Exercises

1. Show, using the formal manipulations for Itô's chain rule discussed in Chapter 1, that

$$Y(t) := e^{W(t)-\frac{t}{2}}$$

solves the stochastic differential equation

$$\begin{cases} dY = Y dW \\ Y(0) = 1. \end{cases}$$

(Hint: If $X(t) := W(t) - \frac{t}{2}$, then $dX = -\frac{dt}{2} + dW$.)

2. Show that

$$S(t) = s_0 e^{\sigma W(t) + \left(\mu - \frac{\sigma^2}{2}\right)t}$$

solves

$$\begin{cases} dS = \mu S dt + \sigma S dW \\ S(0) = s_0. \end{cases}$$

3. (i) Let $(\Omega, \mathcal{U}, P)$ be a probability space and let $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n \subseteq \ldots$ be events. Show that

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{m\to\infty} P(A_m).$$

(Hint: Look at the disjoint events $B_n := A_{n+1} - A_n$.)

(ii) Likewise, show that if $A_1 \supseteq A_2 \supseteq \cdots \supseteq A_n \supseteq \cdots$, then

$$P\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{m\to\infty} P(A_m).$$

4. Let Ω be any set and $\mathcal{A}$ any collection of subsets of Ω. Show that there exists a unique smallest σ-algebra $\mathcal{U}$ of subsets of Ω containing $\mathcal{A}$. We call $\mathcal{U}$ the σ-algebra *generated* by $\mathcal{A}$.

   (Hint: Take the intersection of all the σ-algebras containing $\mathcal{A}$.)

5. Show that if $A_1, \dots, A_n$ are events, then

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i) - \sum_{1 \le i < j \le n} P(A_i \cap A_j) + \sum_{1 \le i < j < k \le n} P(A_i \cap A_j \cap A_k) - \cdots + (-1)^{n-1} P(A_1 \cap A_2 \cap \cdots \cap A_n).$$

   (Hint: Do the case $n = 2$ first and then the general case by induction.)

6. Let $X = \sum_{i=1}^{k} a_i \chi_{A_i}$ be a simple random variable, where the real numbers a_i are distinct, the events A_i are pairwise disjoint, and $\Omega = \bigcup_{i=1}^{k} A_i$. Let $\mathcal{U}(X)$ be the σ-algebra generated by X.

   (i) Describe precisely which sets are in $\mathcal{U}(X)$.

   (ii) Suppose the random variable Y is $\mathcal{U}(X)$-measurable. Show that Y is constant on each set A_i.

   (iii) Show that therefore Y can be written as a function of X.

7. Verify:

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}, \quad \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} x e^{-\frac{(x-m)^2}{2\sigma^2}} dx = m,$$

$$\frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} (x-m)^2 e^{-\frac{(x-m)^2}{2\sigma^2}} dx = \sigma^2.$$

8. Suppose A and B are independent events in some probability space. Show that A^c and B are independent. Likewise, show that A^c and B^c are independent.

9. Suppose we have three cards: one is red on both sides, one is red on one side and white on the other side, and one is white on both sides.

   (i) Pick a card and then one of its sides at random. What is the probability it is red?

   (ii) Given that the side of the card is red, what is the probability that the other side is red?

10. Suppose that $A_1, A_2, \ldots, A_m$ are disjoint events, each of positive probability, such that $\Omega = \bigcup_{j=1}^m A_j$. Prove *Bayes' formula*:

$$P(A_k \mid B) = \frac{P(B \mid A_k)P(A_k)}{\sum_{j=1}^m P(B \mid A_j)P(A_j)}$$

for $k = 1, \ldots, m$ provided $P(B) > 0$.

11. During one fall semester 105 women applied to Miskatonic University, of whom 76 were accepted, and 400 men applied, of whom 230 were accepted.

During the subsequent spring semester, 300 women applied, of whom 100 were accepted, and 112 men applied, of whom 21 were accepted.

Calculate numerically

(a) the probability of a female applicant being accepted during the fall,
(b) the probability of a male applicant being accepted during the fall,
(c) the probability of a female applicant being accepted during the spring,
(d) the probability of a male applicant being accepted during the spring.

Consider now the total applicant pool for both semesters together and calculate

(e) the probability of a female applicant being accepted,
(f) the probability of a male applicant being accepted.

