W_q + .5 = 216.32$

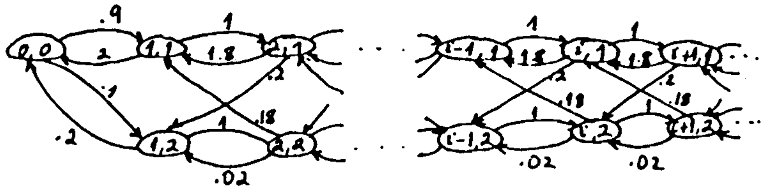
$L_{\text{major repair machines}} = (1)(0.1)(220.82) = 22.082$

$L_{\text{minor repair machines}} = (1)(0.9)(216.32) = 194.69$

(d) state is (n,i) where n is the number of failed machines and i is the type of repair being done on machine under repair ($i=1$ denotes minor repair and $i=2$ denotes major repair).

17.7-11 (CONT'D)

(e)



17.7-12 $\{X_n\}$ is an imbedded Markov chain with states.

And $X_{n+1} = \begin{cases} X_n - 1 + A_{n+1} & X_n \geq 1 \\ A_{n+1} & X_n = 0 \end{cases}$ and $X_{n+1} \leq 3$

A_{n+1} is the number of arrivals in the 10 minutes.

$Pr(A=n) = \frac{e^{-\lambda t}(\lambda t)^n}{n!} = a_n$, and $\lambda t = \frac{60}{50} \cdot \frac{10}{60} = 0.2$

So, the transition matrix is

$$P = \begin{bmatrix} a_0 & a_1 & a_2 & 1-a_0-a_1-a_2 \\ a_0 & a_1 & a_2 & 1-a_0-a_1-a_2 \\ 0 & a_0 & a_1 & 1-a_0-a_1 \\ 0 & 0 & a_0 & 1-a_0 \end{bmatrix}$$

$$= \begin{bmatrix} 0.819 & 0.164 & 0.016 & 0.001 \\ 0.819 & 0.164 & 0.016 & 0.001 \\ 0 & 0.819 & 0.164 & 0.017 \\ 0 & 0 & 0.819 & 0.181 \end{bmatrix}$$

b) Run OR courseware, we get

$p_0 = 0.801$, $p_1 = 0.177$, $p_2 = 0.02$, $p_3 = 0.002$

c) $L = p_1 + 2p_2 + 3p_3 = 0.223$

In M/D/1 model, $L^{\infty} = \rho + \frac{\rho^2}{2(1-\rho)} = 0.2 + \frac{0.04}{2(1-0.2)} = 0.225$

So, $L^{\infty} > L$

17.8-1

a) This system is an example of a nonpreemptive priority queueing system.

b)

n =	2	(# of priority classes)
μ =	20	(mean service rate)
s =	1	(# servers)

17.8-1 (CONT'D)

	λ_i	L	Lq	W	Wq
Priority Class 1	2	0.1666667	0.0666667	0.0833333	0.0333333
Priority Class 2	10	1.3333333	0.8333333	0.1333333	0.0833333
$\lambda =$	12				
$\rho =$	0.6				

c) $\dfrac{W_{q1}}{W_{q2}} = \dfrac{0.033}{0.083} = 0.4$

d) $\rho(12) = 0.6 = 7.2$ hours

17.8-2

	W_{q1}	L_{q1}	W_1	L_1	W_{q2}	L_{q2}	W_2	L_2
$s=1,\ \mu=10$	.133	.533	.233	.933	.667	2.667	.767	3.067
$s=2,\ \mu=5$	.119	.474	.319	1.274	.593	2.370	.793	3.170

If W_1 is the primary concern, one should choose the first alternative (one fast server). On the other hand, if W_{q1} is the primary concern, one should choose the second alternative (two slow servers).

17.8-3

a)

	u	a	b	W
0			1	
1	2.5	0.16	0.6	0.61
2	3.33	0.25	0.3	1.69
3	5	0.29	0.1	9.87

b)

u	3.33
r	0.30
A	4.44

	B	W
0	1	
1	0.7	0.62
2	0.4	1.10
3	0.1	5.93

Conclusion: The approximation is not good for W_2 and W_3.

