Stability

Lecture notes:

Prof. Sérgio Frascino Müller de Almeida

Finite elements

Introduction

Objective:

solve an engineering continuous problem over a complex domain with complex boundary conditions

Discretization:

the finite element method transforms a continuous problem (partial differential equation) into discrete problem (set discrete algebraic or ordinary differential equations)

Introduction

Basic idea:

- **divide the entire domain into a collection of geometric subdomains (elements)**
- **the value of the valor physical variable of interest at an arbitrary point in the domain is computed as an interpolation of the values of a finite set of points defined in the domain (nodes)**

Introduction

- **the domain is divided into a mesh constituted by a set of nodes and elements**
- **the elements are defined by the connectivity for a set of nodes**

Introduction

Types of structural analyses:

1. **Linear**
2. **Non-linear**
    - **material non-linearity**
    - **geometric non-linearity**

Introduction

Types of structural analyses:

1. **Static**
2. **Dynamic**

Introduction

Types of structural analyses:

Coupled analyses

1. **thermoelastic analysis**
2. **eletromechanical analysis**

Introduction

Problem definition:

✓ **Mesh**

1. elements
2. nodes

✓ **Loads**

1. distributed: applied to the elements
2. concentrated: applied to the nodes

Introduction

Problem definition:

✓ **Boundary conditions**

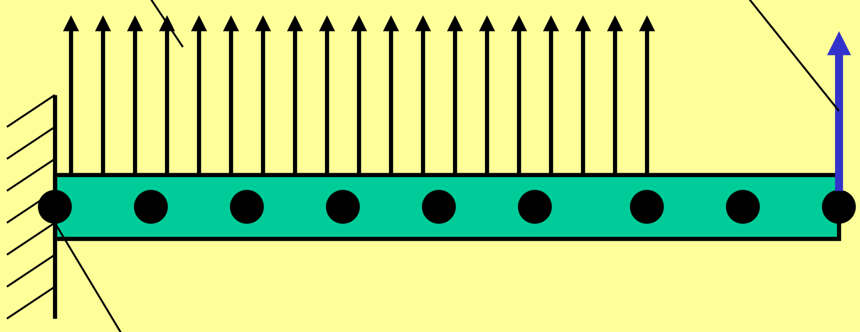
- **prescribed values of the variables of interest in points of the domain**
- **incorrectly defined boundary conditions lead to failure of the solution or wrong results**

Introduction

Example: clamped beam

distributed load

concentrated load



boundary conditions: known displacements and rotations (equal to zero) in this node

Introduction

Formulation:

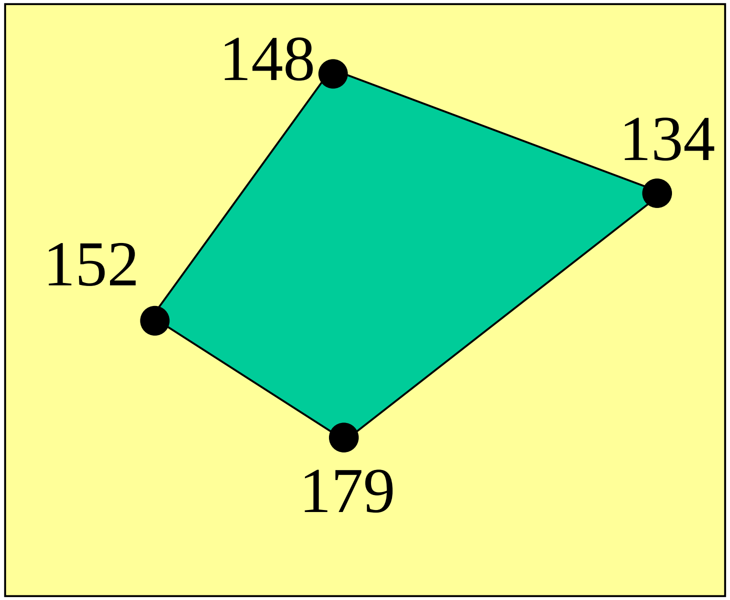
- **a specific element formulation is defined for each type of physical problem**
- **the different element types are arranged in an element library**

Introduction

Element definition:

- element type
- connectivity (element nodes)
- geometrical properties
- material properties

Introduction



- element type: 4 nodes quadrilateral plate element
- connectivity: nodes 134, 148, 152, 179
- geometrical properties: plate thickness = *1 mm*
- material: aluminum

Introduction

Physical problems:

- structural
- thermal
- electromagnetic
- fluid

Introduction

Problem dimension:

- one-dimensional
- two-dimensional
- three-dimensional

Introduction

Order of the interpolation function:

- linear
- quadratic
- cubic
- others

Introduction

Convergence:

- If the element formulation is correct, the more refined the mesh, the closer to the exact solution the model solution is

Strategy:

- ✓ many simple elements
- ✓ relatively few complex elements

Introduction

STRUCTURAL ELEMENTS			
element	**dimension**	**dof**	**geometric prop.**
rod	1D	d	area
beam	1D	d, r	area, mom. of area
transl. spring	1D	d	stiffness
rotation spring	1D	r	stiffness
gap	1D	d	—
membrane	2D	d	thickness
thin shell	2D	d, r	thickness
thick shell	2D	d, r	thickness
solid	3D	d	—

dof = degrees of freedom

d = displacement

r = rotation

Introduction

PRE-PROCESSOR

SOLVER

POST-PROCESSOR

Introduction

PRE-PROCESSOR

mesh generation

load definition

boundary conditions definition

Introduction

SOLVER

element matrices

global matrices

load vector

problem solution

Introduction

POST-PROCESSOR

result visualization

strain and stresses computation

Formulation for buckling of beams

Formulation

Concept:

solve a structural problem based on the minimization of the total potential energy of the beam

Problem:

minimize $\Pi[v]$
subjected to boundary conditions

$\Pi[v]$ is the functional of the total potential energy

v represents the unknown function(s)

Formulation

Total potential energy:

the total potential energy of the beam is given by:

$$\Pi[u,w] = V[u,w] - \overline{W}[u,w]$$

$V[u,w]$ is the elastic strain energy of the beam and $\overline{W}[u,w]$ is the virtual work done by the external forces

Formulation

Discretization:

use of interpolation functions:

$$\{v\} = [\psi]\{a\}$$

$\{v\}$ is the vector of displacements and/or rotations at an arbitrary point in the element domain

$[\psi]$ is the interpolation functions matrix

$\{a\}$ is the element nodal displacement vector

Formulation

strains:
$$\varepsilon(x,z) = \frac{du(x,z)}{dx} + \frac{1}{2}\left(\frac{dw(x)}{dx}\right)^2$$

where:
$$u(x,z) = u_0(x) + z\frac{d^2w}{dx^2}$$
$$w(x,z) = w_0(x)$$

the stresses are computed from the strains by the material constitutive relation

stresses:
$$\sigma(x,z) = E\varepsilon(x,z)$$

Formulation

For an Euler-Bernoulli beam (the transverse shear is zero), the elastic strain energy:

$$V = \frac{1}{2}\int_{vol} \sigma(x,z)\varepsilon(x,z)\,dv = \frac{1}{2}\int_{vol} E\varepsilon^2(x,z)\,dv$$

The linear strains are:

$$\varepsilon(x,z) = \frac{du(x,z)}{dx} = \frac{du_0(x)}{dx} + z\frac{d^2w(x)}{dx^2} = u_0'(x) + zw''(x)$$

Formulation

The elastic strain energy becomes:

$$V=\frac{1}{2}\int_{vol}E\left(\frac{du_0}{dx}+z\theta'\right)^2 dv=\frac{1}{2}\int_0^l\int_A E\left(\frac{du_0}{dx}+z\theta'\right)^2 dAdx$$

$$V=\frac{1}{2}\int_0^l\int_A E\left(\frac{du_0}{dx}\right)^2 dAdx+\int_0^l\int_A E\left(z\frac{du_0}{dx}\theta'\right)^2 dAdx+\frac{1}{2}\int_0^l\int_A Ez^2\theta'^2 dAdx$$

Since the origin of z is the neutral surface, the second term is zero:

$$V=\frac{1}{2}\int_0^l\int_A E\left(\frac{du_0}{dx}\right)^2 dAdx+\frac{1}{2}\int_0^l\int_A Ez^2\theta'^2 dAdx$$

Formulation

u_0 and θ are functions of x and:

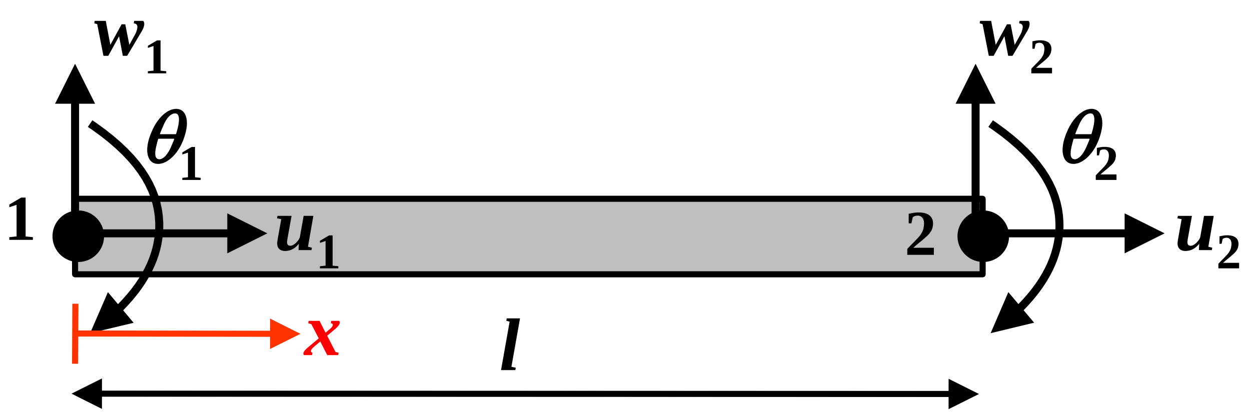
$$\int_A E\,dA = EA(x) \qquad \int_A Ez^2\,dA = EI(x)$$

therefore, for an Euler-Bernoulli beam:

$$V = \frac{1}{2}\int_0^l EA(x)\left(\frac{du_0}{dx}\right)^2 dx + \frac{1}{2}\int_0^l EI(x)\,\theta'^2\,dx$$

