The parachute paradox

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If a freely falling parachutist opens the chute at the local terminal velocity, will the deceleration be larger higher up or lower down? Naïvely (the stuff of paradoxa) one might say "lower down" because of the higher air density there. One soon notices that this typical student tutorial problem on falling bodies requires a second and even a third glance.

The obvious equation of motion is $m_p\dot{v} = m_p g - Fv^2$ with the last term, the viscous drag ($F = 1/2\ C_d \pi r^2\ \rho$, with C_d the drag coefficient, equal to unity for a hemispherical parachute; r, the radius of the parachute; ρ, the local density of the air and v, the parachutist's velocity. The v-squared power law means turbulent flow is assumed). If the parachute opens instantaneously[1] after reaching the local terminal velocity v_i (the initial velocity for the deceleration problem) the dynamic pressure ρv^2, thus the right-hand side—and thus the deceleration is independent of the density, and thus, of the altitude.[2] It is interesting to integrate the equation of motion to calculate the distance z for the opened chute to reach a velocity v. This is

$$z = \frac{1}{2gl^2} \ln\left[\frac{l^2 v_1^2 - 1}{l^2 v^2 - 1}\right], \tag{1}$$

$1/l = \sqrt{(m_p g/F)}$, the terminal velocity of the opened parachute.[3] If we assume the final velocity to be a fixed fraction of this terminal velocity, say 1.1, then the term in the logarithm contains ρ and v_i only in the combination ρv_i^2 so that it does not change with altitude. The distance z is thus inversely proportional to the density and, interestingly, independent of g. Is the above equation of motion really correct?

For when a body is accelerated in a fluid, work must be done against the inertia of the body; the viscosity of the fluid and, in addition, against the inertia of the fluid. Nearby fluid will be more accelerated than distant fluid, and one can imagine the entire change of fluid momentum to take place in an equivalent finite mass—the added mass of the fluid.[4] This mass of fluid associated with the accelerating body has to be added to the body mass in the equation of motion. For a hemispherical parachute the added mass is equal to the mass of air displaced by the corresponding spheres:[5] $\rho 4/3 \pi r^3$, which, for a 4 m (radius) parachute at sea level ($\rho_s = 1.2$ kg/m^3) is 322 kg. Assuming the mass of a parachutist to be 90 kg and that of the parachute, 10 kg, i.e., the combined parachutist mass $m_p = 100$ kg, the added mass is more than three times the mass of the parachutist! What happens if we include the added mass?

Now the equation of motion reads $(m_p + m_a)\dot{v} = m_p g - Fv^2$ and the initial deceleration $\dot{v}_i$ is

$$\dot{v}_i = \frac{m_p g - F v_i^2}{m_p + m_a} = \frac{1 - l^2 v_i^2}{1 + m_a/m_p}.$$

This acceleration is inversely proportional to the sum of the masses, 422 kg at sea level and, since the density of air at 12 000 m is around 1/6 of that at sea level, only 54+100 =154 kg, at 12 000 m. The deceleration at sea level is virtually a third that at 12 000 m and a quarter that not taking the added mass into consideration (24 g at sea level; 65 g at 12 000 m—c.f. 99 g not including the added mass—for a parachutist with an effective radius of 0.4 m). The distance z for the parachute to decelerate is now a factor $1 + m_a/m_p$ greater than that in Eq. (1). We would thus make a nasty error by not incorporating the added mass in estimating the braking distance, for example, to calculate the latest moment at which the parachute should be opened for a safe landing: The value obtained from Eq. (1) is 10 m compared to 43 m with the above factor, a fatal 33 m!

[1] And if ρ does not change significantly over small distances. For problems in these assumptions see, e.g., C. K. Lee, "Modeling of parachute opening: An experimental investigation," J. Aircraft **26**, 444–451 (1989).

[2] A parachutist falling from high up has a better feel for the air pressure than the snatch velocity v_i (at which the parachute opens and begins decelerating—an action which takes little time), which will be smaller lower down ($\propto 1/\sqrt{\rho}$).

[3] There are two length scales in the problem, $1/(gl^2)$—(twice) the distance taken for the parachute falling in a vacuum from rest to reach the terminal velocity $1/l$—and the parachute diameter. Their quotient (formally the Froude number—the ratio of inertial to gravitational forces—known to ship designers) is one of the important scaling parameters in parachute dynamics.

[4] The added mass idea seems to have been born in Bessel's work on corrections to pendulum clocks (F. W. Bessell, see Ref. 5). Although the parachute is perhaps the most extreme example of its importance, F. S. Crawford ["Acceleration of a buoyant sphere in dry water," Am. J. Phys. **54**, 584 (1986)] showed that its inclusion for the initial acceleration of a submerged sphere removed the singularity. It is also important for (classroom) balloon experiments [K. Thompson, "Hydrodynamic mass," Am. J. Phys. **56**, 1043 (1988) and J. Bisquert, P. Ramirez, A. J. Barbero, and S. Mafe," A classroom demonstration on air drag forces," Eur. J. Phys. **12**, 249–252 (1991)]. It is of practical importance, for example, in understanding the flapping flight of birds and insects; the paddling (e.g., fish and whale fins) and whipping swimming motion (e.g., water beetles) as well as periodic flow situations such as blood circulation.

[5] D. J. Cockrell, "The aerodynamics of parachutes," Agardograph No. 295, Agard, Essex, U.K. 1987.