AULA DE EXERCICIOS MAP 2310

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26. Consider the initial value problem (see Example 4)

$$y'' + 5y' + 6y = 0, \qquad y(0) = 2, \quad y'(0) = \beta,$$

where $\beta > 0$.

(a) Solve the initial value problem.

(b) Determine the coordinates t_m and y_m of the maximum point of the solution as functions of β.

(c) Determine the smallest value of β for which $y_m \geq 4$.

(d) Determine the behavior of t_m and y_m as $\beta \to \infty$.

$26(a)$. The characteristic roots are $r = -3, -2$. The solution of the initial value problem is $y(t) = (6+\beta)e^{-2t} - (4+\beta)e^{-3t}$.

(b). The maximum point has coordinates $t_0 = ln\left[\frac{3(4+\beta)}{2(6+\beta)}\right]$, $y_0 = \frac{4(6+\beta)^3}{27(4+\beta)^2}$.

(c). $y_0 = \frac{4(6+\beta)^3}{27(4+\beta)^2} \geq 4$, as long as $\beta \geq 6 + 6\sqrt{3}$.

(d). $\lim_{\beta\to\infty} t_0 = ln\,\frac{3}{2}$. $\lim_{\beta\to\infty} y_0 = \infty$.

24. Consider the initial value problem

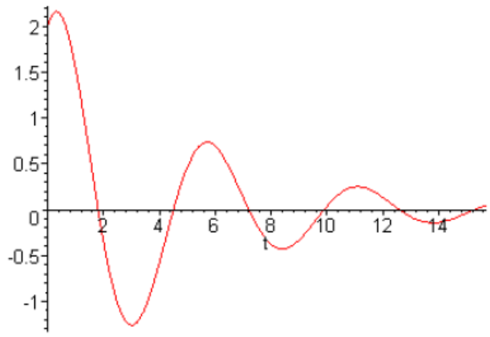
$$5u'' + 2u' + 7u = 0, \qquad u(0) = 2, \quad u'(0) = 1.$$

(a) Find the solution $u(t)$ of this problem.

(b) Find the smallest T such that $|u(t)| \leq 0.1$ for all $t > T$.

24(a). The characteristic equation is $5r^2 + 2r + 7 = 0$, with roots $r = -\frac{1}{5} \pm i\frac{\sqrt{34}}{5}$. The solution is $u = c_1 e^{-t/5} \cos\frac{\sqrt{34}}{5}t + c_2 e^{-t/5} \sin\frac{\sqrt{34}}{5}t$. Invoking the given initial conditions, we obtain the equations for the coefficients: $c_1 = 2$, $-2 + \sqrt{34}\, c_2 = 5$. That is, $c_1 = 2$, $c_2 = 7/\sqrt{34}$. Hence the specific solution is

$$u(t) = 2e^{-t/5} \cos\frac{\sqrt{34}}{5}t + \frac{7}{\sqrt{34}} e^{-t/5} \sin\frac{\sqrt{34}}{5}t.$$



(b). Based on the graph of $u(t)$, T is in the interval $14 < t < 16$. A numerical solution on that interval yields $T \approx 14.5115$.

15. Consider the initial value problem

$$4y'' + 12y' + 9y = 0, \qquad y(0) = 1, \quad y'(0) = -4.$$

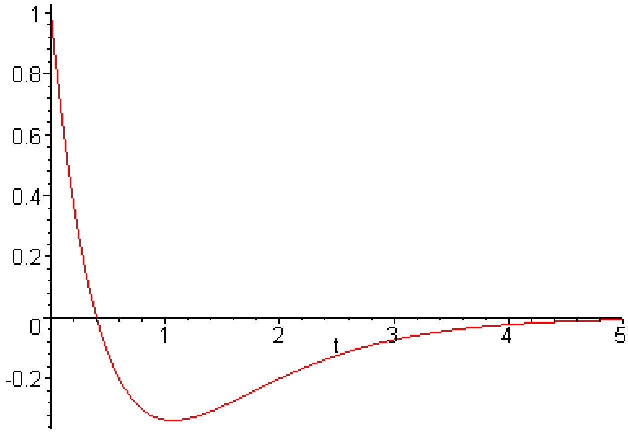
(a) Solve the initial value problem and plot its solution for $0 \le t \le 5$.

