Synchrotron Radiation
Electromagnetism II

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1 Introduction

Synchrotron radiation is the radiation emitted by charged particles when they are accelerated radially to relativistic velocities. The non-relativistic counterpart is called cyclotron radiation.

In order to radially accelerate the particles, it is needed a magnetic field perpendicular to the particles' velocity, which can be produced inside accelerators so called "synchrotrons", or in naturally occurring magnetic fields, for example in the intragalactic medium.

In the astronomical context, this kind of radiation was first detected in 1965, from a jet emitted by Messier 87. Synchrotron radiation is also found in sunspots and in the particle radiation belts in Jupiter. Another important case of synchrotron radiation emission are the pulsar wind nebulae, clouds inside shells of supernova remnants. The blue glow in the center of the Crab nebula is an example.

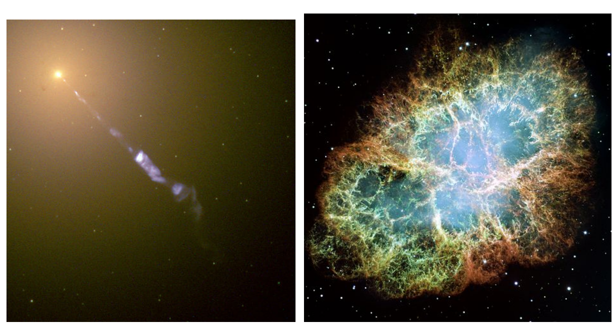

Figure 1: Left: Messier 87's synchrotron radiation. Right: Crab Nebula.

2 Theory

2.1 Liénard-Wiechert potential

Consider the Maxwell equations in the Lorenz gauge ($\partial_\mu A^\mu = 0$)

$$\partial_\mu F^{\mu\nu} = \partial_\mu \partial^\mu A^\nu = \frac{4\pi}{c} J^\nu \tag{1}$$

where $F^{\mu\nu}$ is the electromagnetic tensor, J^ν is the current 4-vector and A^ν is the 4-potential.

In the case of a charged particle in motion, the current is given by

$$J^\nu(x) = ec \int d\tau V^\nu(\tau)\delta(x - r(\tau))$$

where $V^\nu(\tau)$ is the charge's 4-velocity at time τ, and $r^\mu(\tau)$ it's position. The solution to (1) is given by

$$A^\mu(x) = \frac{4\pi}{c} \int d^4x' G(x - x') J^\mu(x')$$

where G is the retarded Green function, defined by $\partial_\mu \partial^\mu G(x, x') = \delta(x - x')$, whose solution is

$$G(x - x') = \frac{1}{2\pi} \theta(x^0 - x'^0)\delta[(x - x')^2]$$

That is, G is only non-null in the past light cone of x, defined by the set of points x' such that $c(x^0 - x'^0) = |\mathbf{x} - \mathbf{x}'|$ and $x^0 > x'^0$. Thus, plugging in the Green function we find the solution

$$A^\mu(x) = 2e \int d\tau V^\mu(\tau)\theta(x^0 - r^0(\tau))\delta([x - r(\tau)]^2)$$

But the trajectory of the particle only intersects the past light cone at one point, which we call $r(\tau_0)$. Using the rule

$$\delta(f(x)) = \sum_i \frac{\delta(x - x_i)}{|f'(x_i)|}$$

where x_i are the roots of $f(x)$, and setting $f(\tau) = (x - r(\tau))^2$, we get

$$\delta([x - r(\tau)]^2) = \frac{\delta(\tau - \tau_0)}{2(x - r(\tau_0))_\mu V^\mu(\tau_0)}$$

And thus, the **Liénard-Wiechert potentials** for a moving charge are given by

$$A^\mu(x) = \left. \frac{eV^\mu(\tau)}{V^\nu[x - r(\tau)]_\nu} \right|_{\tau_0}$$

As $V_\mu = \frac{dr_\mu}{d\tau} = \gamma(c, \mathbf{v})$, if we set $c(x^0 - r^0(\tau_0)) = |\mathbf{x} - \mathbf{r}(\tau_0)| = R$, we can write

$$V^\nu (x - r(\tau))_\nu = \gamma(c, -\mathbf{v}) \cdot (R, R\mathbf{n}) = cR(1 - \beta \cdot \mathbf{n})$$

where $\mathbf{n}$ is an unit vector in the direction of $\mathbf{x} - \mathbf{r}(\tau)$ and $\beta = \mathbf{v}(\tau)/c$.