17.8-4

$\lambda = 8$, $\lambda_1 = 2$, $\lambda_2 = 4$, $\lambda_3 = 2$, $\mu = 10$

(a) First come, first served : $W = \frac{1}{\mu-\lambda} = \frac{1}{2}$ days

(b) Nonpreemptive:

$A = \frac{\mu^2}{\lambda} = \frac{25}{2}$

$B_1 = 1 - (\lambda_1/\mu) = 4/5$

$B_2 = 1 - (\lambda_1+\lambda_2)/\mu = 2/5$

$B_3 = 1 - \lambda/\mu = 1/5$

So $W_1 = \frac{1}{AB_1} + \frac{1}{\mu} = \frac{1}{5} = .20$ days

$W_2 = \frac{1}{AB_1B_2} + \frac{1}{\mu} = \frac{7}{20} = .35$ days

$W_3 = \frac{1}{AB_2B_3} + \frac{1}{\mu} = \frac{11}{10} = 1.1$ days

(c) Preemptive:

$W_1 = \frac{1/\mu}{B_1} = \frac{1}{8} = .125$ days

$W_2 = \frac{1/\mu}{B_1B_2} = \frac{5}{16} = .3125$ days

$W_3 = \frac{1/\mu}{B_2B_3} = \frac{5}{4} = 1.25$ days

17.8-5

$\lambda_1 = 0.1,\ \lambda_2 = 0.4,\ \lambda_3 = 1.5,\quad \lambda = \sum_{i=1}^{3} \lambda_i = 2\ ,\ \mu = 3$

	Preemptive Priorities		Nonpreemptive Priorities	
	s=1	s=2	s=1	s=2
A	...	...	4.5	36
B_1	.967	...	.967	.983
B_2	.833	...	.833	.917
B_3	.333	...	.333	.667
$W_1 - \frac{1}{\mu}$	.011	.00009	.230	.028
$W_2 - \frac{1}{\mu}$	.080	.00289	.276	.031
$W_3 - \frac{1}{\mu}$	.867	.05493	.800	.045

17.8-6

a) The expected number of customers wouldn't change since customers of both types have exactly same arrival pattern and service times. The change of the prority wouldn't affect the total service rate from the server's view and thus the total queue size stays the same

b) Run OR courseware, for Nonpreemptive Prority-Disciplie Queueing model, we have

$\mu = 6$ $\quad \lambda_1 = 5$ $\quad L_1 = 1.37446$

$\lambda_2 = 5$ $\quad L_2 = 4.08009$

$s = 2$

$N = 2$ $\qquad L_p = L_1 + L_2 = 5.45455$

$W_1 = 0.27489 \quad (W_q)_1 = 0.10823$

$W_2 = 0.81602 \quad (W_q)_2 = 0.64935$

for M/M/2 queueing system,

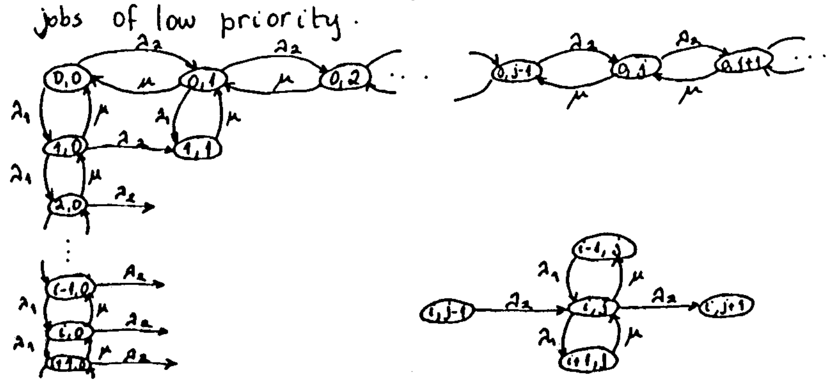
$\lambda = 10$ $\quad L = 5.455$

$\mu = 6$ $\quad L_q = 3.788$ $\qquad$ Thus, $L_p = L$.

$s = 2$ $\quad W = 0.545$

17.8-7

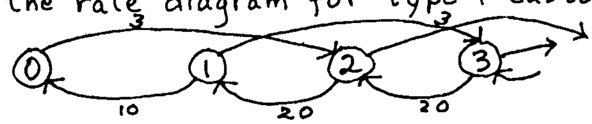
Let state (i,j) denote i jobs of high priority and j jobs of low priority.



