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SOME ASPECTS OF THREE-DIMENSIONAL *TONNETZE*

Edward Gollin

This paper explores how a three-dimensional (3-D) *Tonnetz* enables an interesting spatial representation of tetrachords and the contextual transformations among them. The geometry of the *Tonnetz*, a function of its set-class structure, emphasizes certain operations within a larger group of transformations based on their common-tone retention properties.

After reviewing some features of the traditional two-dimensional (2-D) *Tonnetz*, we will explore features of a 3-D *Tonnetz* based on the familiar dominant-seventh/Tristan tetrachord [0258]. We will then generalize the 3-D *Tonnetz* to accommodate relations among members of any tetrachord class. Lastly, we will investigate the relation of a *Tonnetz*'s geometry to the group structure of the transforms relating its elements.

The Two-dimensional *Tonnetz*

The traditional *Tonnetz* (as manifest in the nineteenth-century writings and theories of Oettingen, Riemann, et al.) is an array of pitches on a (potentially infinite) Euclidean plane.[1] Figure 1a illustrates a region of a traditional 2-D *Tonnetz*. We may summarize several of its essential aspects as follows:

1) Pitches are arranged along two independent axes (not necessarily orthogonal), arranged intervallically by perfect fifth on one axis and by major third on the other (See Figure 1b).[2]
2) The triangular regions whose vertices are defined by some given pitch and those either a perfect fifth and major third above or a perfect fifth and major third below correspond to some triad, either the *Ober-* or *Unterklang* respectively of that given pitch.
3) The edges of the triangular regions correspond to the intervals within the triad. Only two of these lie along axes of the array; the third edge may be defined as a secondary feature resulting from the combination of the primary axes.
4) The dual structure of major and minor triads is visually manifest within the *Tonnetz*. Triangles representing major triads are oriented oppositely from those representing minor triads.
5) Triangles/triads adjacent to any given triangle/triad share two common tones if they share a common edge, or one common tone if they share a common vertex.

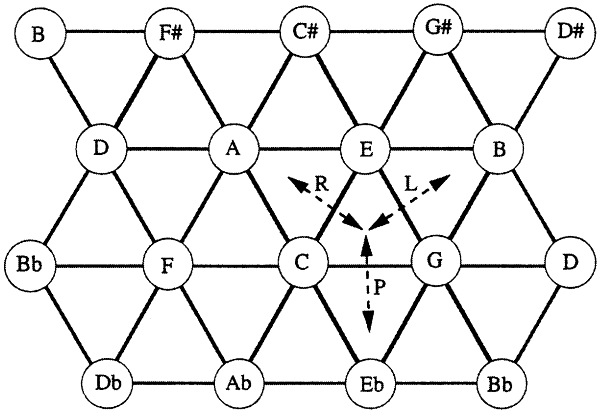
Three particular operations acting on triads, L, P and R (for *Leittonwechsel* or Leading-tone exchange, Parallel, and Relative) which maximize common-tone retention, occupy a privileged position on the *Tonnetz*: they are representable as 'edge-flips' among adjacent triangles/triads.[3] Arrows in Figure 1a illustrate the mappings of L, P and R about a C-major triad.

Several nineteenth-century assumptions underlying the traditional 2-D *Tonnetz* are not obligatory. For instance, if one assumes an arrangement of equally-tempered pitch classes (rather than just-intoned pitches), the *Tonnetz* would be situated not in an infinite Cartesian plane, but on the closed, unbounded surface of a torus.[4] Richard Cohn (1997) and David Lewin (1996) have demonstrated that the *Tonnetz* need not represent relations merely among triads, but may do so among members of any trichordal set class; Cohn (1997) has further shown the extension of the *Tonnetz* in chromatic cardinalities other than c=12.[5] Further, one need not adhere to the restriction of representing trichords in two dimensions, but may extend the *Tonnetz* in three dimensions to accommodate relations among tetrachords.[6]

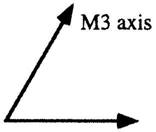
An [0258] *Tonnetz*

Figure 2 illustrates a portion of a 3-D *Tonnetz*, consisting not of points on a plane but of points (representing pitch classes) in a space lattice. Whereas two axes were sufficient to locate points in the 2-D *Tonnetz*, the additional degree of freedom in three dimensions requires an additional axis to describe the location of all points. I have chosen axes of a non-orthogonal coordinate system in Figure 2 (interaxial angles are all 60°),

Lattice = espaço "em rede".