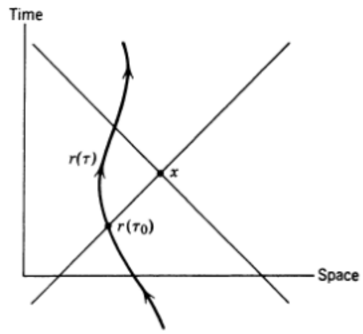

Figure 2: The potential at x is affected by the position of the particle at the "retarded" time τ_0. We defined $R = c(x^0 - r^0(\tau_0)) = |\mathbf{x} - \mathbf{r}(\tau_\mathbf{0})|$

Now these potentials take the more familiar form

$$\phi(\mathbf{x}, t) = \left. \frac{e}{(1 - \beta \cdot \mathbf{n})R} \right|_{\tau_0}$$

$$\mathbf{A}(\mathbf{x}, t) = \left. \frac{e\beta}{(1 - \beta \cdot \mathbf{n})R} \right|_{\tau_0}$$

From this potentials, we can get the fields:

$$\mathbf{E}(\mathbf{x}, t) = e \left[\frac{\mathbf{n} - \beta}{\gamma^2 (1 - \beta \cdot \mathbf{n})^3 R^2} \right]_{\tau_0} + \frac{e}{c} \left[\frac{\mathbf{n} \times \{(\mathbf{n} - \beta) \times \dot{\beta}\}}{(1 - \beta \cdot \mathbf{n})^3 R} \right]_{\tau_0} \tag{2}$$

$$\mathbf{B} = [\mathbf{n} \times \mathbf{E}]_{\tau_0}$$

2.2 Accelerated charges

Let's consider a situation in which the particle is accelerated but observed in a reference frame in which its velocity is small compared to c. Since the first term in (2) falls with R^2, we retain only the second term, in the form

$$\mathbf{E} = \frac{e}{c} \left[\frac{n \times (n \times \dot{\beta})}{R} \right]_{\tau_0}$$

The associated Poynting vector is

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E}\times\mathbf{B} = \frac{c}{4\pi}|\mathbf{E}|^2\mathbf{n}$$

Therefore, the power radiated per unit solid angle, by a particle at position $r(\tau_0)$, through a sphere of radius R is

$$\frac{dP}{d\Omega} = \frac{dP}{dA}R^2 = \mathbf{S}\cdot\mathbf{n}R^2 = \frac{e^2}{4\pi c}|\mathbf{n}\times(\mathbf{n}\times\dot{\beta})|^2$$

If the angle between $\dot{\beta}$ and $\mathbf{n}$ is θ,

$$\frac{dP}{d\Omega} = \frac{e^2}{4\pi c^2}|\mathbf{n}\times\dot{\beta}|^2 = \frac{e^2\dot{\beta}^2}{4\pi c}\sin^2\theta$$

which corresponds to a "donut-shaped" power angular distribution, as seen in figure 3.

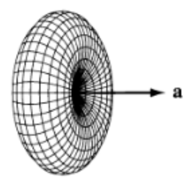

Figure 3: Power per solid angle angular distribution in the non-relativistic picture

Integrating, we get **Larmor's formula** (for an accelerated charge in non-relativistic motion)

$$P = \frac{2}{3}\frac{e^2}{c}\dot{\beta}^2$$

Now consider a particle in relativistic motion. From (2), we see that

$$\mathbf{S}\cdot\mathbf{n} = \frac{e^2}{4\pi c^2}\frac{1}{R^2}\left[\frac{\mathbf{n}\times\{(\mathbf{n}-\beta)\times\dot{\beta}\}}{(1-\beta\cdot\mathbf{n})^3}\right]^2$$

