Professor: Jefferson Bertolai
Monitor: Não há.
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Game Theory
Graduação em Ciências Econômicas
FEARP-USP

Exercises
Dynamic games of complete information

Exercise 1. Suppose a parent and child play the following game. First, the child takes an action, A, that produces income for the child, $I_C(A)$, and income for the parent, $I_P(A)$. (Think of $I_C(A)$ as the child's income net of any costs of the action A.) Second, the parent observes the incomes I_C and I_P and then chooses a bequest, B, to leave to the child. The child's payoff is $U(I_C + B)$; the parent's is $V(I_P - B) + kU(I_C + B)$, where $k > 0$ reflects the parent's concern for the child's well-being. Assume that: the action is a nonnegative number, $A \geqslant 0$; the income functions $I_C(A)$ and $I_P(A)$ are strictly concave and are maximized at $A_C > 0$ and $A_P > 0$, respectively; the bequest B can be positive or negative; and the utility functions U and V are increasing and strictly concave.

(a) Prove the *"Rotten Kid"* Theorem: in the backwards-induction outcome, the child chooses the action that maximizes the family's aggregate income, $I_C(A)+I_P(A)$, even though only the parent's payoff exhibits altruism.

Exercise 2. Now suppose the parent and child play a different game. Let the incomes I_C and I_P be fixed exogenously. First, the child decides how much of the income I_C to save (S) for the future, consuming the rest ($I_C - S$) today. Second, the parent observes the child's choice of S and chooses a bequest , B. The child's payoff is the sum of current and future utilities:

$$U_1(I_C - S) + U_2(S + B)$$

The parent's payoff is

$$V(I_P - B) + k[U_1(I_C - S) + U_2(S + B)]$$

Assume that the utility functions U_1, U_2, and V are increasing and strictly concave.

(a) Show that there is a *"Samaritan's Dilemma"*: in the backwards-induction outcome, the child saves too little, so as to induce the parent to leave a larger bequest (i.e., both the parent's and child's payoffs could be increased if S were suitably larger and B suitably smaller).

Exercise 3. Suppose the players in Rubinstein's infinite-horizon bargaining game have different discount factors: δ_1 for player 1 and δ_2 for player 2. Adapt the argument in the text to show that in backwards-induction outcome, player 1 offers the settlement

$$\left(\tfrac{1-\delta_2}{1-\delta_1\delta_2}, \tfrac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2}\right)$$

to player 2, who accepts.

Exercise 4. Suppose a firm wants a worker to invest in a firm-specific skill, S, but the skill is too nebulous for a court to verify whether the worker has acquired it.[1]) The firm therefore cannot contract to repay the worker's cost of investing: even if the worker invests, the firm can claim that the worker did not invest, and the court cannot tell whose claim is true. Likewise, the worker cannot contract to invest if paid in advance.
It may be that the firm can use the (credible) promise of a promotion as an incentive for the worker to invest, as follows.

- Suppose that there are two jobs in the firm, one easy (E) and the other difficult (D), and that the skill is valuable on both jobs but more so on the difficult job: $y_{D0} < y_{E0} < y_{ES} < y_{DS}$, where y_{ij} is the worker's output in job i ($= E$ or D) when the worker's skill level is j ($= 0$ or S).
- Assume that the firm can commit to paying different wages in the two jobs, w_E and w_D, but that neither wage can be less than the worker's alternative wage, which we normalize to zero.

The timing of the game is as follows:

- At date 0 the firm chooses w_E and w_D and the worker observes these wages.
- At date 1 the worker joins the firm and can acquire the skill S at cost C.[2] Assume that $y_{DS} - y_{E0} > C$, so that it is efficient for the worker to invest.
- At date 2 the firm observes whether the worker has acquired the skill and then decides whether to promote the worker to job D for the worker's second (and last) period of employment.

The firm's second period profit is $y_{ij} - w_i$ when the worker is in job i and has skill level j. The worker's payoff from being in job i in the second period is w_i or $w_i - C$, depending on whether the worker invested in the first period.

(a) Solve for the backwards-induction outcome.

[1] For example, the firm might ask the worker to *"familiarize yourself with how we do things around here"*, or *"become an expert on this new market we might enter"*.
[2] We ignore production and wages during this first period. Since the worker has not yet acquired the skill, the efficient assignment is to job E.