Modeling in Dynamic Analysis

Escolha dos modelos matemáticos em Mecânica das Estruturas

Modelos estruturais

M1 - Treliça

M2 - Estado plano de tensões

M3 - Estado plano de deformações

M4 - Axissimétrico

M5 - Viga de Timoshenko

M6 - Viga de Bernoulli-Euler

M7 - Reissner-Mindlin

M8 - Elasticidade tridimensional

Carregamentos

Condições de contorno

Tipo da análise

A1 – Estática

A2 - Dinâmica

Cinemática

K1 - pequenos deslocamentos e deformações

K2 - grandes deslocamentos e pequenas deformações

K3 - grandes deslocamentos e deformações

Comportamento constitutivo

C1 - Elástico, linear e isotrópico

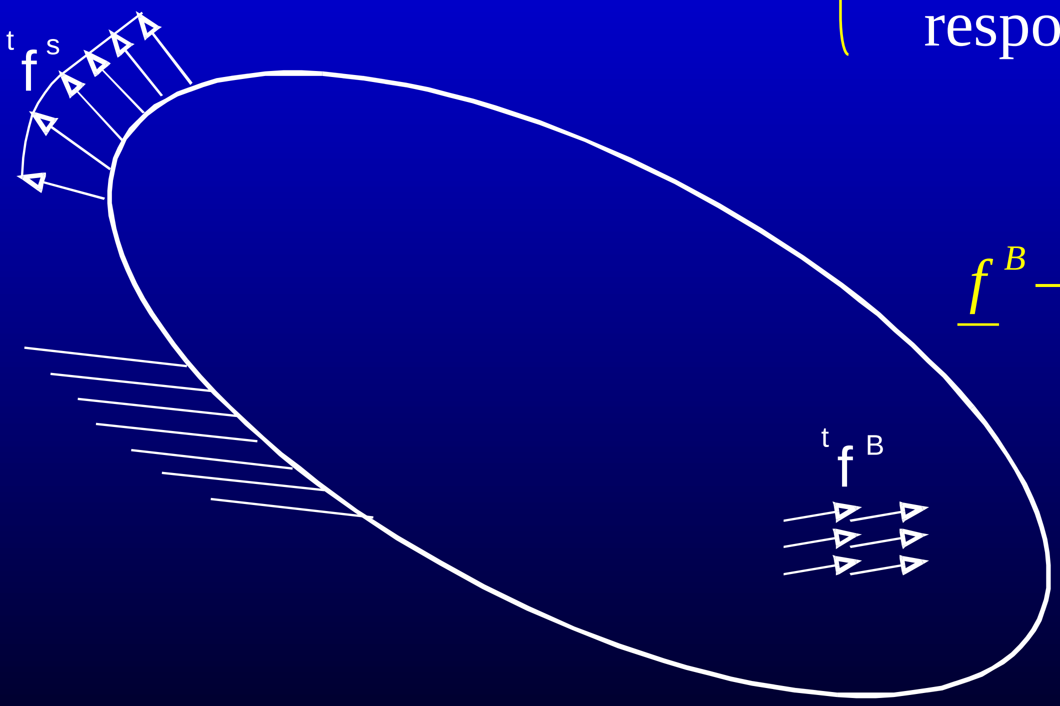
C2 - Elástico, linear e ortotrópico

⋮

Modeling in Dynamic Analysis

Loads varing with time

- "slowly" - quasi-static analysis (resulting acceleration are "small")
- "rapidly" - dynamic analysis (acceleration should be accounted for and significantly influence the response)



$\underline{f}^{B} - {}^{t}\rho \ddot{\underline{u}}(t)$ d´Alembert´s principle

$$\underline{R}_B = \sum_m \int_{V^m} \underline{H}^{(m)^T} \left[\underline{f}^{B(m)} - \rho^{(m)} \underline{H}^{(m)} \underline{\ddot{U}}\right] dV^{(m)}$$

$$\underline{M} = \sum_m \int_{V^m} \rho^{(m)} \underline{H}^{(m)^T} \underline{H}^{(m)} dV^{(m)}$$

$$\underline{M}\,\underline{\ddot{U}} + \underline{K}\,\underline{U} = \underline{R}$$

Linear Analysis

$$\underline{M}\,\underline{\ddot{U}}(t) + \underline{K}\,\underline{U}(t) = \underline{R}(t)$$

If damping is taken into account

$$\underline{f}^B - \rho\underline{\ddot{u}} - k\underline{\dot{u}}$$

$k\underline{\dot{u}}$ → Velocity dependent damping forces

$$\underline{M}\,\underline{\ddot{U}}(t) + \underline{C}\,\underline{\dot{U}}(t) + \underline{K}\,\underline{U}(t) = \underline{R}(t)$$

And initial conditions

$$\underline{U}(0) = \underline{U}_0$$

$$\underline{\dot{U}}(0) = \underline{\dot{U}}_0$$

- Direct Integration Methods
- Mode Superposition

Apoio a PEF-2401 – slides selecionados de apresentação preparada pelo Prof. João Cyro

Sistemas com vários graus de liberdade

$$\mathbf{M}\ddot{\mathbf{U}} + \mathbf{K}\mathbf{U} = \mathbf{0} \qquad \text{com}: \begin{cases} \mathbf{U}(0) = \mathbf{U}_0 \\ \dot{\mathbf{U}}(0) = \dot{\mathbf{U}}_0 \end{cases}$$