This is the energy per unit time per unit area detected at time $x^0 = t$ emitted by the charge at time $r_0(\tau_0) = t' = t - R/c$. We can transform this to the energy per unit *retarded time* per unit area, that is, the emission rate instead of detection rate:

$$\frac{dE}{dAdt'} = \frac{dE}{dAdt}\frac{dt}{dt'}$$

So using that

$$\frac{dt}{dt'} = 1 + \frac{1}{c}\frac{dR}{dt'} = 1 + \frac{1}{c}\frac{d\sqrt{\mathbf{R}\cdot\mathbf{R}}}{dt'} = 1 - \beta\cdot\mathbf{n}$$

we get

$$\frac{dP(t')}{d\Omega} = R^2 \mathbf{S} \cdot \mathbf{n} \frac{dt}{dt'} = \mathbf{S} \cdot \mathbf{n}(1 - \beta \cdot \mathbf{n})$$

$$\frac{dP(t')}{d\Omega} = \frac{e^2}{4\pi c^2} \frac{|\mathbf{n} \times \{(\mathbf{n} - \beta) \times \dot{\beta}\}|^2}{(1 - \beta \cdot \mathbf{n})^5} \tag{3}$$

In the case of linear motion ($\beta \| \dot{\beta}$), we get

$$\frac{dP}{d\Omega}(t') = \frac{e^2 \dot{\beta}^2}{4\pi c} \frac{\sin^2\theta}{(1 - \beta\cos\theta)^5} \tag{4}$$

that is, the angular distribution of the power is tipped towards the direction of motion, as indicated in figure 4.

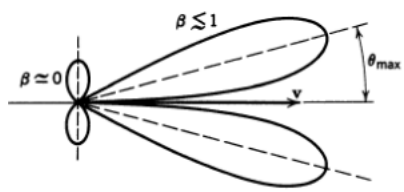

Figure 4: Power per solid angle angular distribution for $\beta \| \dot{\beta}$

The total radiated power, given by integration of (4), is

$$P(t') = \frac{2}{3} \frac{e^2}{c} \dot{\beta}^2 \gamma^6 \tag{5}$$

In the case of instantaneously circular motion ($\beta \perp \dot{\beta}$), defining the angles ϕ and θ as in figure 5 the angular power distribution is

$$\frac{dP}{d\Omega}(t') = \frac{e^2}{4\pi c} \frac{\dot{\beta}^2}{(1 - \beta\cos\theta)^3} \left[1 - \frac{\sin^2\theta\cos^2\phi}{\gamma^2(1 - \beta\cos\theta)^2}\right] \tag{6}$$

Integration of (6) yields the total radiated power

$$P(t') = \frac{2}{3} \frac{e^2}{c} \dot{\beta}^2 \gamma^4 \tag{7}$$

2.3 Frequency spectrum

As seen in the beggining of the previous section, the radiated power is given by

$$\frac{dP}{d\Omega}(t) = |\mathbf{A}|^2$$

where

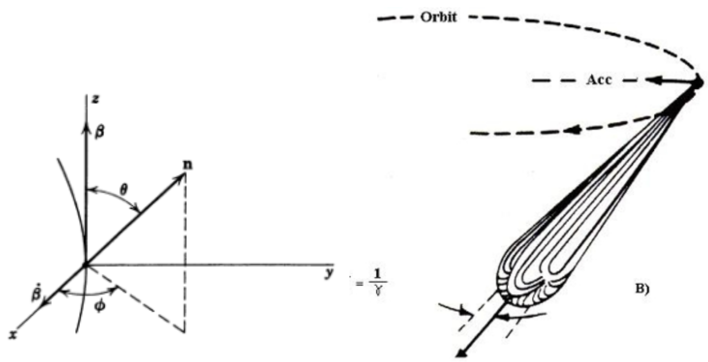

Figure 5: Left: Coordinates θ, ϕ. Right: Power per solid angle distribution for $\beta \perp \dot{\beta}$.

$$\mathbf{A}(t) = \sqrt{\frac{c}{4\pi}}[R\mathbf{E}]_{t_{\mathrm{ret}}}$$

and $\mathbf{E}$ is given by (2). The Fourier transform of $\mathbf{A}(t)$ is

$$\mathbf{A}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathbf{A}(t) e^{i\omega t} dt$$

So the energy radiated per unit solid angle can be written as

$$\frac{dE}{d\Omega} = \frac{1}{2\pi} \int dt d\omega d\omega' \mathbf{A}(\omega')^* \cdot \mathbf{A}(\omega) e^{i(\omega' - \omega)t} = \int_{-\infty}^{\infty} |\mathbf{A}(\omega)|^2 d\omega$$