Beam element formulation

Interpolation functions for a beam element



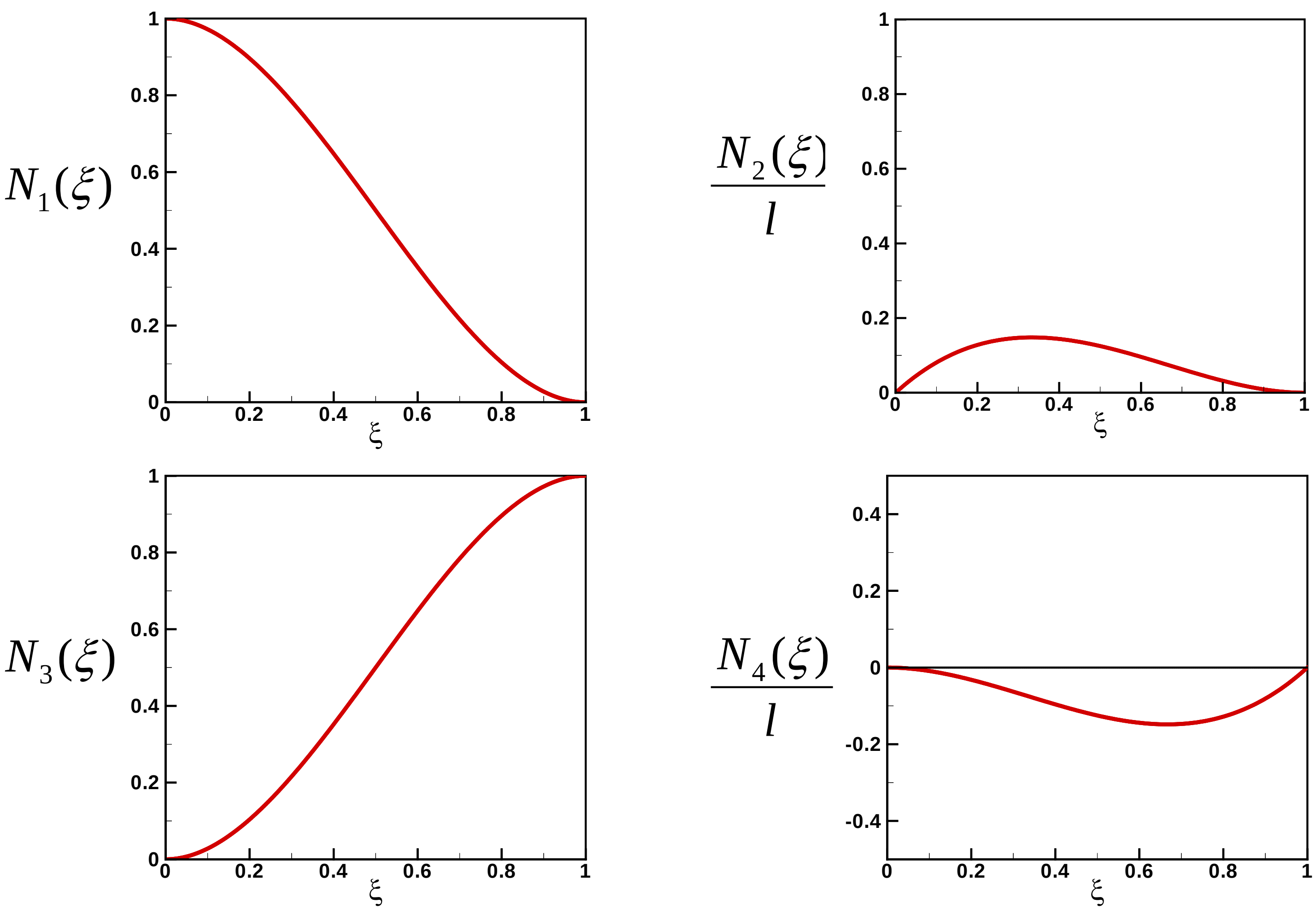
$$\xi = \frac{x}{L}$$

$$\theta(x) = \frac{dw(x)}{dx}$$

$$u_0(\xi) = (1-\xi)\,u_1 + \xi\,u_2$$

$$w(x) = \left(1-3\xi^2+2\xi^3\right)w_1 + \left(\xi-2\xi^2+\xi^3\right)l\theta_1 + \left(3\xi^2-2\xi^3\right)w_2 +$$
$$+\left(-\xi^2+\xi^3\right)l\theta_2 = N_1(\xi)w_1 + N_2(\xi)l\theta_1 + N_3(\xi)w_2 + N_4 l\theta_2(\xi)$$

Interpolation functions



Longitudinal stiffness matrix

The derivatives of the interpolated displacements become:

$$\frac{du_0(\xi)}{dx}=\frac{1}{l}\frac{du_0(\xi)}{d\xi}=\frac{u_2-u_1}{l}$$

$$\frac{d^2w(\xi)}{dx^2}=\frac{1}{l^2}\frac{d^2w(\xi)}{d\xi^2}=\frac{1}{l^2}\left[(12\xi-6)\,w_1+(6\xi-4)\,l\theta_1\right.$$
$$\left.+(6-12\xi)\,w_2+(6\xi-2)\,l\theta_2\right]$$
$$=\frac{1}{l^2}\left[N_1''(\xi)w_1+N_2''(\xi)l\theta_1+N_3''(\xi)w_2+N_4''(\xi)l\theta_2\right]$$

Longitudinal stiffness matrix

The longitudinal strains at mid-surface are:

$$\frac{du_0}{dx} = \frac{u_2 - u_1}{l} = \frac{1}{l}\begin{bmatrix} -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$\left(\frac{du_0}{dx}\right)^2 = \frac{1}{l^2}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}^T \begin{bmatrix} -1 \\ 1 \end{bmatrix}\begin{bmatrix} -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$\left(\frac{du_0}{dx}\right)^2 = \frac{1}{l^2}\{u\}^T \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\{u\}$$

where: $\{u\} = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$

Longitudinal stiffness matrix

Therefore:

$$\frac{1}{2}\int_0^l EA(x)\left(\frac{du_0}{dx}\right)^2 dx = \frac{1}{2}\int_0^1 \frac{EA(x)}{l^2}\{u\}^T\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\{u\} l\, d\xi$$

The nodal displacements $\{u\}$ are constants. Assuming that E and A are constants within the element:

$$\frac{1}{2}\int_0^l EA(x)\left(\frac{du_0}{dx}\right)^2 dx = \frac{1}{2}\frac{EA}{l}\{u\}^T\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\{u\} = \frac{1}{2}\{u\}^T[k_u]\{u\}$$

where $[k_u]$ is the longitudinal stiffness matrix of the element

Longitudinal stiffness matrix

The stiffness matrix are always positive semi-definite matrix

The strain deformation energy is always non-negative

It is always zero for rigid body motions

Bending stiffness matrix

The linear bending strains are:

$$\frac{d^2w(\xi)}{dx^2} = \frac{1}{l^2}\left[N_1''(\xi)w_1 + N_2''(\xi)l\theta_1 + N_3''(\xi)w_2 + N_4''(\xi)l\theta_2\right]$$

$$= \frac{1}{l^2}\begin{bmatrix} N_1''(\xi) & N_2''(\xi) & N_3''(\xi) & N_4''(\xi) \end{bmatrix}\begin{Bmatrix} w_1 \\ l\theta_1 \\ w_2 \\ l\theta_2 \end{Bmatrix}$$

$$= \frac{1}{l^2}[N'']\{w_\theta\}$$

Bending stiffness matrix

and:

$$\left(\frac{dw(\xi)}{dx}\right)^2 = \frac{1}{l^4}\{w_\theta\}^T[N''(\xi)]^T[N''(\xi)]\{w_\theta\}$$

Therefore:

$$\frac{1}{2}\int_0^l EI(x)\theta'^2 dx = \frac{1}{2}\int_0^1 \frac{EI(x)}{l^4}\{w_\theta\}^T\left[N''(\xi)\right]^T\left[N''(\xi)\right]\{w_\theta\}\ l d\xi$$

Bending stiffness matrix

The nodal displacements $\{w_\theta\}$ are constants. Assuming that E and I are constants within the element:

$$\frac{1}{2}\int_0^l EI(x)\,\theta'^2 dx =$$

$$= \frac{1}{2}\frac{EI}{l^3}\{w_\theta\}^T \left[\int_0^1 \begin{bmatrix} N_1''N_1'' & N_1''N_2'' & N_1''N'' & N_1''N_4'' \\ N_1''N_2'' & N_2''N_2'' & N_2''N_3'' & N_2''N_4'' \\ N_1''N_3'' & N_2''N_3'' & N_3''N_3'' & N_3''N_4'' \\ N_1''N_4'' & N_2''N_4'' & N_3''N_4'' & N_4''N_4'' \end{bmatrix} d\xi\right]\{w_\theta\}$$

Bending stiffness matrix

Integrating:

$$\frac{1}{2}\int_0^l EI\left(x\right)\theta'^2 dx = \frac{1}{2}\frac{EI}{l^3}\left\{w_\theta\right\}^T \begin{bmatrix} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{bmatrix} \left\{w_\theta\right\}$$

$$\frac{1}{2}\int_0^l EI\left(x\right)\theta'^2 dx = \frac{1}{2}\frac{EI}{l^3}\left\{w_\theta\right\}^T \left[k_{w\theta}\right]\left\{w_\theta\right\}$$

Work of external forces

work of distributed external forces :

$$\delta\overline{W} = \int_0^l q(x)v(x)dx = \int_0^1 q(\xi)v(\xi)\, l d\xi$$

work of concentrated external forces :

$$\delta\overline{W_i} = P_i v\left(x_i\right)$$

work of concentrated external moments :

$$\delta\overline{W_i} = M_i \theta\left(x_i\right)$$

Work of longitudinal external forces

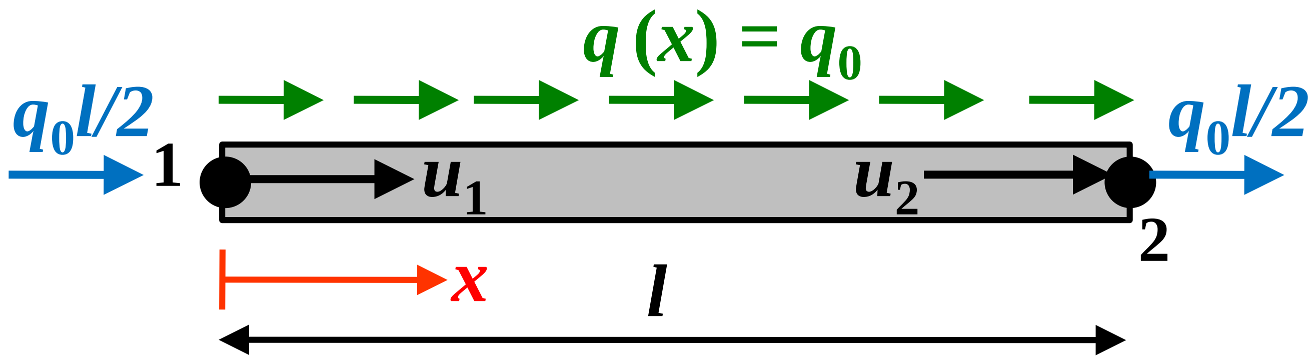
Pre-buckling problem:

- Only the longitudinal external forces are considered
- There are no out-of-plane displacements
- Therefore, the non-linear strain component $(dw/dx)^2 = 0$; only the linear strain is to be considered
- The external forces results in a stress distribution $\sigma(x)$ that will interact with the non-linear strain in the buckling problem

Work of longitudinal external forces

work of longitudinal uniform distributed external forces:

$$\delta\overline{W} = \int_0^l q(x)u(x)dx = q_0 l \int_0^1 \begin{bmatrix} 1-\xi & \xi \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} d\xi = \frac{q_0 l}{2} \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$



Work of bending external forces

work of bending uniform distributed external forces:

$$\delta\overline{W} = \int_0^l q(x)w(x)dx = q_0 l\int_0^1 \begin{bmatrix} N_1 & N_2 & N_3 & N_4 \end{bmatrix} \begin{Bmatrix} w_1 \\ l\theta_1 \\ w_2 \\ l\theta \end{Bmatrix} d\xi$$

$$N_1 = 1 - 3\xi^2 + 2\xi^3$$

$$N_2 = \xi - 2\xi^2 + \xi^3$$

$$N_3 = 3\xi^2 - 2\xi^3$$

$$N_4 = -\xi^2 + \xi^3$$

$$\int_0^1 N_1(\xi)dx = \frac{1}{2}$$

$$\int_0^1 N_3(\xi)dx = \frac{1}{2}$$

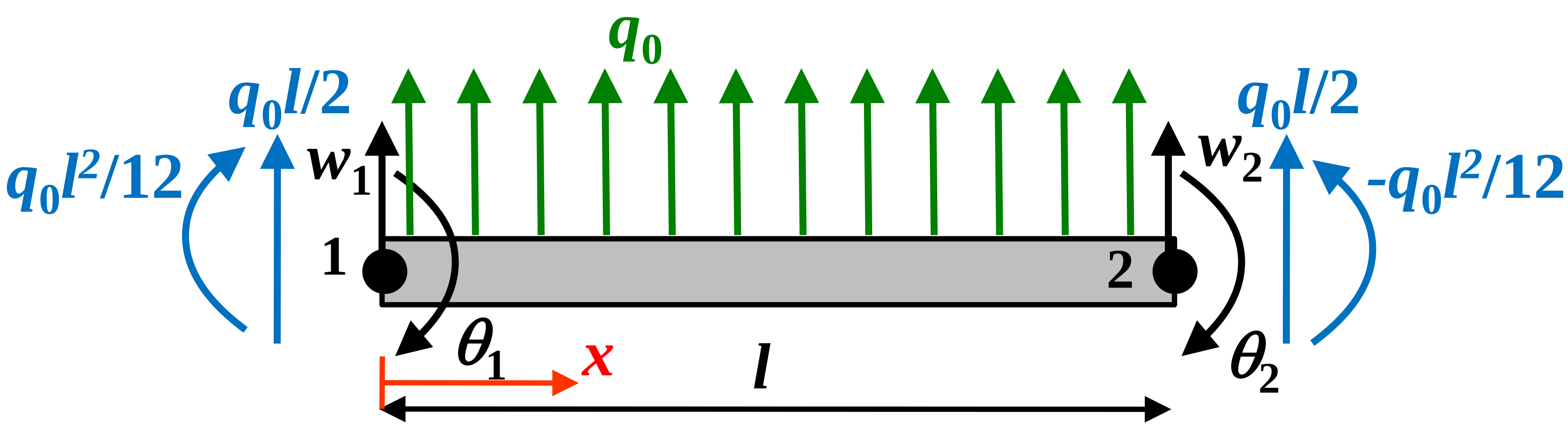
$$\int_0^1 N_2(\xi)dx = \frac{1}{12}$$

$$\int_0^1 N_4(\xi)dx = -\frac{1}{12}$$

Work of bending external forces

therefore:

$$\delta\overline{W}=\int_0^l q(x)w(x)dx=\frac{q_0 l}{2}\int_0^1\left[1 \quad \frac{1}{6} \quad 1 \quad -\frac{1}{6}\right]\begin{Bmatrix} w_1 \\ l\theta_1 \\ w_2 \\ l\theta \end{Bmatrix}d\xi$$