Solução : $\mathbf{U} = \hat{\mathbf{U}} \cos(\omega t - \theta)$

Substituindo a solução na equação do movimento:

$$\left(\mathbf{K} - \omega^2 \mathbf{M}\right)\hat{\mathbf{U}} = \mathbf{0}$$

Sistemas com vários graus de liberdade - sistema não-amortecido

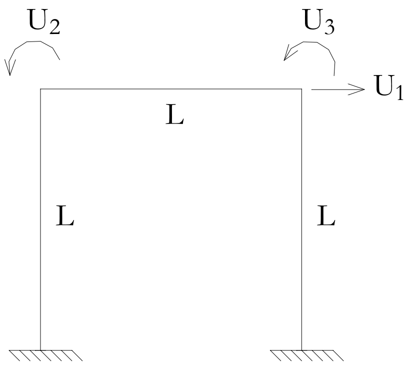
$$\left(\mathbf{K} - \omega^2\mathbf{M}\right)\hat{\mathbf{U}} = \mathbf{0}$$

Problema de autovalores associado :

$$\left|\mathbf{K} - \omega^2\mathbf{M}\right| = 0 \qquad \text{ou}$$

$$\left|\mathbf{A} - \omega^2\mathbf{I}\right| = 0 \quad \textbf{com} \quad \mathbf{A} = \mathbf{M}^{-1}\mathbf{K}$$