So if we define the energy radiated per unit solid angle per unit frequency interval by

$$\frac{dE}{d\Omega} = \int_0^{\infty} \frac{d^2 I(\omega, \mathbf{n})}{d\omega d\Omega} d\omega$$

we find

$$\frac{d^2 I}{d\omega d\Omega} = |\mathbf{A}(\omega)|^2 + |\mathbf{A}(-\omega)|^2 = 2|\mathbf{A}(\omega)|^2$$

and plugging in the field expression, after some algebra (we refer to page 675 of [1]):

$$\frac{d^2 I}{d\omega d\Omega} = \frac{e^2 \omega^2}{4\pi^2 c} \left| \int_{-\infty}^{\infty} \mathbf{n} \times (\mathbf{n} \times \beta) exp(i\omega(t - \mathbf{n} \cdot \mathbf{r}(t)/c)) dt \right|$$

We have seen that the radiation emitted by a charged particle in circular motion is concentrated in a narrow cone in the direction of the tangent of the

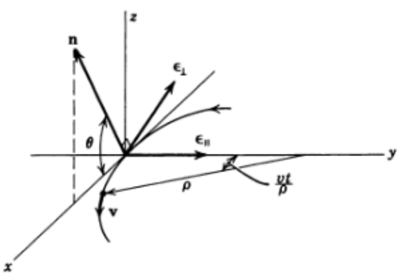

Figure 6: The instantaneous radius of curvature of the trajectory is ρ and θ is the angle between $\mathbf{n}$ and the x axis.

velocity vector (that is, tangent to the trajectory). To calculate this integral, we make use of the coordinate system displayed in figure 6, so that

$$\mathbf{n} \times (\mathbf{n} \times \beta) = \beta \left(-\boldsymbol{\epsilon}_\parallel \sin\left(\frac{vt}{\rho}\right) + \boldsymbol{\epsilon}_\perp \cos\left(\frac{vt}{\rho}\right) \sin\theta \right)$$

Making the approximations $\sin(vt/\rho) = vt/\rho - v^3t^3/6\rho^3$, $\cos\theta = 1 - \theta^2/2$ and $\beta \sim 1$

$$\omega(t - \mathbf{n}\cdot\mathbf{r}(t)/c) = \omega\left[t - \frac{\rho}{c}\sin\left(\frac{vt}{\rho}\right)\cos\theta\right] \simeq \frac{\omega}{2}\left[\left(\frac{1}{\gamma^2}+\theta^2\right)t + \frac{c^2}{3\rho^2}t^3\right]$$

And therefore

$$\frac{d^2I}{d\omega d\Omega} = \frac{e^2\omega^2}{4\pi^2 c}\left|\int_{-\infty}^{\infty}\left(-\boldsymbol{\epsilon}_\parallel\frac{ct}{\rho} + \boldsymbol{\epsilon}_\perp\theta\right) exp\left(i\frac{\omega}{2}\left[\left(\frac{1}{\gamma^2}+\theta^2\right)t + \frac{c^2}{3\rho^2}t^3\right]\right)dt\right|$$

Integration results in

$$\frac{d^2I}{d\omega d\Omega} = \frac{e^2\omega^2\rho^2}{3\pi^2c^3}\left(\frac{1}{\gamma^2}+\theta^2\right)^{1/2}\left[K_{2/3}^2(\xi) + \frac{\theta^2}{(1/\gamma^2)+\theta^2}K_{1/3}^2(\xi)\right] \tag{8}$$

where K are modified Bessel functions of the second kind.

$$K_{2/3}(\xi) = \sqrt{3}\int_0^\infty x\sin\left[\frac{3}{2}\xi\left(x+\frac{1}{3}x^3\right)\right]dx$$

$$K_{1/3}(\xi) = \sqrt{3}\int_0^\infty \cos\left[\frac{3}{2}\xi\left(x+\frac{1}{3}x^3\right)\right]dx$$

$$\xi = \frac{\omega\rho}{3c}\left(\frac{1}{\gamma^2}+\theta^2\right)^{3/2}$$

In (8), the first term (associated to $\boldsymbol{\epsilon}_{\|}$) corresponds to the energy of radiation polarized in the plane of the circular motion, while the second term (associated to $\boldsymbol{\epsilon}_{\perp}$) corresponds to radiation polarized perpendicular to that plane. Integrating over frequency, we find