Are the university's admission policies biased towards females or towards males?

12. Let X be a real-valued, $N(0,1)$ random variable, and set $Y := X^2$. Calculate the density g of the distribution function for Y.

(Hint: You must find g so that $P(-\infty < Y \leq a) = \int_{-\infty}^{a} g\,dy$ for all a.)

13. Take $\Omega = [0,1] \times [0,1]$, with $\mathcal{U}$ the Borel sets and P Lebesgue measure. Let $g : [0,1] \to \mathbb{R}$ be a continuous function.

Define the random variables

$$X_1(\omega) := g(x_1), \quad X_2(\omega) := g(x_2) \quad \text{for } \omega = (x_1, x_2) \in \Omega.$$

Show that X_1 and X_2 are independent and identically distributed.

14. Let $f : [0,1] \to \mathbb{R}$ be continuous and define the *Bernstein polynomial*

$$b_n(x) := \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k(1-x)^{n-k}.$$

Prove that $b_n \to f$ uniformly on $[0,1]$ as $n \to \infty$, by providing the details for the following steps.

(i) Since f is uniformly continuous, for each $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that $|f(x) - f(y)| \le \epsilon$ if $|x - y| \le \delta(\epsilon)$.

(ii) Given $x \in [0,1]$, take a sequence of independent random variables X_k such that $P(X_k = 1) = x, P(X_k = 0) = 1 - x$. Write $S_n = X_1 + \cdots + X_n$. Then $b_n(x) = E(f(\frac{S_n}{n}))$.

(iii) Therefore

$$\begin{aligned}|b_n(x) - f(x)| &\le E(|f(\tfrac{S_n}{n}) - f(x)|) \\ &= \int_A |f(\tfrac{S_n}{n}) - f(x)|\, dP + \textstyle\int_{A^c} |f(\frac{S_n}{n}) - f(x)|\, dP,\end{aligned}$$

for $A := \{\omega \in \Omega \mid |\frac{S_n}{n} - x| \le \delta(\epsilon)\}$.

(iv) Then show that

$$|b_n(x) - f(x)| \le \epsilon + \tfrac{2M}{\delta(\epsilon)^2} V(\tfrac{S_n}{n}) = \epsilon + \tfrac{2M}{n\delta(\epsilon)^2} V(X_1),$$

for $M = \max |f|$. Conclude that $b_n \to f$ uniformly.

15. Let X and Y be independent random variables, and suppose that f_X and f_Y are the density functions for X, Y. Show that the density function for $X + Y$ is

$$f_{X+Y}(z) = \int_{-\infty}^{\infty} f_X(z - y) f_Y(y)\, dy.$$

(Hint: If $g : \mathbb{R} \to \mathbb{R}$, we have

$$E(g(X+Y)) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y) g(x+y)\, dxdy,$$

where $f_{X,Y}$ is the joint density function of X, Y.)

16. Let X and Y be two independent positive random variables, each with density

$$f(x) = \begin{cases} e^{-x} & \text{if } x \ge 0 \\ 0 & \text{if } x < 0. \end{cases}$$

Find the density of $X + Y$.

17. Show that
$$\lim_{n\to\infty}\int_0^1\int_0^1\cdots\int_0^1 f(\tfrac{x_1+\cdots+x_n}{n})\,dx_1dx_2\ldots dx_n = f(\tfrac{1}{2})$$
for each continuous function f.

(Hint: Let $X_1,\ldots,X_n,\ldots$ be independent random variables, each of which has density function $f_i(x) = 1$ if $0 \le x \le 1$ and $= 0$ otherwise. Then $P(|\frac{X_1+\cdots+X_n}{n} - \frac{1}{2}| > \epsilon) \le \frac{1}{\epsilon^2}V(\frac{X_1+\cdots+X_n}{n}) = \frac{1}{12\epsilon^2 n}$.)

18. Prove that

(i) $E(E(X\,|\,\mathcal{V})) = E(X)$,

(ii) $E(X) = E(X\,|\,\mathcal{W})$, where $\mathcal{W} = \{\emptyset,\Omega\}$ is the trivial σ-algebra.