Work of pre-buckling stresses

Buckling problem:

- The work of the pre-buckling stresses with respect to the linear strains has been taken into account in the pre-buckling analysis
- The work of the pre-buckling with respect to the non-linear strain will be analyzed in what follows

$$\delta\overline{W} = \int_{vol} \sigma_0(x)\,\varepsilon^{NL}(x)\,dv = \frac{1}{2}\int_{vol} \sigma_0(x)\left(\frac{dw}{dx}\right)^2 dv$$

Work of pre-buckling stresses

But:

$$\left(\frac{dw}{dx}\right)^2 = \left(\frac{dw}{d\xi}\frac{d\xi}{dx}\right)^2 = \frac{1}{l^2}\left(\frac{dw}{d\xi}\right)^2$$

Therefore:

$$\delta\overline{W} = \frac{1}{2}\int_{vol}\sigma_0(x)\left(\frac{dw}{dx}\right)^2 dv = \frac{1}{2}\int_0^1 \sigma_0(x)A(x)\frac{1}{l^2}\left(\frac{dw}{d\xi}\right)^2 l\,d\xi$$

Work of pre-buckling stresses

If the element is linear in the longitudinal direction then $\sigma_0(x) = \sigma_0$ = constant and $A(x)$ = A_0 = constant.

Therefore:

$$\delta \overline{W} = \frac{P_0}{2l} \int_0^1 \left(\frac{dw}{d\xi} \right)^2 d\xi$$

where: $P_0 = \sigma_0 A_0$

Work of pre-buckling stresses

But:

$$\frac{dw(\xi)}{d\xi} = \left[N_1' w_1 + N_2' l\theta_1 + N_3' w_2 + N_4' l\theta_2 \right]$$

$$= \begin{bmatrix} N_1' & N_2' & N_3' & N_4' \end{bmatrix} \begin{Bmatrix} w_1 \\ l\theta_1 \\ w_2 \\ l\theta_2 \end{Bmatrix} = \left[N' \right] \left\{ w_\theta \right\}$$

Work of pre-buckling stresses

and:

$$\left(\frac{dw(\xi)}{d\xi}\right)^2=\begin{Bmatrix} w_1\\ l\theta_1\\ w_2\\ l\theta_2\end{Bmatrix}^T\begin{bmatrix} N_1'N_1' & N_1'N_2' & N_1'N_3' & N_1'N_4'\\ N_1'N_2' & N_2'N_2' & N_2'N_3' & N_2'N_4'\\ N_1'N_3' & N_2'N_3' & N_3'N_3' & N_3'N_4'\\ N_1'N_4' & N_2'N_4' & N_3'N_4' & N_4'N_4'\end{bmatrix}\begin{Bmatrix} w_1\\ l\theta_1\\ w_2\\ l\theta_2\end{Bmatrix}$$

Work of pre-buckling stresses

Therefore:

$$\delta\overline{W} = \frac{P_0}{2l}\int_0^1 \left(\frac{dw}{d\xi}\right)^2 d\xi$$

$$= \frac{P_0}{2l}\int_0^1 \begin{Bmatrix} w_1 \\ l\theta_1 \\ w_2 \\ l\theta_2 \end{Bmatrix}^T \begin{bmatrix} N_1'N_1' & N_1'N_2' & N_1'N_3' & N_1'N_4' \\ N_1'N_2' & N_2'N_2' & N_2'N_3' & N_2'N_4' \\ N_1'N_3' & N_2'N_3' & N_3'N_3' & N_3'N_4' \\ N_1'N_4' & N_2'N_4' & N_3'N_4' & N_4'N_4' \end{bmatrix} \begin{Bmatrix} w_1 \\ l\theta_1 \\ w_2 \\ l\theta_2 \end{Bmatrix} d\xi$$

Work of pre-buckling stresses

Computing the integrals:

$$\delta\overline{W} = \frac{1}{30}\frac{P_0}{2l}\begin{Bmatrix} w_1 \\ l\theta_1 \\ w_2 \\ l\theta_2 \end{Bmatrix}^T \begin{bmatrix} 36 & 3 & -36 & 3 \\ 3 & 4 & -3 & -1 \\ -36 & -3 & 36 & -3 \\ 3 & -1 & -3 & 4 \end{bmatrix} \begin{Bmatrix} w_1 \\ l\theta_1 \\ w_2 \\ l\theta_2 \end{Bmatrix}$$

$$\delta\overline{W} = P_0 \{w_\theta\}^T [k_G] \{w_\theta\}$$

where $[k_G]$ is the geometric stiffness matrix of the beam element

Buckling problem

$$\Pi = \frac{1}{2}\{w_\theta\}^T [k_w]\{w_\theta\} + \frac{P_0}{2}\{w_\theta\}^T [k_G]\{w_\theta\}$$

Minimizing:

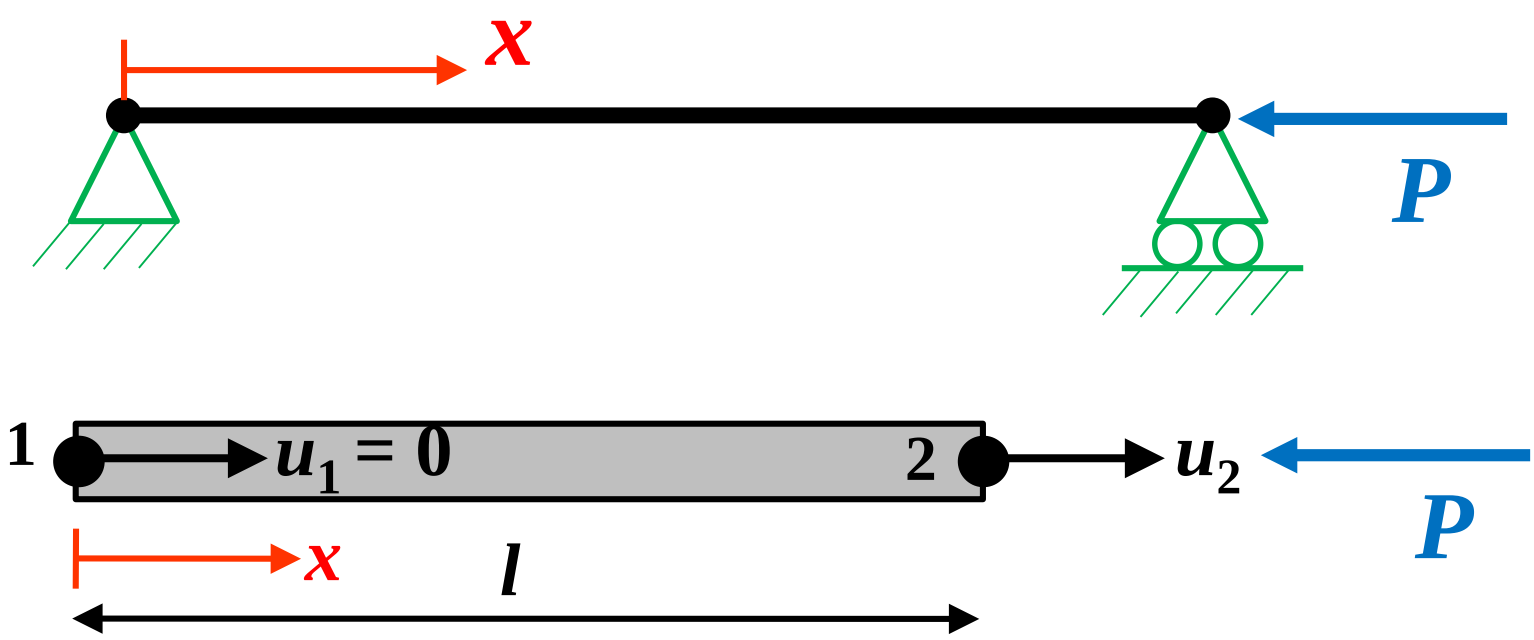
$$\left([k_w] + P_0[k_G]\right)\{w_\theta\} = \{0\}$$

Example 1

Example: Euler beam (1 element)

Use one beam element

Pre-buckling problem



Example: Euler beam (2 elements)

Vectors of nodal displacements

Element 1: $\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ u_2 \end{Bmatrix}$

Element 2: $\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}$

$\begin{Bmatrix} u_2 \\ u_3 \\ u_1 \end{Bmatrix}$ **Global**

u_1 is prescribed: goes to the bottom of the global vector

Example: Euler beam

Pre-buckling problem

$$\frac{1}{2}\int_0^l EA(x)\left(\frac{du_0}{dx}\right)^2 dx = \frac{1}{2}\frac{EA}{l}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}^T \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$\delta\overline{W} = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}^T \begin{Bmatrix} H_1 \\ -P \end{Bmatrix}$$

Minimizing

$$\frac{EA}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} H_1 \\ -P \end{Bmatrix} \xrightarrow{u_1 = 0} \frac{EA}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} H_1 \\ -P \end{Bmatrix}$$

Example: Euler beam

Pre-buckling problem

$$\frac{EA}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} H_1 \\ -P \end{Bmatrix}$$

$\Rightarrow$

$$-\frac{EA}{l}u_2 = H_1$$
$$\frac{EA}{l}u_2 = -P$$

$\Rightarrow$

$$u_2 = -\frac{Pl}{EA}$$
$$H_1 = P$$

$$\varepsilon = \frac{u_2 - u_1}{l} = -\frac{1}{l}\frac{Pl}{EA} = -\frac{P}{EA}$$

$$\sigma = E\varepsilon = -E\frac{P}{EA} = -\frac{P}{A}$$

internal force

$$N = \sigma A = -\frac{P}{A}A = -P$$

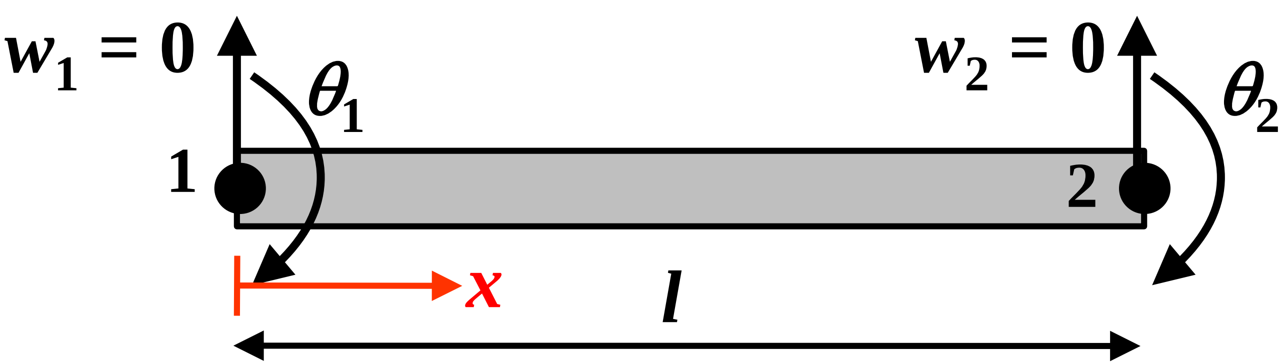
Example: Euler beam (1 element)