VGL: Exemplo 8

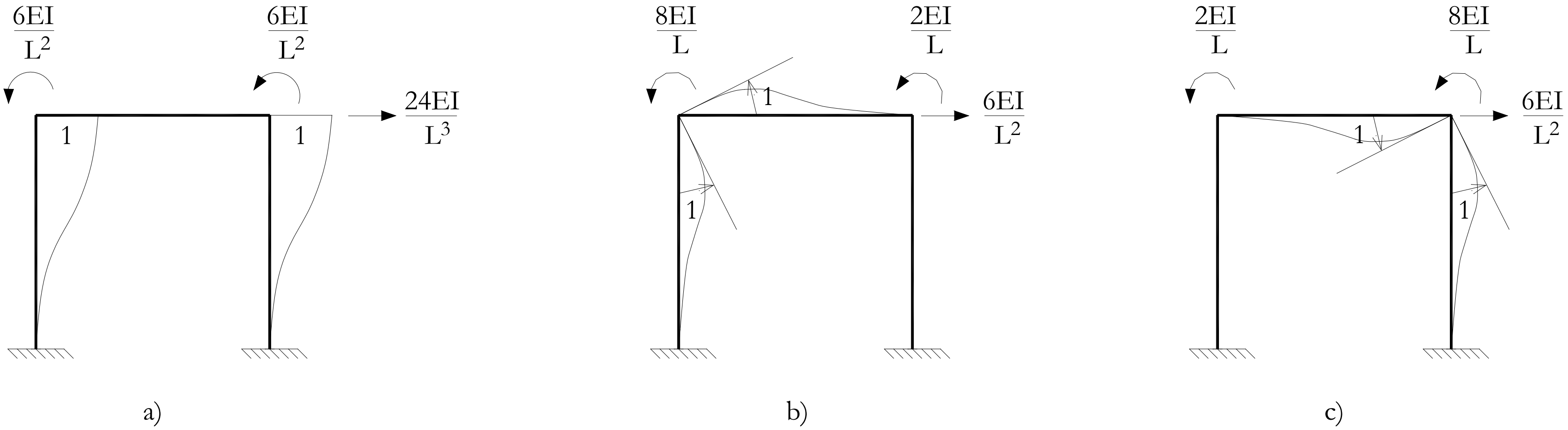


$$EI = 80000 \text{ Nm}^2$$

$$L = 2 \text{ m}$$

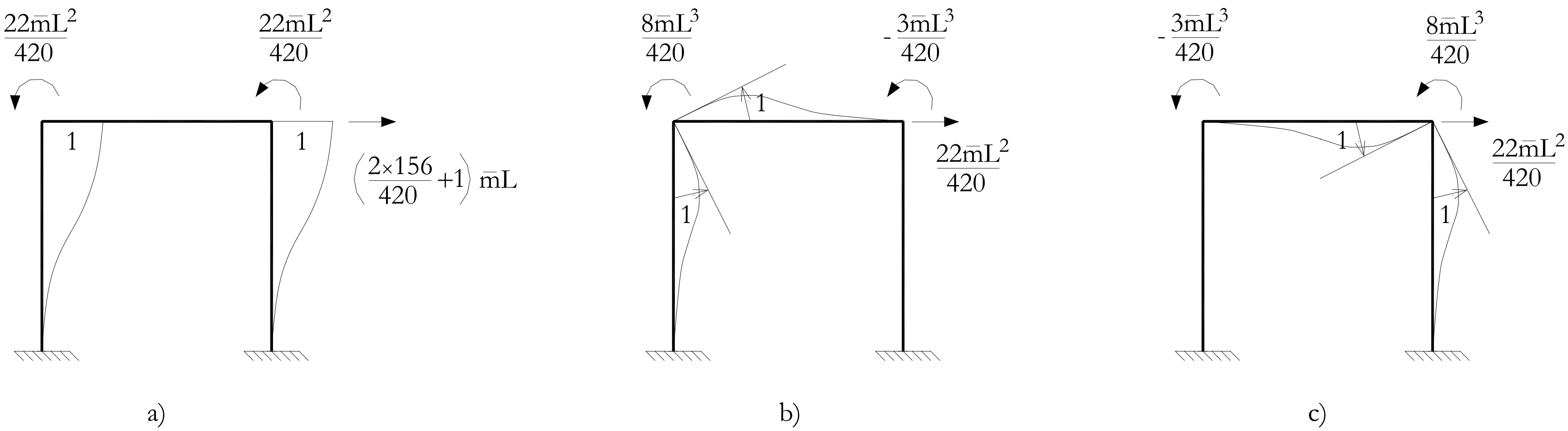
$$m = 50 \text{ kgm}^{-1}$$

VGL - E 8 : Matriz de rigidez



$$\mathbf{K} = \frac{EI}{L^3}\begin{bmatrix} 24 & 6L & 6L \\ 6L & 8L^2 & 2L^2 \\ 6L & 2L^2 & 8L^2 \end{bmatrix}$$

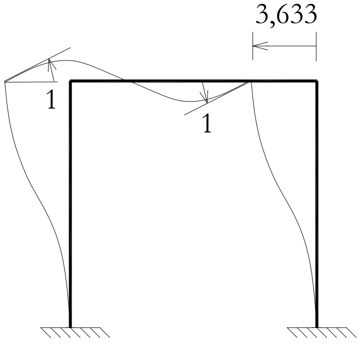
VGL - E 8 : Matriz de massa



$$\mathbf{M} = \frac{mL}{420}\begin{bmatrix} 732 & 22L & 22L \\ 22L & 8L^2 & -3L^2 \\ 22L & -3L^2 & 8L^2 \end{bmatrix}$$

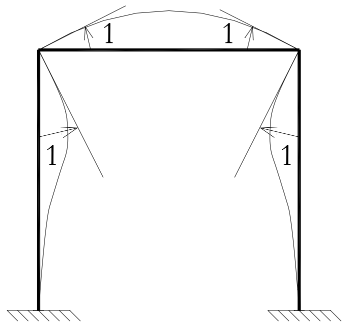
VGL - E 8 :   Modo 1

$$\omega_1 = 32.1\,\mathbf{rd/s} \quad \Rightarrow \quad \hat{\mathbf{U}}_\mathbf{1} = \begin{Bmatrix} -3.633 \\ 1.000 \\ 1.000 \end{Bmatrix}$$



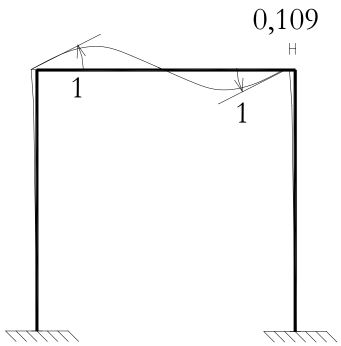
VGL - E 8 :   Modo 2

$$\omega_2 = 151.4\,\mathbf{rd/s} \quad \Rightarrow \quad \hat{\mathbf{U}}_2 = \begin{Bmatrix} 0.000 \\ 1.000 \\ -1.000 \end{Bmatrix}$$



VGL - E 8 : Modo 3

$$\omega_3 = 326.8\,\mathbf{rd/s} \quad \Rightarrow \quad \hat{\mathbf{U}}_3 = \begin{Bmatrix} -0.109 \\ 1.000 \\ 1.000 \end{Bmatrix}$$



Avaliação no ADINA de frequências e modos de vibração

VGL - E 8 : Matriz modal

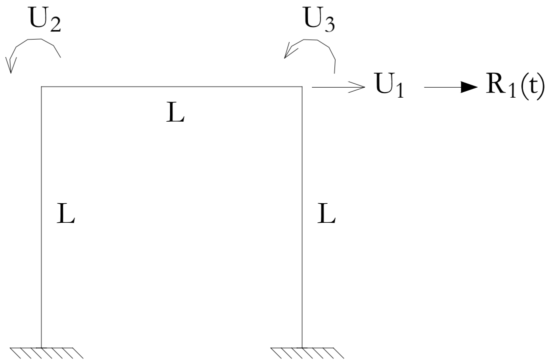
$$\mathbf{\Phi} = \begin{Bmatrix} -3.633 & 0 & -0.109 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{Bmatrix}$$

Propriedades :

$$\mathbf{M}^* = \mathbf{\Phi}^T \mathbf{M} \mathbf{\Phi} = \begin{bmatrix} 2145 & 0 & 0 \\ 0 & 21 & 0 \\ 0 & 0 & 7 \end{bmatrix} \Rightarrow \quad M_{ij}^* = \phi_i^T \mathbf{M} \phi_{\mathbf{j}} = 0$$

$$\mathbf{K}^* = \mathbf{\Phi}^T \mathbf{K} \mathbf{\Phi} = \begin{bmatrix} 2210650 & 0 & 0 \\ 0 & 480535 & 0 \\ 0 & 0 & 751012 \end{bmatrix} \Rightarrow \quad K_{ij}^* = \phi_i^T \mathbf{K} \phi_{\mathbf{j}} = 0$$