(a) A region of traditional 2-D *Tonnetz* and three contextual inversions (sharing a common edge) with a C-major triad



(b) The axis system of the traditional 2-D *Tonnetz*

Figure 1

and have labeled these a, b, and c (avoiding labels x, y, and z to distinguish my axes from those of the Cartesian coordinate system). Axes are labeled according to a right-handed convention (right thumb points in a positive direction along the a-axis, right index finger in a positive direction along the b-axis, middle finger in a positive direction along the c-axis). Points within the lattice are arranged at unit distances in positive and negative directions along the axes from all other points. The regular arrangement of points in this *Tonnetz* constitutes one of two uniform ways of filling space with spheres—crystallographers refer to this arrangement as cubic closest packing (ccp).[7]

Throughout this discussion, we will assume equal temperament. Doing so induces a modular geometry to the 3-D *Tonnetz*—the *Tonnetz* occupies the closed, unbounded volume of a hyper-torus in 4-dimensional space. Units in a positive direction along the a-axis correspond to the musical interval of 4 semitones, units in a positive direction along the b-axis correspond to the interval of 7 semitones, and units in a positive direction along the c-axis correspond to the interval of 10 semitones. Any

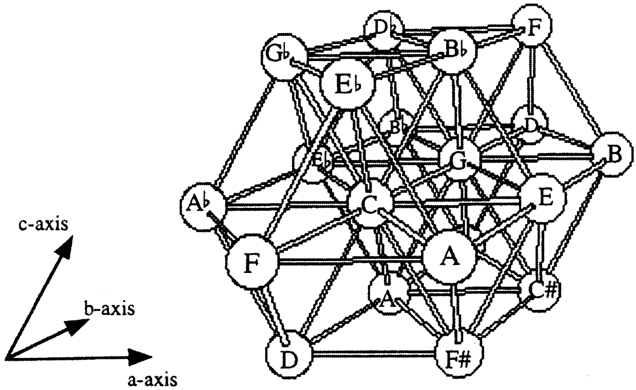


Figure 2. A region within an [0258] *Tonnetz*

point in the lattice may be located in reference to any other point in terms of units traversed along each of the axes, and consequently, if the pitch class of the first point is known, one can determine the pitch class of the second. For instance, in Figure 2, a point one positive unit along the b-axis from C (=0) is 0+7 = G. A point one positive unit along the a-axis, one positive unit along the b-axis, and one negative unit along the c-axis from C is $(1 \cdot 4) + (1 \cdot 7) + (-1 \cdot 10) = 1$ = C♯.

The 60° angle is chosen because it is the angle formed at the edges of a regular tetrahedron meeting at a vertex. Just as triangular areas of the 2-D *Tonnetz* correspond to triadic elements (or trichordal elements, in the case of a generalized 2-D *Tonnetz*), tetrahedral volumes in our 3-D *Tonnetz* correspond to its tetrachordal elements. In Figure 2, one can observe that any point, along with the points lying one positive unit along the a-, b-, and c-axes, describe the vertices of an upward-pointing tetrahedron and that the pitch classes represented by these points constitute a 'dominant-seventh' chord. Similarly, any point and the points lying one negative unit along a-, b-, and c-axes describe the vertices of a downward pointing tetrahedron, and the pitch classes represented by these vertices correspond to some Tristan or half-diminished seventh chord. Analogous to the inverted triangles/triads of the 2-D *Tonnetz*, the dual structures of Tristan and dominant-seventh chords are visually manifest as oppositely oriented tetrahedra in the 3-D *Tonnetz* (i.e., pyramids with peaks pointed upward, versus those with peaks pointed downward).