19. Let X, Y be two real-valued random variables and suppose their joint distribution function has the density $f(x,y)$. Show that
$$E(X|Y) = \Phi(Y) \quad \text{a.s.}$$
for
$$\Phi(y) = \frac{\int_{-\infty}^{\infty} xf(x,y)\,dx}{\int_{-\infty}^{\infty} f(x,y)\,dx}.$$

(Hints: $\Phi(Y)$ is a function of Y and so it is $\mathcal{U}(Y)$-measurable. Therefore we must show that

(*) $$\int_A X\,dP = \int_A \Phi(Y)\,dP \quad \text{for all } A \in \mathcal{U}(Y).$$

Now $A = Y^{-1}(B)$ for some Borel subset of $\mathbb{R}$. So the left-hand side of (*) is

(**) $$\int_A X\,dP = \int_\Omega \chi_B(Y)X\,dP = \int_{-\infty}^{\infty}\int_B xf(x,y)\,dydx.$$

The right-hand side of (*) is
$$\int_A \Phi(Y)\,dP = \int_{-\infty}^{\infty}\int_B \Phi(y)f(x,y)\,dydx,$$
which equals the right-hand side of (**). Fill in the details.)

20. A smooth function $\Phi:\mathbb{R}\to\mathbb{R}$ is called *convex* if $\Phi''(x) \ge 0$ for all $x \in \mathbb{R}$.

(i) Show that if Φ is convex, then
$$\Phi(y) \ge \Phi(x) + \Phi'(x)(y-x) \quad \text{for all } x, y \in \mathbb{R}.$$

(ii) Show that
$$\Phi\left(\frac{x+y}{2}\right) \le \frac{1}{2}\Phi(x) + \frac{1}{2}\Phi(y) \quad \text{for all } x, y \in \mathbb{R}.$$

(iii) A smooth function $\Phi : \mathbb{R}^n \to \mathbb{R}$ is called *convex* if the matrix $D^2\Phi = ((\Phi_{x_i x_j}))$ is nonnegative definite for all $x \in \mathbb{R}^n$. (This means that $\sum_{i,j=1}^n \Phi_{x_i x_j}\xi_i\xi_j \geq 0$ for all $\xi \in \mathbb{R}^n$.) Prove that

$$\Phi(y) \geq \Phi(x) + D\Phi(x) \cdot (y - x),$$

$$\Phi\left(\frac{x+y}{2}\right) \leq \frac{1}{2}\Phi(x) + \frac{1}{2}\Phi(y)$$

for all $x, y \in \mathbb{R}^n$. (Here "D" denotes the gradient.)

21. (i) Prove *Jensen's inequality*:

$$\Phi(E(X)) \leq E(\Phi(X))$$

for a random variable $X : \Omega \to \mathbb{R}$, where Φ is convex. (Hint: Use assertion (iii) from the previous exercise.)

(ii) Prove the *conditional Jensen inequality*:

$$\Phi(E(X|\mathcal{V})) \leq E(\Phi(X)|\mathcal{V}).$$

22. Let $W(\cdot)$ be a one-dimensional Brownian motion. Show that

$$E(W^{2k}(t)) = \frac{(2k)!t^k}{2^k k!} \quad (t > 0).$$

23. Show that if $\mathbf{W}(\cdot)$ is an n-dimensional Brownian motion, then so are

(i) $\mathbf{W}(t+s) - \mathbf{W}(s)$ for all $s \geq 0$,
(ii) $c\mathbf{W}(t/c^2)$ for all $c > 0$ ("Brownian scaling").

24. Let $W(\cdot)$ be a one-dimensional Brownian motion, and define

$$\bar{W}(t) := \begin{cases} tW(\frac{1}{t}) & \text{for } t > 0 \\ 0 & \text{for } t = 0. \end{cases}$$

Show that $\bar{W}(t) - \bar{W}(s)$ is $N(0, t-s)$ for times $0 \leq s \leq t$. ($\bar{W}(\cdot)$ also has independent increments and so is a one-dimensional Brownian motion. You do not need to show this.)

25. Define $X(t) := \int_0^t W(s)\, ds$, where $W(\cdot)$ is a one-dimensional Brownian motion. Show that

$$E(X^2(t)) = \frac{t^3}{3} \quad \text{for each } t > 0.$$

26. Define $X(t)$ as in the previous exercise. Show that

$$E(e^{\lambda X(t)}) = e^{\frac{\lambda^2 t^3}{6}} \quad \text{for each } t > 0.$$

(Hint: $X(t)$ is a Gaussian random variable, the variance of which we know from the previous exercise.)