MSM: Exemplo 9



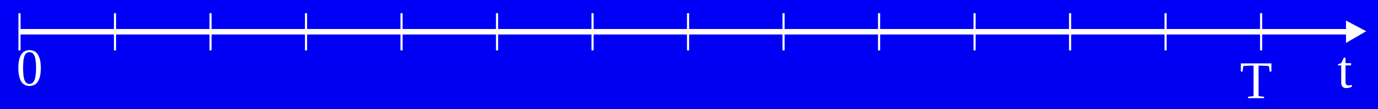
$$EI = 80000 \text{ Nm}^2$$

$$L = 2 \text{ m}$$

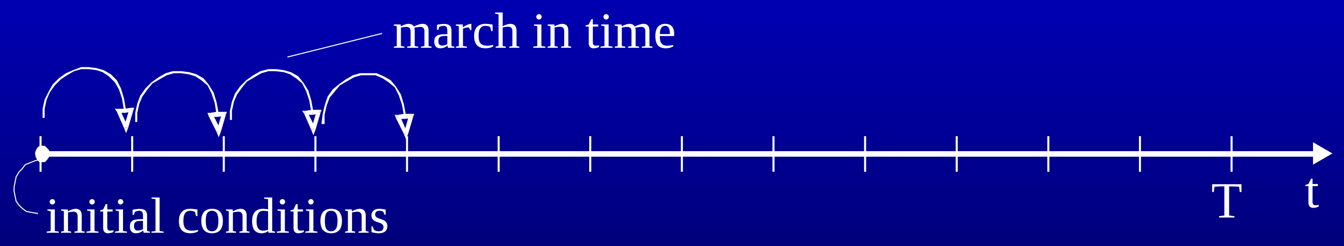
$$m = 50 \text{ kgm}^{-1}$$

$$R_1(t) = 1000 sen(\overline{\omega} t) \quad \text{com} \quad \overline{\omega} = \omega_1$$

Direct Integration Methods

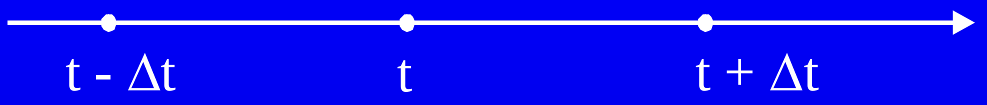


- no transformation of coordinates
- equation is satisfied at discrete times positions
- solution is obtained at these time positions



How we obtain the solution of the “next” time position is dependent of the particular intergration scheme used

Central Difference Method



$$^{t}\ddot{\underline{U}} = \frac{1}{\Delta t^2}\left(^{t-\Delta t}\underline{U} - 2\ ^{t}\underline{U} + {}^{t+\Delta t}\underline{U}\right) \qquad \text{(a)}$$

$$^{t}\dot{\underline{U}} = \frac{1}{2\Delta t}\left(^{t+\Delta t}\underline{U} - {}^{t-\Delta t}\underline{U}\right) \qquad \text{(b)}$$

Substituting (a) and (b) in the “equilibrium” equation at time t

$$\underline{M}\ ^{t}\ddot{\underline{U}} + \underline{C}\ ^{t}\dot{\underline{U}} + \underline{K}\ ^{t}\underline{U} = {}^{t}\underline{R}$$

Then

$$\left(\frac{1}{\Delta t^2}\underline{M} + \frac{1}{2\Delta t}\underline{C}\right) {}^{t+\Delta t}\underline{U} = {}^{t}\underline{R} - \left(\underline{K} - \frac{2}{\Delta t^2}\underline{M}\right) {}^{t}\underline{U} - \left(\frac{1}{\Delta t^2}\underline{M} - \frac{1}{2\Delta t}\underline{C}\right) {}^{t-\Delta t}\underline{U}$$

It is an Explicit Method

- Equilibrium is imposed at time t
- No factorization of the stiffness matrix is required

Since explicit methods requires small time steps they are efficient when one can consider:

- lumped mass matrices
- neglet damping

Then

$$\left(\frac{1}{\Delta t^2}\underline{M}\right)\,{}^{t+\Delta t}\underline{U} = {}^{t}\underline{\hat{R}}$$

$${}^{t}\underline{\hat{R}} = {}^{t}\underline{R} - \left(\underline{K} - \frac{2}{\Delta t^2}\underline{M}\right){}^{t}\underline{U} - \left(\frac{1}{\Delta t^2}\underline{M}\right){}^{t-\Delta t}\underline{U}$$

Resulting

$${}^{t+\Delta t}U_i = {}^{t}\hat{R}_i\left(\frac{\Delta t^2}{m_{ii}}\right)$$

Since triangularization of K is not required, it is not necessary to assemble the stiffness matrix

$$\underline{K}\;{}^{t}\underline{U} = \sum_i \underline{K}^{(i)}\,{}^{t}\underline{U} = \sum_i {}^{t}\underline{F}^{(i)}$$