Before I discuss relations among tetrachordal elements in our *Tonnetz*, it will be helpful to develop a contextual notation for identifying the pitch elements of the tetrachords. For this, we can adapt a contextual notation developed by Moritz Hauptmann to identify the tones of a triad (Hauptmann [1853]1991).

Figure 3a presents Hauptmann's designation for the tones of a major triad, and one of his versions for designating the tones of a minor triad. Roman numeral I designates the *Einheit* of a triad, that chord tone to which the others refer by means of 'directly intelligible intervals' (for Hauptmann, these are the octave, perfect fifth and major third). Roman numeral II refers to the tone in the triad that lies a perfect fifth from the *Einheit*, roman numeral III to the tone that lies a major third from the *Einheit*. The difference between major and minor for Hauptmann is that in the case of major, the *Einheit* lies a perfect fifth and a major third below the other chord tones (i.e. it *has* a perfect fifth and major third), whereas in minor the *Einheit* lies a perfect fifth and major third above the other chord tones (i.e., it *is* a perfect fifth and a major third to the tones below).

One need not accept Hauptmann's acoustical biases nor the dialectical connotations inherent in his reference to II and III as *Zweiheit* and *Verbindung* in order to adopt a similar system of tokens that identifies tones based on their intervallic environment within tetrachords of some Tn/TnI class. In Figure 3b, I illustrate a neo-Hauptmannian system designating two forms of set class [0258]: a 'dominant-seventh' chord [C,E,G,B♭], and a 'Tristan' chord [C♯,E,G,B]. I designate C and B as '*Einheiten*' of their respective chords—altering Hauptmann's symbology, I label these with a lower case roman numeral i. '*Einheit*' here designates a chord tone that lies four semitones, seven semitones, and ten semitones from the other chord tones. In (C,E,G,B♭), '*Einheit*' is the familiar root of the chord; in (C♯,E,G,B), B is the 'dual root'—the upper tone in a traditional third-stacking. Roman numerals ii, iii and iv refer to those pcs that lie four, seven and ten semitones respectively from the *Einheit*, either above (in the nominally 'major' dominant-seventh) or below (in the nominally 'minor' Tristan chord). Our initial choice of '*Einheit*' was arbitrary—any tone could be the basis to which the others refer. What is important is that the system of tokens offers a dual system of naming elements of any asymmetrical tetrachord class.[8]

Our neo-Hauptmannian labels offer a means of identifying the contextual transforms among tetrachords within our lattice, in particular, inversions between tetrachords sharing common tones. Figure 4 illustrates the common-tone mappings about a nexus tetrahedron representing the dominant seventh chord [C, E, G, B♭]. Each mapping self-inverts: the direction of mapping within the *Tonnetz* is dependent upon whether the operation acts on a 'major' dominant seventh or 'minor' Tristan chord. Thus each mapping may be viewed either as a mapping from the dominant seventh chord [C, E, G, B♭] to an adjacent Tristan chord, or conversely a mapping from some Tristan chord to [C, E, G, B♭].

(a) Hauptmann's designation of triadic chord tones

(b) a neo-Hauptmannian system of tokens identifying chord tones in set class [0258]

i	ii	iii	iv	iv	iii	ii	i
(u,v,w)	(u+1,v,w)	(u,v+1,w)	(u,v,w+1)	(u,v,w-1)	(u,v-1,w)	(u-1,v,w)	(u,v,w)

(c) neo-Hauptmannian tokens applied to generalized tetrachords in the 3-D *Tonnetz*

Figure 3

Figure 4a isolates the six transforms that map or exchange tetrahedra sharing a common edge. The exchange is represented spatially as a 'flip' of the two tetrahedra about that common edge. For instance, the operation I^{i}_{ii} exchanges tetrahedra about their common edge bounded by chordal elements i and ii. In Figure 4a, the I^{i}_{ii} operation flips the central, upward-pointing [C, E, G, B♭] tetrachord about the C-E edge, and exchanges it with the downward-pointing [E, C, A, F♯] tetrachord just below and to the front of the illustration. The operation I^{ii}_{iii} similarly 'flips' two tetrahedra about their common elements ii and iii. In Figure 4, I^{ii}_{iii} exchanges the central [C, E, G, B♭] tetrachord (flipping it about the G-E edge) with the downward-pointing [B, G, E, C♯] tetrachord to the lower right in the illustration. Each 'edge-flip' maintains at least the two common tones that constitute the common tetrahedral edge. In the case of I^{ii}_{iv}
!!! in this *Tonnetz*, mapped [0258] tetrachords share three common tones.