27. Define $U(t) := e^{-t}W(e^{2t})$, where $W(\cdot)$ is a one-dimensional Brownian motion. Show that

$$E(U(t)U(s)) = e^{-|t-s|} \quad \text{for all } -\infty < s, t < \infty.$$

28. Let $W(\cdot)$ be a one-dimensional Brownian motion. Show that

$$\lim_{m\to\infty} \frac{W(m)}{m} = 0 \quad \text{almost surely.}$$

(Hint: Fix $\epsilon > 0$ and define the event $A_m := \{|\frac{W(m)}{m}| \geq \epsilon\}$. Then $A_m = \{|X| \geq \sqrt{m}\epsilon\}$ for the $N(0,1)$ random variable $X = \frac{W(m)}{\sqrt{m}}$. Apply the Borel–Cantelli Lemma.)

29. (i) Let $0 < \gamma \leq 1$. Show that if $f : [0,T] \to \mathbb{R}^n$ is uniformly Hölder continuous with exponent γ, it is also uniformly Hölder continuous with each exponent $0 < \delta < \gamma$.

(ii) Show that $f(t) = t^\gamma$ is uniformly Hölder continuous with exponent γ on the interval $[0,1]$.

30. Let $0 < \gamma < \frac{1}{2}$. We showed in Chapter 3 that if $W(\cdot)$ is a one-dimensional Brownian motion, then for almost every ω there exists a constant K, *depending on* ω, such that

$$(*) \qquad |W(t,\omega) - W(s,\omega)| \leq K|t-s|^\gamma \quad \text{for all } 0 \leq s, t \leq 1.$$

Show that there does not exist a constant K such that $(*)$ holds for almost all ω.

31. Prove that if $G, H \in \mathbb{L}^2(0,T)$, then

$$E\left(\int_0^T G\,dW \int_0^T H\,dW\right) = E\left(\int_0^T GH\,dt\right).$$

(Hint: $2ab = (a+b)^2 - a^2 - b^2$.)

32. Let $(\Omega, \mathcal{U}, P)$ be a probability space, and take $\mathcal{F}(\cdot)$ to be a filtration of σ-algebras. Assume X to be an integrable random variable, and define $X(t) := E(X|\mathcal{F}(t))$ for times $t \geq 0$.

Show that $X(\cdot)$ is a martingale.

33. Show directly that $I(t) := W^2(t) - t$ is a martingale.

(Hint: $W^2(t) = (W(t) - W(s))^2 - W^2(s) + 2W(t)W(s)$. Take the conditional expectation with respect to $\mathcal{W}(s)$, the history of $W(\cdot)$, and then condition with respect to the history of $I(\cdot)$.)

34. Suppose $X(\cdot)$ is a real-valued martingale and $\Phi : \mathbb{R} \to \mathbb{R}$ is convex. Assume also that $E(|\Phi(X(t))|) < \infty$ for all $t \geq 0$. Show that

$$\Phi(X(\cdot)) \text{ is a submartingale.}$$

(Hint: Use the conditional Jensen inequality.)

35. Use the Itô chain rule to show that $Y(t) := e^{\frac{t}{2}}\cos(W(t))$ is a martingale.

36. Let $\mathbf{W}(\cdot) = (W^1, \ldots, W^n)$ be an n-dimensional Brownian motion, and write $Y(t) := |\mathbf{W}(t)|^2 - nt$ for times $t \geq 0$. Show that $Y(\cdot)$ is a martingale.
   (Hint: Compute dY.)

37. Show that

$$\int_0^T W^2\, dW = \frac{1}{3}W^3(T) - \int_0^T W\, dt$$

and

$$\int_0^T W^3\, dW = \frac{1}{4}W^4(T) - \frac{3}{2}\int_0^T W^2\, dt.$$

38. Recall from the text that

$$Y := e^{\int_0^t g\, dW - \frac{1}{2}\int_0^t g^2\, ds}$$

satisfies

$$dY = gY\, dW.$$

Use this to prove

$$E(e^{\int_0^T g\, dW}) = e^{\frac{1}{2}\int_0^T g^2\, ds}.$$

39. Let $u = u(x,t)$ be a smooth solution of the *backwards diffusion equation*

$$u_t + \frac{1}{2}u_{xx} = 0,$$

and suppose $W(\cdot)$ is a one-dimensional Brownian motion.
   Show that for each time $t > 0$

$$E(u(W(t),t)) = u(0,0).$$

40. Calculate $E(B^2(t))$ for the Brownian bridge $B(\cdot)$, and show in particular that $E(B^2(t)) \to 0$ as $t \to 1^-$.