Implicit Method

- Newmark - Widely use scheme

$$ {}^{t+\Delta t}\underline{\dot{U}} = {}^{t}\underline{\dot{U}} + [(1-\delta)\,{}^{t}\underline{\ddot{U}} + \delta\,{}^{t+\Delta t}\underline{\ddot{U}}]\Delta t $$

$$ {}^{t+\Delta t}\underline{U} = {}^{t}\underline{U} + {}^{t}\underline{\dot{U}}\Delta t + \left[\left(\frac{1}{2}-\alpha\right){}^{t}\underline{\ddot{U}} + \alpha\,{}^{t+\Delta t}\underline{\ddot{U}}\right]\Delta t $$

For example consider $\delta = \frac{1}{2}, \alpha = \frac{1}{4}$ then

$$ \underbrace{{}^{t+\Delta t}\underline{\dot{U}}}_{v} = \underbrace{{}^{t}\underline{\dot{U}}}_{v_0} + \underbrace{[\overbrace{(1-\delta)}^{1/2}\,{}^{t}\underline{\ddot{U}} + \overbrace{\delta}^{1/2}\,{}^{t+\Delta t}\underline{\ddot{U}}]}_{\alpha\,\Delta t}\Delta t $$

$$ \underbrace{{}^{t+\Delta t}\underline{U}}_{s} = \underbrace{{}^{t}\underline{U}}_{s_0} + \underbrace{{}^{t}\underline{\dot{U}}\Delta t}_{v_0\,\Delta t} + \underbrace{\left[\left(\frac{1}{2}-\alpha\right){}^{t}\underline{\ddot{U}} + \alpha\,{}^{t+\Delta t}\underline{\ddot{U}}\right]\Delta t^2}_{1/2\,\alpha\,(\Delta t)^2} $$

Constant-average accelerations

Integração direta no ADINA

Sistemas com vários graus de liberdade - sistema amortecido

$$\mathbf{M}\ddot{\mathbf{U}} + \mathbf{C}\dot{\mathbf{U}} + \mathbf{K}\mathbf{U} = \mathbf{0} \qquad \textbf{com :} \begin{cases} \mathbf{U}(0) = \mathbf{U}_0 \\ \dot{\mathbf{U}}(0) = \dot{\mathbf{U}}_0 \end{cases}$$

$$\mathbf{C} \ \ \text{qualquer} \ \Rightarrow \begin{cases} \text{problema de autovalores} \\ \text{no campo complexo} \end{cases}$$

$$\mathbf{C}_{tipo\ Rayleigh} = a_0\mathbf{M} + a_1\mathbf{K} \ \Rightarrow \begin{cases} \textbf{problema de autovalores} \\ \textbf{do sistema amortecido} \end{cases}$$

Método da Superposição Modal

$$\mathbf{M\ddot{U}} + \mathbf{C\dot{U}} + \mathbf{KU} = \mathbf{R}(t) \qquad \textbf{com :} \begin{cases} \mathbf{U}(0) = \mathbf{U}_0 \\ \mathbf{\dot{U}}(0) = \mathbf{\dot{U}}_0 \end{cases}$$

Mudança de variáveis: $\mathbf{U}(t) = \mathbf{\Phi Y} = \sum_{i=1}^{n} \phi_i \mathbf{Y_i}(t)$

$$\mathbf{M\Phi\ddot{Y}} + \mathbf{C\Phi\dot{Y}} + \mathbf{K\Phi Y} = \mathbf{R}(t)$$

$$\mathbf{\Phi}^T\mathbf{M\Phi\ddot{Y}} + \mathbf{\Phi}^T\mathbf{C\Phi\dot{Y}} + \mathbf{\Phi}^T\mathbf{K\Phi Y} = \mathbf{\Phi}^T\mathbf{R}(t)$$

$$\mathbf{M}^*\mathbf{\ddot{Y}} + \mathbf{C}^*\mathbf{\dot{Y}} + \mathbf{K}^*\mathbf{Y} = \mathbf{R}^*(t)$$

Método da Superposição Modal

Amortecimento do tipo Rayleigh:

$$\mathbf{C} = a_0\mathbf{M} + a_1\mathbf{K}$$

$$\mathbf{C}^* = \mathbf{\Phi}^T\mathbf{C}\mathbf{\Phi} = \mathbf{\Phi}^T\left(a_0\mathbf{M} + a_1\mathbf{K}\right)\mathbf{\Phi} = a_0\mathbf{M}^* + a_1\mathbf{K}^*$$

$$\mathbf{C}_i^* = a_0\mathbf{M}_i^* + a_1\mathbf{K}_i^*$$

$$\xi_i = \frac{\mathbf{C}_i^*}{2\mathbf{M}_i^*\omega_i}$$

Método da Superposição Modal
para amortecimento do tipo Rayleigh

Problema de n graus de liberdade

=

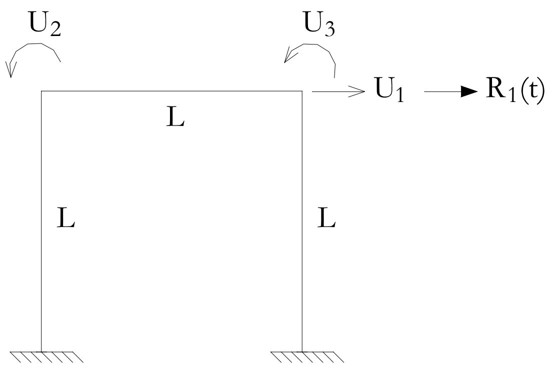
n problemas com 1 grau de liberdade

$$\mathbf{M}_i^*\ddot{\mathbf{Y}}_i + 2\xi_i\mathbf{M}_\mathbf{i}^*\omega_i\dot{\mathbf{Y}}_i + \mathbf{K}_i^*\mathbf{Y}_i = \mathbf{R}_i^*(t)$$