Figure 4b isolates the four transforms that map or exchange tetrahedra sharing a common vertex. The exchange is represented spatially by a 'flip' of the two tetrahedra about that common vertex. For example, the operation I^{i}_{i} exchanges two tetrachords that share a common '*Einheit*.' In Figure 4b, I^{i}_{i} exchanges the central [C, E, G, B♭] tetrachord with the downward-pointing [C, A♭, F, D] tetrachord below and to the forward left in the illustration. Each 'vertex-flip' maintains at least one common tone (the common vertex). However, I^{ii}_{ii} and I^{iv}_{iv} map [0258] tetrachords sharing two common tones, and I^{iii}_{iii} maps [0258] tetrachords sharing three.

We can observe the following degeneracy among transforms in this equal-tempered *Tonnetz*: $I^{ii}_{ii} = I^{iv}_{iv}$ and $I^{iii}_{iii} = I^{ii}_{iv}$. The degeneracy arises from the symmetries inherent in our chosen set class [0258], and from our imposition of equal temperament. Specifically, the equivalence of I^{ii}_{ii} and

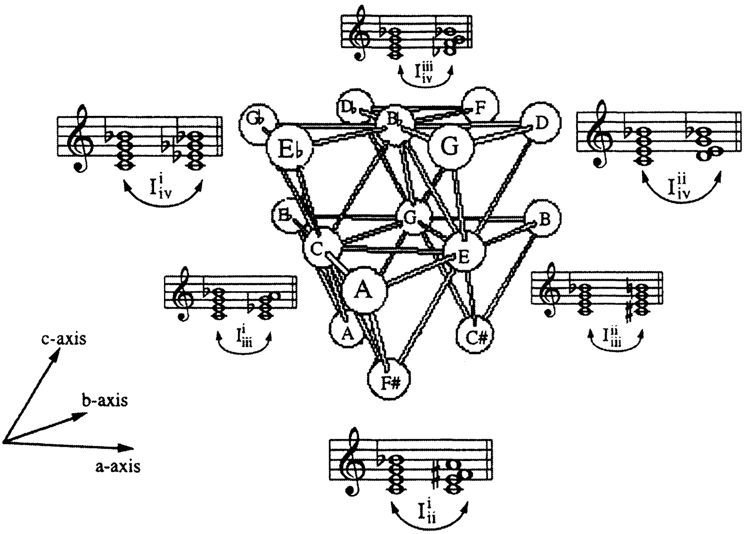


Figure 4a. Six 'edge-flips' about a nexus tetrachord, (C,E,G,B♭), within an [0258] *Tonnetz*

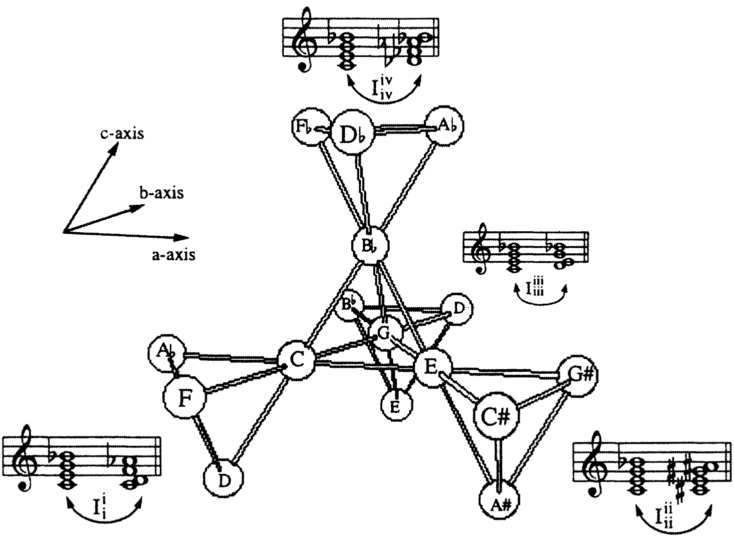


Figure 4b. Four 'vertex-flips' about a nexus tetrachord (C,E,G,B♭), within an [0258] *Tonnetz*