$$\frac{dE}{d\Omega} = \int_0^\infty \frac{d^2 I}{d\omega d\Omega} d\omega = \frac{7e^2}{16\rho} \frac{1}{(1/\gamma^2 + \theta^2)^{5/2}} \left[1 + \frac{5}{7} \frac{\theta^2}{1/\gamma^2 + \theta^2} \right]$$

with a behavior similar to that of (6). Again, the first term corresponds to parallel polarization and the second to perpendicular. We see that the radiation is strongly polarized in the plane of motion.

Furthermore, (8) tells us that the intensity will be negligible for $\xi \gg 1$, which corresponds to larger angles θ. Hence, the radiation is confined to near the plane of motion.

On the other hand, ξ also becomes large as ω grows, independently of θ. We define the critical frequency, beyond which there is negligible radiation, by $\xi = 1/2$ at $\theta = 0$, so that

$$\omega_c = \frac{3}{2}\gamma^3 \left(\frac{c}{\rho} \right) \tag{9}$$

Integrating (8) over all solid angle, we get the frequency spectrum:

$$J(\omega) = \frac{dI}{d\omega} = \sqrt{3}\frac{e^2\gamma}{c}\frac{\omega}{\omega_c}\int_{\omega/\omega_c}^\infty K_{5/3}(x)dx \tag{10}$$

which is plotted in linear and logarithmic scale in figure 7.

In the following section we will discuss certain physical applications where the electrons are accelerated by a magnetic field $\mathbf{B}$. The electrons move with a velocity $\mathbf{v}$ and an angle α between $\mathbf{v}$ and $\mathbf{B}$. In the laboratory frame of reference, its motion is given by

$$\frac{d}{dt}(\gamma m_e \mathbf{v}) = q(\mathbf{v} \times \mathbf{B}) \rightarrow \gamma m_e a = qvB \sin\alpha$$

Equating to the centripetal acceleration (for an instantaneous circular motion with radius r) yields

$$\frac{v_\perp^2}{r} = \frac{qvB\sin\alpha}{\gamma m_e} = \frac{qBv_\perp}{\gamma m_e} \tag{11}$$

Thus, the angular velocity is given by

$$\omega_r = \frac{v_\perp}{r} = \frac{qB}{\gamma m_e} = \frac{\omega_g}{\gamma}$$

where ω_g is the non-relativistic rotation frequency. If we substitute $\rho = v/\omega_r = v_\perp/(\omega_r \sin\alpha)$ in (9)

$$\omega_c = \frac{3}{2}\gamma^3 \frac{c}{v_\perp}\omega_r \sin\alpha = \frac{3}{2}\gamma^2 \left(\frac{c}{v_\perp} \right) \omega_g \sin\alpha$$

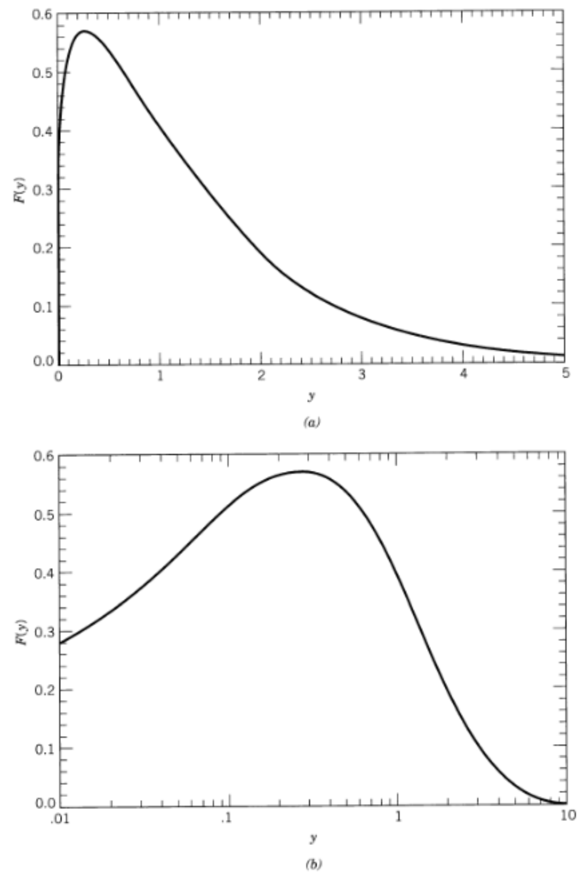

Figure 7: Synchrotron radiation frequency spectrum in (a) linear and (b) logarithmic scale