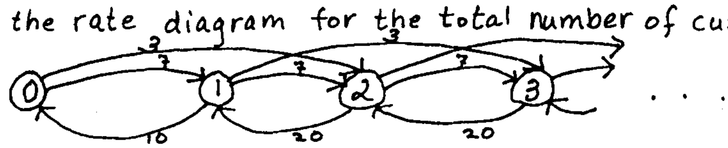
17.8-7 (CONT'D)

state	Rate in = Rate out
(0, 0)	$\mu(P_{1,0} + P_{0,1}) = (\lambda_1 + \lambda_2) P_{0,0}$
(i,0) for $i \geq 1$	$\mu P_{i+1,0} + \lambda_1 P_{i-1,0} = (\mu + \lambda_1 + \lambda_2) P_{i,0}$
(0,j) for $j \geq 1$	$\mu(P_{1,j} + P_{0,j+1}) + \lambda_2 P_{0,j-1} = (\mu + \lambda_1 + \lambda_2) P_{0,j}$
(i,j) for $i,j \geq 1$	$\mu P_{i+1,j} + \lambda_1 P_{i-1,j} + \lambda_2 P_{i,j-1} = (\mu + \lambda_1 + \lambda_2) P_{i,j}$

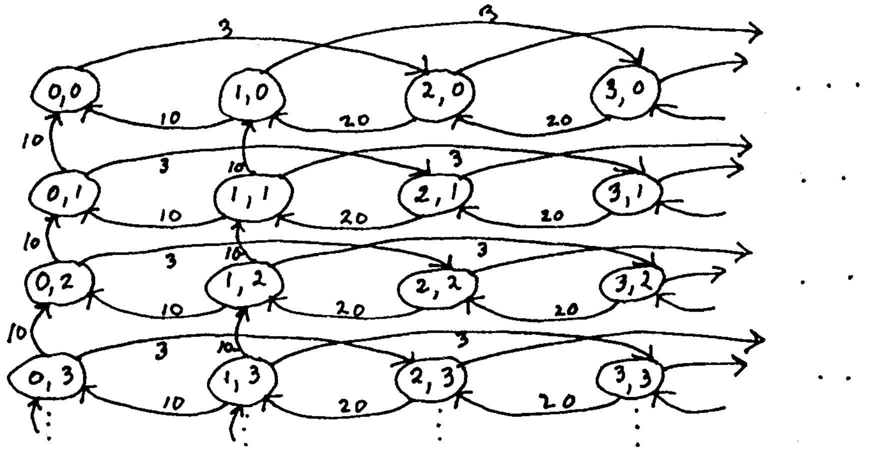
17.9-1. (a) Let the state be n_1 = number of type 1 customers in the system
Then the rate diagram for type 1 customers is:



(b) Let the state be n = number of customers in the system
Then the rate diagram for the total number of customers is:



(c) Let the state be (n_1, n_2) where
n_1 = number of type 1 customers in the system
n_2 = number of type 2 customers in the system
Then the rate diagram is:



17.9-2

(a) $P_{n_1} = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^{n_1}$

$P_{n_2} = \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^{n_2}$

$P\{(N_1, N_2) = (n_1, n_2)\} = P_{n_1} P_{n_2} = \left(\frac{1}{6}\right)\left(\frac{1}{2}\right)^{n_1}\left(\frac{2}{3}\right)^{n_2}$

(b) $P\{(N_1, N_2) = (0,0)\} = \frac{1}{6}$

(c) $L = L_1 + L_2 = 1 + 2 = 3$

$W = W_1 + W_2 = \frac{1}{10} + \frac{2}{10} = .3$ hour = 18 minutes

17.10-1 a) 1 server is optimal.

	B	C	D
3		**Data**	
4	λ =	8	(mean arrival rate)
5	μ =	10	(mean service rate)
6	s =	1	(# servers)
7			
8	Pr(W > t) =	0.90483742	
9	when t =	0.05	
10			
11	Prob(W_q > t) =	0.72386993	
12	when t =	0.05	
13			
14	**Economic Analysis:**		
15	Cs =	$100.00	(cost / server / unit time)
16	Cw =	$10.00	(waiting cost / unit time)
17			
18	Cost of Service	$100.00	
19	Cost of Waiting	$40.00	
20	Total Cost	$140.00	

b) 2 servers are optimal.

	B	C	D
3		**Data**	
4	λ =	8	(mean arrival rate)
5	μ =	10	(mean service rate)
6	s =	2	(# servers)
7			
8	Pr(W > t) =	0.67249526	
9	when t =	0.05	
10			
11	Prob(W_q > t) =	0.12544266	
12	when t =	0.05	
13			
14	**Economic Analysis:**		
15	Cs =	$100.00	(cost / server / unit time)
16	Cw =	$100.00	(waiting cost / unit time)
17			
18	Cost of Service	$200.00	
19	Cost of Waiting	$95.24	
20	Total Cost	$295.24	

c) 3 servers are optimal.