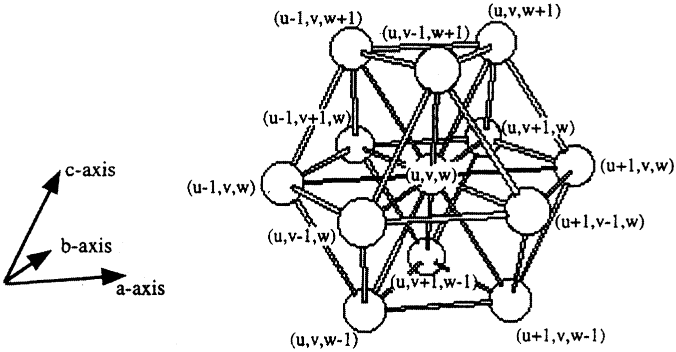


Figure 5. A region of generalized *Tonnetz* about the point (u,v,w)

I_{iv}^{iv} occurs because the interval from ii to iv (6 semitones) spans half the cardinality of our chromatic universe. The equivalence of I_{iii}^{iii} and I_{iv}^{ii} results from the symmetry of the [036] trichord embedded within an [0258].[9] Moreover, in the modular geometry of the hyper-toroidal space in which our equal-tempered *Tonnetz* resides, lattice points identical in pitch are also identical in location. Thus, $(X)I_{ii}^{ii}$ and $(X)I_{iv}^{iv}$, where X is some [0258] tetrachord in this *Tonnetz*, are the *same* tetrachord in our curved modular space.[10]

A generalized 3-D Tonnetz

Figure 5 illustrates a region of lattice points in a generalized 3-D *Tonnetz* adjacent to a point (u,v,w). The triple (u,v,w) indicates a point u units (in a positive direction) along the a-axis, v units along the b-axis and w units along the c-axis from an origin point (0,0,0). An equivalent family of points could be constructed around any point in the space. If we select coefficients α, β, and γ that correspond to intervals in some universe of cardinality N, and posit these as unit lengths along the a-, b-, and c-axes respectively, and if we fix the origin point as 0 in an integer notational system, then the identity of any point (u,v,w) is given by $\alpha u + \beta v + \gamma w$ (mod N). For instance, fixing (0,0,0) = C = 0, and setting α, β, and γ equal to 1, 4 and 8 semitones respectively in a mod 12 pitch-class space, a point (2,-3,5) would represent the pitch class $(1\cdot2)+(4\cdot-3)+(8\cdot5) = 2-12+40 = 30 \bmod 12 = 6$ = F♯ in an [0148] *Tonnetz*.

Figure 3c illustrates how our neo-Hauptmannian tokens may be affixed to points in the generalized 3-D *Tonnetz*. If we let ii refer to a pitch class that lies one unit along the a-axis from some *Einheit*, let iii refer to a pitch class one unit along the b-axis, and let iv refer to a pitch class one unit along the c-axis (in either a positive or negative direction), then given some point in the lattice (u,v,w) as '*Einheit*,' we may define a 'major' form of any tetrachord as the tetrahedron comprising the four vertices (u,v,w), (u+1,v,w), (u,v+1,w), and (u,v,w+1). A tetrahedron representing a 'minor' form of a tetrachord class built from the same *Einheit* would consist of the four vertices (u,v,w), (u-1,v,w), (u,v-1,w) and (u,v,w-1).

Analogously to the [0258] *Tonnetz*, contextual transformations in the generalized 3-D *Tonnetz* may be represented as edge- or vertex-flips among adjacent tetrahedrons. For instance, $\mathrm{I}^{\mathrm{i}}_{\mathrm{ii}}$ is an edge-flip that inverts a tetrachord about its '*Einheit*' and its element ii, mapping [(u,v,w),(u+1,v,w),(u,v+1,w),(u,v,w+1)] to [(u+1,v,w),(u,v,w),(u+1,v-1,w),(u+1,v,w-1)] and vice versa.