$$\textbf{com} \quad : \quad \mathbf{Y}_{i0} = \frac{\phi_i^T\mathbf{M}\mathbf{U}_0}{\mathbf{M}_i^*} \quad \textbf{e} \quad \dot{\mathbf{Y}}_{i0} = \frac{\phi_i^T\mathbf{M}\dot{\mathbf{U}}_0}{\mathbf{M}_i^*}$$

$$\textbf{visto que :} \quad \phi_i^T\mathbf{M}\mathbf{U}_0 = \phi_i^T\mathbf{M}\boldsymbol{\Phi}\mathbf{Y}_0 = \phi_i^T\mathbf{M}\phi_i\mathbf{Y}_{i0}$$

MSM: Exemplo 9



$$EI = 80000 \text{ Nm}^2$$

$$L = 2 \text{ m}$$

$$m = 50 \text{ kgm}^{-1}$$

$$R_1(t) = 100 sen(\overline{\omega} t) \quad \text{com} \quad \overline{\omega} = \omega_1$$

MSM: Exemplo 9

$$\Phi = \begin{bmatrix} -3.6181324775721 4d0 & 1.8558971585539d\text{-}7 & -0.10877823357968d0 \\ 1.0d0 & 1.0d0 & 1.0d0 \\ 1.00000258496357d0 & -1.00000012829729d0 & 0.99999997976364d0 \end{bmatrix}$$

$$\mathbf{M}^* = \begin{bmatrix} 2139.46 & 0 & 0 \\ 0 & 21.0 & 0 \\ 0 & 0 & 7.02 \end{bmatrix} \qquad \mathbf{K}^* = \begin{bmatrix} 2205108 & 0 & 0 \\ 0 & 480000 & 0 \\ 0 & 0 & 750626 \end{bmatrix}$$

$$\mathbf{C}^* = \begin{bmatrix} 6879.94 & 0 & 0 \\ 0 & 1497.6 & 0 \\ 0 & 0 & 2341.95 \end{bmatrix} \Rightarrow \begin{cases} \xi_1 = 0{,}050 \\ \xi_2 = 0{,}235 \\ \xi_3 = 0{,}511 \end{cases}$$

$$\mathbf{R}_0^* = \begin{bmatrix} -3618.13247757214d0 \\ 1.8558971585539d\text{-}4 \\ -108.778233579682d0 \end{bmatrix}$$

MSM: Exemplo 9
parcela permanente da resposta

$$\mathrm{Y}_1(\mathrm{t}) = -\ 0.01638302582894\mathrm{d}0\ \sin(32.1\ \mathrm{t} - 1.568121279932 92\mathrm{d}0)$$

$$\mathrm{Y}_2(\mathrm{t}) = 4.02662053664706\mathrm{d}{-10}\ \sin(32.1\ \mathrm{t} - 0.104291094067 32\mathrm{d}0)$$

$$\mathrm{Y}_3(\mathrm{t}) = -\ 1.45584235043844\mathrm{d}{-4}\ \sin(32.1\ \mathrm{t} - 0.10081750827976\mathrm{d}0)$$

$$\mathbf{U} = \begin{Bmatrix} U_1(t) \\ U_2(t) \\ U_3(t) \end{Bmatrix} = \mathbf{\Phi} \begin{Bmatrix} Y_1(t) \\ Y_2(t) \\ Y_3(t) \end{Bmatrix}$$

Stability and Accuracy

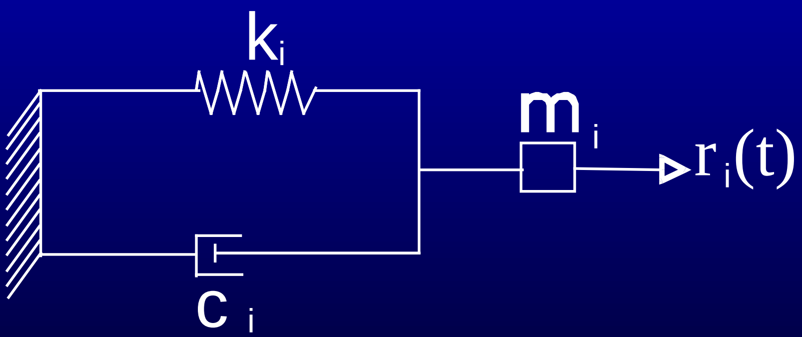
$$\underline{M}\,\underline{\ddot{U}}(t) + \underline{C}\,\underline{\dot{U}}(t) + \underline{K}\,\underline{U}(t) = \underline{R}(t)$$

$$\underline{K}\,\underline{\phi} = w^2\,\underline{M}\,\underline{\phi} \quad \begin{cases} w_i \quad \text{frequencies} \\ \underline{\phi}_1, \underline{\phi}_2, \ldots, \underline{\phi}_n \quad \text{node shapes} \end{cases}$$