(b) Determine where the solution has the value zero.

(c) Determine the coordinates (t_0, y_0) of the minimum point.

(d) Change the second initial condition to $y'(0) = b$ and find the solution as a function of b. Then find the critical value of b that separates solutions that always remain positive from those that eventually become negative.

15(a). The characteristic equation is $4r^2 + 12r + 9 = 0$, with the *double* root $r = -\frac{3}{2}$. The general solution is $y(t) = c_1 e^{-3t/2} + c_2 t\, e^{-3t/2}$. Invoking the first initial condition, it follows that $c_1 = 1$. Now $y'(t) = (-3/2 + c_2)e^{2t/3} - \frac{3}{2}c_2 t\, e^{2t/3}$. The second initial condition requires that $-3/2 + c_2 = -4$, or $c_2 = -5/2$. Hence the specific solution is $y(t) = e^{-3t/2} - \frac{5}{2}t\, e^{-3t/2}$.



(b). The solution crosses the *x-axis* at $t = 0.4$.

(c). The solution has a minimum at the point $(16/15\,,\ -5e^{-8/5}/3)$.

(d). Given that $y'(0) = b$, we have $-3/2 + c_2 = b$, or $c_2 = b + 3/2$. Hence the solution is $y(t) = e^{-3t/2} + \left(b + \frac{3}{2}\right)t\, e^{-3t/2}$. Since the *second* term dominates, the *long-term* solution depends on the *sign* of the coefficient $b + \frac{3}{2}$. The critical value is $b = -\frac{3}{2}$.

In each of Problems 19 through 26:

(a) Determine a suitable form for $Y(t)$ if the method of undetermined coefficients is to be used.

23. $y'' - 4y' + 4y = 2t^2 + 4te^{2t} + t\sin 2t$

23. The characteristic roots are $r = 2, 2$. Hence $y_c(t) = c_1 e^{2t} + c_2 t e^{2t}$. Consider the functions $g_1(t) = 2t^2$, $g_2(t) = 4te^{2t}$, and $g_3(t) = t\, sin\, 2t$. The corresponding forms of the respective parts of the particular solution are $Y_1(t) = A_0 + A_1 t + A_2 t^2$, $Y_2(t) =$ $= e^{2t}(B_2 t^2 + B_3 t^3)$, and $Y_3(t) = t(C_1 cos\, 2t + C_2 sin\, 2t) + (D_1 cos 2t + D_2 sin 2t)$. Substitution into the equation and comparing the coefficients results in

$$Y(t) = \frac{1}{4}(3 + 4t + 2t^2) + \frac{2}{3}t^3 e^{2t} + \frac{1}{8}t\, cos\, 2t + \frac{1}{16}(cos\, 2t - sin\, 2t).$$

In each of Problems 13 through 20 verify that the given functions y_1 and y_2 satisfy the corresponding homogeneous equation; then find a particular solution of the given nonhomogeneous equation. In Problems 19 and 20, g is an arbitrary continuous function.

13. $t^2y'' - 2y = 3t^2 - 1, \quad t > 0; \qquad y_1(t) = t^2, \quad y_2(t) = t^{-1}$

13. Note first that $p(t) = 0$, $q(t) = -2/t^2$ and $g(t) = (3t^2 - 1)/t^2$. The functions $y_1(t)$ and $y_2(t)$ are solutions of the homogeneous equation, verified by substitution. The Wronskian of these two functions is $W(y_1, y_2) = -3$. Using the method of *variation of parameters*, the particular solution is $Y(t) = u_1(t)\, y_1(t) + u_2(t)\, y_2(t)$, in which

$$\begin{aligned} u_1(t) &= -\int \frac{t^{-1}(3t^2-1)}{t^2\, W(t)} dt \\ &= t^{-2}/6 + \ln t \end{aligned}$$

$$\begin{aligned} u_2(t) &= \int \frac{t^2(3t^2-1)}{t^2\, W(t)} dt \\ &= -t^3/3 + t/3 \end{aligned}$$

Therefore $Y(t) = 1/6 + t^2 \ln t - t^2/3 + 1/3$. Hence the general solution is

$$y(t) = c_1 t^2 + c_2 t^{-1} + t^2 \ln t + 1/2\,.$$