	B	C	D
3		**Data**	
4	λ =	8	(mean arrival rate)
5	μ =	10	(mean service rate)
6	s =	3	(# servers)
7			
8	Pr(W > t) =	0.61839666	
9	when t =	0.05	
10			
11	Prob(W_q > t) =	0.01732012	
12	when t =	0.05	
13			
14	**Economic Analysis:**		
15	Cs =	$10.00	(cost / server / unit time)
16	Cw =	$100.00	(waiting cost / unit time)
17			
18	Cost of Service	$30.00	
19	Cost of Waiting	$81.89	
20	Total Cost	$111.89	

17-10.2 Jim should operate 4 cash registers during the lunch hour.

	B	C	D	E	F	G
3		**Data**				**Results**
4	λ =	66	(mean arrival rate)		L =	2.477198599
5	μ =	30	(mean service rate)		L_q =	0.277198599
6	s =	4	(# servers)			
7					W =	0.037533312
8	Pr(W > t) =	0.26733457			W_q =	0.004199979
9	when t =	0.05				
10					ρ =	0.55
11	Prob(W_q > t) =	0.01524213				
12	when t =	0.05			n	P_n
13					0	0.104562001
14	**Economic Analysis:**				1	0.230036403
15	Cs =	$9.00	(cost / server / unit time)		2	0.253040043
16	Cw =	$18.00	(waiting cost / unit time)		3	0.185562698
17					4	0.102059484
18	Cost of Service	$36.00			5	0.056132716
19	Cost of Waiting	$44.59			6	0.030872994
20	Total Cost	$80.59			7	0.016980147

17.10-3 Garrett-Tompkins should have 6 copiers.

	B	C	D	E	F	G
3		**Data**				**Results**
4	$\lambda =$	30	(mean arrival rate)		L =	2.533889152
5	$\mu =$	12	(mean service rate)		$L_q =$	0.033889152
6	s =	6	(# servers)			
7					W =	0.084462972
8	Pr(W > t) =	0.55690297			$W_q =$	0.001129638
9	when t =	0.05				
10					$\rho =$	0.416666667
11	Prob(W_q > t) =	0.00580992				
12	when t =	0.05			n	P_n
13					0	0.081620259
14	**Economic Analysis:**				1	0.204050648
15	Cs =	$1.50	(cost / server / unit time)		2	0.25506331
16	Cw =	$25.00	(waiting cost / unit time)		3	0.212552759
17					4	0.132845474
18	Cost of Service	$9.00			5	0.066422737
19	Cost of Waiting	$63.35			6	0.02767614
20	Total Cost	$72.35			7	0.011531725

17.1 a) Status quo at the presses – 7.52 sheets of in-process inventory.

	A	B	C	D	E	G	H
1	**Template for the M/M/s Queueing Model**						
2							
3			**Data**				**Results**
4		$\lambda =$	7	(mean arrival rate)		L =	7.517372837
5		$\mu =$	1	(mean service rate)		L_q =	0.517372837
6		s =	10	(# servers)			

Status quo at the inspection station – 3.94 wing sections of in-process inventory.

	A	B	C	D	E	F	G
1	**Template for M/D/1 Queueing Model**						
2							
3			**Data**				**Results**
4		$\lambda =$	7	(mean arrival rate)		L =	3.9375
5		$\mu =$	8	(mean service rate)		L_q =	3.0625
6		s =	1	(# servers)			

Inventory cost = (7.52 + 3.94)($8/hour) = $91.68 / hour
Machine cost = (10)($7/hour) = $70 / hour
Inspector cost = $17 / hour

Total cost = $178.68 / hour

b) Proposal 1 will increase the in-process inventory at the presses to 11.05 sheets since the mean service rate has decreased.

	A	B	C	D	E	G	H
1	**Template for the M/M/s Queueing Model**						
2							
3			**Data**				**Results**
4		$\lambda =$	7	(mean arrival rate)		L =	11.04740664
5		$\mu =$	0.83333333	(mean service rate)		L_q =	2.647406638
6		s =	10	(# servers)			

The in-process inventory at the inspection station will not change.

