3. The linear 3-D elasticity mathematical model

In Chapter 2 we examined some fundamental conditions that should be satisfied in the modeling of all deformable solids and structures. The study of truss structures provided an excellent setting to explore how these conditions can be used to formulate a mathematical model in structural mechanics, and how to apply modern procedures to solve this mathematical model. These solution procedures lead to the exact solution of the mathematical model (see examples in Section 2.2 and 2.3).

In engineering analysis, more complex mathematical models need in general to be considered and solved. Indeed, these mathematical models are so complex that exact solutions can mostly not be obtained. The objective in this chapter is to formulate the general mathematical model for three-dimensional (3-D) solids, but still assuming, as for the truss structures, infinitesimally small displacements and the linear constitutive relationship. The conditions to be satisfied, namely equilibrium, compatibility and the stress-strain relationship that we encountered in the analysis of truss structures are also the basic conditions to be satisfied when formulating the three-dimensional mathematical model of a solid. As for truss structures, these conditions need be satisfied for every differential element of the mathematical model – and now of the 3-D solid.

We shall use the discussion of the general three-dimensional mathematical model later in Chapter 4 to derive the various special models useful in engineering analysis. The solution of these models is then presented in the chapters thereafter.

Before embarking on the formulation of the mathematical model of 3-D solids, we motivate its need by means of simple problems which are related to the one-dimensional state of stress and strain of a truss bar, but which already display some 3-D behavior.

3.1 The analysis of a steel sheet problem

In Section 2.2 we considered one-dimensional stress and strain conditions in a truss bar. We now aim in this section to look deeper into the one-dimensional stress/strain conditions and to introduce in a physical manner new phenomena that indeed pertain to two- and three-dimensional conditions. The more

formal physical and mathematical discussions of these phenomena are given in the sections to follow.

3.1.1 One-dimensional conditions

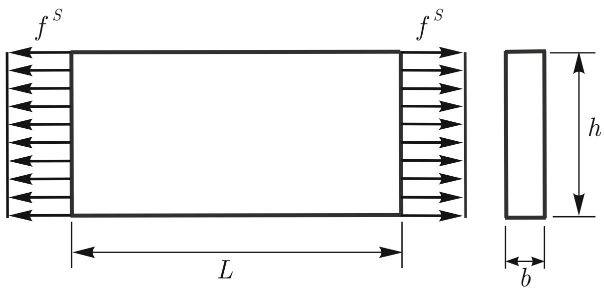

Fig. 3.1. Steel sheet subjected to self-equilibrated constant surface tractions f^S. There are no body forces

Consider the steel sheet shown in Figure 3.1. We note that the external forces are in static equilibrium, *i.e.*, $\mathbf{R} = \mathbf{0}$ and $\mathbf{M_O} = \mathbf{0}$. However, the steel sheet is not properly supported and can undergo rigid body motions. In order to suppress rigid body motions and not to interfere with the straining of the material, we introduce the supports shown in Figure 3.2.

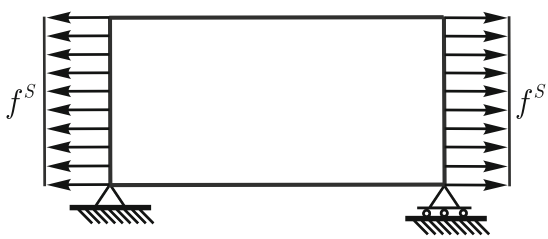

Fig. 3.2. Properly supported steel sheet

If we now cut the sheet by a plane orthogonal to its own plane and the x axis as shown in Figure 3.3a, and introduce the stress τ, see Figure 3.3b, we can, by equilibrium, evaluate the stress magnitude

$$\tau = f^S$$

where τ is constant along the transverse direction y and, of course, independent of the section position given by x (to show that τ is constant as a function of y, we could also cut the sheet with a plane orthogonal to the y–axis). Hence the state of stress is analogous to that in a truss bar. To find the deformed configuration of the sheet we need to consider the relation between the stresses and strains — the constitutive equation. In Section 2.2.4 we introduced Hooke's law for a one-dimensional state of stress and strain. This 1-D constitutive relation is applicable for the steel sheet considering the stress τ and the strain of the longitudinal fibers.

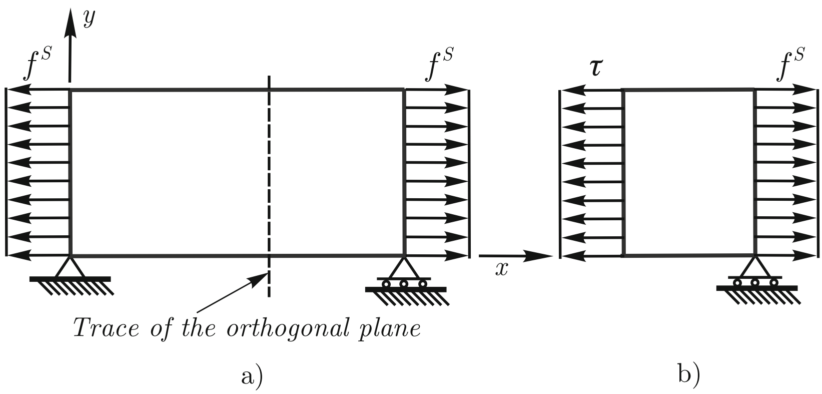

Fig. 3.3. a) Steel sheet cut by an orthogonal plane; b) Equilibrium of the extracted part

An important physical phenomenon in 3-D elasticity is what is referred to as "the Poisson effect". This effect corresponds to a contraction of the fibers that are orthogonal to the direction of the fibers being extended. The deformation of the steel sheet considering Poisson's effect is shown in Figure 3.4: the relative shortenings of the fibers in the transverse directions are proportional to the relative extension of the fibers in the longitudinal direction, that is, $\frac{\Delta h}{h} = \frac{\Delta b}{b} = -\nu \frac{\Delta L}{L}$. The material property constant ν is called Poisson's ratio.

Since in this section we want to concentrate on the longitudinal deformation of the sheet (1-D conditions), we assume, for now, that $\nu = 0$.

Considering Hooke's law

$$\tau = E\varepsilon$$

where

$$\varepsilon = \frac{\Delta L}{L}$$

is the strain of the longitudinal fibers. Hence

$$\tau = E\frac{\Delta L}{L}$$

and, we obtain

$$\Delta L = \tau\frac{L}{E} = f^S\frac{L}{E}.$$

The analogy to the truss model is evident since considering the cross-sectional area $A = hb$ and the axial force $N = \tau A$, we have

$$\Delta L = \tau\frac{L}{E} = \frac{\tau AL}{EA} = N\frac{L}{EA}.$$

As a side-note, this formula is in linear analysis valid for any ν.

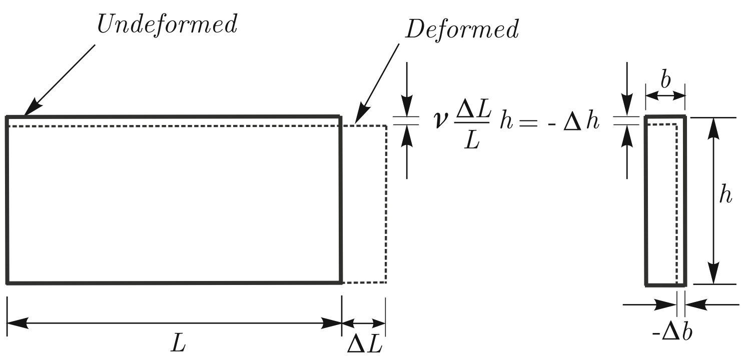

Fig. 3.4. Schematic and magnified deformation of steel sheet

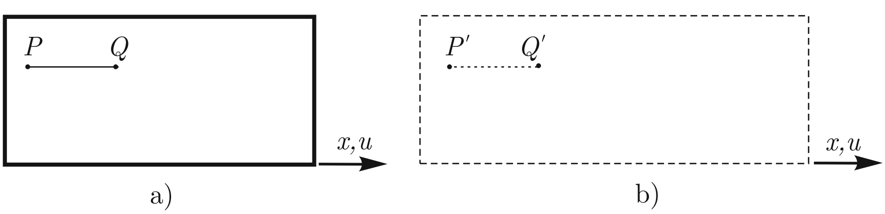

Fig. 3.5. Stretching of fiber **PQ**

Since the longitudinal stress is constant in the sheet the strain of any horizontal fiber such as **PQ**, see Figure 3.5a, is given by

$$\frac{\|\mathbf{P'Q'}\| - \|\mathbf{PQ}\|}{\|\mathbf{PQ}\|} = \frac{\Delta L}{L} \tag{3.1}$$

where $\mathbf{P'Q'}$ is the fiber in the deformed configuration, see Figure 3.5b, and $\|\mathbf{PQ}\|$ indicates the Euclidean norm of the vector defined by $\mathbf{PQ}$, *i.e.*, the vector length[1]. Figure 3.6 illustrates that both sides of (3.1) express the fiber extension measured with respect to its original length induced by the same constant stress field.

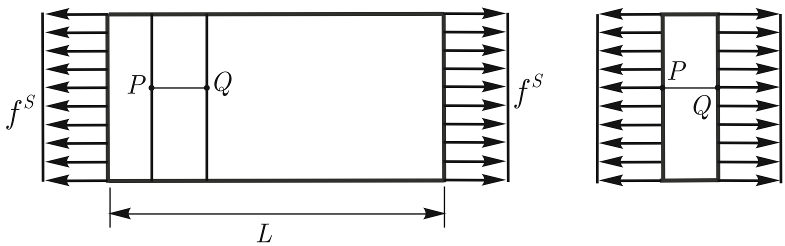

Fig. 3.6. Isolated part of the sheet corresponding to points **P** and **Q**

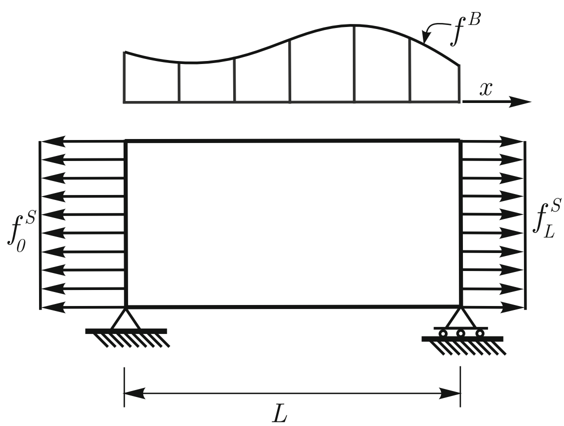

Fig. 3.7. Steel sheet problem considering a field of body forces. The field $\mathrm{f}^B(x)$ acts into the x direction, and $\mathrm{f}^B(x) > 0$ for all x

Next, let us increase the complexity of our loading by considering also a field of body forces acting into the x direction as shown in Figure 3.7. In this

[1] The Euclidian norm of a vector $\mathbf{a}$ can be evaluated by $\|\mathbf{a}\| = \sqrt{\sum_{i=1}^{3} (a_i)^2}$ where a_i are the components of $\mathbf{a}$ in an orthonormal basis (see *e.g.* Bathe, 1996)

figure, the function $f^B(x)$ gives the variation of the magnitude of f^B along the longitudinal direction. We assume that the surface tractions at $x = 0$ and $x = L$ considered together with the body forces are in equilibrium. Again, the stress at a generic section, given by x, can be evaluated by considering a cutting plane to obtain

$$\tau(x) = \frac{1}{A}\left[\int_x^L f^B(z)\, A\, dz + f_L^S A\right] \tag{3.2}$$

where $\tau(x)$ is the stress shown in Figure 3.8. We note that $\tau(x)$ is no longer constant with respect to x and since $f^B(x) > 0$ for all x, $\tau(x)$ decreases with increasing x.

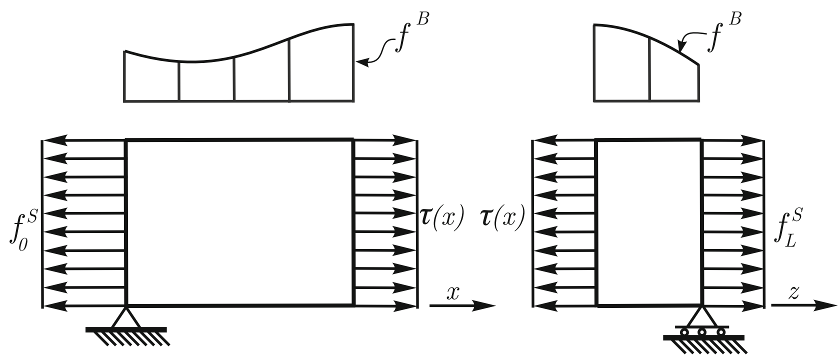

Fig. 3.8. Steel sheet with body forces cut at a generic section

In Figure 3.9 we show the deformation of two fibers: **PQ**, positioned at the same place as in Figure 3.5, and of another fiber originally of the same length as **PQ**, described by **MN**. Since the state of stress varies along the longitudinal direction we have that

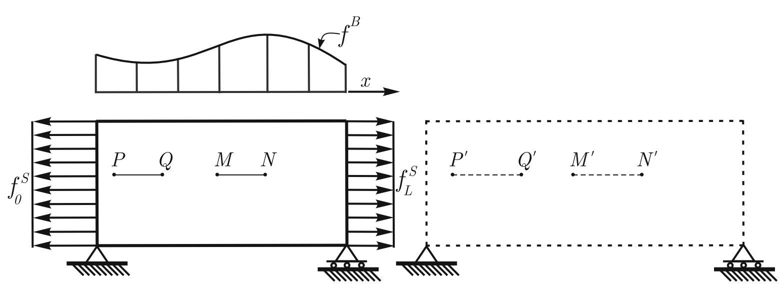

Fig. 3.9. Stretching of two horizontal fibers for problem with body forces

$$\frac{\|\mathbf{P}'\mathbf{Q}'\| - \|\mathbf{PQ}\|}{\|\mathbf{PQ}\|} > \frac{\|\mathbf{M}'\mathbf{N}'\| - \|\mathbf{MN}\|}{\|\mathbf{MN}\|}.$$

There are two pieces of information in the above expression. The first one is that the fiber strains are different since the stress level is different for the regions where the fibers are located. The second piece of information is that since the stress level is greater in the region of the fiber **PQ**, the fiber strain of **PQ** is greater than that of **MN**. Therefore (3.1) only holds for the case considered in Figure 3.2 because a special case is considered: a constant stress leading to an induced constant longitudinal strain over the domain. On the other hand, in the problem of Figure 3.7, the stress and the strain of the fibers vary along the length of the sheet.

Now, if we consider an intermediate point on fiber **PQ**, say $\mathbf{Q}_1$, as shown in Figure 3.10, by the above argument, we should have

$$\frac{\|\mathbf{P}'\mathbf{Q}_1'\| - \|\mathbf{PQ}_1\|}{\|\mathbf{PQ}_1\|} > \frac{\|\mathbf{P}'\mathbf{Q}'\| - \|\mathbf{PQ}\|}{\|\mathbf{PQ}\|}.$$

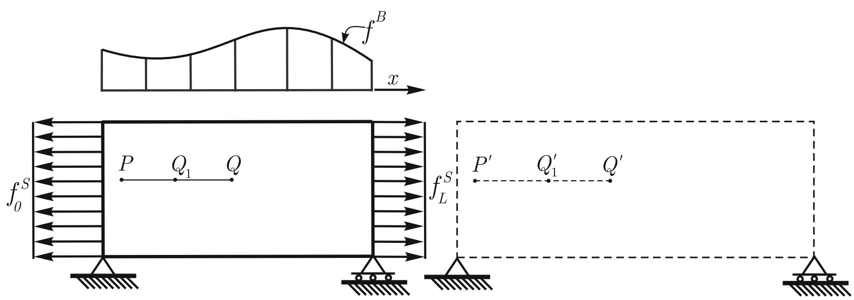

Fig. 3.10. Stretching of fiber **PQ** considering an intermediate point $\mathbf{Q}_1$

However, we want to arrive at a local measure which corresponds to the deformation behavior at a point and hence must be independent of the fiber length considered. We, therefore, define the *normal strain* at point **P** in the horizontal direction by

$$\varepsilon = \lim_{Q \to P} \frac{\|\mathbf{P}'\mathbf{Q}'\| - \|\mathbf{PQ}\|}{\|\mathbf{PQ}\|} \tag{3.3}$$

where point **Q** is always taken on a horizontal line through point **P**. The above quantity gives a measure of the straining of the horizontal fibers of infinitesimal length at point **P**. Hence, using $\|\mathbf{PQ}\| = ds$ and $\|\mathbf{P}'\mathbf{Q}'\| = ds'$, equation (3.3) can be equivalently written as

$$\varepsilon = \frac{ds' - ds}{ds}.$$

Note that as we use a fiber of infinitesimal length to evaluate the normal strain, the direction is given by the direction of the infinitesimal fiber. The terminology "normal" in "normal strain" is due to the fact that the normal stress, normal to a cutting plane, induces this strain[2].

Considering (3.3) it is clear that the change in length of the fiber **PQ** is only due to the displacements of the material particles in the sheet. We denote the displacement of a material particle of the sheet in the x direction by u. In Figure 3.11, we show explicitly the displacements of points **P** and **Q**, and hence we have

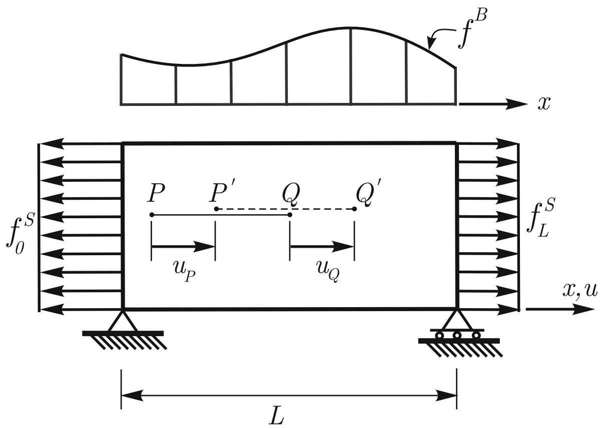

Fig. 3.11. Horizontal displacements of point P and Q

$$\|\mathbf{PQ}\| = x_Q - x_P$$

where x_Q and x_P are the x coordinate of the points **Q** and **P** respectively, and

$$\|\mathbf{PQ}\| + u_Q = u_P + \|\mathbf{P'Q'}\| .$$

The above relation is directly given by measuring the distance between points **P** and **Q′** in two alternative ways, see Figure 3.11. Then, we obtain

$$\lim_{Q \to P} \frac{\|\mathbf{P'Q'}\| - \|\mathbf{PQ}\|}{\|\mathbf{PQ}\|} = \lim_{x_Q \to x_P} \frac{u_Q - u_P}{x_Q - x_P} = \left. \frac{du}{dx} \right|_{x_P} . \tag{3.4}$$

The above limit corresponds to the definition of the derivative of $u(x)$ at x_P. Therefore, (3.4) gives the normal strain of an infinitesimal horizontal fiber with origin at a generic point x

[2] It is also used to distinguish the normal strain from the shear strain which is defined for a pair of fibers, see Section 3.2.2

$$\varepsilon(x) = \frac{du}{dx}.$$

Now we can find the displacements of the material particles in the sheet for the problem of Figure 3.10. Indeed

$$\frac{du}{dx} = \varepsilon(x) = \frac{\tau(x)}{E}$$

which can be integrated using $\tau(x)$ in (3.2) and leads to $u(x)$. While we assumed tensile conditions in the above discussion, this formula is general and also applicable in the case of compression.

3.1.2 Two Dimensional Conditions

Let us next consider the problem of Figure 3.2 with $\nu \neq 0$. Therefore, the deformation of the sheet is as given in Figures 3.4 and 3.12. If we proceed as before for the definition of a local strain measure for the fibers with origin in **P**, but vertical, we would arrive at

$$\varepsilon = \lim_{\mathbf{R}\to\mathbf{P}} \frac{\|\mathbf{P}'\mathbf{R}'\| - \|\mathbf{PR}\|}{\|\mathbf{PR}\|}.$$

This strain value is negative since the vertical fibers contract (due to Pois-

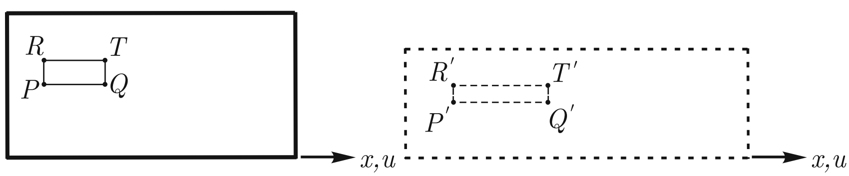

Fig. 3.12. Deformation of fibers **PQ** and **RT** for sheet in Figure 3.4

son's effect). If we take an inclined direction such as that given by **PT** we would again arrive at a different value for the normal strain given by

$$\varepsilon = \lim_{\mathbf{T}\to\mathbf{P}} \frac{\|\mathbf{P}'\mathbf{T}'\| - \|\mathbf{PT}\|}{\|\mathbf{PT}\|}.$$

Therefore, the normal strain depends not only on the fiber location but also on the fiber direction. Hence, we use the terminology "state of strain" at a point and will characterize this state in detail in Section 3.2.

Actually, this more general strain concept is required to solve more complex problems. Referring to Figure 3.2, we recall that the supports were only introduced for the purpose of suppressing rigid body motions. Consider the boundary conditions given in Figure 3.13, *i.e.*,

$$u(0, y) = 0, \ v(0, 0) = 0.$$

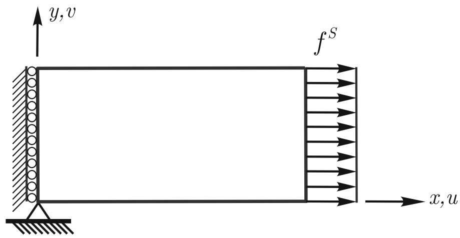

Fig. 3.13. Steel sheet of Figure 3.2 with modified boundary conditions

The quantity $v(0, 0)$ indicates the displacement in the direction of y of the particle at point $(0, 0)$. In general, the field of vertical displacements is denoted by $v(x, y)$. Since the deformation pattern for the problem of Figure 3.2 leads to $u(0, y) = 0$ the additional displacement restrictions introduced in Figure 3.13 do not interfere with the previous deformation. Hence, the solution to this problem can be determined solving for the stresses and strains considering 1-D conditions, as before, and then if $\nu > 0$ evaluating the contraction of the vertical fibers due to Poisson's effect.

Therefore, for any material (either a steel or a rubber sheet) we have the same stresses. Of course, the strain and the displacements depend on the material of the sheet.

The problem of Figure 3.13 represents an analogue, in the context of a 2-D problem, to the statically determinate truss structures (actually a single bar) in the sense that only the equilibrium condition is used to determine the stresses/internal forces in both cases.

Let us modify the boundary conditions of the problem of Figure 3.13 as shown in Figure 3.14a, *i.e.*, we now have the following boundary conditions

$$u(0, y) = 0, \ v(0, y) = 0. \tag{3.5}$$

We see that the shortening of the edge of the sheet given by $x = 0$ is now prevented and this is achieved by the reactions f_x^S and f_y^S at the clamped edge. We could think of solving this problem by starting from the problem of Figure 3.13 and looking for the field of forces f_y^S which should be applied at the edge given by $x = 0$ to obtain $v(0, y) = 0$. Figure 3.14b summarizes this approach.

The state of stress and strain for the problem of Figure 3.14 is no longer unidimensional and in addition to the longitudinal normal stresses, there are normal stresses in the transverse direction and also shear stresses.

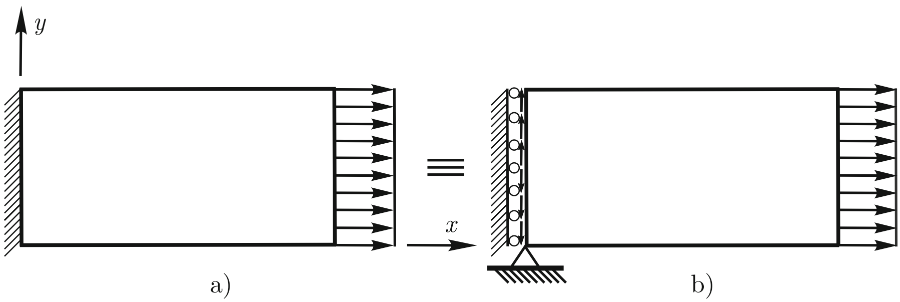

Fig. 3.14. Steel sheet with restrictions to the vertical displacement at the edge $x = 0$

In Figure 3.15, we introduce a notation to distinguish between these stresses, *i.e.*, τ_{xx} and τ_{yy} represent the normal stresses in the directions x and y respectively and τ_{xy} the shear stresses. We also show qualitatively their variations.

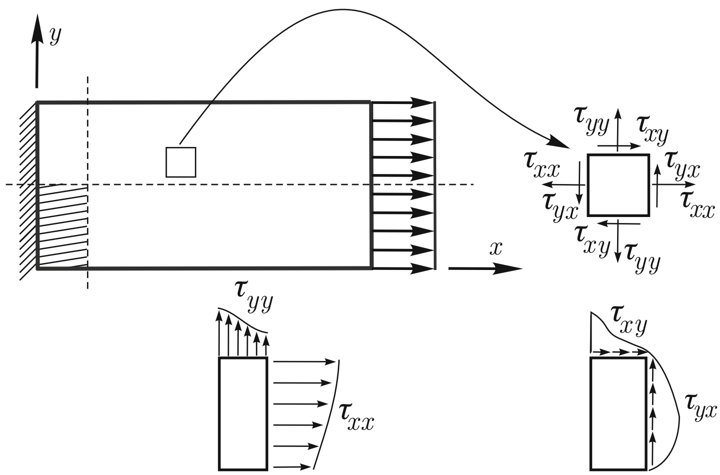

Fig. 3.15. Stresses schematically shown for internal faces defined by the cutting planes

In Figure 3.16, we present the data of a problem as described above. In Figures 3.17 and 3.18 we present the stress predictions for the lines shown in Figure 3.16. The solution was obtained by the finite element method with a mesh sufficiently refined to give a good quantitative description of the solution. We can appreciate that at line 2, far from the edge given by $x = 0$, the solution is very similar to that obtained for the problem of Figure 3.13 in which the supports at the edge prevent only horizontal motions, whereas at line 1, we see that the transverse normal stresses τ_{yy} and the shear stresses τ_{xy} are different from zero and that the normal stresses τ_{xx} are also not constant.