I have been cautious to specify the *Tonnetz* in which a particular tetrahedron-inverting operation occurs, since our neo-Hauptmannian labels (and consequently our inversion labels) are tied to the intervallic layout of our axes. Were we to explore relations in an [0258] *Tonnetz* with differently arranged axes (for example positing units along the a-, b-, and c-axes of 3, 6 and 8 semitones respectively), the same operations of our original [0258] *Tonnetz* would obtain, but with different 'I' labels. The same is not the case if one considers operations within *Tonnetze* of different set classes: the family of common-tone preserving, contextual inversions of one set class is in general a different family of contextual inversions from any other.

Group Structure of Transforms within the *Tonnetz*

If one exhaustively composes the L, P, and R transforms in the triadic 2-D *Tonnetz* of Figure 1, the resulting closed group of operations is a dihedral group known as the Schritt/Wechsel group (S/W).[11] The group, acting on harmonic triads, consists of 12 mode-preserving operations (*Schritte*) and 12 mode-inverting operations (*Wechsel*). The symbols S_0, S_1, S_2, . . . , S_{11} will here indicate 12 *Schritte* that map a triad to the mode-identical triad whose root lies 0, 1, 2, ..., 11 semitones away in the direction of chord 'generation' (up in the case of major triads, down in the case of minor).[12] For example, (E major)S_3= (G major), but (E minor)S_3= (C♯ min.). W_0, W_1, W_2 etc. are *Wechsel* that map mode-inverted triads whose dual roots lie 0, 1, 2, etc., semitones apart in the direction of chord 'generation.' Thus in the triadic 2-D *Tonnetz*, L equals W_{11}, P equals W_7 and R equals W_4.[13]

Analogously, if we exhaustively compose the edge- and vertex-flipping transforms in our [0258] *Tonnetz*, the resulting group of operations is identical in structure (or *isomorphic*) to the S/W group acting on harmonic triads. The isomorphic S/W group acting on [0258] tetrachords maps mode-identical and mode-inverted tetrachords based on the directed intervals among their '*Einheiten*.' Thus in our [0258] *Tonnetz*, $\mathrm{I}^{\mathrm{i}}_{\mathrm{i}}$ is equivalent to W_0, I^{i}_{ii} is equivalent to W_4, etc.[14] Note that if we alter our system of labels (i.e., select some other tone as '*Einheit*') the *Wechsel* associated with each particular contextual inversion are all changed by some constant. The groups resulting from the relabeling of tetrahedral elements in the *Tonnetz* are therefore automorphic images of one another.

The exhaustive composition of the contextual inversions of any asymmetrical tetrachordal (or trichordal) set class will similarly yield an S/W group isomorphic to the S/W groups acting on triads or [0258] tetrachords (or else isomorphic to a dihedral subgroup thereof—contextual inversions among members of [0248] for instance yield only the even *Schritte* and *Wechsel* of the larger group).

This remarkable result, that the same group of operations underlies *Tonnetze* of such drastically different geometries (and even of different dimensions), suggests that the *Tonnetz* is an entity independent of the S/W group. It also suggests an interesting relation between the structure of the *Tonnetz* and the transformations among its members. Elements of mathematical groups are largely indifferent to the music-analytical situations upon which they are called into service in transformational theories. In the context of the S/W group *qua* group, any *Wechsel* is basically like any other (all of order 2, etc.). Yet one could hardly claim that the musical effect of the Relative transform (W_4) acting on a triad (e.g., mapping C maj.↔A min.) is at all the same as that of the *Gegenkleinterzwechsel* (W_3, e.g. mapping C maj.↔A♭ min.). Thus we may view the *Tonnetz* as an independent framework that allows one to make distinctions among the (otherwise indifferent) transforms that underlie its elements. It allows one to posit analytical meaning or value to certain families of transforms from the larger group based on the distance between elements mapped by those transforms within the *Tonnetz*, whose geometry in turn is determined contextually by the intervallic structure of its chordal elements. Distance in the *Tonnetz*, moreover, is a function of common-tone retention. Hence, the function of the *Tonnetz* in neo-Riemannian theory is not unlike that of Lewin's INJ function: it provides a measure of the progressive versus internal quality of a contextual function acting on a '*Klang*' of a trichordal or tetrachordal set class.[15]

