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Exercises

Dynamic games of incomplete information

Exercise 1. Consider the finitely repeated prisoners' dilemma model discussed in section 4.3.C (see Gibbons, 1992). Suppose that the stage game is given by

		Column	
		Cooperate	*Fink*
Row	*Cooperate*	$1, 1$	$-5, 5$
	Fink	$5, -5$	$0, 0$

and that $p = 0.9$.

1. Show that in the two-period case the play described in the following table is the equilibrium path of a perfect Bayesian equilibrium.

	$t = 1$	$t = 2$
$Tit - for - Tat$	C	C
Rational Row	F	F
Column	C	F

2. Show that in the three-period case the play described in the following table is the equilibrium path of a perfect Bayesian equilibrium.

	$t = 1$	$t = 2$	$t = 3$
$Tit - for - Tat$	C	C	C
Rational Row	C	F	F
Column	C	C	F

Exercise 2. Consider a firm and a union bargaining over wages. For simplicity, assume that employment is fixed. The amount that union members earn if not employed, called unions reservation wage, is denoted by w_r. The firm's profit, denoted by π, is a priori uniformly distributed on $[\pi_L, \pi_H]$. The value of π is privately known by the firm. We simplify the analysis by assuming that $w_r = \pi_L = 0$. The bargaining game lasts at most two periods:

- In the first period, the union makes a wage offer, w_1. If the firm accepts this offer then the game ends. The union's payoff is w_1 and the firm's is $\pi - w_1$.

- If the firm rejects this offer the game proceeds to the second period. The union makes a second wage offer, w_2. If the firm accepts this offer then the present values of the players' payoffs are δw_2 for the union and $\delta(\pi - w_2)$ for the firm, where $\delta \in (0, 1)$. If the firm rejects the union's second offer then the game ends and payoffs are zero for both players.

We recall that a strategy for the union is a first-period offer w_1 and a second-period offer function $w_1 \mapsto w_2(w_1)$. There is one first-period information set and the union's first-period belief is denoted by $\mu_1 \in Prob([0, \pi_H])$. After observing the first-period offer w_1 has been rejected, the union's second-period belief is denoted by $\mu_2(\cdot|w_1) \in Prob([0, \pi_H])$. A strategy for the firm involves two decisions. Let $A_1(w_1|\pi)$ equal one if the firm would accept the first-period offer w_1 when its profit is π, and zero if the firm would reject w_1 under these circumstances. Let $A_2(w_2|\pi, w_1)$ equal one if the firm would accept the second-period offer w_2 when its profit is π and the first-period offer was w_1, and zero if the firm would reject w_2 under these circumstances. A strategy for the firm is a pair of functions (A_1, A_2) with

$$A_1 : (w_1, \pi) \mapsto A_1(w_1|\pi) \in \{0, 1\}$$

and

$$A_2 : (w_2, w_1, \pi) \mapsto A_2(w_2|\pi, w_1) \in \{0, 1\}$$

1. Since the firm has complete information throughout the game, its beliefs are trivial. Give the definition of: $\{(w_1^\star, \widetilde{w}_2), (A_1, A_2), (\mu_1, \mu_2)\}$ forms a perfect Bayesian equilibrium. We assume from now on that the family $\{(w_1^\star, \widetilde{w}_2), (A_1, A_2), (\mu_1, \mu_2)\}$ forms a perfect Bayesian equilibrium.

2. Given arbitrary (π, w_1), describe the function $w_2 \mapsto A_2(w_2|\pi, w_1)$.

3. We temporarily consider the following one-period bargaining problem. Suppose the union believes that the firm's profit is uniformly distributed on $[0, \pi_1]$, where for the moment π_1 is arbitrary.

   (a) If the union offers w, what is the firm's best response?

   (b) Deduce the union's optimal wage offer $w^\star(\pi_1)$ as a function of π_1.

4. We return permanently to the two-period problem. Assume that the union offers w_1 in the first period and the firm expects the union to offer w_2 in the second period. The firm's possible strategies at that information set are: accepting w_1, rejecting w_1 and accepting w_2, and rejecting both offers.

   (a) Find $\pi^\star(w_1, w_2)$ (as a function of w_1 and w_2) such that the firm prefers accepting w_1 at the first-period than to accepting w_2 in the second-period, if and only if $\pi \geqslant \pi^\star(w_1, w_2)$.

   (b) Under which necessary and sufficient condition relating π, w_1 and $\pi^\star(w_1, w_2)$, the firm will choose to accept w_1 instead of the two other strategies

   (c) Deduce $A_1(w_1|\pi)$ using the functions $\widetilde{w}_2$ and $\pi^\star$

5. Consider the union's information set of the second period after observing the first-period offer w_1 has been rejected.

   (a) Using the profit $\widehat{\pi}_1(w_1, \widetilde{w}_2)$ defined by

   $$\widehat{\pi}_1(w_1, \widetilde{w}_2) \equiv \max\{\pi^\star(w_1, \widetilde{w}_2(w_1)), w_1\}$$

   describe the union's belief about the types remaining at that information set.

   (b) Given these beliefs, show that the union' optimal second-period offer must be

   $$\widetilde{w}_2(w_1) = \frac{\widehat{\pi}_1(w_1, \widetilde{w}_2)}{2}$$

(c) Solve for $x \geqslant 0$ the fixed-point problem

$$2x = \max\{\pi^\star(w_1, x), w_1\}$$

(d) Deduce that

$$\widetilde{w}_2(w_1) = \frac{w_1}{2-\delta}$$

(e) Describe the union's second-period belief $\mu_2(\cdot|w_1)$, at the information set reached if the first period offer w_1 is rejected, as a function of

$$\widetilde{\pi}(w_1) = \frac{2w_1}{2-\delta}$$

6. We propose now to determine the union's first-period wage offer $w_1^\star$.

(a) Show that, according to the union's first-period belief μ_1, the probability that the firm accepts the union's first-period wage offer w_1 is given by

$$\mu_1[\{\pi \in [0, \pi_H] : A_1(w_1|\pi) = 1\}|w_1] = \frac{\pi_H - \widetilde{\pi}(w_1)}{\pi_H}$$

(b) Show that according to the union's second-period belief $mu_2(\cdot|w_1)$, the probability that the firm accepts the offer $\widetilde{w}_2(w_1)$ after observing that the first-period offer w_1 was rejected, is given by

$$\mu_2[\{\pi \in [0, \pi_H] : A_2(\widetilde{w}_2(w_1)|\pi, w_1) = 1\}|w_1] = \frac{\widetilde{\pi}(w_1) - \widetilde{w}_2(w_1)}{\widetilde{\pi}(w_1)}$$

(c) Deduce that the union's first-period wage offer $w_1^\star$ should be chosen to solve

$$\max_{w_1 \geqslant 0}[w_1 \frac{\pi_H - \widetilde{\pi}(w_1)}{\pi_H} + \delta \widetilde{w}_2(w_1) \frac{\widetilde{\pi}(w_1) - \widetilde{w}_2(w_1)}{\pi_H}]$$

and show that the solution $w_1^\star$ is

$$w_1^\star = \frac{(2-\delta)^2}{2(4-3\delta)} \pi_H$$