41. Let X solve the Langevin equation, and suppose that X_0 is an $N(0, \frac{\sigma^2}{2b})$ random variable. Show that

$$E(X(s)X(t)) = \frac{\sigma^2}{2b}e^{-b|t-s|}.$$

42. (i) Consider the ODE

$$\begin{cases} \dot{x} = x^2 & (t > 0) \\ x(0) = x_0. \end{cases}$$

Show that if $x_0 > 0$, the solution "blows up to infinity" in finite time.

(ii) Next, look at the ODE

$$\begin{cases} \dot{x} = x^{\frac{1}{2}} \quad (t > 0) \\ x(0) = 0. \end{cases}$$

Show that this problem has infinitely many nonnegative solutions.

(Hint: $x \equiv 0$ is a solution. Find also a solution which is positive for times $t > 0$, and then combine these solutions to find ones which are zero for some time and then become positive.)

43. (i) Use the substitution $X = u(W)$ to solve the SDE

$$\begin{cases} dX = -\frac{1}{2}e^{-2X}dt + e^{-X}dW \\ X(0) = x_0. \end{cases}$$

(ii) Show that the solution blows up at a finite, random time.

44. Solve the SDE $dX = -Xdt + e^{-t}dW$.

45. Let $\mathbf{W} = (W^1, W^2, \dots, W^n)$ be an n-dimensional Brownian motion and write

$$R := |\mathbf{W}| = \left(\sum_{i=1}^{n} (W^i)^2 \right)^{\frac{1}{2}}.$$

Show that R solves the *stochastic Bessel equation*

$$dR = \frac{n-1}{2R}dt + \sum_{i=1}^{n} \frac{W^i}{R} dW^i.$$

46. (i) Show that $\mathbf{X} = (\cos(W), \sin(W))$ solves the system of SDE

$$\begin{cases} dX^1 = -\frac{1}{2}X^1 dt - X^2 dW \\ dX^2 = -\frac{1}{2}X^2 dt + X^1 dW. \end{cases}$$

(ii) Show also that if $\mathbf{X} = (X^1, X^2)$ is any other solution, then $|\mathbf{X}|$ is constant in time.

47. Solve the system

$$\begin{cases} dX^1 = dt + dW^1 \\ dX^2 = X^1 dW^2, \end{cases}$$

where $\mathbf{W} = (W^1, W^2)$ is a Brownian motion.

48. Solve

$$\begin{cases} dX^1 = X^2 dt + dW^1 \\ dX^2 = X^1 dt + dW^2. \end{cases}$$

49. Solve

$$\begin{cases} dX = \frac{1}{2}\sigma'(X)\sigma(X)dt + \sigma(X)dW \\ X(0) = 0 \end{cases}$$

where W is a one-dimensional Brownian motion and σ is a smooth, positive function.

(Hint: Let $f(x) := \int_0^x \frac{dy}{\sigma(y)}$ and set $g := f^{-1}$, the inverse function of f. Show that $X = g(W)$.)

50. Let τ be the first time a one-dimensional Brownian motion hits the half-open interval $(a, b]$. Show that τ is a stopping time.

51. Let $\mathbf{W}$ denote an n-dimensional Brownian motion for $n \geq 3$. Write $\mathbf{X} = \mathbf{W} + x_0$, where the point x_0 lies in the region $U = \{0 < R_1 < |x| < R_2\}$. Calculate explicitly the probability that $\mathbf{X}$ will hit the outer sphere $\{|x| = R_2\}$ before hitting the inner sphere $\{|x| = R_1\}$.

(Hint: Check that

$$\Phi(x) = \frac{1}{|x|^{n-2}}$$

satisfies $\Delta\Phi = 0$ for $x \neq 0$. Modify Φ to build a function u which equals 0 on the inner sphere and 1 on the outer sphere.)