3 Power-law distribution of electron energies

In astrophysical applications, we must consider the presence of multiple electrons with a range of energies. In this section we will discuss the synchrotron radiation emitted by a power-law distribution of electron energies, i.e.

$$N(E)dE = \kappa E^{-p} dE$$

where $N(E)dE$ is the number density of electrons in the energy range E to $E + dE$.

This functional form is chosen since the energy spectra in many astrophysical situations can be approximated by a power-law, such as cosmic-ray electrons.

As discussed in the previous section, the intensity of the radiation declines rapidly for frequencies far from ω_c. Therefore, we make the crude approximation that the radiation only occurs at ω_c, which results in

$$\nu \approx \nu_c \approx \gamma^2 \nu_g = \left(\frac{E}{m_e c^2}\right)^2 \left(\frac{eB}{2\pi m_e}\right), \tag{12}$$

where we used the approximations made in the previous section and $\nu = \omega/2\pi$. Hence, the energy radiated in frequencies between ν and $\nu + d\nu$ correspond to the electrons with energies between E and $E + dE$ and the frequency spectrum is

$$J(\nu)d\nu = \left(-\frac{dE}{dt}\right) N(E)dE$$

We also have that

$$E = \gamma m_e c^2 = \sqrt{\nu \frac{2\pi m_e}{eB}}\ m_e c^2 \propto \nu^{1/2} B^{-1/2},$$

where, in the second equality, we used expression 12. Furthermore,

$$dE = \frac{m_e c^2}{2} \sqrt{\nu \frac{2\pi m_e}{eB}}\ d\nu \propto \nu^{1/2} B^{-1/2}$$

Then, using $N(E) = \kappa E^{-p}$ and that the power radiated by each electron is given by (7), substituting the centripetal acceleration (11) and using $\gamma^2 \sim \nu/\nu_g \sim \nu/B$

$$-\frac{dE}{dt} = -P = \frac{2}{3}\frac{e^2}{c}\gamma^4 \dot{\beta}^2 = \frac{2}{3}\frac{e^2}{c}\gamma^4 \frac{e^2 \beta^2 B^2 \sin^2\alpha}{\gamma^2 m_e^2} \propto \nu B$$

Therefore, gathering everything

$$J(\nu) \propto \kappa B^{(p+1)/2} \nu^{-(p-1)/2}$$

The exponent a in $J(\nu) \propto \nu^{-a}$ is called **spectral index**. The full expression for $J(\nu)$ (we refer to (8.130) of [2]) is

$$J(\nu) = 2.344 \times 10^{-25} a(p) B^{(p+1)/2} \kappa \left(\frac{1.253 \times 10^{37}}{\nu}\right)^{(p-1)/2} \quad Wm^{-3}Hz^{-1} \tag{13}$$

where $a(p)$ is a constant that depends on p.

4 Astrophysical Sources

As an application of the ideas discussed above, we consider the radio emission of the Galaxy. Figure 8 exhibits the map of radio emission of the Galaxy at a frequency of 408 MHz, which is dominated by synchrotron radiation.