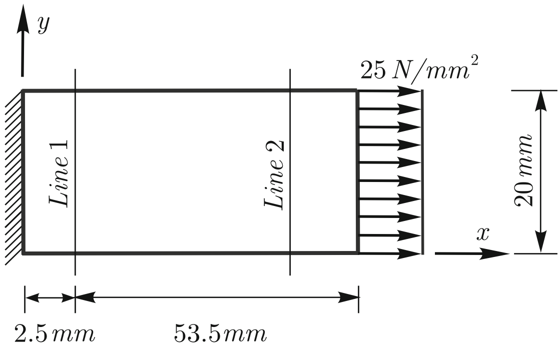

Fig. 3.16. A steel sheet problem. Thickness $t = 1\ mm, E = 70 \times 10^4\ N/mm^2, \nu = 0.25$

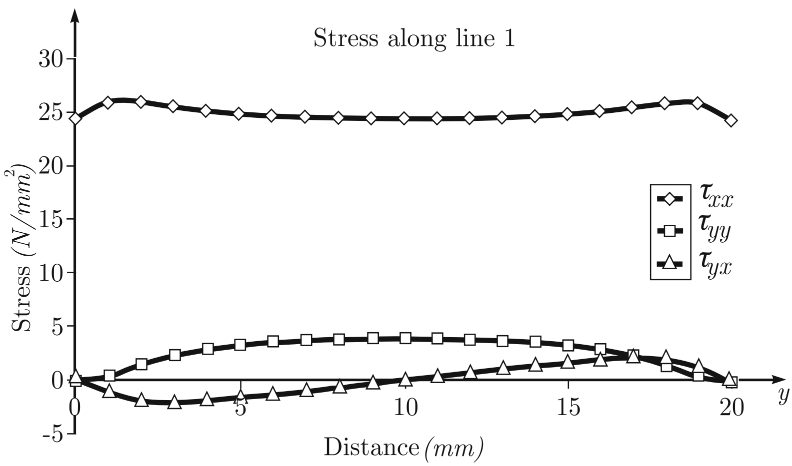

Fig. 3.17. Stress distributions at line 1

We note that this problem is not statically determinate, *i.e.*, we can not find the stress field without imposing also the constitutive equations and compatibility conditions.

When we considered statically indeterminate truss structures, we could find many sets of internal forces that satisfy equilibrium, but only one of these sets led to displacements at the nodes and at the supports that were compatible. The steel sheet is an example of an analogous behavior in the continuum

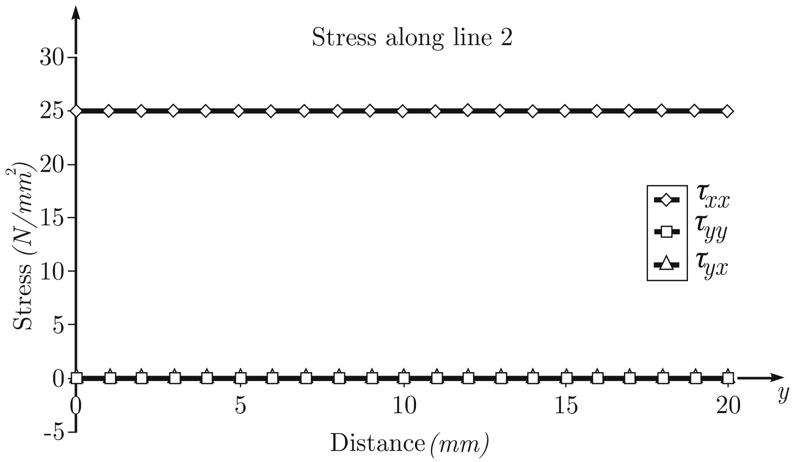

Fig. 3.18. Stress distributions at line 2; the τ_{yy} and τ_{yx} values are practically zero

case, since not all stress fields which satisfy equilibrium lead to fiber extensions/contractions, more generally to strains, that correspond to displacement fields with continuous displacements that satisfy the displacement boundary conditions. Therefore, even to solve the very "simple" case described in Figure 3.16, we need a multi-dimensional description of the strains, stresses and constitutive relations.

Motivated by the above discussion, we can now outline the contents of the remaining sections of this chapter. In Section 3.2 we present the study of the deformations considering first the motion of a three-dimensional body with no restriction on the magnitude of the displacements. Then, we examine the simplifications which are obtained when infinitesimally small displacements are assumed. In Section 3.3 we study the stresses, also in the context of a 3-D body, starting from the concepts already introduced in Chapter 2. In Section 3.4 the relations between the strains and stresses are discussed for a linear isotropic elastic behavior, *i.e.*, the constitutive equations are presented under such conditions. In Section 3.5 the complete formulation of the 3-D linear elasticity problem is presented. Finally, in Section 3.6 the 3-D elasticity solution of the torsion of a prismatic bar is discussed.

3.2 Deformations

In Section 3.1 we showed by means of a very simple example − the analysis of the steel sheet − that the strain of an infinitesimal fiber having origin at a point depends not only on the position of the point but also on the direction of the fiber.

In the analysis of the steel sheet, we also exemplified that the stress in a given direction induces a fiber strain in this direction and in the perpendicular directions as well. Actually, besides these fiber strains, the change in angle between fibers is also related to the stresses and, hence, should be included in the characterization of the complete state of strain at a point.

In order to establish a precise terminology for strains, we call a *normal strain* the strain of *a fiber* as introduced in Section 3.1 and a *shear strain* the change of angle between *pairs of orthogonal fibers.* The shear strain is sometimes referred to as angular distortion.

In this section we assume that the displacements of all material particles are given (from the undeformed to the deformed configuration of a solid) and calculate the normal and shear strains. Of course, the actual displacements can only be calculated once the complete mathematical model has been established.

3.2.1 Displacement field

Considering the motion of a deformable body, we described in Figure 2.1 generic configurations of the body, where ${}^0V$ stood for the initial configuration and ${}^tV$ for a generic configuration at time t. Since we are concerned with statics and, primarily, with linear theories, we can focus on only two configurations: the one prior to the application of external actions – the undeformed configuration – and the one after the introduction of the external actions – the deformed configuration. In order to simplify the notation, we redraw in Figure 3.19 the description presented in Figure 2.1. We choose to label the undeformed configuration as V and the deformed configuration as V_y. We also use x_1, x_2 and x_3 as coordinate axes.

In Figure 3.19, $\mathbf{P}$ is a material particle[3] of the solid in the undeformed configuration and $\mathbf{P}_y$ represents this particle in the deformed configuration. The position of $\mathbf{P}$ is characterized by the vector[4]

---

[3] We refer sometimes to a *material particle* as a *point*

[4] A *Euclidean vector* $\mathbf{v}$ can be defined as the oriented straight line segment that connects two points of the Euclidean three-dimensional space. Hence, the definition of a vector is independent of the choice of a coordinate system. However, to perform calculations with vectors we need to represent them in a coordinate system. For example, $\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3$ where v_i are the components of the vector $\mathbf{v}$ in the coordinate system ($\mathbf{O}$, $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$). When ($\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$) is an orthonormal basis the component v_i is obtained by the scalar product $v_i = \mathbf{v} \cdot \mathbf{e}_i$. Sometimes we represent the vector $\mathbf{v}$ by $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$. Actually, we are defining the vector by a column matrix which collects the components of $\mathbf{v}$ in the coordinate system ($\mathbf{O}$, $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$). Of course, in the coordinate system ($\mathbf{O}'$, $\mathbf{e}'_1$, $\mathbf{e}'_2$, $\mathbf{e}'_3$) the components v'_i are different from v_i and, hence, another column matrix would correspond to the same vector $\mathbf{v}$. Notwithstanding, we adopt this convention to

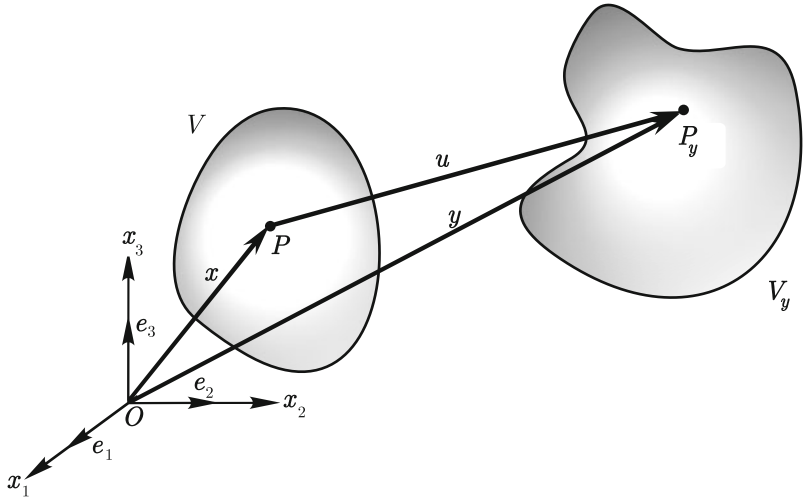

Fig. 3.19. Deformed and undeformed configurations in a stationary Cartesian coordinate system

$$\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3 = \sum_{i=1}^{3} x_i\mathbf{e}_i$$

where the $\mathbf{e}_i$ are the base vectors of the coordinate system. The position of $\mathbf{P}_y$ is given by

$$\mathbf{y} = \sum_{i=1}^{3} y_i\mathbf{e}_i.$$

The displacement vector of the material particle is

$$\mathbf{u} = \mathbf{y} - \mathbf{x}.$$

The deformation of a solid can be fully described by defining the function which gives for every material particle in the undeformed configuration its position in the deformed configuration

$$\mathbf{y} = \mathbf{y}(\mathbf{x})$$

which in components can be written as

keep the notation as simple and as operational as possible with the implicit understanding that the coordinate system used is implied by the context. Note that the calculations are performed with matrix operations. For example, the scalar product (or dot product) between two vectors $\mathbf{v}$ and $\mathbf{w}$ is obtained by $\mathbf{v}^T\mathbf{w}$ (see Bathe, 1996)

$$\begin{aligned} y_1 &= y_1(x_1, x_2, x_3) \\ y_2 &= y_2(x_1, x_2, x_3) \\ y_3 &= y_3(x_1, x_2, x_3)\,. \end{aligned}$$

The displacement field can be also written as

$$\begin{aligned} u_1 &= u_1(x_1, x_2, x_3) \\ u_2 &= u_2(x_1, x_2, x_3) \\ u_3 &= u_3(x_1, x_2, x_3)\,. \end{aligned}$$

In order to allow the reader to become familiar with this notation, let us describe some very simple rigid plane motions in the example below.

Example 3.1

Consider a generic prismatic solid with transverse section parallel to the coordinate plane x_1x_2 as shown in Figure 3.20a. Characterize the displacement field for:

(i) a rigid body translation in the direction of x_1 of intensity Δ
(ii) a rigid body rotation of intensity φ with respect to the x_3 axis

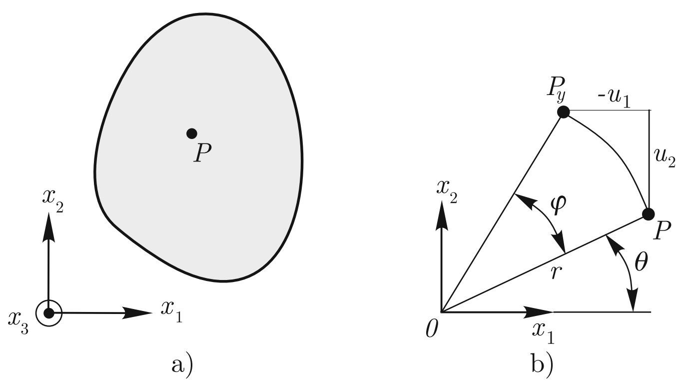

Fig. 3.20. a) Transverse section of a prismatic solid and a generic point **P**; b) Motion of a generic point in the section for an angle of rotation φ.

Solution

(i) It suffices to recognize that for every point of the solid denoting a material particle

$$
\begin{aligned}
u_1\left(x_1, x_2, x_3\right) &= \Delta \\
u_2\left(x_1, x_2, x_3\right) &= 0 \\
u_3\left(x_1, x_2, x_3\right) &= 0.
\end{aligned}
$$

(ii) In Figure 3.20b, $\mathbf{P}$ stands for a generic point denoting a material particle of coordinates x_1 and x_2 in the section of the solid and $\mathbf{P}_y$ represents its position in the deformed configuration, after the rigid body rotation. Note that point $\mathbf{P}$ moves on a circle of radius $r = \|\mathbf{OP}\|$. Therefore the displacements are given by

$$
-u_1 = r\cos\theta - r\cos\left(\theta + \varphi\right), \qquad u_2 = r\sin\left(\theta + \varphi\right) - r\sin\theta
$$

which leads to

$$
u_1 = -r\cos\theta + r\cos\theta\cos\varphi - r\sin\theta\sin\varphi \tag{3.6}
$$

$$
u_2 = r\sin\theta\cos\varphi + r\sin\varphi\cos\theta - r\sin\theta. \tag{3.7}
$$

Considering that

$$
x_1 = r\cos\theta, \qquad x_2 = r\sin\theta \tag{3.8}
$$

equations (3.6) and (3.7) can be re-written as

$$
u_1\left(x_1, x_2, x_3\right) = x_1\cos\varphi - x_2\sin\varphi - x_1 \tag{3.9}
$$

$$
u_2\left(x_1, x_2, x_3\right) = x_1\sin\varphi + x_2\cos\varphi - x_2 \tag{3.10}
$$

and, of course,

$$
u_3\left(x_1, x_2, x_3\right) = 0.
$$

□

3.2.2 Normal and shear strains

As discussed in Section 3.1, the normal strain depends both on the fiber location and its direction. In order to obtain a local measure at a point (independent of the fiber length), a limit process was pursued which means, in essence, to consider a fiber of infinitesimal length. This approach will be used to study deformations in 3-D and from now on, unless stated otherwise, a fiber is assumed to be of infinitesimal length.

Let us consider a deformation as described in Figure 3.19 and consider a fiber with origin at $\mathbf{P}$ given by the vector $\mathbf{dx}$, as shown in Figure 3.21.

Due to the deformation, the fiber $\mathbf{dx}$ becomes $\mathbf{dy}$ in the deformed configuration, V_y, and we can write

$$
\mathbf{dy} = \mathbf{dx} + \mathbf{u}\left(\mathbf{x} + \mathbf{dx}\right) - \mathbf{u}\left(\mathbf{x}\right) \tag{3.11}
$$

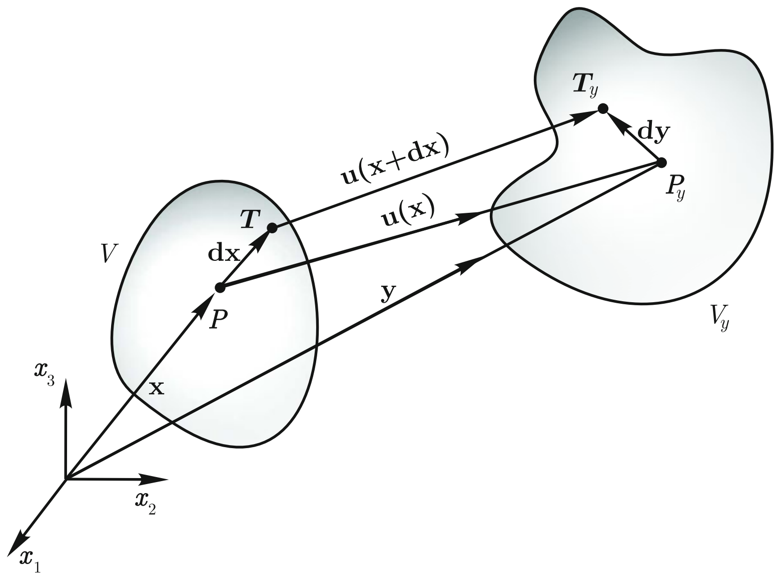

Fig. 3.21. Fibers in deformed and undeformed configurations

or in components

$$dy_i = dx_i + u_i\left(x_1 + dx_1, x_2 + dx_2, x_3 + dx_3\right) - u_i\left(x_1, x_2, x_3\right)$$

for $i = 1$, 2 and 3. Recall from multivariable calculus that

$$u_i\left(x_1 + dx_1, x_2 + dx_2, x_3 + dx_3\right) - u_i\left(x_1, x_2, x_3\right) = \frac{\partial u_i}{\partial x_1}dx_1 + \frac{\partial u_i}{\partial x_2}dx_2 + \frac{\partial u_i}{\partial x_3}dx_3. \tag{3.12}$$

Substituting (3.12) into (3.11) gives

$$dy_i = dx_i + \frac{\partial u_i}{\partial x_1}dx_1 + \frac{\partial u_i}{\partial x_2}dx_2 + \frac{\partial u_i}{\partial x_3}dx_3$$

and the range of indexes, such as i, is implicitly understood to vary from 1 to 3. We can write the above expression in matrix form

$$\begin{bmatrix} dy_1 \\ dy_2 \\ dy_3 \end{bmatrix} = \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix} + \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix} \tag{3.13}$$

or

$$\begin{bmatrix} dy_1 \\ dy_2 \\ dy_3 \end{bmatrix} = \begin{bmatrix} 1+\frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & 1+\frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & 1+\frac{\partial u_3}{\partial x_3} \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix}. \tag{3.14}$$

Let us define the displacement gradient

$$\nabla \mathbf{u} = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{bmatrix} \tag{3.15}$$

and the deformation gradient[5]

$$\mathbf{X} = \begin{bmatrix} 1+\frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & 1+\frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & 1+\frac{\partial u_3}{\partial x_3} \end{bmatrix} = \mathbf{I} + \nabla \mathbf{u}. \tag{3.16}$$

Here, the identity matrix is denoted by $\mathbf{I}$. Equation (3.13) can be re-written as

$$\mathbf{dy} = \mathbf{dx} + \nabla \mathbf{u}\, \mathbf{dx}$$

and (3.14) as

$$\mathbf{dy} = (\mathbf{I} + \nabla \mathbf{u})\, \mathbf{dx} = \mathbf{X}\, \mathbf{dx}. \tag{3.17}$$

[5] Both the displacement gradient and the deformation gradient are linear transformations which relate vectors to vectors (see equation (3.17)). For instance, let $\mathbf{S}$ be a linear transformation. The linearity means that $\mathbf{S}\,(\alpha\mathbf{p}+\beta\mathbf{q}) = \alpha\mathbf{S}\mathbf{p}+\beta\mathbf{S}\mathbf{q}$ for any real numbers α, β and any vectors $\mathbf{p}$, $\mathbf{q}$. Considering a coordinate system ($\mathbf{O}$, $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$), $\mathbf{w} = \mathbf{S}\mathbf{v} = \mathbf{S}\left(\sum_{i=1}^{3} v_i \mathbf{e}_i\right) = \sum_{i=1}^{3} v_i \mathbf{S}\mathbf{e}_i$. Now let $\mathbf{S}\mathbf{e}_i = \sum_{j=1}^{3} S_{ji}\mathbf{e}_j$ then $\mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \mathbf{S}\mathbf{v}$. In this book we call *tensor* a linear transformation which relates vectors to vectors (although a *tensor* is a more general concept). Note that a tensor can be represented in a basis by a 3×3 matrix. Of course, as for vectors, when we change basis the matrix components which represent the tensor will generally change. Nevertheless, we choose to represent a tensor by a 3×3 matrix and the basis is implied by the context (see Bathe, 1996 and Chapelle and Bathe, 2010*a*).

We also note that in many instances we will use matrices in their ordinary sense, that is, not as representing either a vector or a tensor. We mention that no special notation is used to distinguish between matrices when used in these distinct situations

Relation (3.17) is a very important result because it gives the change in any fiber due to the deformation. That is, the deformation gradient **X** relates a fiber **dx** in the undeformed configuration to the same fiber **dy** in the deformed configuration. Of course, **X** and $\nabla\mathbf{u}$ depend on the position (but not the direction) of the fiber in the undeformed configuration. However, if we choose a point, any fiber deformation at this point is obtained by the same **X** (or $\nabla\mathbf{u}$).

The relation (3.17) can now be used to calculate both the normal and shear strains.

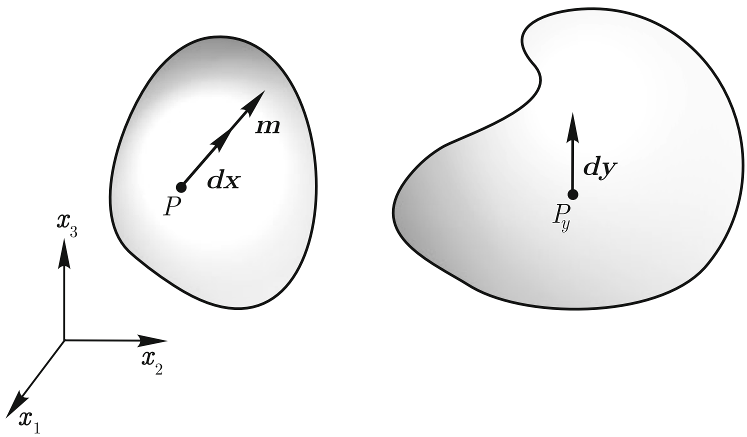

Fig. 3.22. Infinitesimal fibers in undeformed and deformed configurations

Let us first derive the expression that gives the normal strain of any fiber. Let **m** be the unit vector in the direction of **dx** as shown in Figure 3.22 and ds and ds_y be the lengths of the fibers **dx** and **dy**, which are given by

$$\begin{aligned} ds &= \|\mathbf{dx}\| = \left((dx_1)^2 + (dx_2)^2 + (dx_3)^2\right)^{\frac{1}{2}} \\ ds_y &= \|\mathbf{dy}\| = \left((dy_1)^2 + (dy_2)^2 + (dy_3)^2\right)^{\frac{1}{2}} \end{aligned} \tag{3.18}$$

The normal strain[6] of the fiber **dx** is given by

$$\varepsilon_\ell = \frac{ds_y - ds}{ds} = \frac{ds_y}{ds} - 1.$$

It is usual to define the stretch of a fiber by

[6] For other strain measures, useful in nonlinear formulations, see *e.g.* Bathe, 1996

$$\lambda = \frac{ds_y}{ds}$$

i.e., the ratio between the deformed and undeformed lengths. Hence,

$$\varepsilon_\ell = \lambda - 1.$$

From equation (3.18)

$$ds_y = \|\mathbf{dy}\| = \sqrt{\mathbf{dy}^T\mathbf{dy}} = \sqrt{\mathbf{dx}^T\mathbf{X}^T\mathbf{X}\mathbf{dx}}.$$

which leads to

$$\varepsilon_\ell = \frac{ds_y}{ds} - 1 = \frac{\sqrt{\mathbf{dx}^T\mathbf{X}^T\mathbf{X}\mathbf{dx}}}{\sqrt{\mathbf{dx}^T\mathbf{dx}}} - 1$$

and since $\mathbf{dx} = ds\ \mathbf{m}$ we obtain

$$\varepsilon_\ell = \frac{ds\sqrt{\mathbf{m}^T\mathbf{X}^T\mathbf{X}\mathbf{m}}}{ds\sqrt{\mathbf{m}^T\mathbf{m}}} - 1$$

or equivalently

$$\varepsilon_\ell = \sqrt{\mathbf{m}^T\mathbf{X}^T\mathbf{X}\mathbf{m}} - 1 \tag{3.19}$$

since $\mathbf{m}$ is a unit vector.

The above expression gives the normal strain of a fiber, in the direction $\mathbf{m}$, *i.e.*, $\varepsilon_\ell(\mathbf{m})$. There is no restriction on the magnitude of the displacements involved in the deformation. However, when the displacements are assumed to be infinitesimally small, we can introduce simplifications (this is the case in linear elasticity).