$$\underline{U}(t) = \sum_{i=1}^{n} x_i(t)\,\underline{\phi}_i$$

and defining $r_i = \underline{\phi}_i^{\,T}\,\underline{R}$

$$\ddot{x}_i + 2\xi_i\omega_i\dot{x}_i + (\omega_i)^2 x_i = r_i$$



$$m_i = \underline{\phi}_i^{\,T}\,\underline{M}\,\underline{\phi}_i$$

$$c_i = \underline{\phi}_i^{\,T}\,\underline{C}\,\underline{\phi}_i$$

$$k_i = \underline{\phi}_i^{\,T}\,\underline{K}\,\underline{\phi}_i$$

$$w_i = \sqrt{\frac{k_i}{m_i}} \quad ; \quad \xi_i = \frac{c_i}{2m_i w_i}$$

If all equation (i=1,2, ..., n) are integrated numerically with the same Δt then the solution of the mode superposition approach is exactly the same as integrating directly

$$\underline{M}\,\ddot{\underline{U}}(t)+\underline{C}\,\dot{\underline{U}}(t)+\underline{K}\,\underline{U}(t)=\underline{R}(t)$$

with the same numerical scheme and with this time step Δt.

Therefore the considerations of accurancy and stability can be made in the uncoupled system.

Accuracy

- How many modes we want to represent accurately?
  - Depends on the frequency content of the load

Finite element discretization

n degrees of fredom

p is the number of modes to be represented accurately

In general: p << n

The period of the p^{th} mode $T_p = \dfrac{2\pi}{w_p}$

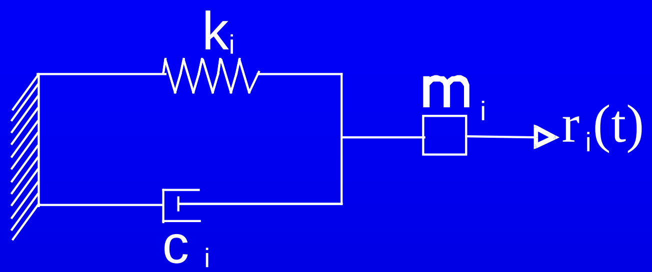
Accuracy $\Delta t = \dfrac{T_p}{20} = \dfrac{\pi}{10 w_p}$

However for mode n

$$\frac{\Delta t}{T_n} = \frac{\Delta t}{T_n}\frac{T_p}{T_p} = \underbrace{\frac{\Delta t}{T_p}}_{\approx \frac{1}{20}}\underbrace{\frac{T_p}{T_n}}_{\text{can be} \approx 1000} \underbrace{=}_{\text{can be}} 50$$

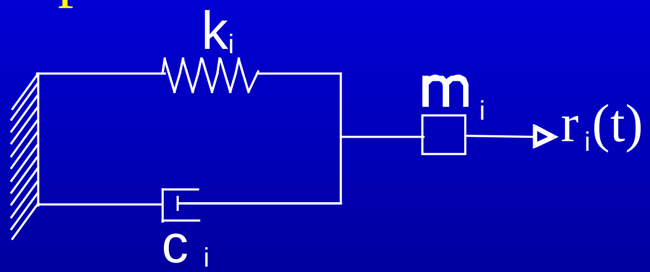
No accurancy for modes $\rightarrow$ n. But the response on these higher modes need to be bounded $\Rightarrow$ Stability.

Analytically



$$u(0) = 0$$
$$\dot{u}(0) = 0$$

In the computer



$$u(0) \neq 0$$
$$\dot{u}(0) \neq 0$$

(finite digit computations)

Unconditionally stable - the solution remains bounded for any Δt (r_i=0)

Conditionally stable - the solution remains bounded as long as $\Delta t \leq \Delta t_{cr}$ (r_i=0)

Δt_{cr} - critical time step

Central Difference Method

$$\Delta t_{cr} = \frac{T_n}{\pi}$$

Newmark Method

Unconditionally stable as long as $\delta \geq 0{,}5 \quad \alpha \geq 0{,}25(\delta + 0{,}5)^2$