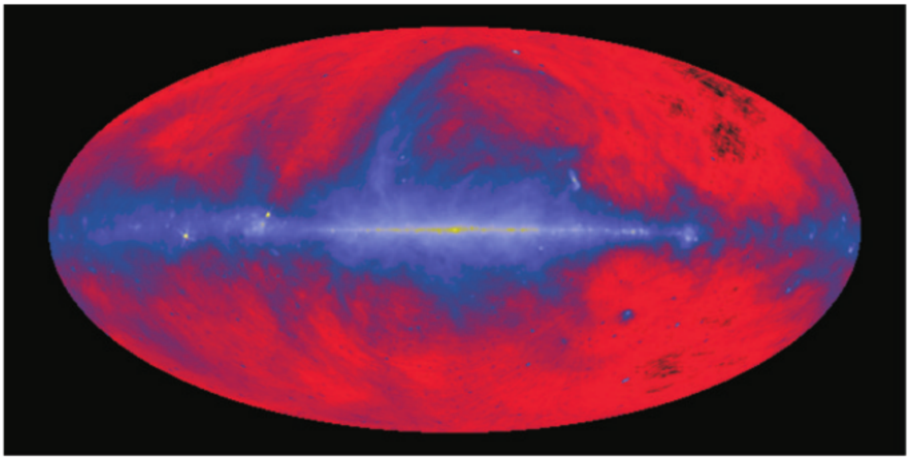

Figure 8: Radio emission map at 408MHz

Figure 9(top) shows the spectrum of radio emission in the direction of the anticentre (that is, outwards from the center of the galaxy), for lower (I, close to the North Galactic Pole) and higher (II, close to the interarm region) latitudes.

For $\nu < 200MHz$, the spectrum is well described by a power law $J(\nu) \sim \nu^{-0.4}$, and for $\nu > 400MHz$, the spectral index is about 0.8-0.9.

For cosmic ray electrons at energies greater than 10 GeV, we assume a distribution ([2])

$$N(E)dE = 2.9 \times 10^{-5} E^{-3.3} dE \text{ electrons} m^{-3}$$

Electrons with energy $E = \gamma m_e c^2$ will radiate most of their energy at frequency

$$\nu \sim \gamma^2 \nu_g = \gamma^2 \frac{qB}{2\pi m_e} \sim 28\gamma^2 B \ GHz$$

where B is measured in Tesla.

Now we calculate the spectrum using (13), $\kappa = 2.9 \times 10^{-5}$ and $p = 3.3$ (for which $a(p) = 0.238$. The result is compared to the measured spectrum in figure 9(bottom).

We see that for $B = 0.6\ nT$, the observed and predicted spectrum join smoothly. However, in general it is assumed that the magnetic field is about $0.15 - 0.3\ nT$. This discrepancy may arise for different reasons, for example, the Earth may be in a low electron density region relative to the interstellar medium.

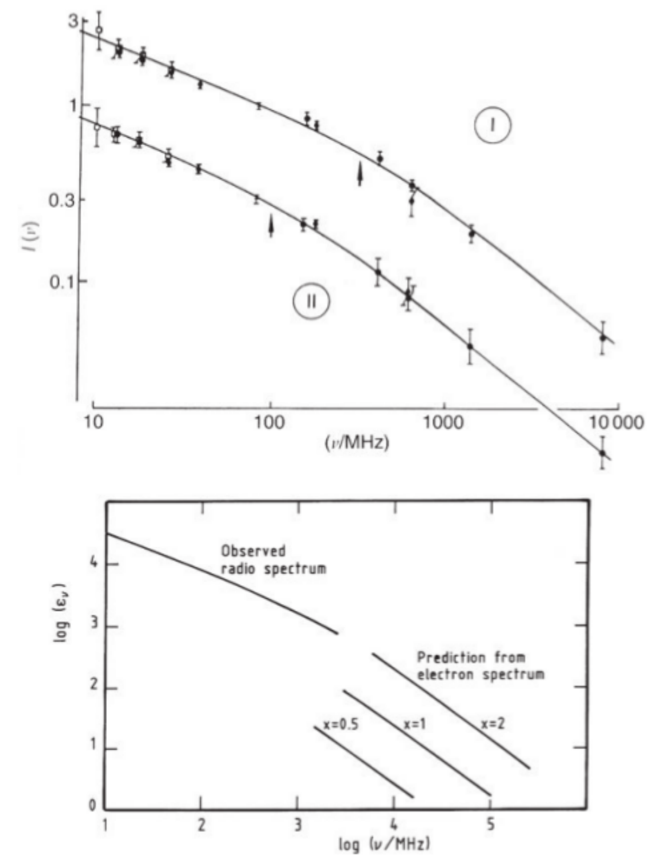

Figure 9: (Top) Galactic radio emission in the anticenter direction (I) and interarm region (II), (bottom) Comparison of observed and predicted spectrum for different values of $B = 0.3x$ nT $(x = 0.5, 1, 2)$. Frequency distribution $J(\nu)$ here is denoted ϵ_ν and is normalized with respect to the local spectrum of the interstellar medium.

References

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