Using equation (3.16) we can write

$$\begin{aligned} \mathbf{X}^T\mathbf{X} &= (\mathbf{I} + \nabla\mathbf{u})^T(\mathbf{I} + \nabla\mathbf{u}) \\ &= \mathbf{I} + \nabla\mathbf{u} + \nabla\mathbf{u}^T + \nabla\mathbf{u}^T\nabla\mathbf{u} \end{aligned} \tag{3.20}$$

Note that the components of $\nabla\mathbf{u}^T\nabla\mathbf{u}$ involve sums of terms of the form $\frac{\partial u_i}{\partial x_k}\frac{\partial u_j}{\partial x_k}$ (refer to relation (3.15)). When the displacements are considered infinitesimally small, the terms $\frac{\partial u_i}{\partial x_k}$ are also considered infinitesimally small and therefore we can neglect $\frac{\partial u_i}{\partial x_k}\frac{\partial u_j}{\partial x_k}$ with respect to $\frac{\partial u_i}{\partial x_k}$. Hence we can neglect $\nabla\mathbf{u}^T\nabla\mathbf{u}$ with respect to $\nabla\mathbf{u}$ and use

$$\mathbf{X}^T\mathbf{X} = \mathbf{I} + \nabla\mathbf{u} + \nabla\mathbf{u}^T. \tag{3.21}$$

and can define the infinitesimal strain tensor $\mathbf{E}$ by

$$\mathbf{E} = \frac{1}{2}\left(\nabla\mathbf{u} + \nabla\mathbf{u}^T\right). \tag{3.22}$$

The components of $\mathbf{E}$ are

$$\mathbf{E} = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) \\ \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) & \frac{\partial u_2}{\partial x_2} & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) \\ \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) & \frac{\partial u_3}{\partial x_3} \end{bmatrix}. \quad (3.23)$$

The strain tensor $\mathbf{E}$ is symmetric, $\mathbf{E} = \mathbf{E}^T$, whereas $\mathbf{X}$ is in general not symmetric. Hence assuming infinitesimally small displacements we have

$$\mathbf{X}^T\mathbf{X} = \mathbf{I}+2\mathbf{E} \quad (3.24)$$

and equation (3.19) can be re-written as

$$\varepsilon_\ell = \sqrt{\mathbf{m}^T(\mathbf{I}+2\mathbf{E})\mathbf{m}} - 1 = (1 + 2\mathbf{m}^T\mathbf{E}\mathbf{m})^{\frac{1}{2}} - 1. \quad (3.25)$$

Using (3.23) we can verify that

$$\mathbf{m}^T\mathbf{E}\mathbf{m} = \sum_{i=1}^{3}\sum_{j=1}^{3} \frac{\partial u_i}{\partial x_j} m_i m_j. \quad (3.26)$$

Of course, $\mathbf{m}^T\mathbf{E}\mathbf{m}$ is a real number. We next recall the useful mathematical identity

$$(1+\delta)^s = 1 + s\delta + (\text{higher-order terms in } \delta) \quad (3.27)$$

for δ and s real numbers and δ small. Substituting $\delta = 2\mathbf{m}^T\mathbf{E}\mathbf{m}$ and $s = \frac{1}{2}$, we can write

$$\left(1 + 2\mathbf{m}^T\mathbf{E}\mathbf{m}\right)^{\frac{1}{2}} = 1 + \frac{1}{2}\, 2\mathbf{m}^T\mathbf{E}\mathbf{m} + \left(\text{higher-order terms in } \mathbf{m}^T\mathbf{E}\mathbf{m}\right). \quad (3.28)$$

If we neglect higher-order terms in $\frac{\partial u_i}{\partial x_j}$, we arrive at

$$\left(1 + 2\mathbf{m}^T\mathbf{E}\mathbf{m}\right)^{1/2} = 1 + \mathbf{m}^T\mathbf{E}\mathbf{m} \quad (3.29)$$

which is exact for infinitesimally small displacements. Introducing (3.29) into (3.25), we finally obtain for infinitesimally small displacements

$$\varepsilon_\ell(\mathbf{m}) = \mathbf{m}^T\mathbf{E}\mathbf{m}. \quad (3.30)$$

Therefore in analyses assuming infinitesimally small displacements, the normal strain ε_ℓ of any fiber with origin at point $\mathbf{P}$ and direction $\mathbf{m}$ can be

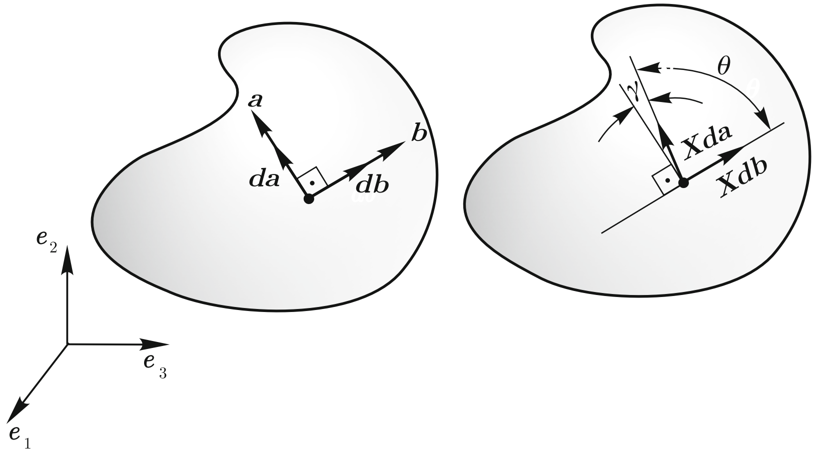

Fig. 3.23. Shear strain γ for fibers **da** and **db**

determined by the strain tensor **E** at this point, and is given by equation (3.30).

As we mentioned earlier, the change in angle between fibers due to the deformation needs also to be evaluated.

Hence, let us consider two orthogonal fibers **da** and **db** in the undeformed configuration whose directions are given by the unit vectors **a** and **b** as indicated in Figure 3.23.

In general, these fibers are no longer orthogonal in the deformed configuration. Let θ be the angle between the fibers in the deformed configuration. We define the shear strain (angular distortion) for this pair of fibers by the angle $\gamma = \frac{\pi}{2} - \theta$. Hence, the shear strain measures the deviation from the orthogonality and γ is positive when the angle between fibers, originally equal to $\pi/2$ has decreased. To obtain γ, using the definition of the scalar product, we write

$$\mathbf{X}\,\mathbf{da} \cdot \mathbf{X}\,\mathbf{db} = \|\mathbf{X}\,\mathbf{da}\|\,\|\mathbf{X}\,\mathbf{db}\| \cos\theta$$

and using that $\mathbf{da} = da\,\mathbf{a}$ and $\mathbf{db} = db\,\mathbf{b}$, we obtain

$$\mathbf{Xa} \cdot \mathbf{Xb} = \|\mathbf{Xa}\|\,\|\mathbf{Xb}\| \cos\theta .$$

Then also

$$\mathbf{Xa} \cdot \mathbf{Xb} = \mathbf{a}^T\,\mathbf{X}^T\mathbf{Xb}, \quad \|\mathbf{Xa}\| = \sqrt{\mathbf{a}^T\,\mathbf{X}^T\mathbf{Xa}}, \quad \|\mathbf{Xb}\| = \sqrt{\mathbf{b}^T\,\mathbf{X}^T\mathbf{Xb}}$$

leading to

$$\sin\gamma = \frac{\mathbf{a}^T\,\mathbf{X}^T\mathbf{Xb}}{\sqrt{\mathbf{a}^T\,\mathbf{X}^T\mathbf{Xa}}\sqrt{\mathbf{b}^T\,\mathbf{X}^T\mathbf{Xb}}} \tag{3.31}$$

where we used $\cos\theta = \sin\gamma$. From (3.19)

$$1+\varepsilon_\ell(\mathbf{a}) = \sqrt{\mathbf{a}^T\mathbf{X}^T\mathbf{X}\mathbf{a}}, \qquad 1+\varepsilon_\ell(\mathbf{b}) = \sqrt{\mathbf{b}^T\mathbf{X}^T\mathbf{X}\mathbf{b}}$$

where $\varepsilon_\ell(\mathbf{a})$ and $\varepsilon_\ell(\mathbf{b})$ are the normal strains of the fibers in the directions of $\mathbf{a}$ and $\mathbf{b}$. Therefore (3.31) can be written as

$$\sin\gamma = \frac{\mathbf{a}^T\mathbf{X}^T\mathbf{X}\mathbf{b}}{(1+\varepsilon_\ell(\mathbf{a}))(1+\varepsilon_\ell(\mathbf{b}))}. \tag{3.32}$$

Both expressions (3.31) and (3.32) give the shear strain without any approximation no matter how large the displacements might be. However, considering infinitesimally small displacements and using (3.24), the numerator of expression (3.31) can be written as

$$\mathbf{b}^T(\mathbf{I}+2\mathbf{E})\mathbf{a} = 2\mathbf{b}^T\mathbf{E}\mathbf{a} + \mathbf{b}^{\mathbf{T}}\mathbf{a} = 2\mathbf{b}^T\mathbf{E}\mathbf{a}$$

since $\mathbf{a}$ and $\mathbf{b}$ are orthogonal to each other. Using (3.26) and (3.30) we can see that ε_ℓ will always be given by a sum of terms of the type $\frac{\partial u_i}{\partial x_j}$. Hence, we can neglect ε_ℓ with respect to 1 in the denominator of (3.32). Also, using $\sin\gamma = \gamma$ since we are considering infinitesimally small displacements, we obtain

$$\gamma = 2\mathbf{b}^T\mathbf{E}\mathbf{a}. \tag{3.33}$$

Hence, in summary, the infinitesimal strain tensor $\mathbf{E}$ gives, through the very simple expressions (3.30) and (3.33), the normal and shear strains. Writing

$$\mathbf{E} = \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{21} & E_{22} & E_{23} \\ E_{31} & E_{32} & E_{33} \end{bmatrix}$$

the normal strain of a fiber in the direction of $\mathbf{e}_1$ is given by

$$\varepsilon_\ell = \mathbf{e}_1^T\mathbf{E}\mathbf{e}_1$$

or

$$\varepsilon_\ell = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{21} & E_{22} & E_{23} \\ E_{31} & E_{32} & E_{33} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

which leads to

Table 3.1. Normal and shear strains for finite and infinitesimally small displacements

	Finite displacements	Infinitesimally small displacements
Normal strain of fiber with direction **m**	$\varepsilon_\ell = \sqrt{\mathbf{m}^T\mathbf{X}^T\mathbf{X}\mathbf{m}} - 1$	$\varepsilon_\ell = \mathbf{m}^T\mathbf{E}\mathbf{m}$
Shear strain of a pair of orthogonal fibers with direction **a** and **b**	$\sin\gamma = \frac{\mathbf{a}^T\mathbf{X}^T\mathbf{X}\mathbf{b}}{(1+\varepsilon_\ell(\mathbf{a}))(1+\varepsilon_\ell(\mathbf{b}))}$	$\gamma = 2\mathbf{b}^T\mathbf{E}\mathbf{a}$

$$\varepsilon_\ell(\mathbf{e}_1) = E_{11}$$

i.e., E_{11} gives the normal strain of a fiber in the direction of $\mathbf{e}_1$. Analogously,

$$\varepsilon_\ell(\mathbf{e}_2) = E_{22}, \qquad \varepsilon_\ell(\mathbf{e}_3) = E_{33}.$$

The shear strain of two fibers whose directions are $\mathbf{e}_1$ and $\mathbf{e}_2$ is given by

$$\begin{aligned}\gamma &= 2\mathbf{e}_1^T\mathbf{E}\mathbf{e}_2 \\ &= 2\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{21} & E_{22} & E_{23} \\ E_{31} & E_{32} & E_{33} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\end{aligned}$$

which leads to

$$\gamma(\mathbf{e}_1, \mathbf{e}_2) = 2E_{12}$$

and allows for the physical interpretation of the component E_{12} (see Example 3.2). We note that $\gamma(\mathbf{e}_1, \mathbf{e}_2) = 2E_{12} = 2E_{21} = \gamma(\mathbf{e}_2, \mathbf{e}_1)$ due to the symmetry of $\mathbf{E}$. Analogously

$$\gamma(\mathbf{e}_1, \mathbf{e}_3) = 2E_{13} = 2E_{31} = \gamma(\mathbf{e}_3, \mathbf{e}_1), \gamma(\mathbf{e}_2, \mathbf{e}_3) = 2E_{23} = 2E_{32} = \gamma(\mathbf{e}_3, \mathbf{e}_2).$$

In Table 3.1 we summarize the equations that give the normal and shear strains considering finite and infinitesimally small displacements. Note that the term "finite" (as opposed to "infinitesimal") is used to characterize displacements of arbitrary magnitude (which of course includes the infinitesimally small displacement case). The terminology "large displacements" and "small displacements" is also used to describe these conditions.

There is one important point to recognize. While we have six strain components in $\mathbf{E}$, see (3.23), these are clearly not linearly independent since they are all calculated from the three independent displacement fields u, v and w. However, as long as the strain components are calculated from the (continuous) displacements as discussed above, the strains will be compatible. Throughout the presentations in this book we shall assume continuous displacements for all problems considered, and hence strain compatibility reduces to calculating the strain components appropriately from the applicable strain-displacement relationships.

Example 3.2

Consider a block which has a generic section parallel to the x_1x_2 plane as shown in Figure 3.24a. The deformed configuration to be studied is defined by

$$\begin{aligned} y_1 &= x_1 + \tan\beta\, x_2 \\ y_2 &= x_2 \\ y_3 &= x_3 \end{aligned}$$

with β large as shown in Figure 3.24b.

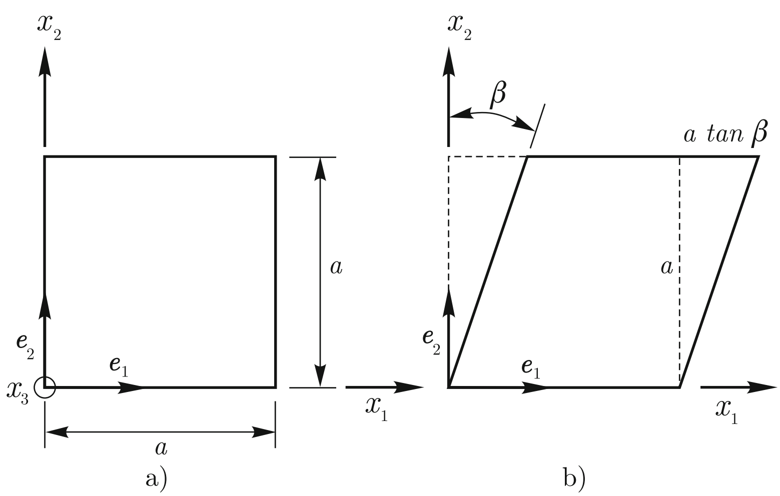

Fig. 3.24. Section of block and deformation

(i) Calculate the displacement field and the normal strains of the fibers with directions $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{m}_1 = \frac{\sqrt{2}}{2}\mathbf{e}_1 + \frac{\sqrt{2}}{2}\mathbf{e}_2$ and $\mathbf{m}_2 = -\frac{\sqrt{2}}{2}\mathbf{e}_1 + \frac{\sqrt{2}}{2}\mathbf{e}_2$.

(ii) Calculate the shear strains of the pairs of fibers with directions $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{m}_1$, $\mathbf{m}_2$.
(iii) Repeat items (i) and (ii) assuming that β is small. Obtain the results first directly using small displacement theory, and then show that these same results are also obtained from (i) and (ii).

Solution

(i) The displacement field is given by

$$u_i = y_i - x_i$$

which leads to

$$\begin{aligned} u_1 &= y_1 - x_1 = \tan\beta\, x_2 \\ u_2 &= y_2 - x_2 = 0 \\ u_3 &= y_3 - x_3 = 0. \end{aligned}$$

Considering large displacements, we need to evaluate the deformation gradient

$$\mathbf{X} = \begin{bmatrix} 1 & \tan\beta & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where we note that throughout the block the elements in $\mathbf{X}$ are constant (independent of x_1, x_2, x_3). Then

$$\mathbf{X}^T\mathbf{X} = \begin{bmatrix} 1 & \tan\beta & 0 \\ \tan\beta & 1 + (\tan\beta)^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Now using the left column in Table 3.1

$$\varepsilon_\ell(\mathbf{e}_1) = \sqrt{\mathbf{e}_1^T\mathbf{X}^T\mathbf{X}\mathbf{e}_1} - 1 = 0 \tag{3.34}$$

$$\varepsilon_\ell(\mathbf{e}_2) = \sqrt{\mathbf{e}_2^T\mathbf{X}^T\mathbf{X}\mathbf{e}_2} - 1 = \sqrt{1 + (\tan\beta)^2} - 1 \tag{3.35}$$

$$\varepsilon_\ell(\mathbf{m}_1) = \sqrt{\mathbf{m}_1^T\mathbf{X}^T\mathbf{X}\mathbf{m}_1} - 1 = \sqrt{1 + \tan\beta + \frac{(\tan\beta)^2}{2}} - 1 \tag{3.36}$$

$$\varepsilon_\ell(\mathbf{m}_2) = \sqrt{\mathbf{m}_2^T\mathbf{X}^T\mathbf{X}\mathbf{m}_2} - 1 = \sqrt{1 - \tan\beta + \frac{(\tan\beta)^2}{2}} - 1. \tag{3.37}$$

We can interpret these results with the aid of Figure 3.25, where we can see that the fibers in the directions given by AB and AC extended, whereas the fibers along the diagonal DB shortened. Of course, the fibers parallel to AD did not change their lengths.

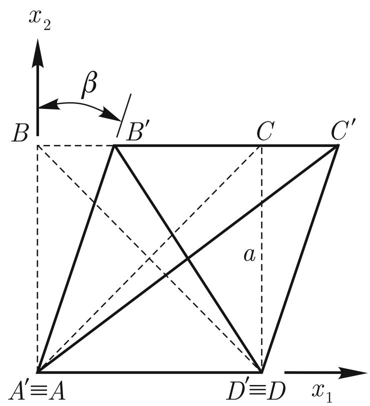

Fig. 3.25. Deformed and undeformed configurations with diagonals represented

(ii) The shear strains can be evaluated using again Table 3.1

$$\sin\gamma\left(\mathbf{e}_1,\mathbf{e}_2\right)=\frac{\mathbf{e}_1^T\mathbf{X}^T\mathbf{X}\mathbf{e}_2}{\left(1+\varepsilon_\ell(\mathbf{e}_1)\right)\left(1+\varepsilon_\ell(\mathbf{e}_2)\right)}=\frac{\tan\beta}{\sqrt{1+\left(\tan\beta\right)^2}}=\sin\beta \quad (3.38)$$

leading to $\gamma=\beta$, and

$$\begin{aligned}\sin\gamma\left(\mathbf{m}_1,\mathbf{m}_2\right) &= \frac{\mathbf{m}_1^T\mathbf{X}^T\mathbf{X}\mathbf{m}_2}{\left(1+\varepsilon_\ell(\mathbf{m}_1)\right)\left(1+\varepsilon_\ell(\mathbf{m}_2)\right)}\\ &= \frac{\left(\tan\beta\right)^2}{2\sqrt{1+\tan\beta+\frac{(\tan\beta)^2}{2}}\sqrt{1-\tan\beta+\frac{(\tan\beta)^2}{2}}}. \quad (3.39)\end{aligned}$$

(iii) When the displacements are infinitesimally small, we can evaluate the normal and shear strains using the infinitesimal strain tensor whose components are given by

$$E_{ij}=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right).$$

Then,

$$E_{11} = \frac{\partial u_1}{\partial x_1} = 0, \qquad E_{22} = \frac{\partial u_2}{\partial x_2} = 0, \qquad E_{33} = \frac{\partial u_3}{\partial x_3} = 0$$

and since for infinitesimally small displacements $\tan\beta = \beta$

$$\begin{aligned} E_{12} &= E_{21} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) = \frac{\beta}{2} \\ E_{13} &= E_{31} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) = 0 \\ E_{23} &= E_{32} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) = 0 \end{aligned}$$

leading to

$$\mathbf{E} = \begin{bmatrix} 0 & \beta/2 & 0 \\ \beta/2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and, hence, we obtain

$$\varepsilon_\ell(\mathbf{e}_1) = E_{11} = 0, \qquad \varepsilon_\ell(\mathbf{e}_2) = E_{22} = 0, \qquad \gamma(\mathbf{e}_1, \mathbf{e}_2) = 2E_{12} = \beta.$$

The normal strains of the fibers in the directions of $\mathbf{m}_1$ and $\mathbf{m}_2$ can be evaluated by

$$\begin{aligned} \varepsilon_\ell(\mathbf{m}_1) &= \mathbf{m}_1^T \mathbf{E} \mathbf{m}_1 \\ &= \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \end{bmatrix} \begin{bmatrix} 0 & \beta/2 & 0 \\ \beta/2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix} = \frac{\beta}{2} \\ \varepsilon_\ell(\mathbf{m}_2) &= \mathbf{m}_2^T \mathbf{E} \mathbf{m}_2 \\ &= \begin{bmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \end{bmatrix} \begin{bmatrix} 0 & \beta/2 & 0 \\ \beta/2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix} = -\frac{\beta}{2}. \end{aligned}$$

The shear strain of the fibers with directions $\mathbf{m}_1$, $\mathbf{m}_2$ is given by

$$\gamma(\mathbf{m}_1, \mathbf{m}_2) = 2\mathbf{m}_1^T \mathbf{E} \mathbf{m}_2$$
$$= 2 \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \end{bmatrix} \begin{bmatrix} 0 & \beta/2 & 0 \\ \beta/2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix} = 0.$$

In order to show that we obtain the above results from the values calculated in (i) and (ii), we need to consider β infinitesimally small in the expressions (3.35) to (3.39). Using the mathematical identity (3.27) we obtain

$$\varepsilon_\ell(\mathbf{e}_2) = 1 + \frac{1}{2} (\tan\beta)^2 + \left(\text{higher order terms in } (\tan\beta)^2\right) - 1$$

leading to

$$\varepsilon_\ell(\mathbf{e}_2) = 0. \tag{3.40}$$

Similarly

$$\varepsilon_\ell(\mathbf{m}_1) = 1 + \frac{1}{2}\left(\tan\beta + \frac{(\tan\beta)^2}{2}\right) + \left(\text{higher order terms in } \left(\tan\beta + \frac{(\tan\beta)^2}{2}\right)\right) - 1$$

leading to

$$\varepsilon_\ell(\mathbf{m}_1) = \frac{1}{2}\tan\beta. \tag{3.41}$$

Analogously, we obtain

$$\varepsilon_\ell(\mathbf{m}_2) = -\frac{1}{2}\tan\beta. \tag{3.42}$$

Note that to obtain these relations we neglected the terms $(\tan\beta)^2$ and of higher–order. And now using that for infinitesimally small displacements $\tan\beta = \beta$ gives

$$\varepsilon_\ell(\mathbf{e}_2) = 0, \quad \varepsilon_\ell(\mathbf{m}_1) = \frac{\beta}{2} \quad \text{and} \quad \varepsilon_\ell(\mathbf{m}_2) = -\frac{\beta}{2}.$$

Introducing the same approximations for (3.38) and (3.39), we directly obtain

$$\gamma(\mathbf{e}_1, \mathbf{e}_2) = \beta$$

and

$$\gamma(\mathbf{m}_1, \mathbf{m}_2) = 0.$$

Referring to Figure 3.25, we note that fibers that are parallel to the diagonals AC and DB, hence orthogonal in the undeformed configuration, remain orthogonal for infinitesimally small displacements.

□

3.2.3 Finite and infinitesimal rigid deformations

The kinematic description presented so far is of course very general and therefore includes a rigid body motion/deformation of a solid.

We start by studying a rigid body rotation which is always referred to an axis – the axis of rotation which is shown in Figure 3.26a. The rotation can

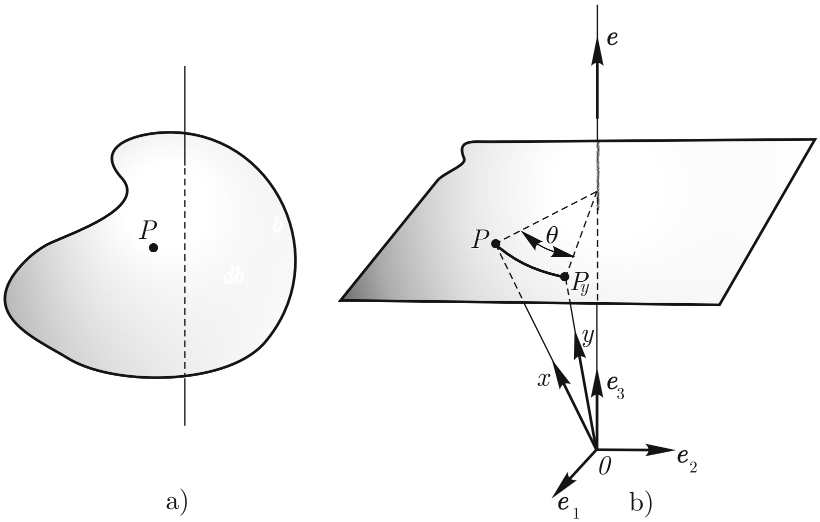

Fig. 3.26. Kinematic description of a rigid rotation

be kinematically characterized by describing the motion of a generic point **P** of the solid as shown in Figure 3.26b. We see that point **P** moves on the plane which is orthogonal to the axis of rotation and passes through **P**, and on the circle centered on the axis of rotation. The angle θ defines the magnitude of the rotation. Hence, a rotation is fully characterized by the vector $\theta = \theta\mathbf{e}$, where $\mathbf{e}$ is a unit vector in the direction of the axis of rotation.

Let us choose a reference system such that $\mathbf{e}_3 = \mathbf{e}$ as shown in Figure 3.26b. Referring to Example 3.1, the deformation corresponding to the rotation described above is given by

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{3.43}$$

$$\mathbf{y} = \mathbf{Q}\mathbf{x} \tag{3.44}$$

where $\mathbf{Q}$ is an orthogonal tensor. Orthogonal tensors are defined by either one of the following equivalent statements:

$$\|\mathbf{Q}\mathbf{w}\| = \|\mathbf{w}\| \quad \text{for every vector } \mathbf{w} \tag{3.45}$$

$$\mathbf{Q}\mathbf{Q}^T = \mathbf{Q}^T\mathbf{Q} = \mathbf{I} \quad \Rightarrow \quad \mathbf{Q}^{-1} = \mathbf{Q}^T. \tag{3.46}$$

From (3.46), we can write $\det(\mathbf{Q}\mathbf{Q}^T) = \det\mathbf{Q}\,\det\mathbf{Q}^T = (\det\mathbf{Q})^2 = 1$. Hence $\det\mathbf{Q} = \pm 1$.

It can be shown that every orthogonal tensor with positive determinant ($\det\mathbf{Q} = 1$) describes a rotation through equation (3.44). The example below is an illustration.