Inventory cost = (11.05 + 3.94)($8/hour) = $119.92 / hour
Machine cost = (10)($6.50) = $65 / hour
Inspector cost = $17 / hour

Total cost = $201.92 / hour

This total cost is higher than for the status quo so should not be adopted. The main reason for the higher cost is that slowing down the machines won't change in-process inventory for the inspection station.

c) Proposal 2 will increase the in-process inventory at the inspection station to 4.15 wing sections since the variability of the service rate has increased.

	B	C	D	E	F	G
3		**Data**				**Results**
4	λ =	7	(mean arrival rate)		L =	4.1475
5	μ =	8.33333333	(mean service rate)		L_q =	3.3075
6	k =	2	(shape parameter)			
7	s =	1	(# servers)		W =	0.5925
8					W_q =	0.4725

The in-process inventory at the presses will not change.

Inventory cost = (7.52 + 4.15)($8/hour) = $93.36 / hour
Machine cost = (10)($7/hour) = $70 / hour
Inspector cost = $17 / hour

Total cost = $180.36 / hour

This total cost is higher than for the status quo so should not be adopted. The main reason for the higher cost is the increase in the service rate variability (Erlang rather than constant) and the resulting increase in the in-process inventory.

d) They should consider *increasing* power to the presses (increasing there cost to $7.50 per hour but reducing their average time to form a wing section to 0.8 hours). This would decrease the in-process inventory at the presses to 5.69.

	A	B	C	D	E	G	H
1	**Template for the M/M/s Queueing Model**						
2							
3			**Data**				**Results**
4		λ =	7	(mean arrival rate)		L =	5.688419945
5		μ =	1.25	(mean service rate)		L_q =	0.088419945
6		s =	10	(# servers)			

Inventory cost = (5.69 + 3.94)($8/hour) = $77.04 / hour
Machine cost = (10)($7.50/hour) = $75 / hour
Inspector cost = $17 / hour

Total cost = $169.04 / hour

This total cost is lower than the status quo and both proposals.

Case

17.2 The operations of the records and benefits call center can be modeled as an M/M/s queueing system. We, therefore, use the template for the M/M/s queueing model throughout this case. The mean arrival rate equals 70 per hour, and the mean service rate of every representative equals 6 per hour. Mark needs at least s = 12 representatives answering phone calls to ensure that the queue does not grow indefinitely.

a) In order to solve this problem we have to determine the number of servers by "trial and error" until we find a number s such that the probability of waiting more than 4 minutes in the queue is above 35%.

For 13 servers we obtain the following results:

Template for M/M/s Queueing Model

Data

l =	70	(mean arrival rate)
m =	6	(mean service rate)
s =	13	(# servers)

Pr(w>t) =	0.825608
when t =	0.066667

Prob(w_q>t) =	0.362914
when t =	0.066667

Results

L =	17.07963527
L_q =	5.4129686
W =	0.24399479
W_q =	0.077328123
r =	0.897435897
P_0 =	5.32592E-06
P_1 =	6.21358E-05
P_2 =	0.000362459
P_3 =	0.001409561
P_4 =	0.004111221
P_5 =	0.009592849
P_6 =	0.018652761
P_7 =	0.031087935
P_8 =	0.045336573
P_9 =	0.058769631
P_{10} =	0.06856457
P_{11} =	0.072719998
P_{12} =	0.070699998
P_{13} =	0.063448716
P_{14} =	0.056941166
P_{15} =	0.051101037
P_{16} =	0.045859905
P_{17} =	0.041156325
P_{18} =	0.036935163
P_{19} =	0.033146942
P_{20} =	0.029747255
P_{21} =	0.026696255
P_{22} =	0.023958177
P_{23} =	0.021500928
P_{24} =	0.019295705
P_{25} =	0.017316658

For 13 servers, the probability that a customer has to wait more than 4 minutes equals 36.3%.

If there are 12 servers, this probability would be 78%:

Template for M/M/s Queueing Model

	Data	
l =	70	(mean arrival rate)
m =	6	(mean service rate)
s =	12	(# servers)

Pr(w>t) =	0.944173
when t =	0.066667

Prob(w_q>t) =	0.779968
when t =	0.066667

If there are 14 servers, this probability would be less than 16.4%:

Template for M/M/s Queueing Model

	Data	
l =	70	(mean arrival rate)
m =	6	(mean service rate)
s =	14	(# servers)

Pr(w>t) =	0.75683
when t =	0.066667

Prob(w_q>t) =	0.163704
when t =	0.066667

It appears that Mark currently employs 13 servers.