Buckling problem

$$\left(\left[k_w\right]-P_0\left[k_G\right]\right)\left\{w_\theta\right\}=\left\{0\right\}$$

where: $P_0 = P$

$$\left(\frac{EI}{l^3}\begin{bmatrix} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{bmatrix} - \frac{P_0}{30l}\begin{bmatrix} 36 & 3 & -36 & 3 \\ 3 & 4 & -3 & -1 \\ -36 & -3 & 36 & -3 \\ 3 & -1 & -3 & 4 \end{bmatrix}\right)\begin{Bmatrix} w_1 \\ l\theta_1 \\ w_2 \\ l\theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$



Example: Euler beam (1 element)

$w_1 = w_2 = 0$:

$$\left(\frac{EI}{l^3}\begin{bmatrix} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{bmatrix} - \frac{P_0}{30l}\begin{bmatrix} 36 & 3 & -36 & 3 \\ 3 & 4 & -3 & -1 \\ -36 & -3 & 36 & -3 \\ 3 & -1 & -3 & 4 \end{bmatrix}\right)\begin{Bmatrix} 0 \\ l\theta_1 \\ 0 \\ l\theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\left(\frac{EI}{l^3}\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} - \frac{P_0}{30l}\begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix}\right)\begin{Bmatrix} l\theta_1 \\ l\theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\left(\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} - \lambda\begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix}\right)\begin{Bmatrix} l\theta_1 \\ l\theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

where: $\lambda = \dfrac{P_0 l^2}{30EI}$

Example: Euler beam (1 element)

$$\left(\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix}\right)\begin{Bmatrix} l\theta_1 \\ l\theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} 4-4\lambda & 2+\lambda \\ 2+\lambda & 4-\lambda \end{bmatrix}\begin{Bmatrix} l\theta_1 \\ l\theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \Rightarrow \begin{vmatrix} 4-4\lambda & 2+\lambda \\ 2+\lambda & 4-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \begin{matrix} \lambda_1 = 0.4 \\ \lambda_2 = 2.0 \end{matrix} \Rightarrow \lambda_{crit} = \frac{P_{crit} l^2}{30EI} = 0.4 \Rightarrow P_{crit} = \frac{12EI}{l^2}$$

exact solution: $$\boxed{P_{crit} = \frac{\pi^2 EI}{l^2}}$$

Example: Euler beam

first eigenmode:

$$\begin{bmatrix} 4-4\lambda & 2+\lambda \\ 2+\lambda & 4-\lambda \end{bmatrix} \begin{Bmatrix} l\theta_1 \\ l\theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \xrightarrow{\lambda_{crit} = 0.4} \quad \begin{bmatrix} 2.4 & 2.4 \\ 2.4 & 2.4 \end{bmatrix} \begin{Bmatrix} l\theta_1 \\ l\theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$\theta_1 = -\theta_2$ **and** $w_1 = w_2 = 0$

therefore:

$$w(x) = l\theta_1 N_2(x) + l\theta_2 N_4(x) = A_1\left(\frac{x}{l} - 2\left(\frac{x}{l}\right)^2 + \left(\frac{x}{l}\right)^3 + \left(\frac{x}{l}\right)^2 - \left(\frac{x}{l}\right)^3\right)$$

Example: Euler beam

first eigenmode:

$$w(x) = A_1\left(\frac{x}{l} - \left(\frac{x}{l}\right)^2\right)$$

second eigenmode:

$$\begin{bmatrix} 4-4\lambda & 2+\lambda \\ 2+\lambda & 4-\lambda \end{bmatrix}\begin{Bmatrix} l\theta_1 \\ l\theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \xrightarrow{\lambda_2 = 2.0} \quad \begin{bmatrix} -4 & 4 \\ 4 & -4 \end{bmatrix}\begin{Bmatrix} l\theta_1 \\ l\theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$\theta_1 = -\theta_2$ **and** $w_1 = w_2 = 0$

$$\lambda_2 = \frac{P_2 l^2}{30EI} = 2.0 \quad \Rightarrow \quad P_2 = \frac{60EI}{l^2}$$

Example: Euler beam (1 element)

second eigenmode:

$$w(x) = l\theta_1 N_2(x) + l\theta_2 N_4(x) = A_2\left(\frac{x}{l} - 2\left(\frac{x}{l}\right)^2 + \left(\frac{x}{l}\right)^3 - \left(\frac{x}{l}\right)^2 + \left(\frac{x}{l}\right)^3\right)$$

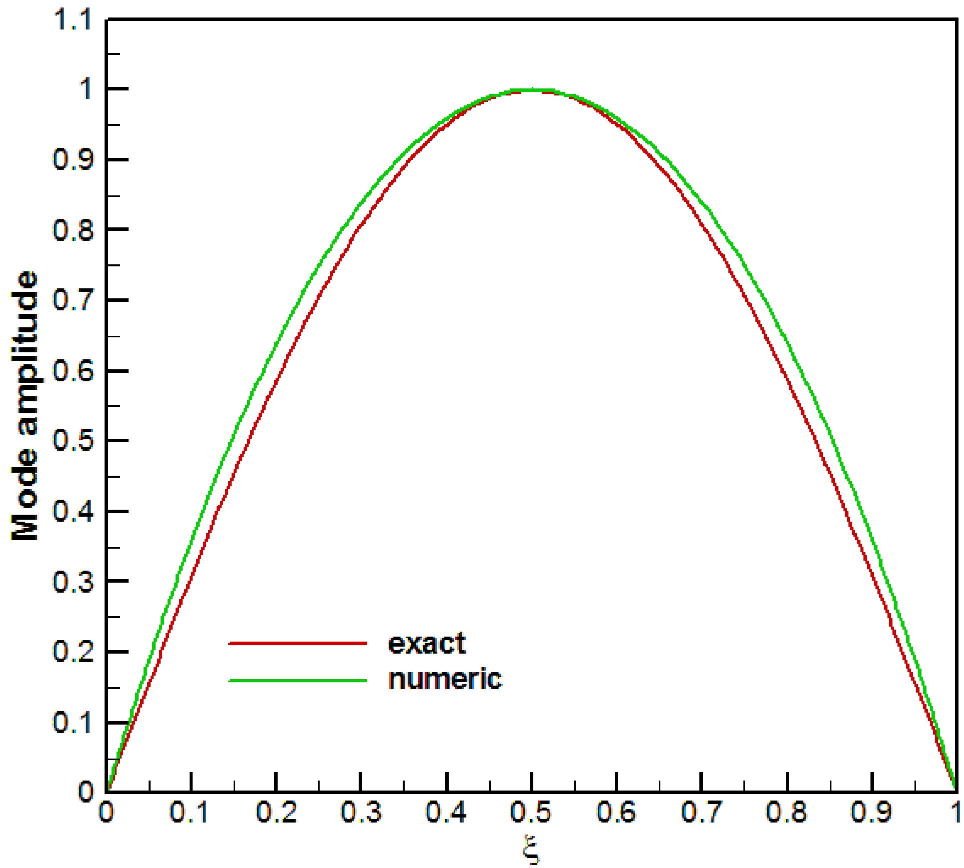
$$w(x) = A_2\left(\frac{x}{l} - 3\left(\frac{x}{l}\right)^2 + 2\left(\frac{x}{l}\right)^3\right)$$

Example: Euler beam (1 element)

mode	λ	FEM $P_i l^2/EI$	exact $P_i l^2/EI$	error(%)
1	0.400	12.000	9.867	21.59
2	2.000	60.000	39.478	51.98

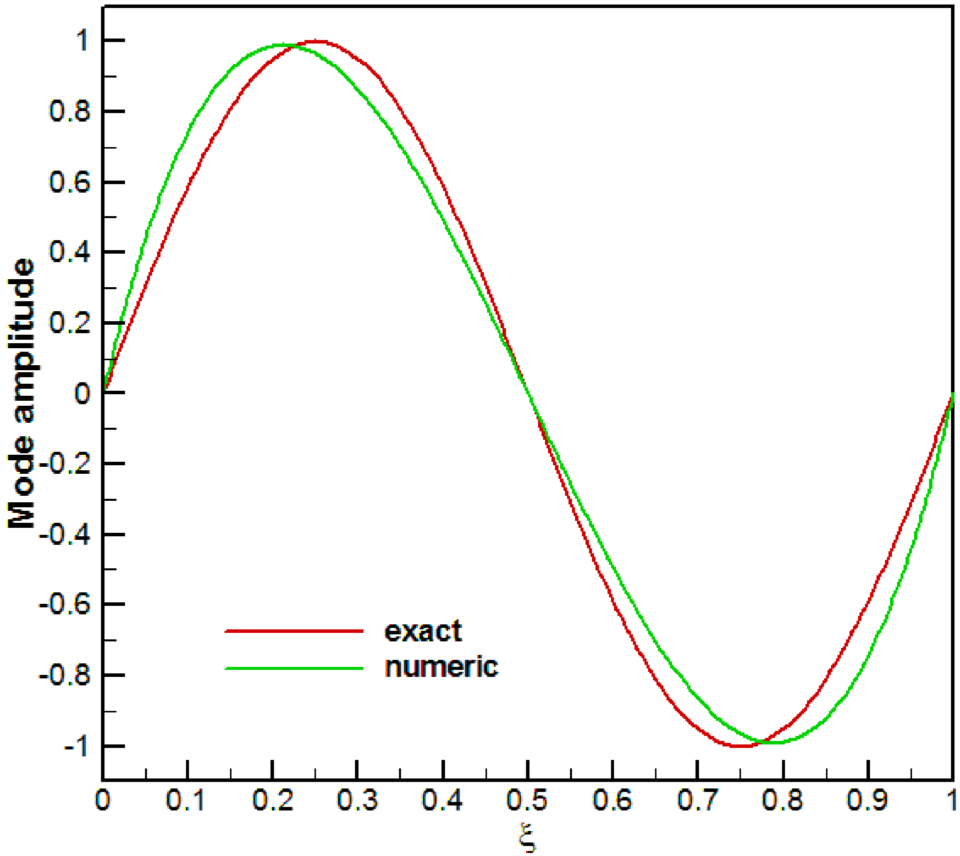
Example: Euler beam (1 element)

First mode



Example: Euler beam (1 element)

Second mode



Example: Euler beam

Convergence:

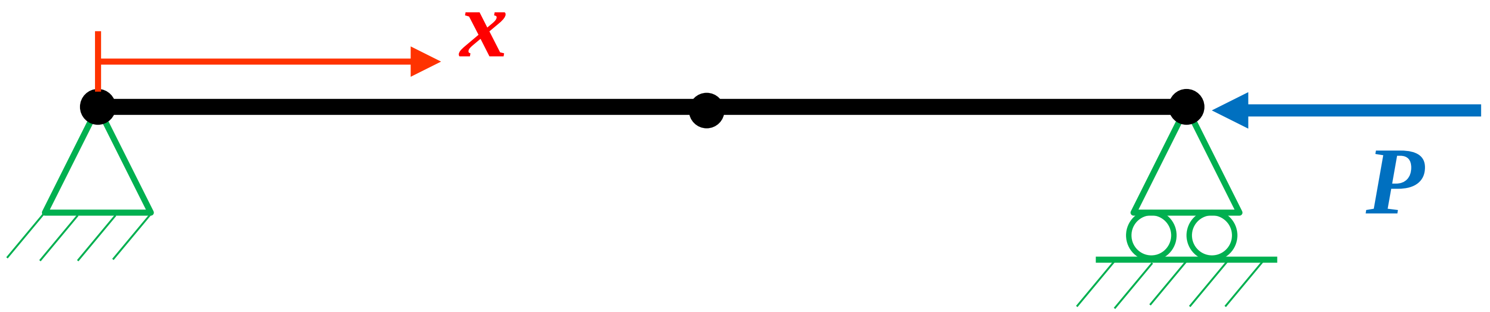
- A proper mesh refinement leads to a better the approximation
- The eigenvalues converge faster than the eigenmodes
- The eigenvalues of a more refined mesh are always smaller or equal to the less refined mesh