Example 3.3

Consider a tensor $\mathbf{Q}$ given in the basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ by

$$\mathbf{Q} = \begin{bmatrix} \frac{4}{9} + \frac{5\sqrt{3}}{18} & \frac{5}{18} - \frac{2\sqrt{3}}{9} & \frac{5}{9} - \frac{\sqrt{3}}{9} \\ \frac{11}{18} - \frac{2\sqrt{3}}{9} & \frac{4}{9} + \frac{5\sqrt{3}}{18} & -\frac{1}{9} - \frac{\sqrt{3}}{9} \\ -\frac{1}{9} - \frac{\sqrt{3}}{9} & \frac{5}{9} - \frac{\sqrt{3}}{9} & \frac{1}{9} + \frac{4\sqrt{3}}{9} \end{bmatrix}.$$

Verify that $\mathbf{Q}$ is an orthogonal tensor. Then, obtain the axis and the magnitude of the rotation given by $\mathbf{Q}$.

Solution

It suffices to verify that $\mathbf{Q}^T\mathbf{Q} = \mathbf{I}$ to conclude that $\mathbf{Q}$ is an orthogonal tensor.

Let $\mathbf{e}$ be a unit vector in the direction of the axis of rotation defined by $\mathbf{Q}$. Then

$$\mathbf{Q}\mathbf{e} = \mathbf{e}.$$

We can determine $\mathbf{e}$ solving

$$(\mathbf{Q} - \mathbf{I})\,\mathbf{e} = \mathbf{0} \tag{3.47}$$

to obtain

$$\mathbf{e} = \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}.$$

Let us define a unit vector $\mathbf{g}$ which is orthogonal to the axis of rotation using

$$\mathbf{g} = \frac{\mathbf{e} \times \mathbf{e}_2}{\|\mathbf{e} \times \mathbf{e}_2\|} = \begin{bmatrix} -\frac{\sqrt{5}}{5} \\ 0 \\ \frac{2\sqrt{5}}{5} \end{bmatrix}.$$

Note that

$$\mathbf{f} = \mathbf{Q}\mathbf{g} = \begin{bmatrix} -\frac{\sqrt{15}}{10} + \frac{2\sqrt{5}}{15} \\ -\frac{\sqrt{5}}{6} \\ \frac{\sqrt{15}}{5} + \frac{\sqrt{5}}{15} \end{bmatrix}$$

gives a unit vector which is rotated with respect to $\mathbf{g}$ by the angle of rotation θ. Then

$$\mathbf{g} \cdot \mathbf{f} = \|\mathbf{g}\| \, \|\mathbf{f}\| \cos\theta = \frac{\sqrt{3}}{2}$$

and since

$$(\mathbf{g} \times \mathbf{f}) \cdot \mathbf{e} = \frac{1}{2}$$

we have

$$\theta = \frac{\pi}{6}.$$

□

Now let us show that the deformation given by

$$\mathbf{y}(\mathbf{x}) = \mathbf{y}_0 + \mathbf{Q}(\mathbf{x} - \mathbf{x}_0) \tag{3.48}$$

is rigid for any given vectors $\mathbf{y}_0$ and $\mathbf{x}_0$, and any orthogonal tensor $\mathbf{Q}$. As detailed later (see Figure 3.27), $\mathbf{y}_0$ gives the translation part of the rigid deformation while $\mathbf{Q}(\mathbf{x} - \mathbf{x}_0)$ gives the rotation part and $\mathbf{x}_0$ is a point on the axis of rotation.

The fundamental property of a rigid deformation is that the distance between any two points of the solid does not change with the deformation. Let us verify that this property holds for any deformation defined by (3.48). Let $\mathbf{x}_1$ and $\mathbf{x}_2$ be the position vectors of two arbitrarily selected points of the solid. Then, their positions in the deformed configuration are given by

$$\mathbf{y}_1 = \mathbf{y}_0 + \mathbf{Q}(\mathbf{x}_1 - \mathbf{x}_0) \tag{3.49}$$

$$\mathbf{y}_2 = \mathbf{y}_0 + \mathbf{Q}(\mathbf{x}_2 - \mathbf{x}_0). \tag{3.50}$$

Subtracting (3.49) from (3.50), we obtain

$$\mathbf{y}_2 - \mathbf{y}_1 = \mathbf{Q}\mathbf{x}_2 - \mathbf{Q}\mathbf{x}_1 = \mathbf{Q}\left(\mathbf{x}_2 - \mathbf{x}_1\right). \tag{3.51}$$

Taking norms in both sides of (3.51) and using (3.45), we can write

$$\|\mathbf{y}_2 - \mathbf{y}_1\| = \|\mathbf{Q}\left(\mathbf{x}_2 - \mathbf{x}_1\right)\| = \|\mathbf{x}_2 - \mathbf{x}_1\|$$

which shows that the distance between any two points of the solid is maintained and, indeed, (3.48) defines a rigid deformation or a *finite rigid deformation* since the magnitude of the rotation may be arbitrarily large.

Of course, we can select $\mathbf{x}_1$ and $\mathbf{x}_2$ close enough such that $\mathbf{dx} = \mathbf{x}_2 - \mathbf{x}_1$ defines a fiber. Then, (3.51) leads to

$$\mathbf{dy} = \mathbf{Qdx}$$

showing that the deformation gradient of a rigid deformation given by (3.48) is $\mathbf{Q}$, that is,

$$\mathbf{X} = \mathbf{Q} \tag{3.52}$$

and due to (3.16)

$$\nabla \mathbf{u} = \mathbf{Q} - \mathbf{I}. \tag{3.53}$$

Note that $\mathbf{X}$ and $\nabla \mathbf{u}$ as given in equations (3.52) and (3.53) are the same for every point of the solid.

Since

$$\|\mathbf{dy}\| = \|\mathbf{Qdx}\| = \|\mathbf{dx}\| \tag{3.54}$$

the normal strain in any direction is zero. Also, there is no shear strain between any pair of orthogonal fibers. Indeed, using (3.32)

$$\sin\gamma = \frac{\mathbf{a}^T \mathbf{Q}^T \mathbf{Q} \mathbf{b}}{(1 + \varepsilon_\ell(\mathbf{a}))(1 + \varepsilon_\ell(\mathbf{b}))} = \mathbf{a}^T \mathbf{b} = 0. \tag{3.55}$$

The results given by (3.54) and (3.55) actually confirm our expectation regarding strains for a rigid deformation.

Defining $\mathbf{u}_0 = \mathbf{y}_0 - \mathbf{x}_0$, we obtain the displacement field for a rigid deformation

$$\mathbf{u} = \mathbf{y}(\mathbf{x}) - \mathbf{x} = \mathbf{u}_0 + \mathbf{Q}(\mathbf{x} - \mathbf{x}_0) - (\mathbf{x} - \mathbf{x}_0) = \mathbf{u}_0 + (\mathbf{Q} - \mathbf{I})(\mathbf{x} - \mathbf{x}_0) \tag{3.56}$$

and using (3.53), we obtain

$$\mathbf{u} = \mathbf{u}_0 + \nabla \mathbf{u}(\mathbf{x} - \mathbf{x}_0). \tag{3.57}$$

We emphasize that $\nabla \mathbf{u}$ in (3.57) is the displacement gradient for the finite rigid deformation given by (3.53).

In order to motivate the definition of an infinitesimal rigid deformation we consider the following derivations.

Using (3.20) and (3.52), we obtain

$$\mathbf{X}^T \mathbf{X} = \mathbf{Q}^T \mathbf{Q} = \mathbf{I} = \mathbf{I} + \nabla \mathbf{u} + \nabla \mathbf{u}^T + \nabla \mathbf{u}^T \nabla \mathbf{u}$$

hence,

$$\frac{\nabla\mathbf{u}+\nabla\mathbf{u}^T}{2} = -\frac{1}{2}\nabla\mathbf{u}^T\nabla\mathbf{u}$$

and the infinitesimal strain tensor for the rigid deformation is given by (see (3.22))

$$\mathbf{E} = \frac{1}{2}(\nabla\mathbf{u}+\nabla\mathbf{u}^T) = -\frac{1}{2}\nabla\mathbf{u}^T\nabla\mathbf{u} \tag{3.58}$$

which is not exactly zero. This fact motivates the definition of an *infinitesimal rigid deformation* as the deformation which gives $\mathbf{E} = \mathbf{0}$.

Consider

$$\mathbf{u}(\mathbf{x}) = \mathbf{u}_0 + \mathbf{W}(\mathbf{x}-\mathbf{x}_0) \tag{3.59}$$

where $\mathbf{W}$ is a *skew tensor*, that is, a tensor for which $\mathbf{W}^T = -\mathbf{W}$. Then the displacement gradient for the $\mathbf{u}(\mathbf{x})$ defined in (3.59) is

$$\nabla\mathbf{u} = \mathbf{W}$$

and

$$\mathbf{E} = \frac{1}{2}\left(\nabla\mathbf{u}+\nabla\mathbf{u}^T\right) = \frac{1}{2}\left(\mathbf{W}+\mathbf{W}^T\right) = \mathbf{0}. \tag{3.60}$$

Equation (3.60) shows that every displacement field defined by (3.59) with $\mathbf{W}$ skew leads to $\mathbf{E} = \mathbf{0}$, *i.e.*, defines an infinitesimal rigid deformation.

Since for every skew tensor $\mathbf{W}$ there is a unique vector $\mathbf{w}$, called the axial vector, such that

$$\mathbf{W}\mathbf{a} = \mathbf{w}\times\mathbf{a} \quad \text{for every vector } \mathbf{a}$$

and vice-versa (see Crandall, Dahl and Lardner, 1978), we can re-write (3.59) as

$$\mathbf{u}(\mathbf{x}) = \mathbf{u}_0 + \mathbf{w}\times(\mathbf{x}-\mathbf{x}_0). \tag{3.61}$$

Now, let us interpret the finite and infinitesimal rigid deformations. In Figure 3.27a, we detail the displacement of a generic point for a finite rigid deformation. The unit vector $\mathbf{e}$ gives the direction of the axis of rotation associated with the orthogonal tensor $\mathbf{Q}$ and $\mathbf{x}_0$ is a point on the axis of rotation. According to equation (3.56) there are two contributions for the displacement. The first, $\mathbf{Q}(\mathbf{x}-\mathbf{x}_0)-(\mathbf{x}-\mathbf{x}_0)$, is due to the rotation and the second, $\mathbf{u}_0$, is due to the translation. In Figure 3.27b, we consider the displacement of a generic point $\mathbf{x}$ for an infinitesimal rigid deformation. The vector $\mathbf{w}$ is the axial vector associated with $\mathbf{W}$ and according to (3.61) the term $\mathbf{w}\times(\mathbf{x}-\mathbf{x}_0)$ gives the contribution of the rotation and $\mathbf{u}_0$ of the translation.

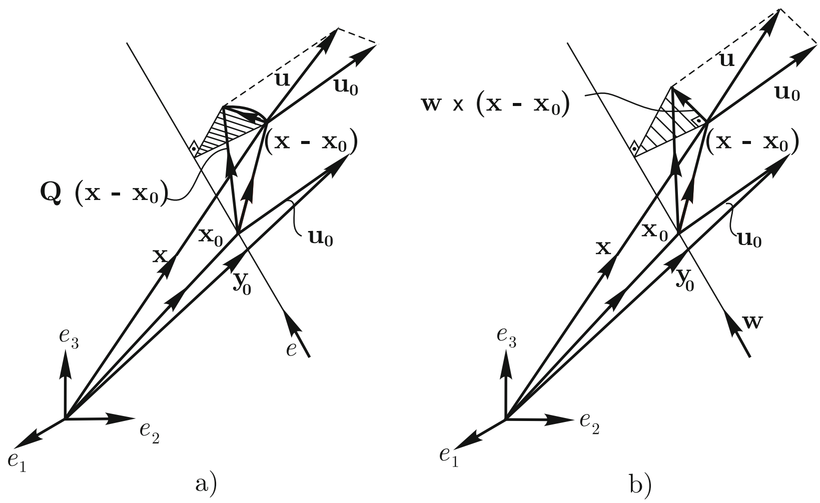

Fig. 3.27. Displacement of a point for finite and infinitesimal rigid deformations

Comparing both rigid deformations, we note that the difference lies on how the rotation is described. While for the finite rotation the point displaces along the circumference centered on the axis of rotation, for the infinitesimal rotation the point displaces over the straight line which makes a right angle with the radius of this circumference, that is, over the tangent to this circumference. We recall that this distinction in describing the rotation was also addressed in Section 2.2.5 for the rotation of a truss bar.

Example 3.4

Consider the rigid body rotation described in Example 3.1 (ii). Let the angle of rotation be infinitesimally small, that is, $d\varphi$ instead of φ. Calculate the displacement field associated with this infinitesimal rigid deformation. Then, verify that $\mathbf{E} = \mathbf{0}$.

Solution

In Figure 3.28a we repeat Figure 3.20b which describes the finite rotation of a generic point of the solid and in Figure 3.28b we show the analogous situation for an infinitesimal rotation. Since the rotation is infinitesimal, we should take the displacements over the tangent to the circumference which has center in O and radius $r = \|\mathbf{OP}\|$. Therefore

$$
\begin{aligned}
-u_1 &= d\varphi\, r \sin\theta \\
u_2 &= d\varphi\, r \cos\theta .
\end{aligned}
$$

Using (3.8) , we obtain

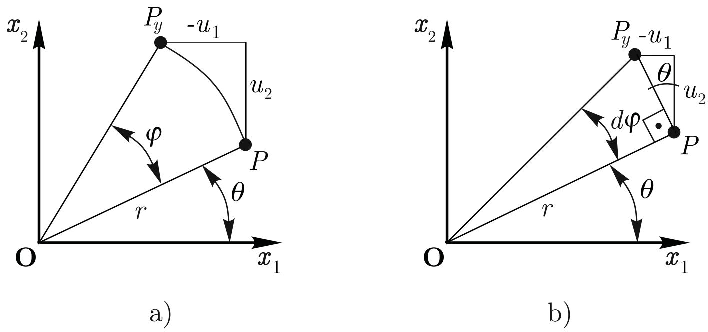

Fig. 3.28. Motion of a generic point in the section. a) Finite rotation φ; b) Infinitesimal rotation $d\varphi$

$$
\begin{aligned}
u_1 &= -d\varphi\ x_2 \\
u_2 &= d\varphi\ x_1
\end{aligned}
$$

Finally

$$
\begin{aligned}
u_1\left(x_1, x_2, x_3\right) &= -d\varphi\ x_2 \\
u_2\left(x_1, x_2, x_3\right) &= d\varphi\ x_1 \\
u_3\left(x_1, x_2, x_3\right) &= 0.
\end{aligned}
$$

Using the displacement field above and (3.23) we obtain $\mathbf{E} = \mathbf{0}$.

□

3.2.4 Technical or engineering notation for the strains

In the engineering literature it is very common to use what is called technical or engineering notation in which the coordinate axes are denoted by x, y, and z, *i.e.*, $x \equiv x_1$, $y \equiv x_2$ and $z \equiv x_3$ and the displacements by u, v, and w, *i.e.*, $u \equiv u_1$, $v \equiv u_2$ and $w \equiv u_3$.

In this notation the components of the strain tensor are given and denoted by

$$
\begin{aligned}
\varepsilon_{xx} &= \varepsilon_x = \frac{\partial u}{\partial x} \\
\varepsilon_{yy} &= \varepsilon_y = \frac{\partial v}{\partial y}
\end{aligned}
$$

$$\begin{aligned}
\varepsilon_{zz} &= \varepsilon_z = \frac{\partial w}{\partial z} \\
\varepsilon_{xy} &= \varepsilon_{yx} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \\
\varepsilon_{yz} &= \varepsilon_{zy} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \\
\varepsilon_{zx} &= \varepsilon_{xz} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right).
\end{aligned}$$

These relations are called strain-displacement relations. We can also write the shear strain for the pairs of fibers parallel to the coordinate axes by

$$\gamma_{xy} = 2\varepsilon_{xy}, \qquad \gamma_{yz} = 2\varepsilon_{yz}, \qquad \gamma_{zx} = 2\varepsilon_{zx}. \tag{3.62}$$

Note that these shear strains are twice the corresponding tensorial components and we call γ_{xy}, γ_{yz}, γ_{zx} *the engineering shear strains* and ε_{xy}, ε_{yz}, ε_{zx} the *tensorial shear strains.*

3.2.5 Deformation in the vicinity of a point

In order to obtain more insight into the deformation, let us examine the deformation in a small region around a point – its vicinity. Consider a deformation with infinitesimally small displacements and $w = 0$, $u = u(x, y)$ and $v = v(x, y)$. Under such conditions the deformation observed in any plane parallel to the plane xy is identical, and the displacement gradient and strain tensor are given by

$$\nabla \mathbf{u} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & 0 \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{E} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & 0 \\ \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & \frac{\partial v}{\partial y} & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Assume that at point **P** in Figure 3.29 $u = 0$, $v = 0$, $\frac{\partial u}{\partial x} = 0$ and $\frac{\partial v}{\partial y} = 0$, *i.e.*, the normal strains in the directions of x and y are zero. In Figure 3.30a, we show the deformed and undeformed configurations for the selected part. We note that the geometrical interpretation of the engineering shear strain $\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$ is evident. Now, in Figure 3.30b we show a special case in which $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$. Then, of course, $\gamma_{xy} = 0$ and we have a rigid body rotation of the part of intensity $\alpha = \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$. We note that despite the fact we have a rigid rotation, the components of the displacement gradient are not zero since $\frac{\partial u}{\partial y} \neq 0$ and $\frac{\partial v}{\partial x} \neq 0$.

Considering the deformation of Figure 3.30a, we can also write

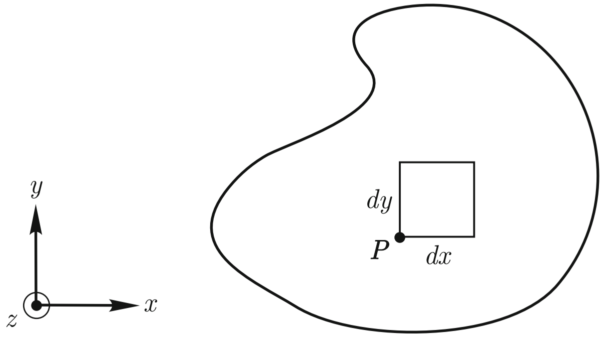

Fig. 3.29. Schematic representation of a generic square part defining the region of interest

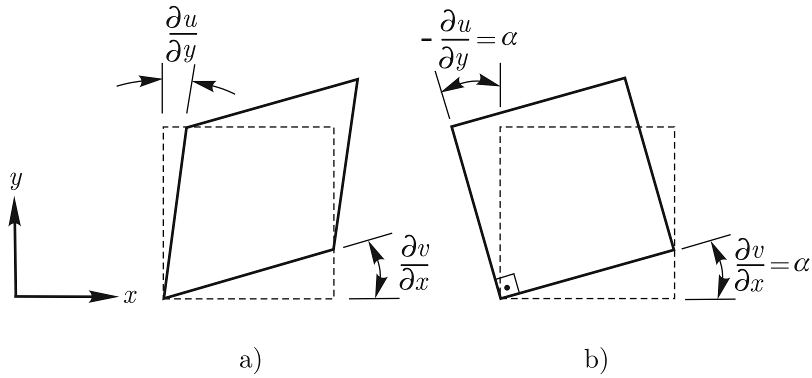

Fig. 3.30. a) Deformation of the selected part; b) Geometrical interpretation of an infinitesimal rigid rotation

$$\frac{\partial u}{\partial y} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + \frac{1}{2}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) \tag{3.63}$$

$$\frac{\partial v}{\partial x} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) - \frac{1}{2}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) \tag{3.64}$$

The left hand sides of equations (3.63) and (3.64) give components of the displacement gradient and the right hand sides represent one half of the engineering shear strain plus a term which can be interpreted as a rigid body rotation. The geometrical interpretation of equations (3.63) and (3.64) is given in Figure 3.31[7].

Although in the above discussion some assumptions were made to simplify the visualization of the deformation, the interpretation given is quite general. Let us define

[7] In Section 3.4 we will relate γ_{xy} by a material constant to a shear stress. Note that the rotation in (3.63) and (3.64) does not cause a stress

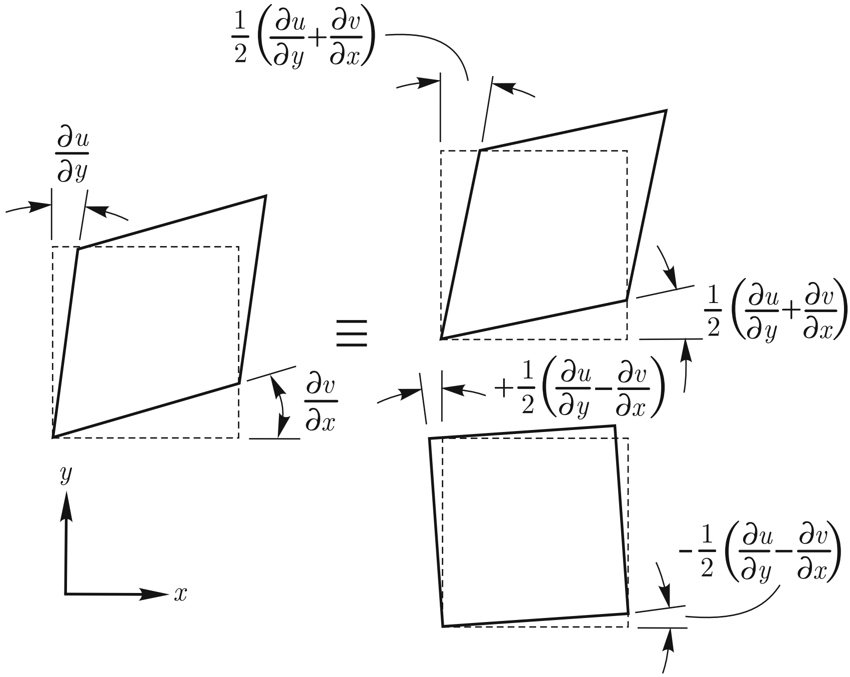

Fig. 3.31. Geometrical interpretation of the decomposition of the displacement gradient. The inherent rotation does not cause a stress (see Section 3.4)

$$
\begin{aligned}
\mathbf{W} &= \nabla\mathbf{u} - \mathbf{E} \\
\mathbf{W} &= \frac{1}{2}\left(\nabla\mathbf{u} - \nabla\mathbf{u}^T\right)
\end{aligned}
$$

or

$$
\mathbf{W} = \begin{bmatrix} 0 & \frac{1}{2}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) \\ -\frac{1}{2}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) & 0 & \frac{1}{2}\left(\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\right) \\ -\frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) & -\frac{1}{2}\left(\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\right) & 0 \end{bmatrix}.
$$

Of course,

$$\nabla\mathbf{u} = \mathbf{E} + \mathbf{W}. \tag{3.65}$$

We can also write

$$\mathbf{u}(\mathbf{x} + \mathbf{dx}) - \mathbf{u}(\mathbf{x}) = \nabla\mathbf{u}\mathbf{dx} = (\mathbf{E} + \mathbf{W})\,\mathbf{dx}. \tag{3.66}$$

Since $\mathbf{W}$ is skew, $\mathbf{Wdx}$ represents the displacement due to an infinitesimal rigid rotation (refer to equation (3.59)). Therefore, (3.66) allows the interpretation that, locally, the increment in displacements has a contribution of a rigid rotation plus that of the straining of the material fibers.

Equation (3.65) is referred to as the additive decomposition of the displacement gradient $\nabla \mathbf{u}$ into its symmetric part – the infinitesimal strain tensor – and into its skew part $\mathbf{W}$ – the infinitesimal rotation tensor.

Example 3.5

Consider the cylinder shown in Figure 3.32. Suppose that the transverse sections rotate without deformation in the plane yz around the cylinder's axis by an angle $\theta(x)$ with the constant rate of rotation $\frac{d\theta}{dx} = \alpha$. Calculate, assuming infinitesimal displacement conditions:

(i) The displacement field.
(ii) The strain tensor within the cylinder.

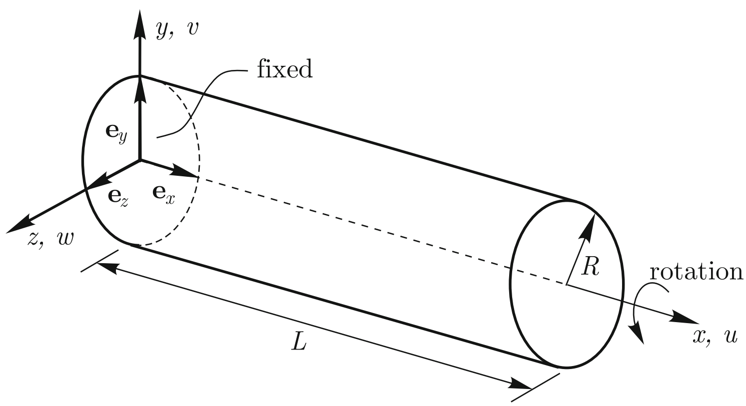

Fig. 3.32. Cylinder under study

Solution

(i) We obtain by integration

$$\theta(x) = \alpha x + C.$$

Since the rotation at $x = 0$ is prevented

$$\theta(0) = 0 \;\Rightarrow\; C = 0$$

and therefore the rotation of a generic section is given by

$$\theta(x) = \alpha x.$$

Since the section rotations are infinitesimal, we can use directly the results derived in Example 3.4. Therefore, considering $x_1 \equiv y$, $x_2 \equiv z$, $x_3 \equiv x$, we obtain

$$\begin{aligned} u &= 0 \\ v &= -\theta z = -\alpha x z \\ w &= \theta y = \alpha x y. \end{aligned}$$

(ii) The strain components are

$$\begin{aligned} \varepsilon_{xx} &= \frac{\partial u}{\partial x} = 0, \quad \varepsilon_{yy} = \frac{\partial v}{\partial y} = 0, \quad \varepsilon_{zz} = \frac{\partial w}{\partial z} = 0 \\ \varepsilon_{xy} &= \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = -\frac{1}{2}\alpha z \\ \varepsilon_{xz} &= \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) = \frac{1}{2}\alpha y \\ \varepsilon_{yz} &= \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) = \frac{1}{2}(-\alpha x + \alpha x) = 0. \end{aligned}$$

Therefore

$$\mathbf{E} = \begin{bmatrix} 0 & -\frac{1}{2}\alpha z & \frac{1}{2}\alpha y \\ -\frac{1}{2}\alpha z & 0 & 0 \\ \frac{1}{2}\alpha y & 0 & 0 \end{bmatrix}$$

and the engineering shear strains are

$$\begin{aligned} \gamma_{xy} &= -\alpha z \\ \gamma_{xz} &= \alpha y \\ \gamma_{yz} &= 0. \end{aligned}$$

In Figure 3.33 a geometrical interpretation of γ_{xz} is given. Referring to Figure 3.33 we can calculate γ_{xz} for a point of coordinates x, $y = R$, $z = 0$ as the ratio

$$\gamma_{xz} = \frac{d\theta R}{dx} = \alpha R$$

which is in accordance with the derived expression.