b) Using the same procedure as in part (a) we find that for s = 18 servers the probability of waiting more than 1 minute drops below 5%:

Template for M/M/s Queueing Model

Data

l =	70	(mean arrival rate)
m =	6	(mean service rate)
s =	18	(# servers)

Pr(w>t) =	0.909075
when t =	0.016667

Prob(w_q>t) =	0.032078
when t =	0.016667

Results

L =	11.77798802
L_q =	0.111321353
W =	0.168256972
W_q =	0.001590305
r =	0.648148148
P_0 =	8.49029E-06
P_1 =	9.90534E-05
P_2 =	0.000577812
P_3 =	0.002247045
P_4 =	0.006553882
P_5 =	0.015292391
P_6 =	0.029735204
P_7 =	0.049558673
P_8 =	0.072273065
P_9 =	0.093687307
P_{10} =	0.109301858
P_{11} =	0.115926213
P_{12} =	0.11270604
P_{13} =	0.101146446
P_{14} =	0.084288705
P_{15} =	0.065557882
P_{16} =	0.047802622
P_{17} =	0.032805721
P_{18} =	0.021262967
P_{19} =	0.013781553
P_{20} =	0.008932488
P_{21} =	0.005789576
P_{22} =	0.003752503
P_{23} =	0.002432178
P_{24} =	0.001576411
P_{25} =	0.001021748

c) Using the same "trial and error" method as before, we find the minimal number of servers necessary to ensure that 80% of customers wait one minute or less to be s = 15

Template for M/M/s Queueing Model

	Data	
l =	70	(mean arrival rate)
m =	6	(mean service rate)
s =	15	(# servers)

Pr(w>t) =	0.926712
when t =	0.016667

Prob(w_q>t) =	0.194213
when t =	0.016667

The minimal number of servers to ensure that 95% of customers wait 90 seconds or less is s = 17.

Template for M/M/s Queueing Model

	Data	
l =	70	(mean arrival rate)
m =	6	(mean service rate)
s =	17	(# servers)

Pr(w>t) =	0.870524
when t =	0.025

Prob(w_q>t) =	0.046459
when t =	0.025

When an employee of Cutting Edge calls the benefits center from work and has to wait on the phone, the company loses valuable work time for this customer. Mark should try to estimate the amount of work time employees lose when they have to wait on the phone. Then he could determine the cost of this waiting time and try to choose the number of representatives in such a fashion that he reaches a reasonable trade-off between the cost of employees waiting on the phone and the cost of adding new representatives.

Clearly, Mark's criteria would be different if he were dealing with external customers. While the internal customers might become disgruntled when they have to wait on the phone, they cannot call somewhere else. Effectively, the benefits center holds monopolistic power. On the contrary, if Mark were running a call center dealing with external customers, these customers could decide to do business with a competitor if they become angry from waiting on the phone.

d) If the representatives can only handle 6 calls per hour, then Mark needs to employ 18 representatives (see part b). If a representative can handle 8 calls per hour, then the minimal number of representatives equals 14:

Template for M/M/s Queueing Model

	Data	
l =	70	(mean arrival rate)
m =	8	(mean service rate)
s =	14	(# servers)

Pr(w>t) =	0.881748
when t =	0.016667

Prob(w_q>t) =	0.036649
when t =	0.016667

The cost of training 14 employees equals 14*$2500 = $35000 and saves Mark 4*$30000 = $120000 in annual salary. In the first year alone Mark would save $85000 if he chose to train all his employees so that they can handle 8 instead of 6 phone calls per hour.

e) Mark needs to carefully check the number of calls arriving at the call center per hour. In this case we have made the simplifying assumption that the arrival rate is constant. That assumption is unrealistic; clearly we would expect more calls during certain times of the day, during certain days of the week, and during certain weeks of the year. We might want to collect data on the number of calls received depending on the time. This data could then be used to forecast the number of calls the center will receive in the near future, which in turn would help to forecast the number of representatives needed.

Also, Mark should carefully check the number of phone calls a representative can answer per hour. Clearly, the length of a call will depend on the issue the caller wants to discuss. We might want to consider training representatives for special issues. These representatives could then always answer those particular calls. Using specialized representatives might increase the number of phone calls the entire center can handle.

Finally, using an M/M/s model is clearly a great simplification. We need to evaluate whether the assumptions for an M/M/s model are at least approximately satisfied. If this is not the case, we should consider more general models such as M/G/s or G/G/s.