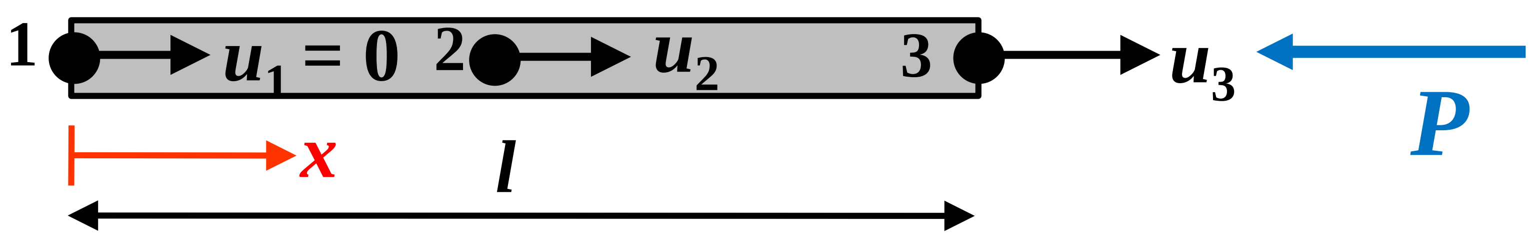
Example 2

Example: Euler beam (2 elements)

Use two beam elements

Pre-buckling problem





Example: Euler beam (2 elements)

Vectors of nodal longitudinal displacements

Element 1: $\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ u_2 \end{Bmatrix}$

Element 2: $\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}$

Global $\begin{Bmatrix} u_2 \\ u_3 \\ u_1 \end{Bmatrix}$

u_1 is prescribed: goes to the bottom of the global vector

Example: Euler beam (2 elements)

Stiffness matrices of the longitudinal elements

Element 1: nodes 1 and 2

$$\begin{array}{cc} & \begin{array}{cc} u_1 & u_2 \end{array} \\ \dfrac{EA}{l_e} & \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{array}{c} u_1 \\ u_2 \end{array} \end{array}$$

Element 2: nodes 2 and 3

$$\begin{array}{cc} & \begin{array}{cc} u_2 & u_3 \end{array} \\ \dfrac{EA}{l_e} & \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{array}{c} u_2 \\ u_3 \end{array} \end{array}$$

Element lengths: $l_e = l/2$

Example: Euler beam (2 elements)

Assembly of global matrix

Step 1: initialize with zeroes

$$\frac{EA}{l_e}\begin{array}{cc} \begin{array}{ccc} u_2 & u_3 & u_1 \end{array} & \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{array}{c} u_2 \\ u_3 \\ u_1 \end{array} \end{array}$$

Example: Euler beam (2 elements)

Assembly of global matrix

Step 2: add stiffness matrix of element 1

$$
\frac{EA}{l_e}\begin{array}{c c}
\begin{array}{cc} u_1 & u_2 \end{array} & \\
\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} & \begin{array}{c} u_1 \\ u_2 \end{array}
\end{array}
\qquad
\frac{EA}{l_e}\begin{array}{c c}
\begin{array}{ccc} u_2 & u_3 & u_1 \end{array} & \\
\begin{bmatrix} 0+1 & 0 & 0-1 \\ 0 & 0 & 0 \\ 0-1 & 0 & 0+1 \end{bmatrix} & \begin{array}{c} u_2 \\ u_3 \\ u_1 \end{array}
\end{array}
$$

Example: Euler beam (2 elements)

Assembly of global matrix

Step 3: add stiffness matrix of element 2

$$
\frac{EA}{l_e}\begin{array}{c c}
u_2 & u_3 \\
\left[\begin{array}{cc} 1 & -1 \\ -1 & 1 \end{array}\right] &
\begin{array}{c} u_2 \\ u_3 \end{array}
\end{array}
\qquad
\frac{EA}{l_e}\begin{array}{c c}
u_2 \quad u_3 \quad u_1 & \\
\left[\begin{array}{ccc} 1+1 & 0-1 & -1 \\ 0-1 & 0+1 & 0 \\ -1 & 0 & 1 \end{array}\right] &
\begin{array}{c} u_2 \\ u_3 \\ u_1 \end{array}
\end{array}
$$

Example: Euler beam (2 elements)

Assembly of global matrix

Final result

$$\frac{EA}{l_e}\begin{array}{c} \begin{array}{ccc} u_2 & u_3 & u_1 \end{array} \\ \begin{bmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ u_2 \\ u_3 \\ u_1 \end{array}$$

Example: Euler beam (2 elements)

Assembly of force vector

Element force vectors (same procedure)

Element 1:
$$\begin{array}{cc} \begin{Bmatrix} H_1 \\ 0 \end{Bmatrix} & \begin{matrix} u_1 \\ u_2 \end{matrix} \end{array}$$

Element 2:
$$\begin{array}{cc} \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} & \begin{matrix} u_2 \\ u_3 \end{matrix} \end{array}$$

Nodal:
$$\begin{Bmatrix} -P \end{Bmatrix} \; u_3$$

$$\begin{array}{cc} \begin{Bmatrix} 0+0+0 \\ 0+0-P \\ 0+H_1 \end{Bmatrix} & \begin{matrix} u_2 \\ u_3 \\ u_1 \end{matrix} \end{array}$$

Example: Euler beam (2 elements)

Assembly of force vector

Final result

$$\begin{Bmatrix} 0 \\ -P \\ H_1 \end{Bmatrix} \begin{matrix} u_2 \\ u_3 \\ u_1 \end{matrix}$$

Equilibrium equations

$$\frac{EA}{l_e}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -P \\ H_1 \end{Bmatrix}$$

Example: Euler beam (2 elements)

Equilibrium equations

$$2u_2 - u_3 = 0 \quad\Rightarrow\quad u_3 = 2u_2 \quad\Rightarrow\quad \boxed{u_2 = -\frac{Pl_e}{EA}} \quad\Rightarrow\quad \boxed{u_3 = -\frac{2Pl_e}{EA}}$$

$$-u_2 + u_3 = -\frac{Pl}{2EA}$$

$$-\frac{EA}{l_e}u_2 = H_1 \quad\Rightarrow\quad \boxed{H_1 = \frac{Pl_e}{EA}}$$

Element 1: $\varepsilon = \dfrac{(u_2 - u_1)}{l_e} = -\dfrac{1}{l_e}\dfrac{Pl_e}{EA} = -\dfrac{P}{EA} \quad\Rightarrow\quad \sigma = -\dfrac{P}{A}$

Element 2: $\varepsilon = \dfrac{(u_3 - u_2)}{l_e} = -\dfrac{1}{l_e}\dfrac{Pl_e}{EA} = -\dfrac{P}{EA} \quad\Rightarrow\quad \sigma = -\dfrac{P}{A}$

Example: Euler beam (2 elements)

Vector of global bending displacements

Element 1: $\begin{Bmatrix} w_1 \\ l_e\theta_1 \\ w_2 \\ l_e\theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ l_e\theta_1 \\ w_2 \\ l_e\theta_2 \end{Bmatrix}$

Element 2: $\begin{Bmatrix} w_2 \\ l_e\theta_2 \\ w_3 \\ l_e\theta_3 \end{Bmatrix} = \begin{Bmatrix} w_2 \\ l_e\theta_2 \\ 0 \\ l_e\theta_3 \end{Bmatrix}$

$$\begin{Bmatrix} l_e\theta_1 \\ w_2 \\ l_e\theta_2 \\ l_e\theta_3 \\ w_1 \\ w_3 \end{Bmatrix}$$

$$l_e = \frac{l}{2}$$ element length

Global

w_1 and w_3 are prescribed: go to the bottom of the global vector

Example: Euler beam (2 elements)

Element stiffness bending matrices

Element 1:

$$\frac{EI}{l_e^3}\begin{array}{c} \begin{array}{cccc} w_1 & l_e\theta_1 & w_2 & l_e\theta_2 \end{array} \\ \begin{bmatrix} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{bmatrix} \end{array} \begin{array}{c} \\ w_1 \\ l_e\theta_1 \\ w_2 \\ l_e\theta_2 \end{array}$$

Element 2:

$$\frac{EI}{l_e^3}\begin{array}{c} \begin{array}{cccc} w_2 & l_e\theta_2 & w_3 & l_e\theta_3 \end{array} \\ \begin{bmatrix} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{bmatrix} \end{array} \begin{array}{c} \\ w_2 \\ l_e\theta_2 \\ w_3 \\ l_e\theta_3 \end{array}$$

Example: Euler beam (2 elements)

Add element 1 to global stiffness bending matrix

$$
\frac{EI}{l_e^3}
\begin{array}{c}
\begin{array}{cccc} w_1 & l_e\theta_1 & w_2 & l_e\theta_2 \end{array} \\
\begin{bmatrix}
12 & 6 & -12 & 6 \\
6 & 4 & -6 & 2 \\
-12 & -6 & 12 & -6 \\
6 & 2 & -6 & 4
\end{bmatrix}
\end{array}
\begin{array}{c} \\ w_1 \\ l_e\theta_1 \\ w_2 \\ l_e\theta_2 \end{array}
$$

$$
\frac{EI}{l_e^3}
\begin{array}{c}
\begin{array}{cccccc} l_e\theta_1 & w_2 & l_e\theta_2 & l_e\theta_3 & w_1 & w_3 \end{array} \\
\begin{bmatrix}
0+4 & 0-6 & 0+2 & 0 & 0+6 & 0 \\
0-6 & 0+12 & 0-6 & 0 & 0-12 & 0 \\
0+2 & 0-6 & 0+4 & 0 & 0+6 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0+6 & 0-12 & 0+6 & 0 & 0+12 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\end{array}
\begin{array}{c} \\ l_e\theta_1 \\ w_2 \\ l_e\theta_2 \\ l_e\theta_3 \\ w_1 \\ w_3 \end{array}
$$

Example: Euler beam (2 elements)

Add element 2 to global stiffness bending matrix

$$
\frac{EI}{l_e^3}
\begin{array}{c}
\begin{array}{cccc} w_2 & l_e\theta_2 & w_3 & l_e\theta_3 \end{array} \\
\begin{bmatrix}
12 & 6 & -12 & 6 \\
6 & 4 & -6 & 2 \\
-12 & -6 & 12 & -6 \\
6 & 2 & -6 & 4
\end{bmatrix}
\end{array}
\begin{array}{c} w_2 \\ l_e\theta_2 \\ w_3 \\ l_e\theta_3 \end{array}
\qquad
\frac{EI}{l_e^3}
\begin{array}{c}
\begin{array}{cccccc} l_e\theta_1 & w_2 & l_e\theta_2 & l_e\theta_3 & w_1 & w_3 \end{array} \\
\begin{bmatrix}
4 & -6 & 2 & 0 & 6 & 0 \\
-6 & 12+12 & -6+6 & 0+6 & -12 & 0-12 \\
2 & -6+6 & 4+4 & 0+2 & 6 & 0-6 \\
0 & 0+6 & 0+2 & 0+4 & 0 & 0-6 \\
6 & -12 & 6 & 0 & 12 & 0 \\
0 & 0-12 & 0-6 & 0-6 & 0 & 0+12
\end{bmatrix}
\end{array}
\begin{array}{c} l_e\theta_1 \\ w_2 \\ l_e\theta_2 \\ l_e\theta_3 \\ w_1 \\ w_3 \end{array}
$$

Example: Euler beam (2 elements)

Global stiffness bending matrices

$$
\frac{EI}{l_e^3}
\begin{array}{c}
\begin{array}{cccccc} l_e\theta_1 & w_2 & l_e\theta_2 & l_e\theta_3 & w_1 & w_3 \end{array} \\
\begin{bmatrix}
4 & -6 & 2 & 0 & 6 & 0 \\
-6 & 24 & 0 & 6 & -12 & -12 \\
2 & 0 & 8 & 2 & 6 & -6 \\
0 & 6 & 2 & 4 & 0 & -6 \\
6 & -12 & 6 & 0 & 12 & 0 \\
0 & -12 & -6 & -6 & 0 & 12
\end{bmatrix}
\end{array}
\begin{array}{c} \\ l_e\theta_1 \\ w_2 \\ l_e\theta_2 \\ l_e\theta_3 \\ w_1 \\ w_3 \end{array}
$$

Example: Euler beam (2 elements)