□

3.3 Stresses

In Section 2.1.3 we introduced the concept of stress, see Figure 2.3. In this figure, a field of forces per unit area – the field of stresses – is acting on the internal surface of the part $\Delta^t V$.

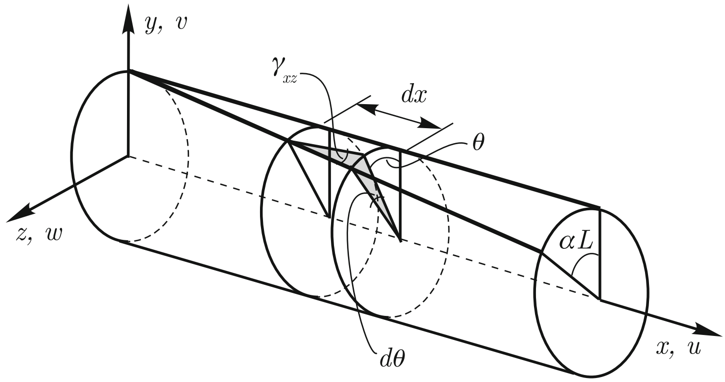

Fig. 3.33. Deformation of the cylinder

In Section 2.1.4 we arrived at the conditions for a solid, subjected to external forces, to be in static equilibrium. These conditions are given by ${}^t\mathbf{R} = \mathbf{0}$ and ${}^t\mathbf{M_O} = \mathbf{0}$ where ${}^t\mathbf{R}$ is the resultant of all externally applied forces and ${}^t\mathbf{M_O}$ is the moment of these forces about the system origin[8]. We also pointed out that the static equilibrium condition should hold for any part $\Delta^t V$ extracted from the body when, of course, we consider the stresses and the external actions on $\Delta^t V$.

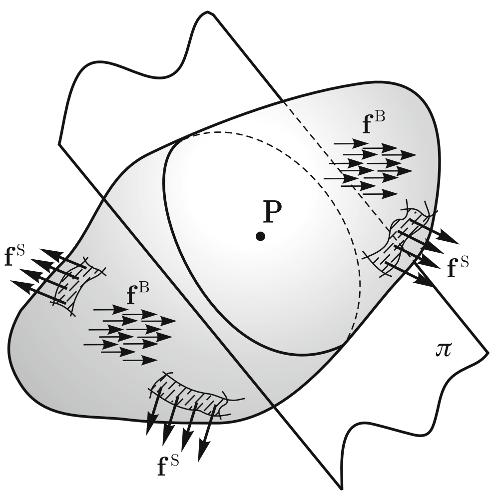

Fig. 3.34. Solid in equilibrium

[8] Actually the moment has to be zero about any point since ${}^t\mathbf{R} = \mathbf{0}$ as discussed in Chapter 2

The objective of this section is to examine in detail the concept of stress at a point, the stress field and the relations which link the stress field to the external field of forces acting on the solid.

We start in the next section by introducing in a classical manner the concept of stress.

3.3.1 Classical concept of stress

Let V_y in Figure 3.19 represent the deformed configuration of a solid which is in static equilibrium subjected to the field of body forces $\mathbf{f}^B$ and surface tractions $\mathbf{f}^S$. We suppose that $\mathbf{f}^S$ is defined over the entire external surface of the solid, therefore it includes possible reactions associated with motion restraints. Since we consider only this deformed configuration in the following discussion, we use simply V to denote V_y and this situation is summarized in Figure 3.34. Let $\mathbf{P}$ be a point in the interior of the solid and let π be a plane that passes through $\mathbf{P}$ and which sections the solid into two parts, ΔV_I and ΔV_{II}. These two parts are shown separately in Figure 3.35.

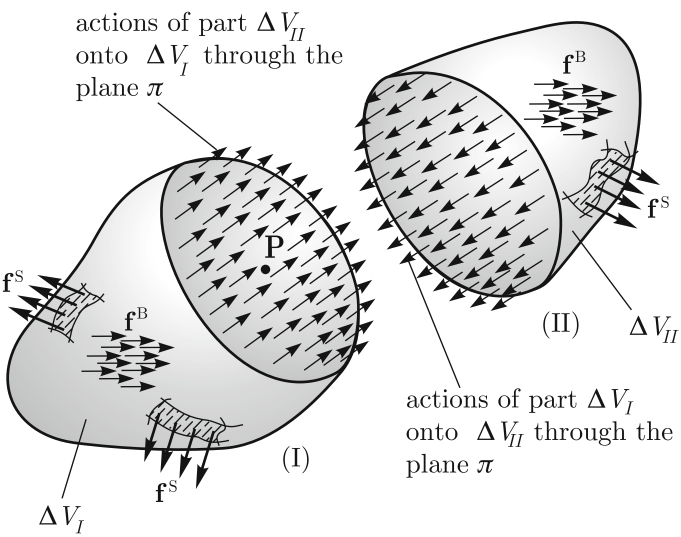

Fig. 3.35. Two parts of solid V sectioned by plane π (part rotated merely for better visualization)

The fields of internal forces that act on ΔV_I and ΔV_{II} on that plane are also shown. We note that the force actions of part ΔV_{II} onto part ΔV_I are such that ΔV_I considered alone is in static equilibrium, and similarly for part ΔV_{II}.

Let ΔS be a surface region around $\mathbf{P}$ as shown in Figure 3.36 and let $\Delta\mathbf{F}$ be the resultant of the internal forces acting on ΔS by the action of part

ΔV_{II} onto ΔV_I. We can define

$$\mathbf{t}_m = \frac{\Delta\mathbf{F}}{\Delta S}$$

which is an average force per unit of surface area. This vector quantity $\mathbf{t}_m$ is called the average stress at $\mathbf{P}$ considering the plane π and the area ΔS.

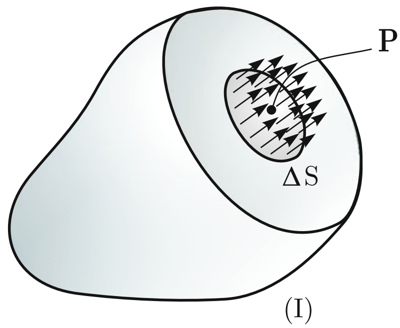

Fig. 3.36. Forces transmitted through ΔS

Since we want to obtain a local measure of the force transmitted per unit of area, we define the stress $\mathbf{t}$ at $\mathbf{P}$ on the plane π as

$$\mathbf{t} = \lim_{\Delta S \to 0} \frac{\Delta\mathbf{F}}{\Delta S} = \lim_{\Delta S \to 0} \mathbf{t}_m .$$

Since the plane π is well defined by its normal unit vector $\mathbf{n}$ we have

$$\mathbf{t} = \mathbf{t}\left(\mathbf{P}, \mathbf{n}\right). \tag{3.67}$$

Here $\mathbf{n}$ is taken to point outward from the face of the solid on which the stress is acting. Hence, the stress at the same point $\mathbf{P}$ and considering the same plane π but representing the action of part ΔV_I onto ΔV_{II} is given by

$$\mathbf{t} = \mathbf{t}\left(\mathbf{P}, -\mathbf{n}\right).$$

It is usual to decompose the stress into two vector components: $\mathbf{t}_n$ normal to the plane, *i.e.*, in the direction $\mathbf{n}$, and $\mathbf{t}_s$ tangential to the plane, *i.e.*, normal to $\mathbf{n}$, see Figure 3.37. Therefore

$$\begin{aligned} \mathbf{t} &= \mathbf{t}_n + \mathbf{t}_s \\ \mathbf{t}_n &= (\mathbf{t} \cdot \mathbf{n})\,\mathbf{n} = t_n \mathbf{n}, \qquad \mathbf{t}_s = \mathbf{t} - \mathbf{t}_n . \end{aligned}$$

Then $\mathbf{t}_n$ is called the normal stress and $\mathbf{t}_s$ the shear stress. This decomposition is not merely formal, since the physical effects associated with the action of these components are very different. Indeed, the normal stress when $t_n > 0$

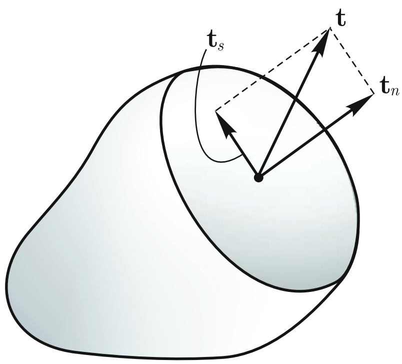

Fig. 3.37. Decomposition of the stress

(tensile stress) induces extension of the fiber along the normal direction, and when $t_n < 0$ (compressive stress) induces shortening of the same fiber. Also, as detailed later, the shear stress induces shear strain.

In the discussion above, we have considered the plane π as a generic plane to pass through $\mathbf{P}$. Therefore for each plane defined by a generic normal $\mathbf{n}$ there is a different value of $\mathbf{t}$ representing the action of part ΔV_{II} on ΔV_I (of course, as $\mathbf{n}$ changes ΔV_I and ΔV_{II} also change). This fact was acknowledged in equation (3.67) as we explicitly indicated the dependency on the plane through the normal $\mathbf{n}$. As there are infinitely many planes that pass through $\mathbf{P}$ (each time characterized by the corresponding $\mathbf{n}$), it is usual to refer to a "state of stress" at a point.

3.3.2 Characterization of the state of stress at a point

Our next objective is to better characterize the state of stress at a point.

First consequence of equilibrium

Let us first establish a relation between the stresses $\mathbf{t}(\mathbf{P},\mathbf{n})$ and $\mathbf{t}(\mathbf{P},-\mathbf{n})$, *i.e.*, they are defined for the same point and for the same plane but act on different parts of the sectioned solid, see Figure 3.34. Consider as a part ΔV of the solid of Figure 3.34, the parallelepiped shown in Figure 3.38. It has two faces parallel to the plane π which also sections the parallelepiped into two equal halves. This parallelepiped is also shown in Figure 3.39. Its thickness is $(\delta)^q$, $q \geq 2$, with δ infinitesimally small.

Equilibrium requires

$$\mathbf{t}(\mathbf{P},\mathbf{n})\,\delta^2 + \mathbf{t}(\mathbf{P},-\mathbf{n})\,\delta^2 + \mathbf{t}(\mathbf{P},\mathbf{n}_1)\,\delta^{q+1} + \mathbf{t}(\mathbf{P},-\mathbf{n}_1)\,\delta^{q+1} +$$

$$\mathbf{t}(\mathbf{P},\mathbf{n}_2)\,\delta^{q+1} + \mathbf{t}(\mathbf{P},-\mathbf{n}_2)\,\delta^{q+1} + \mathbf{f}^B(\mathbf{P})\,\delta^{q+2} = \mathbf{0}$$

Since δ is infinitesimal, the stress is constant over each face and therefore

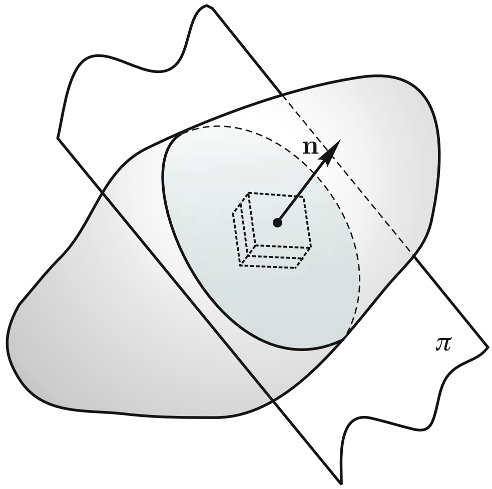

Fig. 3.38. Selection of a parallelepiped part around P

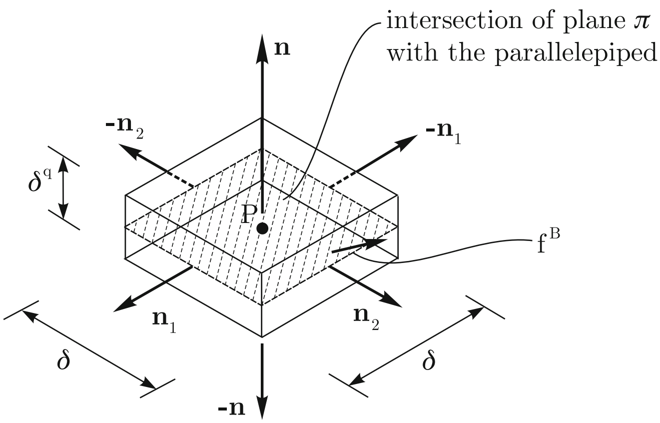

Fig. 3.39. Definition of the parallelepiped part, $\mathbf{f}^B$ is the body force per unit of volume

can be evaluated at **P**. Neglecting terms of higher-order in δ, we obtain

$$\mathbf{t}\left(\mathbf{P},\mathbf{n}\right) = -\mathbf{t}\left(\mathbf{P},-\mathbf{n}\right). \tag{3.68}$$

Of course, taking the height to be δ^q and the base to be δ was an appropriate choice and certainly allowed since every part of the solid should be in equilibrium.

The relation (3.68) simply states that the stress acting on a plane but on the two different parts of the sectioned solid is of same magnitude and direction but of opposite orientation.

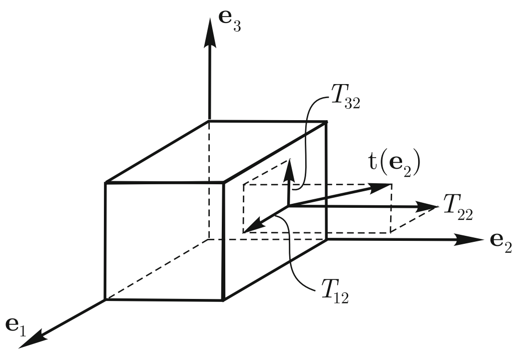

Fig. 3.40. Decomposition of the stress on the plane with unit normal $\mathbf{e}_2$

Stress components

Consider the situation in Figure 3.40 where we isolated a parallelepiped with infinitesimally small sides from the solid to examine the stress acting on a plane with normal $\mathbf{e}_2$. Let T_{22} be the component of $\mathbf{t}(\mathbf{e}_2)$ in the direction of $\mathbf{e}_2$ and T_{12} and T_{32} be the components in the directions $\mathbf{e}_1$ and $\mathbf{e}_3$ respectively, *i.e.*,

$$\mathbf{t}(\mathbf{e}_2) = T_{12}\mathbf{e}_1 + T_{22}\mathbf{e}_2 + T_{32}\mathbf{e}_3 = \sum_{i=1}^{3} T_{i2}\mathbf{e}_i. \tag{3.69}$$

where $T_{i2} = \mathbf{t}(\mathbf{e}_2) \cdot \mathbf{e}_i$.

In Figure 3.41 we still represent the same part but now we indicate explicitly the stresses and their components, defined as above, also for planes whose normals are $\mathbf{e}_1$ and $\mathbf{e}_3$. We can write

$$\mathbf{t}(\mathbf{e}_j) = \sum_{i=1}^{3} T_{ij}\mathbf{e}_i. \tag{3.70}$$

The stress acting at the same point as $\mathbf{t}(\mathbf{e}_2)$, but considering a plane defined by $-\mathbf{e}_2$, is also represented in Figure 3.41. This stress represents the action of part (I) onto part (II) whereas $\mathbf{t}(\mathbf{e}_2)$ represents the action of part (II) onto part (I) (note that part (II) was displaced vertically only to permit visualization). Using (3.68) and (3.69) we have that

$$\mathbf{t}(-\mathbf{e}_2) = -\mathbf{t}(\mathbf{e}_2) = -T_{12}\mathbf{e}_1 - T_{22}\mathbf{e}_2 - T_{32}\mathbf{e}_3$$

and the orientation indicated in Figure 3.41 are those corresponding to positive values of T_{i2}. Here T_{ii} represents the normal stress to the plane given by $\mathbf{e}_i$ and -$\mathbf{e}_i$ and a positive value indicates tension, T_{ij}, $i \neq j$, represents a shear stress component in the direction of $\mathbf{e}_i$ for the plane defined by the

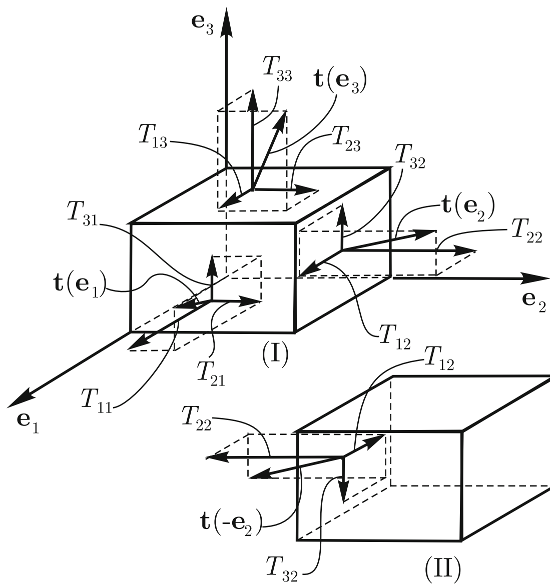

Fig. 3.41. Stress components in planes parallel to coordinate planes

normal $\mathbf{e}_j$ and a positive value indicates that the orientation is the same as $\mathbf{e}_i$. However, when the plane is defined by $-\mathbf{e}_j$, then T_{ij} with a positive value indicates that the orientation is opposite to that given by $\mathbf{e}_i$.

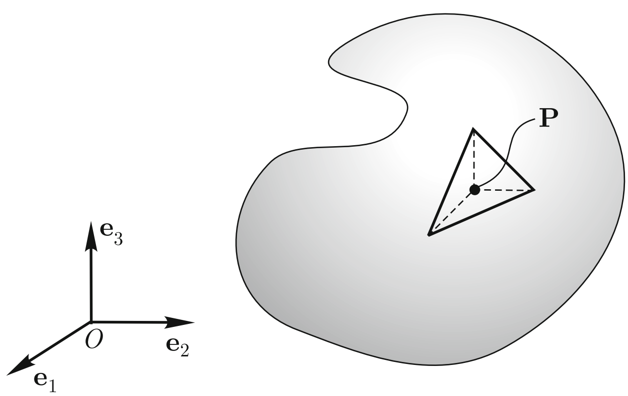

Fig. 3.42. Tetrahedral part to be isolated from the solid

Second consequence of equilibrium

We are now ready to demonstrate an important result which helps to further characterize the state of stress at a point. Consider a solid in static equilibrium and a part of the solid, the tetrahedron, shown in Figure 3.42. The tetrahedron has one vertex denoted by **P**, and as shown three edges are parallel to the coordinate axes. The edges are assumed to be of infinitesimal length. We can isolate the tetrahedron from the solid, and the action of the rest of the solid onto the tetrahedron is represented by the stresses acting on its faces.

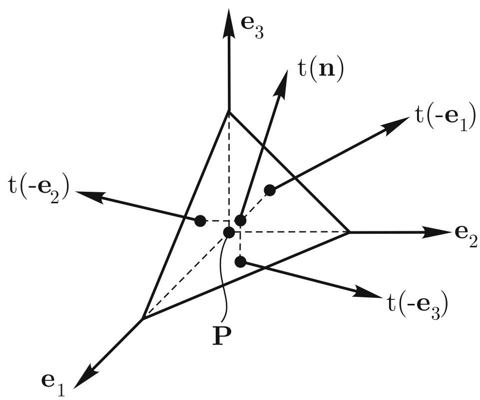

Fig. 3.43. Stress on tetrahedral faces, **n** is the unit outward normal vector on the face S

Therefore, in Figure 3.43, we are representing by $\mathbf{t}\,(-\mathbf{e}_i)$ the stress on the face with exterior normal $-\mathbf{e}_i$ and area $S_i\ (i = 1, 2, 3)$. The stress acting on the inclined face, which has normal $\mathbf{n}$ and area S, is given by $\mathbf{t}\,(\mathbf{n})$. The stresses are taken to be constant over each face since the edges are of infinitesimal lengths and are given by $\mathbf{t}\,(\mathbf{n})$ and $\mathbf{t}\,(-\mathbf{e}_i)$ $i = 1, 2, 3$ (representing $\mathbf{t}\,(\mathbf{P}, \mathbf{n})$, $\mathbf{t}\,(\mathbf{P}, -\mathbf{e}_i)$ respectively). Similarly, we do not include the spatial variation of the stress from **P** to the inclined face at the distance h from **P** since this variation is given by an infinitesimal of higher order and its contribution would drop out in the derivation not affecting the final result.

Some geometric properties of the tetrahedron are shown in Figure 3.44. Note that $n_i = \mathbf{n}\cdot\mathbf{e}_i = \|\mathbf{n}\|\,\|\mathbf{e}_i\|\cos\alpha_i = \cos\alpha_i$, where α_i is the angle between $\mathbf{n}$ and $\mathbf{e}_i$; $\cos\alpha_1$, $\cos\alpha_2$ and $\cos\alpha_3$ are usually called the direction cosines of $\mathbf{n}$. Therefore the volume of the tetrahedron can be evaluated either taking S as the base and h as the height or S_i as the base and $\frac{h}{n_i}$ as the height leading to

$$V = \frac{1}{3}Sh = \frac{1}{3}S_i\left(\frac{h}{n_i}\right) \tag{3.71}$$

from which

$$S_i = n_i S. \tag{3.72}$$

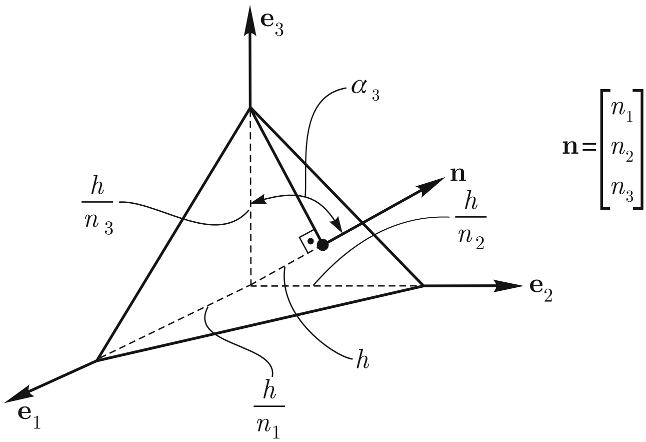

Fig. 3.44. Geometric properties of the tetrahedron of volume V

The tetrahedron must be in equilibrium and, therefore, the resultant of all forces acting on the tetrahedron should be zero, *i.e.*,

$$\mathbf{R} = \mathbf{t}(\mathbf{n})\,S + \mathbf{t}(-\mathbf{e}_1)\,S_1 + \mathbf{t}(-\mathbf{e}_2)\,S_2 + \mathbf{t}(-\mathbf{e}_3)\,S_3 + \mathbf{f}^B V = \mathbf{0} \tag{3.73}$$

where $\mathbf{f}^B$ is the externally applied body force for points inside the tetrahedron which can also be taken as constant, since the edges are of infinitesimal lengths. Dividing (3.73) by S and using (3.71) and (3.72) gives

$$\mathbf{t}(\mathbf{n}) + \mathbf{t}(-\mathbf{e}_1)\,n_1 + \mathbf{t}(-\mathbf{e}_2)\,n_2 + \mathbf{t}(-\mathbf{e}_3)\,n_3 + \frac{1}{3}\mathbf{f}^B h = \mathbf{0}. \tag{3.74}$$

Since h is also infinitesimal (the edges are infinitesimal) we can neglect the term $\frac{1}{3}\mathbf{f}^B h$ with respect to the others terms in (3.74). Using (3.68) we obtain

$$\mathbf{t}(\mathbf{n}) = \mathbf{t}(\mathbf{e}_1)\,n_1 + \mathbf{t}(\mathbf{e}_2)\,n_2 + \mathbf{t}(\mathbf{e}_3)\,n_3. \tag{3.75}$$

The fundamental result is that the stress on the inclined face, defined by the normal $\mathbf{n}$, can be obtained from the stresses on the three planes parallel to the coordinate planes.

Substituting (3.70) into (3.75) results into

$$\mathbf{t}(\mathbf{n}) = \left(\sum_{i=1}^{3} T_{i1}\mathbf{e}_i\right) n_1 + \left(\sum_{i=1}^{3} T_{i2}\mathbf{e}_i\right) n_2 + \left(\sum_{i=1}^{3} T_{i3}\mathbf{e}_i\right) n_3. \tag{3.76}$$

Since $\mathbf{t}(\mathbf{n}) = \sum_{i=1}^{3} t_i \mathbf{e}_i$, we obtain

$$\begin{aligned} t_1 &= T_{11}n_1 + T_{12}n_2 + T_{13}n_3 \\ t_2 &= T_{21}n_1 + T_{22}n_2 + T_{23}n_3 \\ t_3 &= T_{31}n_1 + T_{32}n_2 + T_{33}n_3 \end{aligned}$$

or in matrix form

$$\begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}. \tag{3.77}$$

This relation defines implicitly the *Cauchy stress tensor* $\mathbf{T}$ given by

$$\mathbf{T} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}.$$

We note that the Cauchy stress components listed in $\mathbf{T}$ correspond to the base vectors $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$. Equation (3.77) shows that for any point in the solid we can use $\mathbf{T}$ to calculate the stresses acting on any plane through the point, that is we have

$$\mathbf{t}(\mathbf{n}) = \mathbf{T}\mathbf{n}. \tag{3.78}$$

Third consequence of equilibrium

For the tetrahedron to be in equilibrium it is also necessary to have that the moment of the forces acting on it about any point is equal to zero.