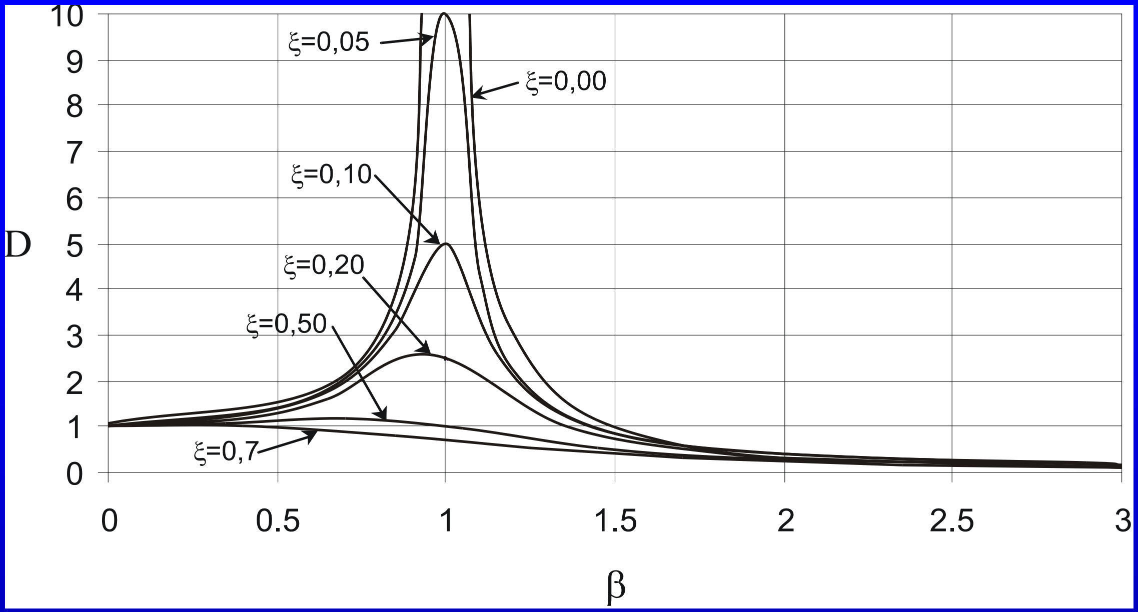
Note that for CDM

$$\frac{\Delta t_{cr}}{T_n} \leq \frac{1}{\pi}$$

$$\frac{\Delta t_{cr}}{T_n}\frac{T_p}{T_p} = \frac{\Delta t_{cr}}{T_p}\underbrace{\frac{T_n}{T_p}}_{\approx 1000} \leq \frac{1}{\pi}$$

$$\frac{\Delta t_{cr}}{T_p}1000 \leq \frac{1}{\pi} \Rightarrow \Delta t_{cr} \leq \frac{T_p}{\pi}\frac{1}{1000} \Rightarrow \Delta t_{cr} \leq \frac{T_p}{3141{,}6}$$

(much smaller than T_p /20)



$$\ddot{x}_i + 2\xi_i \omega_i \dot{x}_i + (\omega_i)^2 x_i = r_i$$

$$\beta = \frac{\overline{\omega}}{\omega}$$

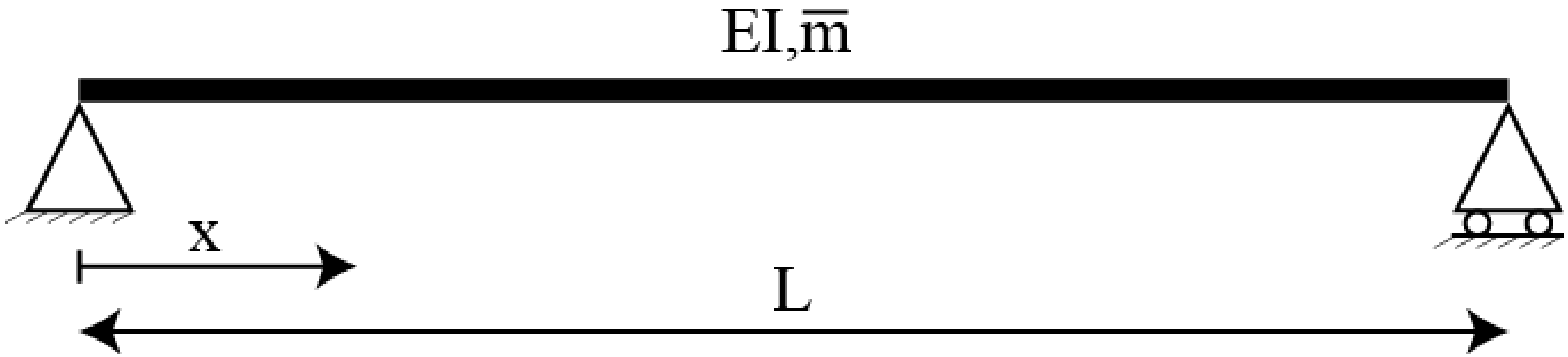
$$\ddot{x}_i + 2\xi_i \omega_i \dot{x}_i + (\omega_i)^2 x_i = r_i$$

$$r_i = \underline{\phi}_i^T \underline{R}$$

- $\dfrac{\overline{\omega}}{\omega}$ grande, resposta bem pequena
- $\dfrac{\overline{\omega}}{\omega}$ pequeno, resposta praticamente estática

Qual é o conteúdo em freqüência de r_i ?

Selecionar Δt para representar p modos com precisão.



Modos de vibrar: $\emptyset_n = A \sin \frac{n\pi x}{L}, n = 1,2,3, \ldots$

Frequências naturais: $\omega_n = n^2 \pi^2 \sqrt{\frac{EI}{\bar{m} L^4}}, n = 1,2,3, \ldots$

Frequência natural	BE (4 elementos)	Chapa (20 elementos)	Valor analítico
1	3,9482	3,9560	3,9478
2	15,8433	15,9240	15,7914
3	36,1262	36,2169	35,5306
4	69,9223	65,3914	63,1655
5	109,5290	104,2780	98,6960
6	110,9720	108,1190	142,1223
7	175,1270	153,9630	193,4442
8	261,6290	215,5530	252,6619
9	317,7380	288,9910	319,7752
10	345,5230	324,3660	394,7842
11	627,6540	367,2850	477,6889
12	907,6420	411,3270	568,4892
13	-	540,7190	667,1853
...	-	...	...
207	-	54 878,5000	16 9161,1000
Obs.: Valores de frequências em rad/s.			