Element geometric stiffness matrices

Element 1:

$$-\frac{P_0}{30 l_e}\begin{array}{cc} \begin{matrix} w_1 & l_e\theta_1 & w_2 & l_e\theta_2 \end{matrix} & \\ \begin{bmatrix} 36 & 3 & -36 & 3 \\ 3 & 4 & -3 & -1 \\ -36 & -3 & 36 & -3 \\ 3 & -1 & -3 & 4 \end{bmatrix} & \begin{matrix} w_1 \\ l\theta_1 \\ w_2 \\ l\theta_2 \end{matrix} \end{array}$$

Element 2:

$$-\frac{P_0}{30 l_e}\begin{array}{cc} \begin{matrix} w_2 & l_e\theta_2 & w_3 & l_e\theta_3 \end{matrix} & \\ \begin{bmatrix} 36 & 3 & -36 & 3 \\ 3 & 4 & -3 & -1 \\ -36 & -3 & 36 & -3 \\ 3 & -1 & -3 & 4 \end{bmatrix} & \begin{matrix} w_2 \\ l\theta_2 \\ w_3 \\ l\theta_3 \end{matrix} \end{array}$$

Example: Euler beam (2 elements)

Add element 1 to global geometric stiffness matrix

$$
-\frac{P_0}{30l_e}
\begin{array}{c c}
 & \begin{array}{cccc} w_1 & l_e\theta_1 & w_2 & l_e\theta_2 \end{array} \\
 & \left[\begin{array}{cccc} 36 & 3 & -36 & 3 \\ 3 & 4 & -3 & -1 \\ -36 & -3 & 36 & -3 \\ 3 & -1 & -3 & 4 \end{array}\right]
\begin{array}{c} w_1 \\ l\theta_1 \\ w_2 \\ l\theta_2 \end{array}
\end{array}
\qquad
-\frac{P_0}{30l_e}
\begin{array}{c c}
 & \begin{array}{cccccc} l_e\theta_1 & w_2 & l_e\theta_2 & l_e\theta_3 & w_1 & w_3 \end{array} \\
 & \left[\begin{array}{cccccc} 0+4 & 0-3 & 0-1 & 0 & 0+3 & 0 \\ 0-3 & 0+36 & 0-3 & 0 & 0-36 & 0 \\ 0-1 & 0-3 & 0+4 & 0 & 0+3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0+3 & 0-36 & 0+3 & 0 & 0+36 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]
\begin{array}{c} l_e\theta_1 \\ w_2 \\ l_e\theta_2 \\ l_e\theta_3 \\ w_1 \\ w_3 \end{array}
\end{array}
$$

Example: Euler beam (2 elements)

Add element 2 to global geometric stiffness matrix

$$
-\frac{P_0}{30l_e}
\begin{array}{c}
\begin{array}{cccc} w_2 & l_e\theta_2 & w_3 & l_e\theta_3 \end{array} \\
\begin{bmatrix} 36 & 3 & -36 & 3 \\ 3 & 4 & -3 & -1 \\ -36 & -3 & 36 & -3 \\ 3 & -1 & -3 & 4 \end{bmatrix}
\end{array}
\begin{array}{c} w_2 \\ l\theta_2 \\ w_3 \\ l\theta_3 \end{array}
$$

$$
-\frac{P_0}{30l_e}
\begin{array}{c}
\begin{array}{cccccc} l_e\theta_1 & w_2 & l_e\theta_2 & l_e\theta_3 & w_1 & w_3 \end{array} \\
\begin{bmatrix}
4 & -3 & -1 & 0 & 3 & 0 \\
-3 & 36+36 & -3+3 & 0+3 & -36 & 0-36 \\
-1 & -3+3 & 4+4 & 0-1 & 3 & 0-3 \\
0 & 0+3 & 0-1 & 0+4 & 0 & 0-3 \\
3 & -36 & 3 & 0 & 36 & 0 \\
0 & 0-36 & 0-3 & 0-3 & 0 & 0+36
\end{bmatrix}
\end{array}
\begin{array}{c} l_e\theta_1 \\ w_2 \\ l_e\theta_2 \\ l_e\theta_3 \\ w_1 \\ w_3 \end{array}
$$

Example: Euler beam (2 elements)

Global geometric stiffness matrix

$$
-\frac{P_0}{30 l_e}
\begin{array}{c}
\begin{array}{cccccc} l_e\theta_1 & w_2 & l_e\theta_2 & l_e\theta_3 & w_1 & w_3 \end{array} \\
\begin{bmatrix}
4 & -3 & -1 & 0 & 3 & 0 \\
-3 & 72 & 0 & 3 & -36 & -36 \\
-1 & 0 & 8 & -1 & 3 & -3 \\
0 & 3 & -1 & 4 & 0 & -3 \\
3 & -36 & 3 & 0 & 36 & 0 \\
0 & -36 & -3 & -3 & 0 & 36
\end{bmatrix}
\end{array}
\begin{array}{c} \\ l_e\theta_1 \\ w_2 \\ l_e\theta_2 \\ l_e\theta_3 \\ w_1 \\ w_3 \end{array}
$$

Example: Euler beam (2 elements)

Buckling problem

$$\left(\left[k_w\right]-P_0\left[k_G\right]\right)\left\{w_\theta\right\}=\left\{0\right\}$$

where: $P_0 = P$

$$\left(\frac{EI}{l_e^3}\begin{bmatrix} 4 & -6 & 2 & 0 & 6 & 0 \\ -6 & 24 & 0 & 6 & -12 & -12 \\ 2 & 0 & 8 & 2 & 6 & -6 \\ 0 & 6 & 2 & 4 & 0 & -6 \\ 6 & -12 & -6 & 0 & 12 & 0 \\ 0 & -12 & -6 & -6 & 0 & 12 \end{bmatrix} - \lambda\frac{P_0}{30l_e}\begin{bmatrix} 4 & -3 & -1 & 0 & 3 & 0 \\ -3 & 72 & 0 & 3 & -36 & -36 \\ -1 & 0 & 8 & -1 & 3 & -3 \\ 0 & 3 & -1 & 4 & 0 & -3 \\ 3 & -36 & 3 & 0 & 36 & 0 \\ 0 & -36 & -3 & -3 & 0 & 36 \end{bmatrix}\right)\begin{Bmatrix} l_e\theta_1 \\ w_2 \\ l_e\theta_2 \\ l_e\theta_3 \\ w_1 \\ w_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Example: Euler beam (2 elements)

Buckling problem

$w_1 = w_3 = 0$, therefore:

$$\left(\frac{EI}{l_e^3}\begin{bmatrix} 4 & -6 & 2 & 0 \\ -6 & 24 & 0 & 6 \\ 2 & 0 & 8 & 2 \\ 0 & 6 & 2 & 4 \end{bmatrix} - \lambda\frac{P_0}{30 l_e}\begin{bmatrix} 4 & -3 & -1 & 0 \\ -3 & 72 & 0 & 3 \\ -1 & 0 & 8 & -1 \\ 0 & 3 & -1 & 4 \end{bmatrix}\right)\begin{Bmatrix} l_e\theta_1 \\ w_2 \\ l_e\theta_2 \\ l_e\theta_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\left(\begin{bmatrix} 4 & -6 & 2 & 0 \\ -6 & 24 & 0 & 6 \\ 2 & 0 & 8 & 2 \\ 0 & 6 & 2 & 4 \end{bmatrix} - \lambda\begin{bmatrix} 4 & -3 & -1 & 0 \\ -3 & 72 & 0 & 3 \\ -1 & 0 & 8 & -1 \\ 0 & 3 & -1 & 4 \end{bmatrix}\right)\begin{Bmatrix} l_e\theta_1 \\ w_2 \\ l_e\theta_2 \\ l_e\theta_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

where:

$$\lambda = \frac{P_0 l_e^2}{30EI}$$

Example: Euler beam (2 elements)

Buckling problem

$$\begin{bmatrix} 4-4\lambda & -6+3\lambda & 2+\lambda & 0 \\ -6+3\lambda & 24-72\lambda & 0 & 6-3\lambda \\ 2+\lambda & 0 & 8-8\lambda & 2+\lambda \\ 0 & 6-3\lambda & 2+\lambda & 4-4\lambda \end{bmatrix} \begin{Bmatrix} l_e\theta_1 \\ w_2 \\ l_e\theta_2 \\ l_e\theta_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Eigenvalues:

$$\lambda_i = \frac{P_{0i} l_e^2}{30EI}$$

$$l_e = \frac{l}{2}$$

$$\Rightarrow \quad \boxed{P_{0i} = 120\lambda_i \frac{EI}{l^2}}$$

Example: Euler beam (2 elements)

Eigenvalues:

mode	λ	$P_i l^2/EI$ FEM	$P_i l^2/EI$ exact	error(%)
1	0.08287	9.944	9.867	0.76
2	0.4000	48.000	39.478	21.59
3	1.0727	128.724	88.826	44.92
4	2.0000	240.000	157.914	51.98

Example: Euler beam (2 elements)

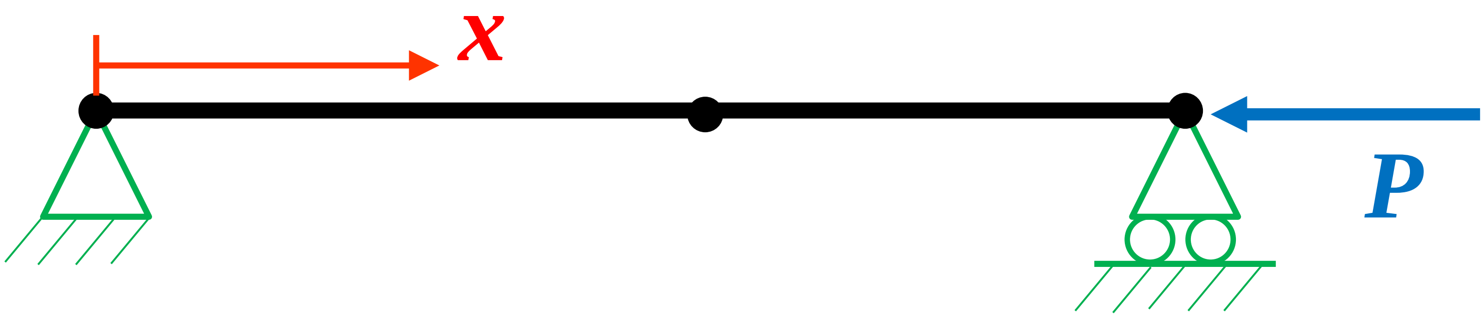
Eigenvectors:

	1st	2nd	3rd	4th
w_1	0.0000	0.0000	0.0000	0.0000
$l\theta_1$	-1.0000	1.0000	-1.0000	1.0000
w_2	-0.6379	0.0000	0.1045	0.0000
$l\theta_2$	0.0000	-1.0000	0.0000	1.0000
w_3	0.000	0.0000	0.0000	0.0000
$l\theta_3$	1.0000	1.0000	1.0000	1.0000

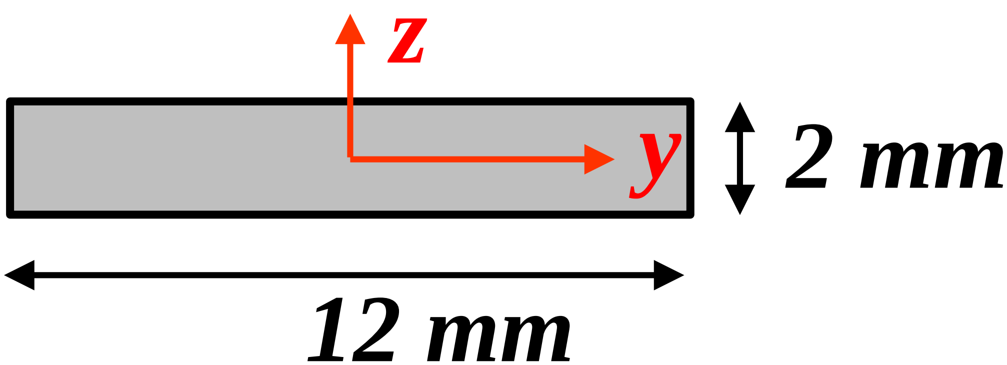
Example 3

Example: Euler beam using ANSYS

Use from 1 to 16 beam elements



Cross section



$$E = 100000\ MPa$$

$$I = 8\ mm^4$$

$$l = 200\ mm$$

Example: Euler beam (2 elements)

Buckling loads:

# elem	1st	2nd	3rd	4th
1	239.97	24130	—	—
2	200.34	959.53	4588.90	24130
4	197.57	800.13	1899.60	3832.60
8	197.39	790.06	1783.80	3200.70
16	197.37	789.31	1775.50	3156.50
exact	197.39	789.57	1776.53	3158.27

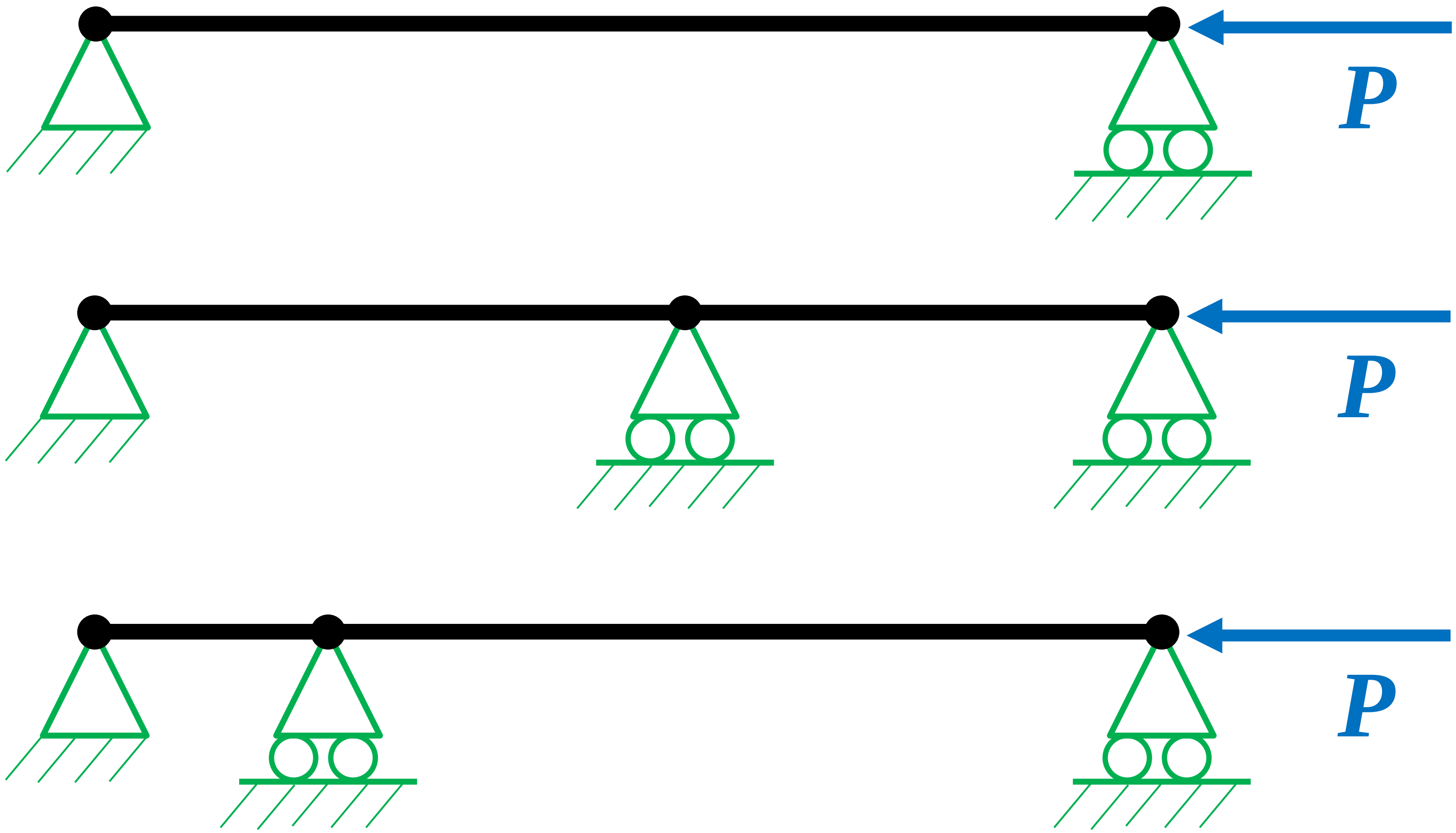
Discussions

Increasing the buckling load

- Change the material
  - ✓ variation is proportional to E
- Increase the moment of inertia
  - ✓ variation is proportional to I (t^3)
- Reducing the length
  - ✓ variation is proportional to $1/l^2$
- Changing the boundary conditions

Change of boundary conditions

Best configuration?

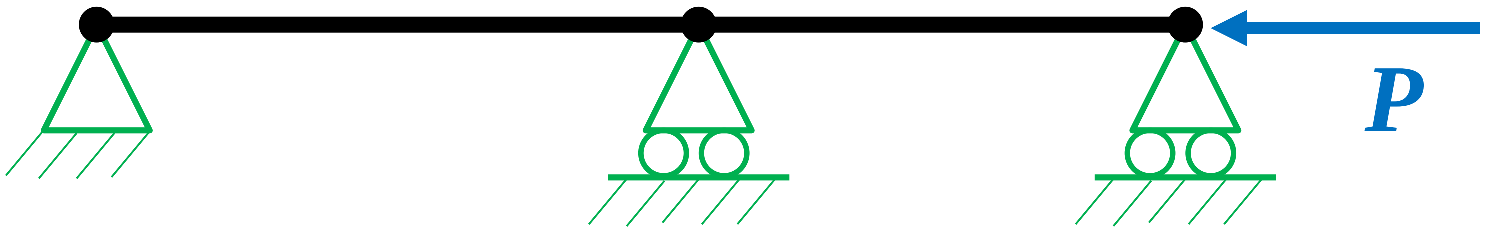


Change of boundary conditions

In this type of configuration, both beam sections buckle

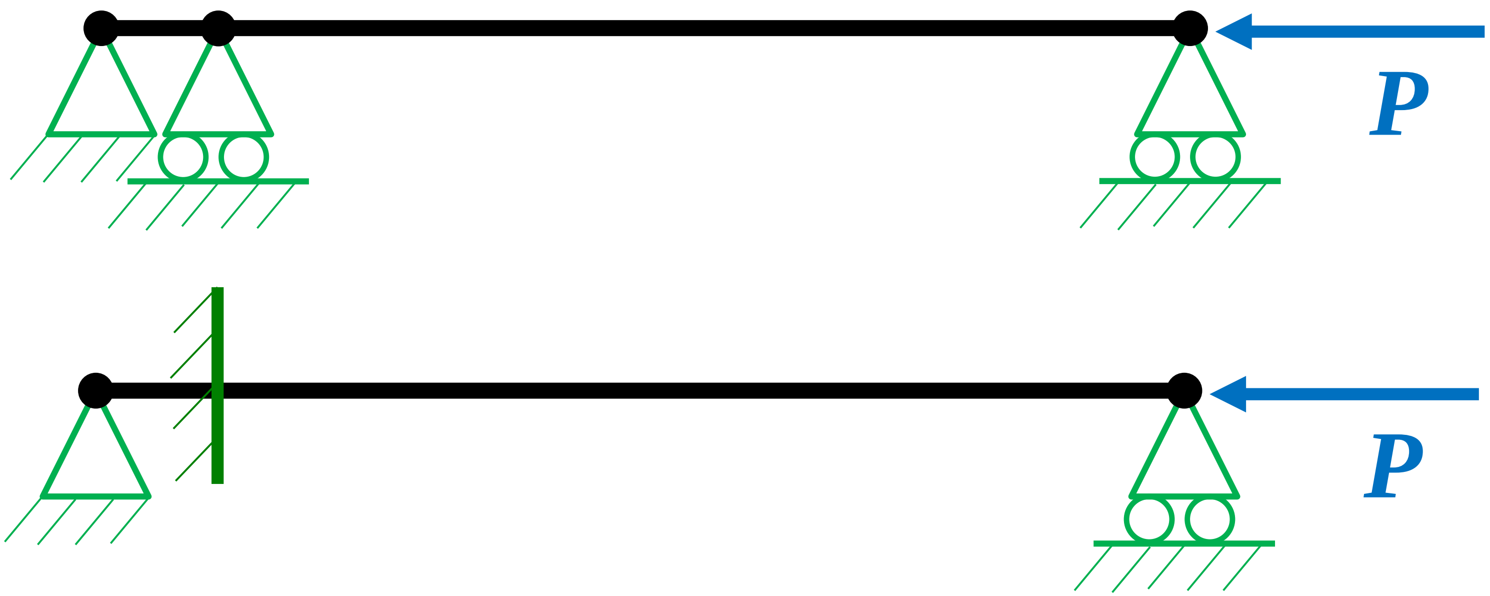
the critical buckling load depends on the position of the mid support

closed form solution is quite difficult



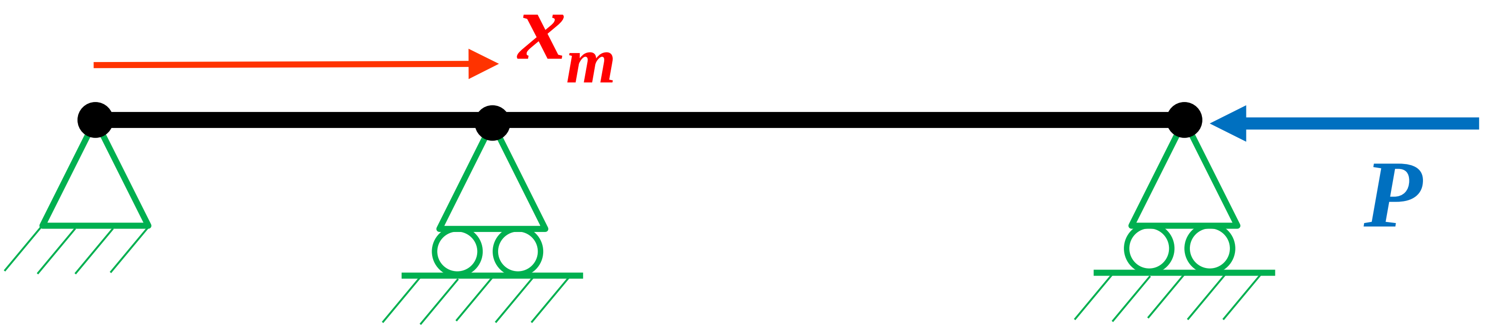
Change of boundary conditions

Approximately equivalent!