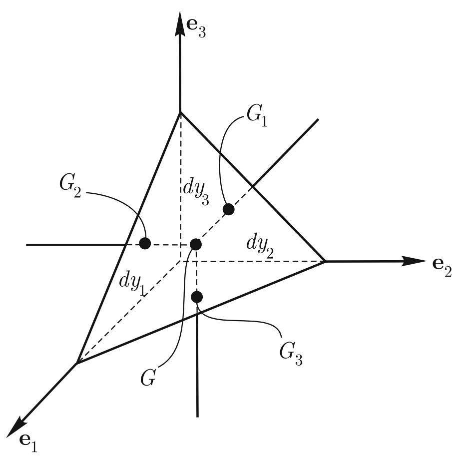

Fig. 3.45. Geometric property that holds for the centers of gravity of tetrahedron faces

We note that since the stresses can be taken as constant over each face of the tetrahedron, the resultant of the stresses on each face should be applied to the center of gravity of the face. Let G_i be the center of gravity of the face S_i and G the center of gravity of the inclined face. It is a geometric property that straight lines parallel to the coordinate axes through the points G_i meet at the center of gravity of the inclined face, as shown in Figure 3.45.

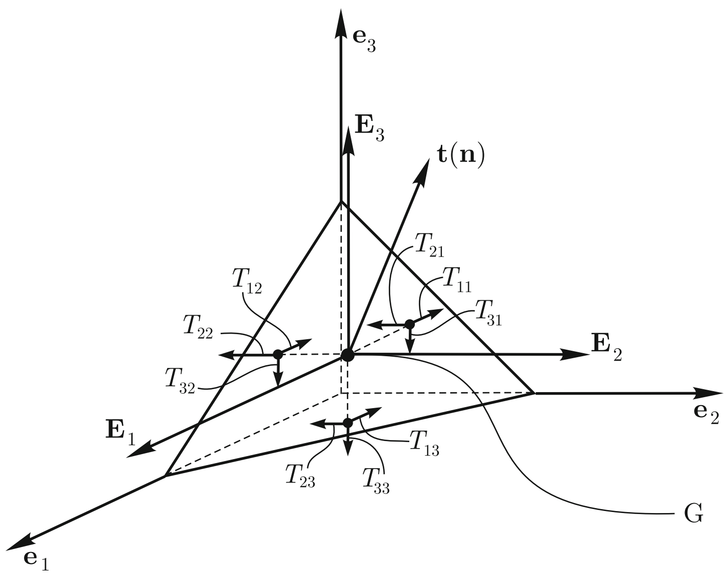

Fig. 3.46. Quantities used for moment balance

Referring to Figure 3.46 and imposing moment equilibrium we obtain

$$\sum M_{E_1} = 0 = T_{32} S_2 \frac{dy_2}{3} - T_{23} S_3 \frac{dy_3}{3}$$

therefore

$$T_{32} \left(\frac{dy_1 dy_3}{2} \right) \frac{dy_2}{3} = T_{23} \left(\frac{dy_1 dy_2}{2} \right) \frac{dy_3}{3}$$

which leads to

$$T_{23} = T_{32}.$$

Note that the moment produced by the body forces was not considered since it corresponds to an infinitesimal quantity of higher–order.

Also

$$\sum M_{E_2} = 0 = -T_{31} S_1 \frac{dy_1}{3} + T_{13} S_3 \frac{dy_3}{3}$$

$$T_{31}\left(\frac{dy_2dy_3}{2}\right)\frac{dy_1}{3}=T_{13}\left(\frac{dy_1dy_2}{2}\right)\frac{dy_3}{3}$$

leading to

$$T_{31}=T_{13}$$

and

$$\sum M_{E_3}=0=T_{21}S_1\frac{dy_1}{3}-T_{12}S_2\frac{dy_2}{3}$$

$$T_{21}\left(\frac{dy_2dy_3}{2}\right)\frac{dy_1}{3}=T_{12}\left(\frac{dy_1dy_3}{2}\right)\frac{dy_2}{3}$$

resulting into

$$T_{21}=T_{12}.$$

Hence, based on the moment equilibrium condition we showed that the Cauchy stress tensor $\mathbf{T}$ is symmetric.

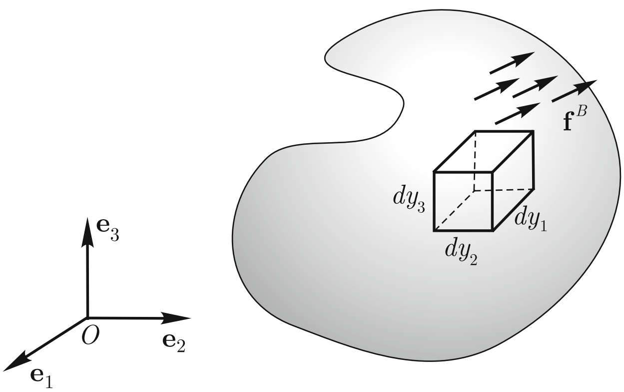

Fig. 3.47. Generic parallelepiped isolated from the solid

3.3.3 Differential equilibrium equations

So far we did not consider any spatial variations in the stresses – which however, of course, exist in almost all analyses. As we mentioned, if the spatial variation of the stresses were included in all above derivations, they would result into higher–order infinitesimal contributions and the final results would be the same as those given. However, we now want to study how every differential element of sides dy_1, dy_2, dy_3 (see Figure 3.47) is in equilibrium when body forces are present and then need to include the spatial variation of the stresses.

Let us isolate the parallelepiped in Figure 3.47 as a free body and consider the stresses on its faces and the body forces in its volume. We show the stresses in three separate figures, 3.48, 3.49 and 3.50 merely to facilitate the visualization; each figure shows the stresses on opposing parallel faces.

Imposing equilibrium in the $\mathbf{e}_1$ direction

$$\begin{aligned}
&-T_{11}dy_2dy_3+T_{11}dy_2dy_3+\frac{\partial T_{11}}{\partial y_1}dy_1dy_2dy_3\\
&-T_{12}dy_1dy_3+T_{12}dy_1dy_3+\frac{\partial T_{12}}{\partial y_2}dy_2dy_1dy_3\\
&-T_{13}dy_1dy_2+T_{13}dy_1dy_2+\frac{\partial T_{13}}{\partial y_3}dy_3dy_1dy_2\\
&+f_1^Bdy_1dy_2dy_3=0
\end{aligned} \tag{3.79}$$

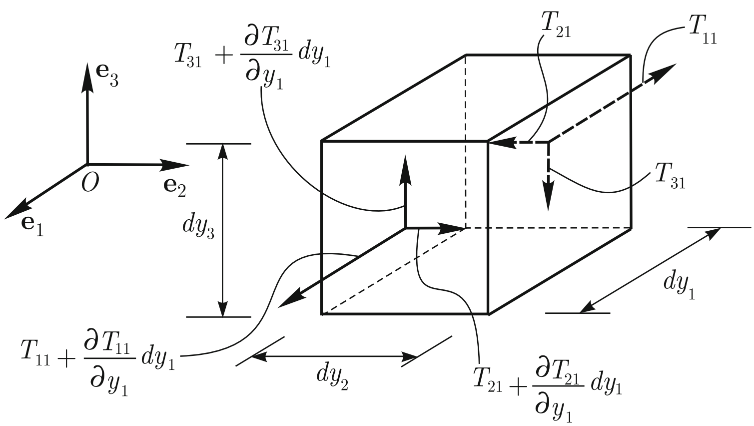

Fig. 3.48. Components of stress in planes dy_1 apart

Since the edges are infinitesimal we may assume the stresses to be constant on each face and the body force $\mathbf{f}_i^B = f_1^B\mathbf{e}_1 + f_2^B\mathbf{e}_2 + f_3^B\mathbf{e}_3$ to be constant inside the parallelepiped. If spatial variations for these quantities were included, they would result in infinitesimal contributions of higher-order and not affect the final equations[9]. Simplifying (3.79) and dividing it by the parallelepiped's volume $dy_1dy_2dy_3$ we obtain

[9] The general approach here, and in Section 3.3.2, is to include only those variations that lead to infinitesimal quantities of low-order and need to be included to extract the final result

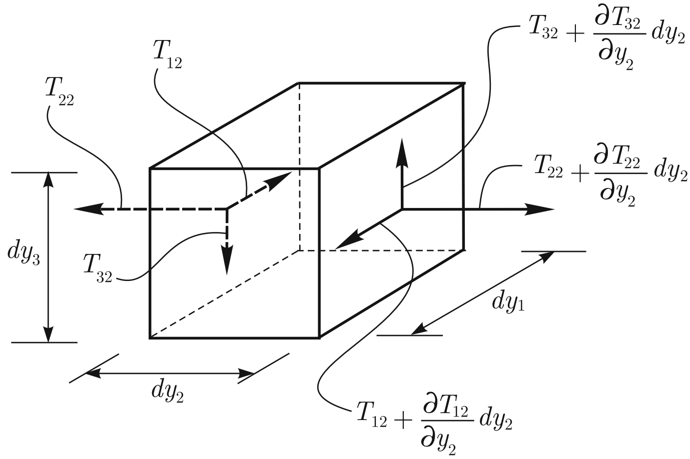

Fig. 3.49. Components of stress in planes dy_2 apart

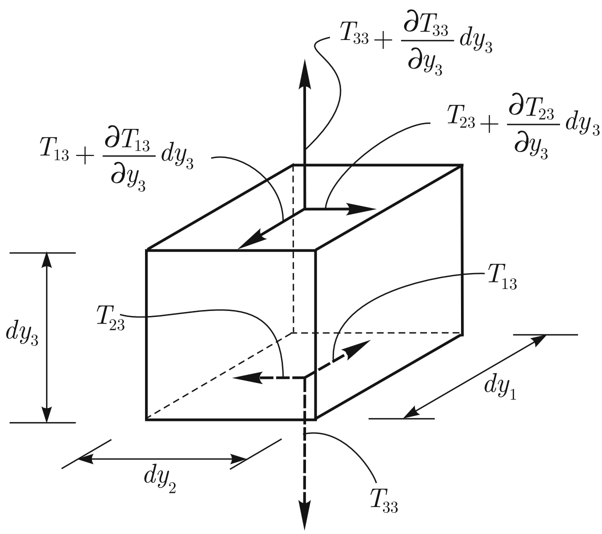

Fig. 3.50. Components of stress in planes dy_3 apart

$$\frac{\partial T_{11}}{\partial y_1} + \frac{\partial T_{12}}{\partial y_2} + \frac{\partial T_{13}}{\partial y_3} + f_1^B = 0 \tag{3.80}$$

Analogously, imposing equilibrium in the directions $\mathbf{e}_2$ and $\mathbf{e}_3$ gives

$$\frac{\partial T_{21}}{\partial y_1} + \frac{\partial T_{22}}{\partial y_2} + \frac{\partial T_{23}}{\partial y_3} + f_2^B = 0 \tag{3.81}$$

$$\frac{\partial T_{31}}{\partial y_1} + \frac{\partial T_{32}}{\partial y_2} + \frac{\partial T_{33}}{\partial y_3} + f_3^B = 0. \tag{3.82}$$

The equations (3.80) to (3.82) are referred to as the differential equilibrium equations, and these need to be satisfied throughout the body.

3.3.4 Principal stresses

Since the stress $\mathbf{t}$ at a point depends on the plane it is acting given by the unit normal vector $\mathbf{n}$, see Figure 3.43 and equation (3.78), natural questions are:

- On which plane do we have the maximum and minimum normal stresses?
- What values do the maximum and minimum normal stresses have?
- On which plane do we have the maximum and minimum shear stresses?
- What values do the maximum and minimum shear stresses have?
- Are there planes on which the shear stress is zero, and if yes, on which planes?

Mathematically, since the stress vector is given by equation (3.78), this last question is answered by solving for t_n and $\mathbf{n}$, the equation[10]

$$\mathbf{Tn} = t_n \mathbf{n}. \tag{3.83}$$

Namely, only when this equation holds, are the shear stresses zero. Equation (3.83) can be re-written as

$$(\mathbf{T} - t_n \mathbf{I})\,\mathbf{n} = \mathbf{0} \tag{3.84}$$

or as

$$\begin{bmatrix} T_{11}-\lambda & T_{12} & T_{13} \\ T_{21} & T_{22}-\lambda & T_{23} \\ T_{31} & T_{32} & T_{33}-\lambda \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{3.85}$$

In equation (3.85) we have replaced t_n by λ.

A trivial solution of (3.85) is $n_1 = n_2 = n_3 = 0$, but since $\mathbf{n}$ should be a unit vector, *i.e.*, $\|\mathbf{n}\| = 1$, the trivial solution is of no value. The non-trivial solutions of (3.85) are obtained by enforcing that the coefficient matrix be singular, that is, we enforce

$$\det \begin{bmatrix} T_{11}-\lambda & T_{12} & T_{13} \\ T_{21} & T_{22}-\lambda & T_{23} \\ T_{31} & T_{32} & T_{33}-\lambda \end{bmatrix} = 0 \tag{3.86}$$

which leads to an equation for λ

---

[10] The relation (3.83) is an eigenvalue problem (see *e.g.* Bathe, 1996)

$$\lambda^3 - I_1\lambda^2 + I_2\lambda - I_3 = 0 \tag{3.87}$$

where

$$\begin{aligned}
I_1 &= T_{11} + T_{22} + T_{33} \\
I_2 &= \begin{vmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{vmatrix} + \begin{vmatrix} T_{22} & T_{23} \\ T_{32} & T_{33} \end{vmatrix} + \begin{vmatrix} T_{11} & T_{13} \\ T_{31} & T_{33} \end{vmatrix} \\
I_3 &= \begin{vmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{vmatrix}.
\end{aligned}$$

The scalar quantities I_1, I_2 and I_3 are called the stress invariants since they are independent of the reference system used.

The roots of equation (3.87) are the eigenvalues and are denoted as λ_i, $i = 1, 2$ and 3. For each λ_i we can find an eigenvector $\mathbf{n}_i$ by solving[11] (3.85) with $\lambda = \lambda_i$. It can be proven that, since $\mathbf{T}$ is symmetric, the solutions of (3.87) are always real values and three situations can arise (see Bathe, 1996):

(i) The three values of λ which satisfy (3.87) are distinct, *i.e.*, $\lambda_1 \neq \lambda_2 \neq \lambda_3$. Then the vectors $\mathbf{n}_1$, $\mathbf{n}_2$ and $\mathbf{n}_3$ are automatically orthogonal. In fact, considering $\mathbf{n}_1$ and $\mathbf{n}_2$, for example, we have

$$\begin{aligned}
\mathbf{T}\mathbf{n}_1 &= \lambda_1\mathbf{n}_1 \\
\mathbf{T}\mathbf{n}_2 &= \lambda_2\mathbf{n}_2
\end{aligned}$$

and pre multiplying by $\mathbf{n}_2^T$ and $\mathbf{n}_1^T$, respectively, we obtain

$$\mathbf{n}_2^T\mathbf{T}\mathbf{n}_1 = \lambda_1\mathbf{n}_2^T\mathbf{n}_1 \tag{3.88}$$

$$\mathbf{n}_1^T\mathbf{T}\mathbf{n}_2 = \lambda_2\mathbf{n}_1^T\mathbf{n}_2. \tag{3.89}$$

Since $\mathbf{T}$ is symmetric, we have

$$\mathbf{n}_1^T\mathbf{T}\mathbf{n}_2 = \mathbf{n}_2^T\mathbf{T}\mathbf{n}_1 \tag{3.90}$$

and obtain

$$(\lambda_1 - \lambda_2)\,\mathbf{n}_1^T\mathbf{n}_2 = 0. \tag{3.91}$$

Since $\lambda_1 \neq \lambda_2$, $\mathbf{n}_1^T\mathbf{n}_2 = 0$ and hence $\mathbf{n}_1$ and $\mathbf{n}_2$ are orthogonal vectors.

(ii) Assume $\lambda_1 = \lambda_2$, $\lambda_3 \neq \lambda_1$. Then there is a unique $\mathbf{n}_3$ corresponding to λ_3. Also, it can be proven that any unit vector in a plane orthogonal to $\mathbf{n}_3$ is a vector that satisfies (3.85) for $\lambda = \lambda_1 = \lambda_2$. Hence we can choose two such vectors, orthogonal to each other, and assign them to correspond to λ_1 and λ_2.

[11] In this solution we enforce $\|\mathbf{n}_i\|$ to be equal to 1

(iii) Assume $\lambda_1 = \lambda_2 = \lambda_3$. In this case any unit vector $\mathbf{n}$ satisfies (3.85) for $\lambda = \lambda_1 = \lambda_2 = \lambda_3$ and we can choose three orthogonal vectors and assign them to correspond to λ_1, λ_2 and λ_3.

From these properties, it is easy to see that we can always find three orthogonal unit vectors, $\mathbf{n}_1$, $\mathbf{n}_2$ and $\mathbf{n}_3$ which satisfy (3.85) and which are associated with the solutions of (3.87), *i.e.*, λ_1, λ_2 and λ_3. We order λ_1, λ_2 and λ_3 according to their algebraic values

$$\lambda_1 \geq \lambda_2 \geq \lambda_3.$$

If instead of using the orthonormal unit vectors $\mathbf{e}_i$, we now use the orthonormal vectors $\mathbf{n}_i$ to define our reference system[12], then the stress tensor in such a system reads

$$\mathbf{T} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}. \tag{3.92}$$

Of course, the stress tensor given in (3.92), tells that there are no shear stresses on the planes with normals $\mathbf{n}_1$, $\mathbf{n}_2$ and $\mathbf{n}_3$.

Let us now address the question of determining the maximum and minimum values of the normal stresses. Recall that the normal stress acting on a plane with normal $\mathbf{n}$ is given by

$$t_n = \mathbf{n} \cdot \mathbf{t} = \mathbf{n} \cdot \mathbf{Tn} = \mathbf{n}^T\mathbf{Tn}. \tag{3.93}$$

To find the maximum and minimum values of t_n that can be reached by varying $\mathbf{n}$, it is effective to evaluate t_n as a function of the normal $\mathbf{n}$ using the reference system defined by $\mathbf{n}_1$, $\mathbf{n}_2$ and $\mathbf{n}_3$, *i.e.*,

$$t_n(\mathbf{n}) = \mathbf{n}^T\mathbf{Tn} = \begin{bmatrix} n_1 & n_2 & n_3 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}.$$

Hence

$$t_n = \lambda_1 (n_1)^2 + \lambda_2 (n_2)^2 + \lambda_3 (n_3)^2 \tag{3.94}$$

Since $\mathbf{n}$ is a unit vector

$$(n_1)^2 + (n_2)^2 + (n_3)^2 = 1$$

[12] Orthogonal vectors $\mathbf{n}_i$ and $\mathbf{n}_j$ are also orthonormal if $\mathbf{n}_i^T\mathbf{n}_j = \delta_{ij}$ (δ_{ij}, the Kronecker delta, $= 1$ for $i = j$ and $= 0$ for $i \neq j$)

and substituting in (3.94)

$$(n_1)^2 = 1 - (n_2)^2 - (n_3)^2$$

gives

$$t_n = \lambda_1 \left(1 - (n_2)^2 - (n_3)^2\right) + \lambda_2 (n_2)^2 + \lambda_3 (n_3)^2$$

or

$$t_n = \lambda_1 + (\lambda_2 - \lambda_1)(n_2)^2 + (\lambda_3 - \lambda_1)(n_3)^2 . \tag{3.95}$$

Since $\lambda_1 \geq \lambda_2 \geq \lambda_3$ we can conclude that

$$t_n \leq \lambda_1$$

and hence the maximum value of t_n that can be reached by varying the plane at the point (varying $\mathbf{n}$) is λ_1 and, of course, is reached for the plane with normal $\mathbf{n}_1$. Analogously, substituting $(n_3)^2 = 1 - (n_1)^2 - (n_2)^2$ into (3.94) leads to

$$t_n = \lambda_3 + (\lambda_1 - \lambda_3)(n_1)^2 + (\lambda_1 - \lambda_2)(n_2)^2$$

which implies that

$$t_n \geq \lambda_3$$

and hence λ_3 is the minimum value[13] of t_n and, of course, is reached for the plane with normal $\mathbf{n}_3$.

Since λ_1 and λ_3 are the maximum and minimum stress values that can be reached, we call λ_1, λ_2 and λ_3 principal values and denote these principal stresses as τ_1, τ_2 and τ_3 respectively, and the associated normals $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ the principal stress directions.

Example 3.6

Consider the cylinder as shown in Figure 3.32. The solid cylinder is subjected to self-equilibrating torsional moments $\mathbf{M}_t$ at the end sections, *i.e.*, $\mathbf{M}_t = M_t \mathbf{e}_x$ at $x = L$ and $\mathbf{M}_t = -M_t \mathbf{e}_x$ at $x = 0$. The stresses at a generic cross-section are given in Figure 3.51 (see Section 3.6). Therefore, at a generic cross-section there is only a shear stress distribution and the maximum value τ is related to the twisting moment by $\tau = \frac{M_t}{I_t} R$ where $I_t = \frac{\pi R^4}{2}$. Let us consider the point $\mathbf{P}$ with coordinates $y = R$, $z = 0$, $0 \leq x \leq L$. Evaluate the principal stresses and the principal stress directions.

[13] The expression $t_n(\mathbf{n}) = \mathbf{n}^T \mathbf{T} \mathbf{n}$ is actually a Rayleigh quotient, and we have in general for the Rayleigh quotient $\rho(\mathbf{v}) = \mathbf{v}^T \mathbf{A} \mathbf{v}$, with $\mathbf{v}^T \mathbf{v} = 1$ and $\mathbf{A}$ a symmetric matrix of order n, that $\lambda_n \leq \rho(\mathbf{v}) \leq \lambda_1$, where λ_n and λ_1 are the smallest and largest eigenvalues of $\mathbf{A}$, see Bathe, 1996 for a proof

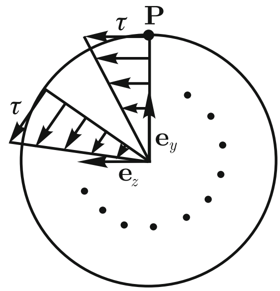

Fig. 3.51. Shear stress for a cross-section of a cylinder in torsion

Solution
The stress tensor at point **P** is given by

$$\mathbf{T} = \begin{bmatrix} 0 & 0 & \tau \\ 0 & 0 & 0 \\ \tau & 0 & 0 \end{bmatrix}$$

and the principal stresses can be obtained solving

$$\det(\mathbf{T}-\lambda\mathbf{I}) = \det\begin{bmatrix} -\lambda & 0 & \tau \\ 0 & -\lambda & 0 \\ \tau & 0 & -\lambda \end{bmatrix} = 0$$

which leads to

$$-\lambda\left(\lambda^2 - \tau^2\right) = 0.$$

The roots of this equation are $\lambda_1 = \tau$, $\lambda_2 = 0$ and $\lambda_3 = -\tau$. Hence, the principal stresses are

$$\tau_1 = \tau,\ \tau_2 = 0,\ \tau_3 = -\tau.$$

To obtain the principal stress directions, we need to solve

$$(\mathbf{T}-\tau_i\mathbf{I})\mathbf{n}_i = \mathbf{0} \qquad \text{with} \quad \|\mathbf{n}_i\| = 1$$

for $i = 1$ to 3. For the first principal stress, $\tau_1 = \tau$ we have

$$\begin{bmatrix} -\tau & 0 & \tau \\ 0 & -\tau & 0 \\ \tau & 0 & -\tau \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{3.96}$$

or

$$\begin{cases} -\tau n_1 + \tau n_3 = 0 \\ -\tau n_2 = 0 \\ \tau n_1 - \tau n_3 = 0 \end{cases} .$$

Of course, the first and third equations in the above system are the same. Considering the first and the second equations, we obtain

$$n_1 = n_3, \qquad n_2 = 0$$

and imposing that

$$n_1^2 + n_2^2 + n_3^2 = 1 \tag{3.97}$$

we obtain

$$2n_1^2 = 1 \quad \Rightarrow \quad n_1 = \pm\frac{\sqrt{2}}{2}.$$

Choosing $n_1 = \frac{\sqrt{2}}{2}$ we have

$$\mathbf{n}_1 = \frac{\sqrt{2}}{2}\mathbf{e}_x + \frac{\sqrt{2}}{2}\mathbf{e}_z.$$

Note that if we had used $n_1 = -\frac{\sqrt{2}}{2}$, then we would have simply obtained $-\mathbf{n}_1$ as the solution. Considering τ_2 we have

$$\begin{bmatrix} 0 & 0 & \tau \\ 0 & 0 & 0 \\ \tau & 0 & 0 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or

$$\begin{cases} \tau n_3 = 0 \\ 0 = 0 \\ \tau n_1 = 0 \end{cases}$$

which yields

$$n_1 = 0, \qquad n_3 = 0$$

and equation (3.97) gives

$$n_2 = \pm 1.$$

Therefore we can select

$$\mathbf{n}_2 = \mathbf{e}_y.$$

Finally, for τ_3 we have

$$\begin{bmatrix} \tau & 0 & \tau \\ 0 & \tau & 0 \\ \tau & 0 & \tau \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or

$$\begin{cases} \tau n_1 + \tau n_3 = 0 \\ \tau n_2 = 0 \\ \tau n_1 + \tau n_3 = 0 \end{cases}$$

which gives

$$n_1 = -n_3, \qquad n_2 = 0$$

and substituting in (3.97)

$$n_1 = \pm\frac{\sqrt{2}}{2}$$

Selecting $n_1 = -\frac{\sqrt{2}}{2}$ yields $n_3 = \frac{\sqrt{2}}{2}$ and we have in summary the following orthonormal vectors defining the principal directions, *i.e.*,

$$\begin{aligned} \mathbf{n}_1 &= \frac{\sqrt{2}}{2}\mathbf{e}_x + \frac{\sqrt{2}}{2}\mathbf{e}_z \\ \mathbf{n}_2 &= \mathbf{e}_y \\ \mathbf{n}_3 &= -\frac{\sqrt{2}}{2}\mathbf{e}_x + \frac{\sqrt{2}}{2}\mathbf{e}_z. \end{aligned}$$

These can be used as new base vectors of a new reference coordinate system, and in this system,

$$\mathbf{T} = \begin{bmatrix} \tau & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\tau \end{bmatrix}.$$

Of course, since τ_1 and τ_3 are the maximum and minimum normal stresses, we have

$$\begin{aligned} \max(t_n) &= \tau \\ \min(t_n) &= -\tau. \end{aligned}$$

□

Mohr's circles

In order to obtain further insight into the state of stress at a point, we study below the stresses for planes which contain one of the principal directions. For example, let us consider a generic plane π which contains $\mathbf{n}_3$. This situation is summarized in Figure 3.52.