NOTES

1. For a history of the *Tonnetz* and its evolution from Euler through Riemann, see Mooney 1996. A study that discusses the *Tonnetz* and its relation to *Funktionstheorie* in Riemann and his successors is Imig 1970.
2. The nineteenth-century basis of interval selection was acoustical: the intervals corresponded to what Riemann (along with numerous theorists before him) believed to be the only intervals (along with the octave) given by nature.
3. Richard Cohn (1997) has recently explored the *Tonnetz* representation of these operations, noting their potential for common-tone retention as well as their parsimonious voice-leading. The abbreviations derive from Brian Hyer 1989 and 1995 (101–38).
4. Cohn 1997 cites Lubin 1974 as the first recognition of the toroidal structure of an equally tempered *Tonnetz*. Hyer's Figure 3 (1995, 119) presents a *Klangnetz*, the geometric dual of a *Tonnetz* (see Douthett and Steinbach 1998), in pitch-class space, which similarly induces a toroidal structure.
5. Lewin's hexagonal graph of a 3-valued Cohn function (Lewin 1998; 1996, 189), is, like Hyer's, a dual of an [013] *Tonnetz*.
6. Other writings that extend the *Tonnetz* into three dimensions include Vogel 1993 and Lindley and Turner-Smith 1993.
7. It is so called because the unit cells of the lattice (the smallest units that can fill space by translation alone) are face centered cubes (lattice points exist at the six vertices and at the center of each cube face).
8. Henry Klumpenhouwer (1991, chap. 2) explores a similar system of contextual labeling.
9. The embedded [036] is also the reason that I^{iii}_{iii} and I^{ii}_{iv} map tetrachords in our [0258] *Tonnetz* with three common tones. It is interesting to observe that none of the 29 tetrachord-classes in a chromatic universe of cardinality 12 are without degenerate common-tone-preserving contextual transforms: any tetrachord in c=12 either (a) contains an ic 6 dyad, (b) is symmetrical or embeds a symmetrical trichordal subset, or (c) satisfies conditions (a) and (b).
10. This would not be true in a just-intoned version of our *Tonnetz*, where (C,E,G,B♭) I^{ii}_{ii} = (G♯,E,C♯,A♯) ≠ (C,E,G,Bb) I^{iv}_{iv} = (A♭,F♭,D♭,B♭). Such a Tonnetz would, however, occupy an infinite Euclidean space. See Vogel 1993, 123 ff., and Lindley and Turner-Smith 1993, 66–68, for discussions of unequally tempered [0258] *Tonnetze*.
11. Riemann expounds his system of *Schritte* and *Wechsel* as directed root-interval relations among triads in *Skizze einer neuen Methode der Harmonielehre* (Riemann 1880). Klumpenhouwer 1994 reinterprets and extends Riemann's *Schritte* and *Wechsel* into a group of operations acting on the set of major and minor triads.
12. This is a vestige of Riemann's belief that major triads were acoustically generated from the root up (via overtones), and that minor triads were acoustically generated from their upper, dual root down (via undertones).
13. One can derive the complete S/W group from the L and R transforms alone. Since $LR = W_{11}W_4 = S_7$ is an element of order 12, it may be combined with L or R or any *Wechsel* (all are of order 2) to generate the complete group.

14. Analogously, we can produce generators for our [0258] S/W group. $\mathrm{I}^{ii}_{iii}\,\mathrm{I}^{i}_{ii} = \mathrm{W}_{11}\mathrm{W}_{4} = \mathrm{S}_{7}$ is an element of order 12, which may then be combined with any edge- or vertex-flip to generate the complete group.
15. INJ is discussed in Lewin 1987, 123 ff.