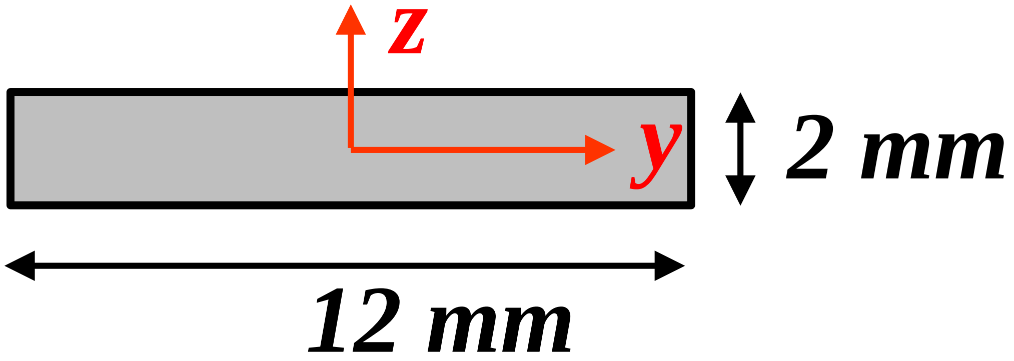


Change of boundary conditions

FEM solutions with 20 elements



Cross section



$$E = 100000\ MPa$$

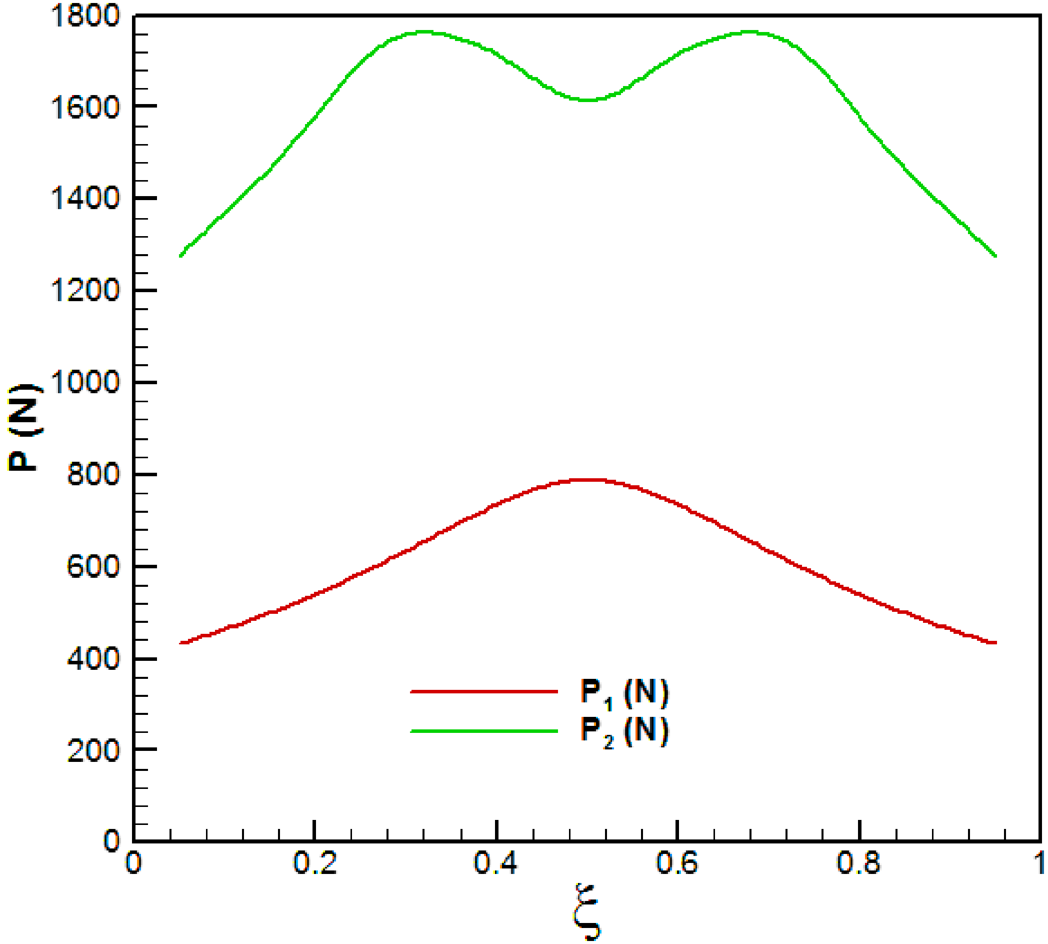
$$I = 8\ mm^4$$

$$l = 200\ mm$$

$$\xi = x_m / l$$

Change of boundary conditions

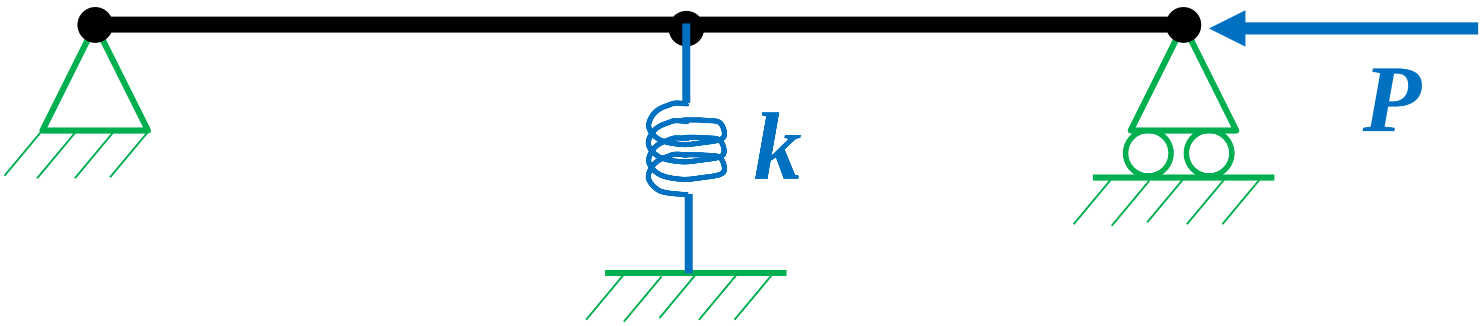
FEM solutions with 20 elements



Optimum position: **Center of the beam**

Elastic foundation

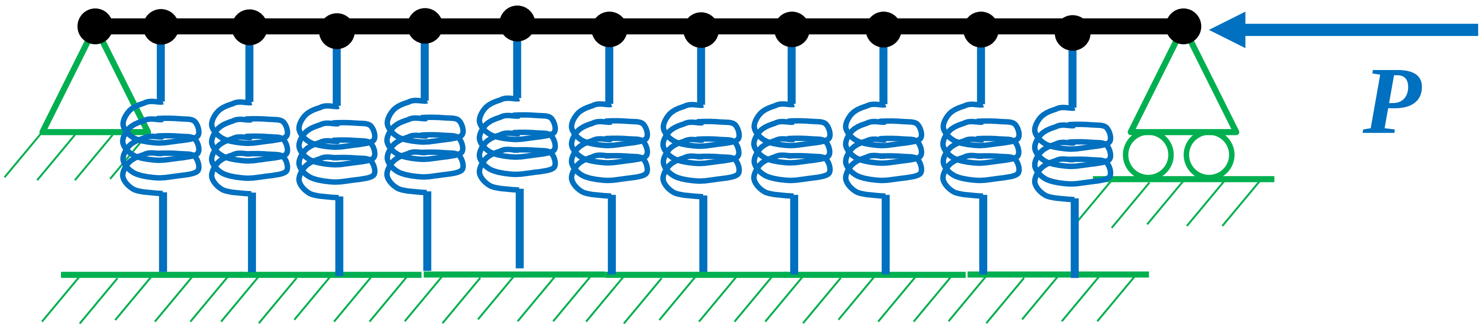
Elastic foundation



- the spring increase the buckling load
- k small tend to Euler-beam
- k very large tend to a rigid support

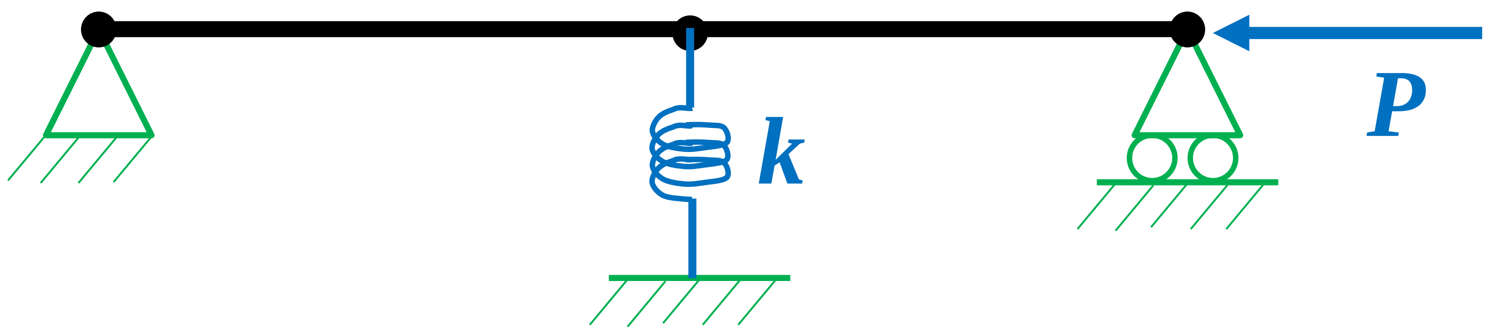
Elastic foundation

Beam over an elastic foundation

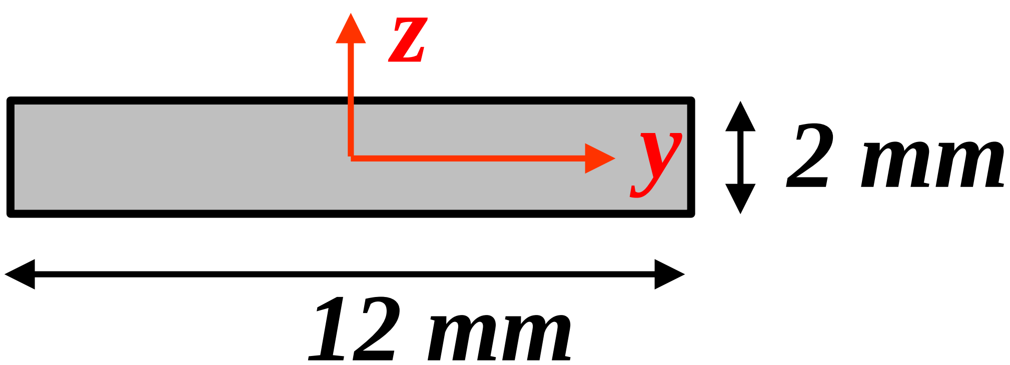


$$q(x) = k(x)\, w(x)$$

Euler beam with a center spring



Cross section



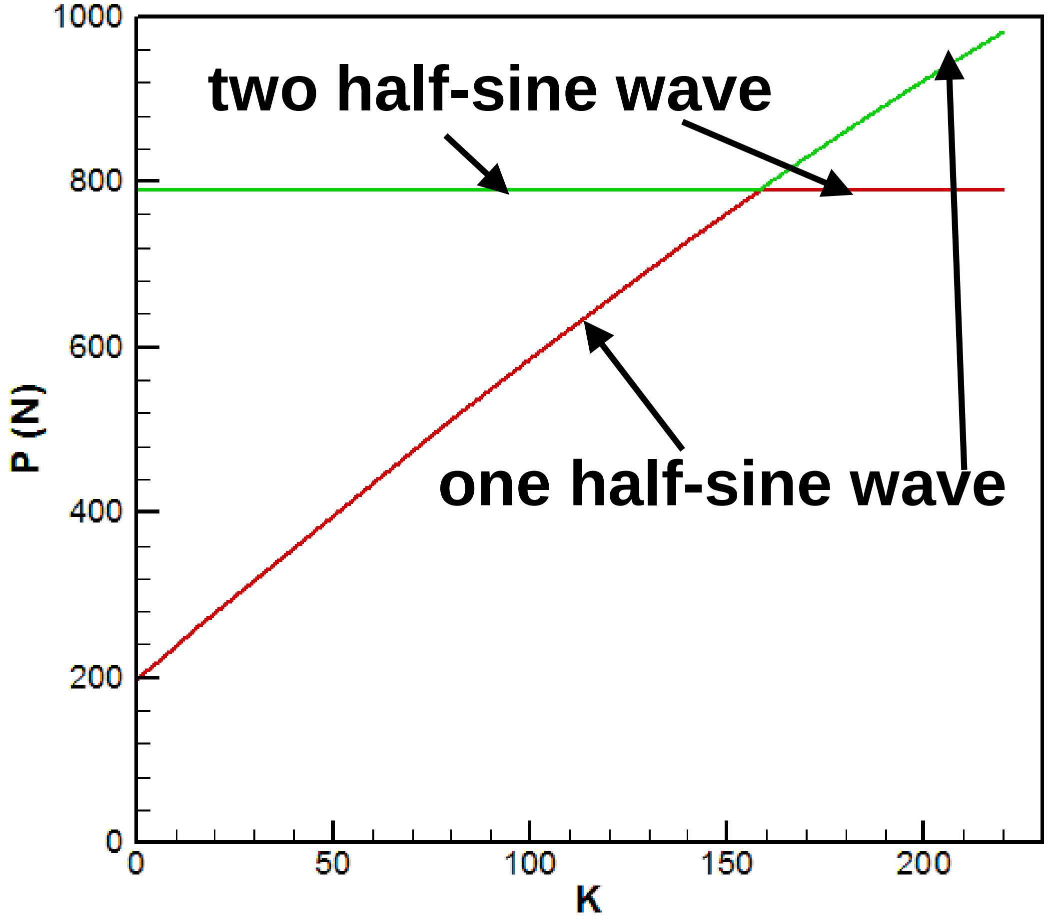
$$E = 100000\ MPa$$

$$I = 8\ mm^4$$

$$l = 200\ mm$$

$$k = KEI / l^3$$

Euler beam with a center spring



For stiff spring (K = 157.85) there is a “mode shift” between the two lowest eigenvalues