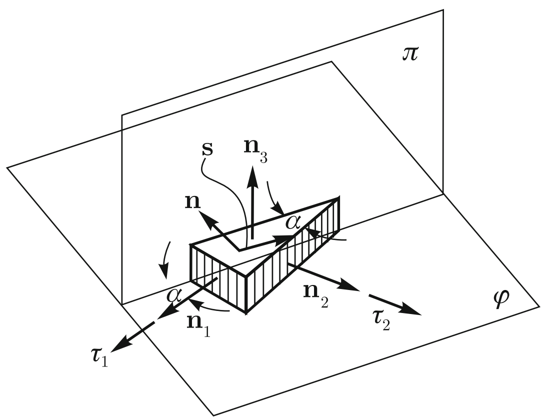

Fig. 3.52. Definition of a generic plane π which contains $\mathbf{n}_3$; the angle α is varying; the stress state in the plane π is analyzed

Since $\mathbf{n}_1$ and $\mathbf{n}_2$ are orthogonal to $\mathbf{n}_3$, they lie in a plane orthogonal to π which we call φ. Let α be the angle between π and $\mathbf{n}_1$ measured in the clockwise sense from $\mathbf{n}_1$, see Figure 3.52. The stress vector acting on π can be decomposed in the normal and tangential directions, *i.e.*,

$$\mathbf{t} = \mathbf{t}_n + \mathbf{t}_s = t_n \mathbf{n} + t_s \mathbf{s}$$

where $\mathbf{n}$ is the normal to π and we let $\mathbf{s}$ be a unit vector in the direction of $\mathbf{t}_s$. The sense of $\mathbf{s}$ is chosen such that when t_s is positive it tends to rotate the prism of triangular base shown in Figure 3.52 in the clockwise direction. The unit vectors $\mathbf{n}$ and $\mathbf{s}$ and hence $\mathbf{t}$ lie in plane φ and the stress on the plane π is

$$\mathbf{t} = \mathbf{T}\mathbf{n} = \begin{bmatrix} \tau_1 & 0 & 0 \\ 0 & \tau_2 & 0 \\ 0 & 0 & \tau_3 \end{bmatrix} \begin{bmatrix} -\sin\alpha \\ -\cos\alpha \\ 0 \end{bmatrix} = \begin{bmatrix} -\tau_1 \sin\alpha \\ -\tau_2 \cos\alpha \\ 0 \end{bmatrix}$$

where we used the reference system defined by $\mathbf{n}_1$, $\mathbf{n}_2$ and $\mathbf{n}_3$. Here α is an angle that we consider to vary. We can evaluate t_n using

$$t_n = \mathbf{n}^T\mathbf{T}\mathbf{n} = \begin{bmatrix} -\sin\alpha & -\cos\alpha & 0 \end{bmatrix} \begin{bmatrix} \tau_1 & 0 & 0 \\ 0 & \tau_2 & 0 \\ 0 & 0 & \tau_3 \end{bmatrix} \begin{bmatrix} -\sin\alpha \\ -\cos\alpha \\ 0 \end{bmatrix}$$

which leads to

$$t_n = \tau_1 \sin^2\alpha + \tau_2 \cos^2\alpha$$

and using the identities $\sin^2\alpha + \cos^2\alpha = 1$ and $\cos^2\alpha = \frac{1+\cos 2\alpha}{2}$, we obtain

$$t_n = \frac{1}{2}(\tau_1 + \tau_2) - \frac{1}{2}(\tau_1 - \tau_2)\cos 2\alpha. \tag{3.98}$$

We can also evaluate t_s

$$t_s = = \mathbf{s}^T\mathbf{T}\mathbf{n} = \begin{bmatrix} -\cos\alpha & \sin\alpha & 0 \end{bmatrix} \begin{bmatrix} \tau_1 & 0 & 0 \\ 0 & \tau_2 & 0 \\ 0 & 0 & \tau_3 \end{bmatrix} \begin{bmatrix} -\sin\alpha \\ -\cos\alpha \\ 0 \end{bmatrix}$$

which yields

$$t_s = (\tau_1 - \tau_2)\cos\alpha \sin\alpha = \frac{1}{2}(\tau_1 - \tau_2)\sin 2\alpha. \tag{3.99}$$

Equations (3.98) and (3.99) allow a graphical representation of the pair (t_n, t_s) as shown in Figure 3.53.

This representation is referred to as Mohr's Circle and represents the variation of stress in a plane orthogonal to a principal stress direction (this principal stress might be zero, as in a plane stress analysis, see Section 4.1.2).

If we consider the variation of the stress in planes that contain the principal directions $\mathbf{n}_1$ and $\mathbf{n}_2$, we obtain the additional results summarized in Figure 3.54. It is possible to show that the pairs (t_n, t_s) for planes which do not contain any of the principal directions are in the dashed region of Figure 3.54. This result permits to determine the maximum absolute value of the shear stress, *i.e.*,

$$t_{s_{\max}} = \max(\|t_s\|) = \frac{(\tau_1 - \tau_3)}{2}.$$

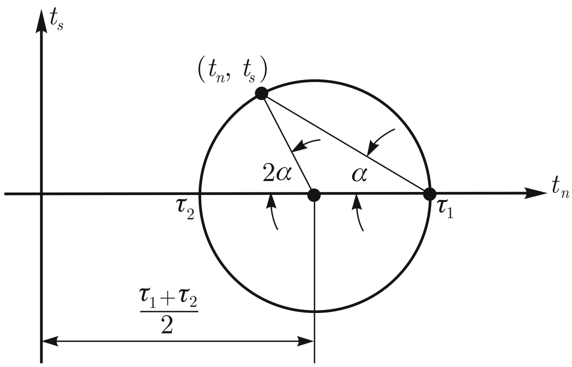

Fig. 3.53. Mohr's circle graphical representation for pairs (t_n, t_s)

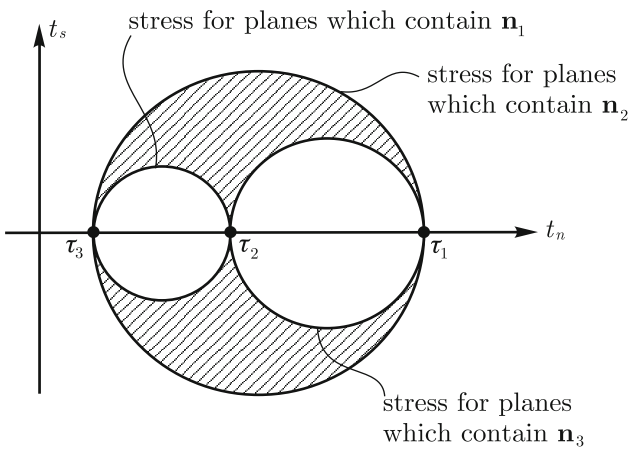

Fig. 3.54. Mohr's Circles for planes which contain one of the principal stress directions. Dashed region represents allowed pairs (t_n, t_s)

3.3.5 Principal strains

The results derived in the study of the stress at a point can directly be used to also study the strains at a point and evaluate the principal strain values.

As we mentioned, an eigenvalue problem was considered, see (3.83) with $\mathbf{T}$ a symmetric tensor. The infinitesimal strain tensor $\mathbf{E}$ is also symmetric and hence we can consider the problem of finding the eigenvalues and eigenvectors of

$$\mathbf{Ex} = \lambda \mathbf{x}.$$

The eigenvalues are represented by

$$\varepsilon_1 \geq \varepsilon_2 \geq \varepsilon_3$$

and are called the principal strains. The eigenvectors which can be denoted also by $\mathbf{h}_1$, $\mathbf{h}_2$ and $\mathbf{h}_3$ define the principal strain directions. In the reference system defined by the principal strain directions, $\mathbf{E}$ is given by

$$\mathbf{E} = \begin{bmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{bmatrix}.$$

We note that the off–diagonal terms are zero hence there are no shear strains between fibers aligned with the principal strain directions. We also recall that the strain of a fiber whose direction is defined by the unit vector $\mathbf{m}$ is given by

$$\varepsilon(\mathbf{m}) = \mathbf{m}^T \mathbf{E} \mathbf{m}.$$

Therefore, referring to (3.93) we can conclude that ε_1 is the maximum normal strain and ε_3 is the minimum normal strain and these occur for fibers in the directions of $\mathbf{h}_1$ and $\mathbf{h}_3$ respectively.

3.3.6 Infinitesimally small displacements

The discussion presented so far in Section 3.3 is valid for arbitrarily large displacements. In fact, the concept of stress was introduced for a generic deformed configuration of the solid. However, as discussed in Section 2.1.5, when we consider infinitesimally small displacements, the principles of linear and angular momenta are written for the undeformed configuration. Then, the conditions of equilibrium apply for the undeformed configuration, that is, as if the solid had not displaced.

Hence, when we consider infinitesimally small displacements all the discussion presented in Section 3.3 is valid considering the undeformed configuration. In particular, the differential equilibrium equations now read

$$\begin{aligned} \frac{\partial T_{11}}{\partial x_1} + \frac{\partial T_{12}}{\partial x_2} + \frac{\partial T_{13}}{\partial x_3} + f_1^B &= 0 \\ \frac{\partial T_{21}}{\partial x_1} + \frac{\partial T_{22}}{\partial x_2} + \frac{\partial T_{23}}{\partial x_3} + f_2^B &= 0 \\ \frac{\partial T_{31}}{\partial x_1} + \frac{\partial T_{32}}{\partial x_2} + \frac{\partial T_{33}}{\partial x_3} + f_3^B &= 0. \end{aligned}$$

The formulation of the mathematical model for three-dimensional solids is greatly simplified for infinitesimally small displacement conditions. One of the reasons is that equilibrium is enforced in the undeformed configuration which is known.

3.3.7 Technical or engineering notation for the stresses

In the technical or engineering notation, which is commonly used in the engineering literature, the coordinate axes are denoted by x, y and z, *i.e.*, $x \equiv x_1$, $y \equiv x_2$ and $z \equiv x_3$ and the stress components are represented by τ_{xx}, τ_{yy}, τ_{zz}, $\tau_{xy} = \tau_{yx}$, $\tau_{xz} = \tau_{zx}$ and $\tau_{yz} = \tau_{zy}$.

3.4 Constitutive equations

In this section, we introduce the characterization of the material of deformable solids. Recall that, both, the study of the deformations in Section 3.2 and the study of the stresses in Section 3.3 were carried out without having specified a material behavior for the solid. But in the analysis of the truss structures (see Section 2.2.4) and the analysis of the steel sheet (Section 3.1) we used the fact, justified by experimental observations, that the stresses induce strains.

For the one-dimensional stress conditions in the bars, the truss structures could be completely analyzed using Hooke's law relating the one-dimensional normal stress to the one-dimensional normal strain.

For the analysis of the steel sheet, we introduced the Poisson's ratio effect and the problem could be solved only for very simple conditions. Actually, the observed interaction between the strains and the stresses prompted to undertake a 3-D study of the deformations and stresses.

When we consider a three-dimensional solid undergoing arbitrarily large displacements, the constitutive equation is defined as the relation that gives the Cauchy stress at a point in a given configuration as a function of the complete history of deformation, that is, from the initial configuration of the solid up to the current configuration (see Bathe, 1996 and Kojic and Bathe, 2005). Such a general constitutive equation can be used to describe not only usual structural engineering materials but also very unusual ones.

The concept explored for the 1-D problem, that for an elastic material the normal stress depends only on the current state of deformation given by the current normal strain, can be generalized to define a constitutive relation for elastic materials in 3-D conditions. Considering three-dimensional solids undergoing arbitrarily large displacements, we can define a constitutive relation for elastic behavior by

$$\mathbf{T} = \mathcal{F}(\mathbf{X}) \qquad \text{for every point of the solid}$$

that is, the Cauchy stress at a point in the current configuration depends on the deformation gradient $\mathbf{X}$ at that point through a function $\mathcal{F}$.

However, our present objective is to introduce the elastic behavior considering infinitesimally small displacements. This is achieved by relating, for a generic point in the solid, the state of stress in the undeformed configuration to the state of strain for infinitesimally small displacement conditions.

Our discussion is restricted to linear elastic materials, *i.e.*, to those materials for which the deformation is immediate (with loading) and reversible, and moreover the relation between the stresses and strains is linear. We also assume that the material is isotropic, *i.e.*, it behaves in the same manner in all directions, and homogenous, *i.e.*, its behavior does not change from point to point. Under these conditions the relation between the stresses and strains can be represented by the generalized Hooke's law which we detail below.

3.4.1 Hooke's law for three-dimensional isotropic material conditions

Consider a parallelepiped of differential volume $dx_1dx_2dx_3$ subjected to a uniform one-dimensional state of stress as shown in Figure 3.55a. Since this is a one-dimensional state of normal stress we can use Hooke's law

$$\varepsilon = \frac{t_n}{E}$$

where E is the Young modulus. In the two directions orthogonal to the di-

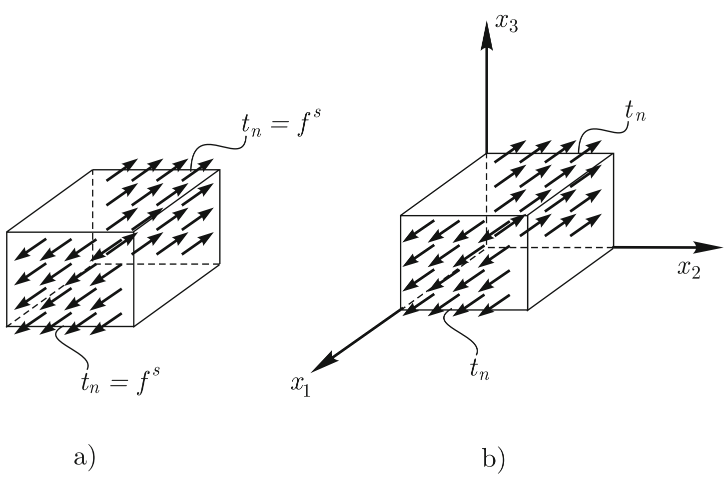

Fig. 3.55. a) Surface forces inducing an uniform state of stress; b) Placement of the parallelepiped with respect to the coordinate axes

rection of the applied stress a contraction takes place due to Poisson's effect given by $-\nu\frac{t_n}{E}$, where ν is the Poisson ratio. If we place the parallelepiped with respect to the coordinate axes shown in Figure 3.55b the Cauchy stress tensor would be

$$\mathbf{T} = \begin{bmatrix} t_n & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

i.e.,

$$T_{11} = t_n$$

and the infinitesimal strain tensor would be given by

$$\mathbf{E} = \begin{bmatrix} \frac{t_n}{E} & 0 & 0 \\ 0 & -\nu\frac{t_n}{E} & 0 \\ 0 & 0 & -\nu\frac{t_n}{E} \end{bmatrix}$$

or

$$E_{11} = \frac{t_n}{E} = \frac{T_{11}}{E}$$

and

$$E_{22} = E_{33} = -\nu\frac{t_n}{E} = -\nu\frac{T_{11}}{E} = -\nu E_{11}.$$

If we consider analogous situations for the directions $\mathbf{e}_2$ and $\mathbf{e}_3$, and use superposition, we can write

$$E_{11} = \frac{T_{11}}{E} - \frac{\nu}{E}\left(T_{22} + T_{33}\right) \tag{3.100}$$

$$E_{22} = \frac{T_{22}}{E} - \frac{\nu}{E}\left(T_{11} + T_{33}\right) \tag{3.101}$$

$$E_{33} = \frac{T_{33}}{E} - \frac{\nu}{E}\left(T_{11} + T_{22}\right) \tag{3.102}$$

where T_{22} and T_{33} are normal stresses for the planes with normals $\mathbf{e}_2$ and $\mathbf{e}_3$, respectively. We note that an isotropic behavior was implicitly assumed since the same behavior was assumed for the three directions (expressed by E and ν). For material with non-isotropic behavior, the Young's modulus and Poisson's ratios depend on the material direction considered, see Crandall, Dahl and Lardner, 1978 and Jones, 1975. We will focus in this book on isotropic material behaviors.

Now let us suppose that shearing tractions shown in Figure 3.56a are applied to the surface of the parallelepiped. The stress tensor is given by

$$\mathbf{T} = \begin{bmatrix} 0 & t_s & 0 \\ t_s & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

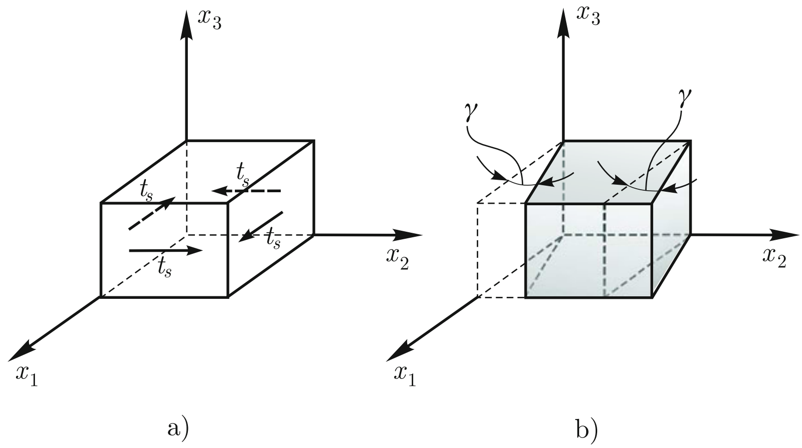

Fig. 3.56. a) Uniform shearing state of stress; b) Deformation associated with uniform shearing state of stress

It is an experimentally supported observation that the deformation induced by this state of stress is that given in Figure 3.56b[14], *i.e.*, a shear strain is induced and the strain tensor is given by

$$\mathbf{E} = \begin{bmatrix} 0 & \frac{\gamma}{2} & 0 \\ \frac{\gamma}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

as detailed in Example 3.2.

The relation between the shear stress and the shear strain is

$$\gamma = \frac{t_s}{G} \tag{3.103}$$

where G is a material constant called the shear modulus. Relation (3.103) expressed in components reads

$$2E_{12} = \frac{T_{12}}{G} \tag{3.104}$$

and considering analogous uniform shearing stress states corresponding to T_{13} and T_{23}, we have

$$2E_{13} = \frac{T_{13}}{G} \tag{3.105}$$

[14] Of course, an infinitesimal rigid rotation could be added to the deformation which would not affect the resulting $\mathbf{E}$ (see Section 3.2.5)

$$2E_{23} = \frac{T_{23}}{G}. \tag{3.106}$$

Collecting equations (3.100) to (3.102) and (3.104) to (3.106), we have the (so-called) *Generalized Hooke's law* which relates the stress components to the strain components

$$\begin{aligned}
E_{11} &= \frac{T_{11}}{E} - \frac{\nu}{E}\left(T_{22} + T_{33}\right)\\
E_{22} &= \frac{T_{22}}{E} - \frac{\nu}{E}\left(T_{11} + T_{33}\right)\\
E_{33} &= \frac{T_{33}}{E} - \frac{\nu}{E}\left(T_{11} + T_{22}\right)\\
E_{12} &= \frac{T_{12}}{2G} \quad \text{or} \quad \gamma_{12} = \frac{T_{12}}{G}\\
E_{13} &= \frac{T_{13}}{2G} \quad \text{or} \quad \gamma_{13} = \frac{T_{13}}{G}\\
E_{23} &= \frac{T_{23}}{2G} \quad \text{or} \quad \gamma_{23} = \frac{T_{23}}{G}.
\end{aligned}$$

When the solid is subjected to changes in its temperature field, thermal strains are induced. Let α_{th} be the thermal expansion coefficient and

$$\Delta\theta(\mathbf{x}) = \theta_f - \theta_i$$

be the field of temperature changes, where θ_i and θ_f are the initial and final temperature fields respectively. The induced thermal strain field is given by

$$\mathbf{E}^{th} = \begin{bmatrix} \alpha_{th}\Delta\theta & 0 & 0 \\ 0 & \alpha_{th}\Delta\theta & 0 \\ 0 & 0 & \alpha_{th}\Delta\theta \end{bmatrix}.$$

Then, the Generalized Hooke's law reads

$$\begin{aligned}
E_{11} &= \frac{T_{11}}{E} - \frac{\nu}{E}\left(T_{22} + T_{33}\right) + \alpha_{th}\Delta\theta\\
E_{22} &= \frac{T_{22}}{E} - \frac{\nu}{E}\left(T_{11} + T_{33}\right) + \alpha_{th}\Delta\theta\\
E_{33} &= \frac{T_{33}}{E} - \frac{\nu}{E}\left(T_{11} + T_{22}\right) + \alpha_{th}\Delta\theta\\
E_{12} &= \frac{T_{12}}{2G} \quad \text{or} \quad \gamma_{12} = \frac{T_{12}}{G}
\end{aligned}$$

$$E_{13} = \frac{T_{13}}{2G} \quad \text{or} \quad \gamma_{13} = \frac{T_{13}}{G}$$

$$E_{23} = \frac{T_{23}}{2G} \quad \text{or} \quad \gamma_{23} = \frac{T_{23}}{G}.$$

Of course, the strains above are the total strains, that is, the strains due to the stress field – the mechanical strains – plus those due to the temperature changes – the thermal strains.

3.4.2 Relation between G and E, ν

The following analysis shows that there is a relation between the shear modulus G, Young's modulus E and Poisson's ratio ν.

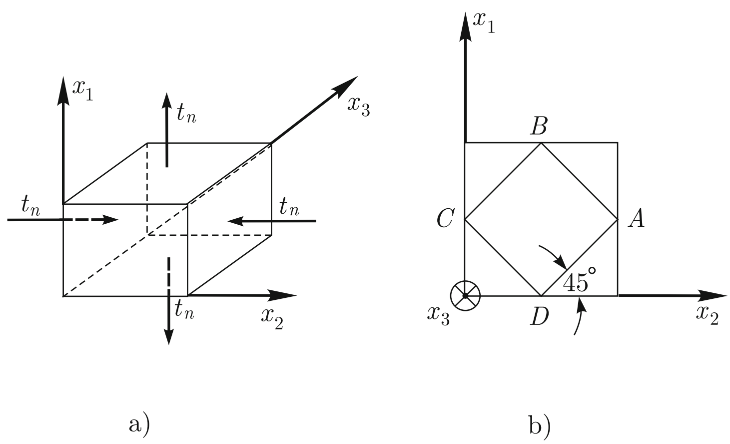

Fig. 3.57. a) Uniform state of stress in a cube; b) Section of the cube by a cutting plane parallel to x_1x_2. The square $ABCD$ represents the trace of the parallelepiped on the cutting plane

Consider the uniform state of stress in the cube shown in Figure 3.57a. The stress tensor is given by

$$\mathbf{T} = \begin{bmatrix} t_n & 0 & 0 \\ 0 & -t_n & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{3.107}$$

Let us select a parallelepiped in the interior of the cube of Figure 3.57a as shown in Figure 3.57b.

The stress acting on the face defined by AB can be evaluated by

$$\mathbf{t} = \mathbf{T}\mathbf{n} = \begin{bmatrix} t_n & 0 & 0 \\ 0 & -t_n & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix} = t_n \begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \\ 0 \end{bmatrix}.$$

This stress is tangential to the parallelepiped's face since

$$\mathbf{n}^T\mathbf{T}\mathbf{n} = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \end{bmatrix} t_n \begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \\ 0 \end{bmatrix} = 0$$

and its direction is given by the unit vector $\mathbf{s}$, $\mathbf{s}^T = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \end{bmatrix}$. It is therefore a shear stress given by

$$\mathbf{t}_s = t_s\mathbf{s}$$

with $t_s = t_n$.

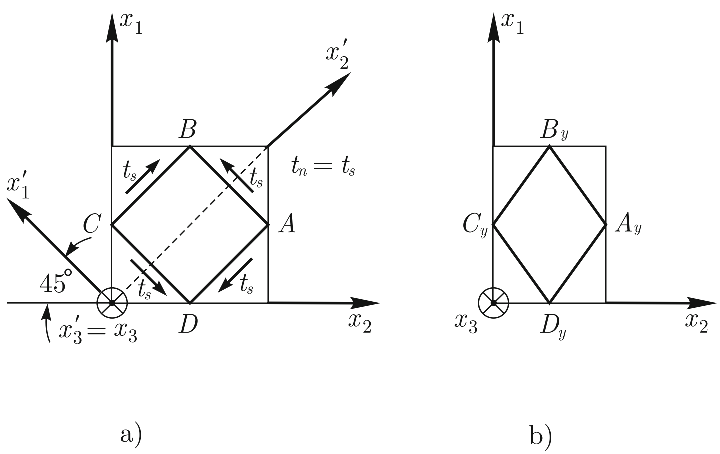

Fig. 3.58. a) Stress acting on the parallelepiped faces; b) Induced deformation

Analogous derivations for the other three faces of the parallelepiped lead to the situation shown in Figure 3.58a. If we choose a new set of axes to represent the stress tensor denoted by x_1', x_2' and x_3' with base vector ($\mathbf{e}_1'$, $\mathbf{e}_2'$, $\mathbf{e}_3'$) (see Figure 3.58a), we have

$$\mathbf{T}_{e'} = \begin{bmatrix} 0 & t_s & 0 \\ t_s & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and by the generalized Hooke's law, we obtain

$$\mathbf{E}_{e'} = \begin{bmatrix} 0 & \frac{t_s}{2G} & 0 \\ \frac{t_s}{2G} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The notation $\mathbf{T}_{e'}$ $\mathbf{E}_{e'}$ is being used in the above expressions to emphasize that we are referring the stresses/strains to the reference system defined by $\mathbf{e}'$, that is, $(\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3)$.

The strain tensor for the reference system $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ denoted by $\mathbf{e}$ can also be evaluated by Hooke's law. From (3.107)

$$\begin{aligned} E_{11} &= \frac{T_{11}}{E} - \nu (T_{22} + T_{33}) = \frac{t_n}{E} - \frac{\nu}{E}(-t_n) = \frac{t_n (1+\nu)}{E} \\ E_{22} &= \frac{T_{22}}{E} - \nu (T_{11} + T_{33}) = -\frac{t_n}{E} - \frac{\nu}{E} t_n = -\frac{t_n (1+\nu)}{E} \\ E_{33} &= \frac{T_{33}}{E} - \nu (T_{11} + T_{22}) = -\frac{\nu}{E}(t_n - t_n) = 0 \\ E_{12} &= E_{13} = E_{23} = 0 \end{aligned}$$

leading to

$$\mathbf{E}_e = \begin{bmatrix} \frac{t_n(1+\nu)}{E} & 0 & 0 \\ 0 & -\frac{t_n(1+\nu)}{E} & 0 \\ 0 & 0 & 0 \end{bmatrix}. \qquad (3.108)$$

We note that the normal strain of a fiber in the direction of $\mathbf{e}_1$ is $\varepsilon_l(\mathbf{e}_1) = \frac{t_n(1+\nu)}{E}$ (refer to (3.108)). This strain can also be calculated by using the strain tensor $\mathbf{E}_{e'}$. The unit vector $\mathbf{e}_1$ can be written as $\frac{\sqrt{2}}{2}\mathbf{e}'_1 + \frac{\sqrt{2}}{2}\mathbf{e}'_2$ and therefore the strain in this direction can also be evaluated by

$$\begin{aligned} \varepsilon_\ell\left(\frac{\sqrt{2}}{2}\mathbf{e}'_1 + \frac{\sqrt{2}}{2}\mathbf{e}'_2\right) &= \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \end{bmatrix} \mathbf{E}_{e'} \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \end{bmatrix} \begin{bmatrix} 0 & \frac{t_s}{2G} & 0 \\ \frac{t_s}{2G} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix} \\ &= \frac{t_s}{2G}. \end{aligned}$$

Equating the values, we obtain

$$\frac{t_s}{2G} = \frac{t_n}{E}(1+\nu)$$

since $t_s = t_n$, we arrive at

$$G = \frac{E}{2(1+\nu)}. \tag{3.109}$$

Hence, of the three elastic constants E, ν and G only two are independent.

In Figure 3.58b we show schematically the deformed configuration of the section. The fiber considered above is given by DB in the undeformed configuration and D_yB_y in the deformed configuration. We note that the only way this fiber displays the same deformation when interpreted as being in the cube subjected to normal stresses and in the parallelepiped subjected to pure shear is when relation (3.109) holds.

3.4.3 Generalized Hooke's law for an isotropic material in matrix notation

We may define column matrices to collect the independent components of the stress and strain tensors denoting them by τ and ε respectively

$$\begin{aligned} \tau^T &= \begin{bmatrix} \tau_{11} & \tau_{22} & \tau_{33} & \tau_{12} & \tau_{13} & \tau_{23} \end{bmatrix} \\ \varepsilon^T &= \begin{bmatrix} \varepsilon_{11} & \varepsilon_{22} & \varepsilon_{33} & \gamma_{12} & \gamma_{13} & \gamma_{23} \end{bmatrix} \end{aligned}$$

where $\tau_{ij} = T_{ij}$, $\varepsilon_{ii} = E_{ii}$ for $i,j = 1,2,3$ and $\gamma_{12} = 2E_{12}$, $\gamma_{13} = 2E_{13}$, $\gamma_{23} = 2E_{23}$. Considering the above definitions the generalized Hooke's law reads

$$\varepsilon = \mathbf{D}\tau \tag{3.110}$$

where

$$\mathbf{D} = \frac{1}{E}\begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1+\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1+\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+\nu) \end{bmatrix}. \tag{3.111}$$

We also have

$$\tau = \mathbf{D}^{-1}\varepsilon = \mathbf{C}\varepsilon \tag{3.112}$$

where

$$\mathbf{C} = \frac{E\,(1-\nu)}{(1+\nu)\,(1-2\nu)} \begin{bmatrix} 1 & \frac{\nu}{1-\nu} & \frac{\nu}{1-\nu} & 0 & 0 & 0 \\ \frac{\nu}{1-\nu} & 1 & \frac{\nu}{1-\nu} & 0 & 0 & 0 \\ \frac{\nu}{1-\nu} & \frac{\nu}{1-\nu} & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2(1-\nu)} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2(1-\nu)} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2(1-\nu)} \end{bmatrix}. \tag{3.113}$$

We note that the Hooke's law written in the form of (3.112) leads directly to the expected result that an infinitesimally small rigid motion produces no stresses, since this motion results in zero strains.

We can also conclude either from (3.110) or (3.112) that for an isotropic elastic material the principal directions of the stress and strain tensors coincide, since zero shear stresses lead to zero shear strains and vice-versa and, hence, the principal stresses and strains are also related by (3.110) and (3.112).

3.5 Formulation of the linear elasticity problem

In Chapter 2 we examined, in the context of truss structures, the *Three Fundamental Conditions* to be satisfied in the formulation of a structural mechanics problem:

- Equilibrium.
- Compatibility.
- Constitutive equations.

In Section 3.1, by means of a simple problem – the analysis of the steel sheet – we motivated the need for expressing these conditions for a three-dimensional deformable body.

Compatibility was addressed in Section 3.2 in which the deformations of a 3-D body were studied, strain measures were defined and the relations between the strains and displacements were derived.

In Section 3.3, the state of stress at a point was characterized by means of the Cauchy stress tensor and the equations of differential equilibrium were established.

In Section 3.4, the relations between the strains and stresses were detailed for an elastic, isotropic and homogeneous material. Now we are ready to formulate the elasticity problem.

Consider a deformable solid subjected to the action of a field of body forces $\mathbf{f}^B$ defined for every point $\mathbf{x}$ within the solid. On part of its surface, S_u, prescribed displacements $\hat{\mathbf{u}}$ are imposed and on the complementary part of the surface, S_f, with unit normal $\mathbf{n}$, prescribed surface tractions $\mathbf{f}^S$ are imposed[15]. We note that $S_u \cup S_f = S$ which represents the surface of the body and $S_u \cap S_f = \emptyset$. This situation is schematically represented in Figure 3.59 where the undeformed configuration V is shown. The linear elasticity problem can be posed as follows.

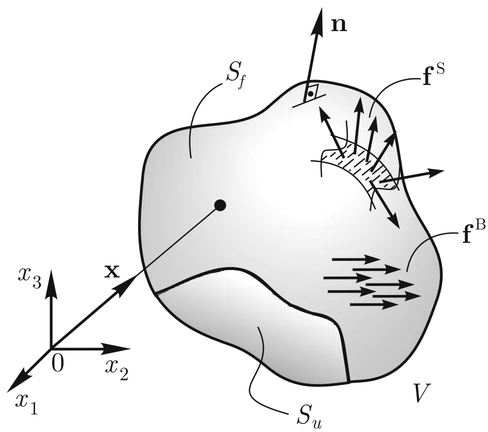

Fig. 3.59. Schematic representation of actions and boundary conditions for linear elasticity problem

Formulation of the 3-D elasticity problem

Find the displacement field $\mathbf{u}(\mathbf{x})$, the stress field $\tau(\mathbf{x})$ and the strain field $\varepsilon(\mathbf{x})$ such that for every point $\mathbf{x}$ within V

$$\sum_{j=1}^{3} \frac{\partial \tau_{ij}}{\partial x_j} + f_i^B = 0 \qquad i = 1, 2, 3 \tag{3.114}$$

[15] There are situations for which at a point on the surface the displacements are restricted in some directions and the surface tractions are prescribed in the remaining directions. Then S_u and S_f should be defined for each component of surface displacement and surface traction (see Section 2.1.1)

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \qquad i, j = 1, 2, 3 \tag{3.115}$$

$$\tau = \mathbf{C}\varepsilon \tag{3.116}$$

with the boundary conditions for points $\mathbf{x}$ on S_f

$$\sum_{j=1}^{3} \tau_{ij} n_j = f_i^S, \qquad i = 1, 2, 3, \tag{3.117}$$

where the n_i are the components of $\mathbf{n}$, and for the points $\mathbf{x}$ on S_u

$$\mathbf{u} = \hat{\mathbf{u}} \tag{3.118}$$

or

$$u_i = \hat{u}_i, \qquad i = 1, 2, 3.$$

The equations (3.114) to (3.118) are field equations in the volume V and on the surfaces S_f and S_u, as applicable. The *Equilibrium Condition* corresponds to satisfying the differential equilibrium in (3.114) and (3.117); the *Compatibility Condition* corresponds to finding continuous displacements $\mathbf{u}(\mathbf{x})$ that satisfy (3.118) and from which the strains are calculated as given in (3.115); and the *Constitutive Condition* is given in (3.116). Therefore all the requirements for the solution of the 3-D problem are contained in the above equations and the formulation is complete.

Note that when the solid is also subjected to known temperature changes, we should use in the above formulation the generalized Hooke's law which includes the thermal strains as given in Section 3.4.1.

The search for analytical solutions of the above problem when specialized to specific geometries, loading and boundary conditions has challenged mathematicians and engineers alike for a number of centuries. A great deal of solutions have been derived – albeit frequently based on simplifying assumptions – and they are available in the literature. These solutions represent a very important resource for understanding the behavior of elastic solids subjected to external actions.

In the past, prior to the availability of computer resources and numerical methods for the solution of (3.114) to (3.118), these analytical solutions to special problems, derived and idealized from the full 3-D problem, were the only solutions available for analysis. While, at present, full 3-D problems with arbitrary boundary conditions can be solved using the finite element method, the solution of the full 3-D problem is in many cases not an effective way to perform engineering analysis.

The 3-D elasticity model is of great importance, since it is our highest order hierarchical model assuming linear elastic behavior. Therefore, it provides a conceptual reference model from which other specific and frequently more effective models can be derived. The solutions of these models can then be compared with the solution of the full 3-D model giving good insight into the modeling of physical problems – all within the objective of using hierarchical mathematical modeling in engineering analysis.

Before we close this chapter, we present two illustrative cases of analytical solutions of the 3-D elasticity model which are important for structural analysis: the pure bending of a prismatic bar of rectangular cross-section (given in Example 3.7) and the uniform torsion of a prismatic bar (given in Section 3.6).

Example 3.7

Consider the undeformed configuration of a solid as shown in Figure 3.60.

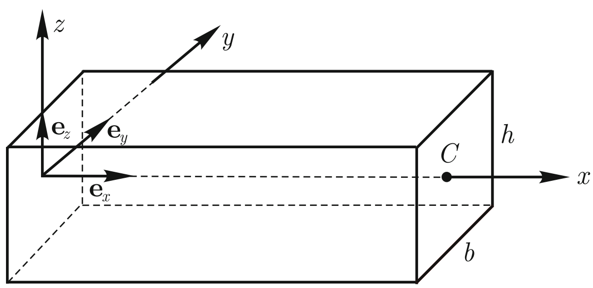

Fig. 3.60. Definition of the solid under study

Let the displacement field be given by

$$u = -\frac{M}{EI}xz \tag{3.119}$$

$$v = \nu\frac{M}{EI}yz \tag{3.120}$$

$$w = \frac{M}{2EI}\left[x^2 + \nu\left(z^2 - y^2\right)\right]. \tag{3.121}$$

where M is a positive constant, E and ν are Young's modulus and Poisson's ratio of the material and I is the moment of inertia about the y axis, $I = \frac{bh^3}{12}$ (see Section 4.2.2).

(i) Find the stress field associated with the given displacement field.

(ii) Identify the problem for which the given displacement field represents the exact solution of the 3-D elasticity model.

Solution

(i) The strains are given as

$$\begin{aligned}
\varepsilon_{xx} &= \frac{\partial u}{\partial x} = -\frac{M}{EI}z \\
\varepsilon_{yy} &= \frac{\partial v}{\partial y} = \nu\frac{M}{EI}z \\
\varepsilon_{zz} &= \frac{\partial w}{\partial z} = \nu\frac{M}{EI}z \\
\varepsilon_{xy} &= \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = 0 \\
\varepsilon_{xz} &= \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) = \frac{1}{2}\left(-\frac{M}{EI}x + \frac{M}{EI}x\right) = 0 \\
\varepsilon_{yz} &= \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) = \frac{1}{2}\left(\nu\frac{M}{EI}y - \nu\frac{M}{EI}y\right) = 0.
\end{aligned}$$

The stresses can be obtained using the generalized Hooke's law (see equation (3.112))

$$\tau_{xx} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\left[\varepsilon_{xx} + \frac{\nu}{1-\nu}(\varepsilon_{yy} + \varepsilon_{zz})\right].$$

Noting that for our problem

$$\varepsilon_{yy} = \varepsilon_{zz} = -\nu\varepsilon_{xx} \tag{3.122}$$

we obtain

$$\tau_{xx} = E\varepsilon_{xx} = -\frac{M}{I}z.$$

We also have

$$\begin{aligned}
\tau_{yy} &= \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\left[\varepsilon_{yy} + \frac{\nu}{1-\nu}(\varepsilon_{xx} + \varepsilon_{zz})\right] \\
\tau_{zz} &= \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\left[\varepsilon_{zz} + \frac{\nu}{1-\nu}(\varepsilon_{xx} + \varepsilon_{yy})\right]
\end{aligned}$$

and considering (3.122) we obtain

$$\tau_{yy} = \tau_{zz} = 0.$$

Of course, since $\gamma_{xy} = \gamma_{xz} = \gamma_{yz} = 0$, we have

$$\tau_{xy} = \tau_{xz} = \tau_{yz} = 0.$$

This completes part (i) of the problem.

(ii) To identify the 3-D problem actually solved, we need to determine which fields of body forces and surface tractions are in equilibrium with this stress field. Considering the differential equilibrium equations and introducing this stress field, we have

$$\begin{aligned}\frac{\partial \tau_{xx}}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}+\frac{\partial \tau_{xz}}{\partial z}+f_x^B &= 0 \Rightarrow f_x^B=0\\ \frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \tau_{yy}}{\partial y}+\frac{\partial \tau_{yz}}{\partial z}+f_y^B &= 0 \Rightarrow f_y^B=0\\ \frac{\partial \tau_{xz}}{\partial x}+\frac{\partial \tau_{yz}}{\partial y}+\frac{\partial \tau_{zz}}{\partial z}+f_z^B &= 0 \Rightarrow f_z^B=0.\end{aligned}$$

Therefore, there should be no body forces. Since the only nonzero stress component is τ_{xx}, the surface tractions should be zero on all four lateral surfaces. Further, for the surface defined by $x = L$

$$\mathbf{Tn}=\mathbf{f}^S \;\Rightarrow\; \tau_{xx}\mathbf{e}_x=\mathbf{f}^S \;\Rightarrow\; \mathbf{f}^S=-\frac{M}{I}z\mathbf{e}_x$$

and for $x = 0$

$$\mathbf{Tn}=\mathbf{f}^S \;\Rightarrow\; \text{-}\tau_{xx}\mathbf{e}_x=\mathbf{f}^S \;\Rightarrow\; \mathbf{f}^S=\frac{M}{I}z\mathbf{e}_x.$$

If we reduce the surface tractions at the section given by $x = L$ to its center of gravity we obtain for the force resultant

$$\mathbf{R}=\int_A \mathbf{f}^S\, dA=\int_A -\frac{M}{I}z\mathbf{e}_x\, dA=0$$

and for the moment resultant with respect to the center of the section, represented by $\mathbf{C}$

$$\begin{aligned}\mathbf{M_C} &= \int_A (y\mathbf{e}_y+z\mathbf{e}_z)\times\mathbf{f}^S\, dA\\ &= \int_A\left(\frac{M}{I}yz\mathbf{e}_z-\frac{M}{I}z^2\mathbf{e}_y\right)dA=-\frac{M}{I}\int_A z^2 dA\;\mathbf{e}_y=-M\;\mathbf{e}_y.\end{aligned}$$

Hence the applied surface tractions at $x = L$ correspond to a bending moment, see Figure 3.61. Analogously, for the end section defined by $x = 0$ we obtain $\mathbf{R} = \mathbf{0}$ and $M\mathbf{e}_y$ leading to a self-equilibrated force system.

Note also that the displacement field given in (3.119) – (3.121) satisfies $u = 0$ for $x = 0$, and $u = v = w = 0$ for $x = y = z = 0$.

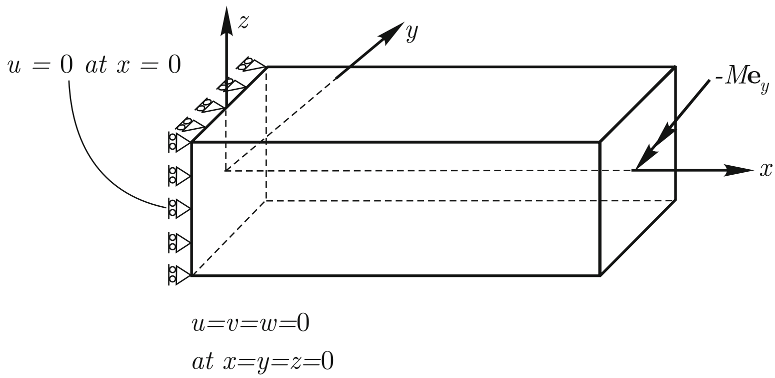

Fig. 3.61. Problem summary. Body forces are zero

We can conclude that the displacements given in (3.119) to (3.121), the derived strains and stresses correspond to the exact solution of the cantilever beam subjected to a pure bending moment at the tip, as summarized in Figure 3.61, as long as the bending moment at $x = L$ is introduced by the surface traction field $\mathbf{f}^S = -\frac{M}{I} z \mathbf{e}_x$ and the displacement restrictions at the "built-in section" $x = 0$ are as shown in Figure 3.61.

In Figure 3.62, we show the deformed and undeformed configuration of the solid. We see that a line parallel to the y axis in the undeformed configuration is deformed into an arc, whose curvature is opposite to the curvature of the deformed axis. This transverse curvature is known as *anticlastic curvature* and it is due to the Poisson effect.

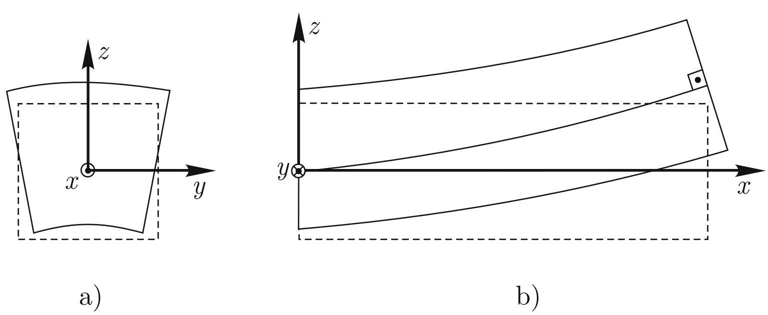

Fig. 3.62. Deformations for selected planes. a) Plane given by $x = 0$; b) Plane given by $y = 0$. The magnitude of the displacements is chosen for visualization purposes; the shown displacements are much larger than those for which the linear model would be adequate

□

3.6 Torsion of a prismatic bar

We present below the exact solution of a 3-D elasticity problem – the Saint Venant torsion solution for a prismatic bar – which is of considerable practical importance. It is also a classical example of the so-called semi-inverse method for deriving solutions. In this method, some assumptions on the functional form of the displacements are made, either motivated by experimental observations or by intuition. These displacement assumptions are the starting point towards obtaining a solution.

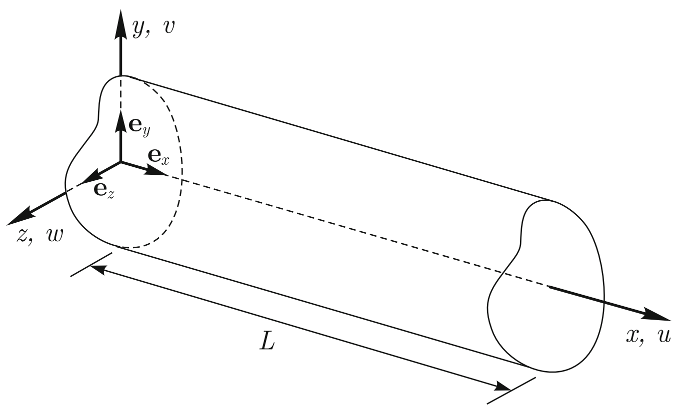

Fig. 3.63. Generic prismatic bar which will be subjected to torsion. Solid section with no holes

Consider a prismatic bar with a generic cross-section as shown in Figure 3.63. Suppose that the bar is subjected to self-equilibrated torsional moments at the end sections, *i.e.*, $\mathbf{M}_t = M_t\mathbf{e}_x$ at $x = L$ and $-\mathbf{M}_t$ at $x = 0$. It is an experimental observation that for a bar subjected to such loading the cross-sections rotate as rigid bodies in their own plane (see Example 3.5 for the in-plane displacements). However, these sections do not remain plane, they display some warping. Motivated by these observations, we will seek a solution of the 3-D elasticity mathematical model of this problem using the following displacement assumptions

$$u = \theta'\psi(y,z) \tag{3.123}$$

$$v = -\theta' xz \tag{3.124}$$

$$w = \theta' xy \tag{3.125}$$

where θ' is the rate of rotation of the cross-sections with respect to the x axis which is assumed to be constant. Hence, denoting by $\theta(x)$ the angle of rotation of a generic section, we have

$$\theta' = \frac{d\theta}{dx} = \text{constant}.$$

Referring to Example 3.5, v and w as given by equations (3.124) and (3.125) are the displacements associated with the section rotations given by $\theta(x) = \theta' x$. Note that we are implicitly assuming that there is no section rotation at $x = 0$. The functional form of u reflects the warping of the section. In fact, $\psi(y,z)$ gives the spatial variation of the out of section displacements and is called the warping function. Its precise form should be determined based on additional conditions as will be seen shortly. We also note that the intensity of warping is assumed to be proportional to the rate of rotation of the sections, θ'.

Starting from the displacement assumptions given in equations (3.123) to (3.125), we derive the associated strain field

$$\begin{aligned}
\varepsilon_{xx} &= \frac{\partial u}{\partial x} = 0, \quad \varepsilon_{yy} = \frac{\partial v}{\partial y} = 0, \quad \varepsilon_{zz} = \frac{\partial w}{\partial z} = 0 \\
\gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \theta' \frac{\partial \psi}{\partial y} - \theta' z = \theta' \left(\frac{\partial \psi}{\partial y} - z \right) \\
\gamma_{xz} &= \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = \theta' \frac{\partial \psi}{\partial z} + \theta' y = \theta' \left(\frac{\partial \psi}{\partial z} + y \right) \\
\gamma_{yz} &= \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = -\theta' x + \theta' x = 0.
\end{aligned}$$

The stresses can be obtained using the generalized Hooke's law. Since we are considering a homogenous and isotropic material we have

$$\begin{aligned}
\tau_{xx} &= \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \left[\varepsilon_{xx} + \frac{\nu}{1-\nu} (\varepsilon_{yy} + \varepsilon_{zz}) \right] \\
\tau_{yy} &= \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \left[\varepsilon_{yy} + \frac{\nu}{1-\nu} (\varepsilon_{xx} + \varepsilon_{zz}) \right] \\
\tau_{zz} &= \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \left[\varepsilon_{zz} + \frac{\nu}{1-\nu} (\varepsilon_{xx} + \varepsilon_{yy}) \right]
\end{aligned}$$

which lead to

$$\tau_{xx} = \tau_{yy} = \tau_{zz} = 0 \tag{3.126}$$

since $\varepsilon_{xx} = \varepsilon_{yy} = \varepsilon_{zz} = 0$. For the shear stresses we have

$$\tau_{xy} = G\gamma_{xy} = G\theta' \left(\frac{\partial \psi}{\partial y} - z \right) \tag{3.127}$$

$$\tau_{xz} = G\gamma_{xz} = G\theta' \left(\frac{\partial \psi}{\partial z} + y \right) \tag{3.128}$$

$$\tau_{yz} = G\gamma_{yz} = 0. \tag{3.129}$$

Consider now the equilibrium conditions. Assuming that we have no body forces, the differential equilibrium relations read

$$\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = 0 \tag{3.130}$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} = 0 \tag{3.131}$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} = 0. \tag{3.132}$$

Referring to the stresses given in equations (3.126) to (3.129), we verify that the equilibrium conditions (3.131) and (3.132) are identically satisfied and equation (3.130) leads to

$$\frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} = 0 \tag{3.133}$$

which is the condition on the warping function that guarantees equilibrium.

Next, we consider the boundary conditions. Since the lateral surfaces are free from any surface tractions, we should have

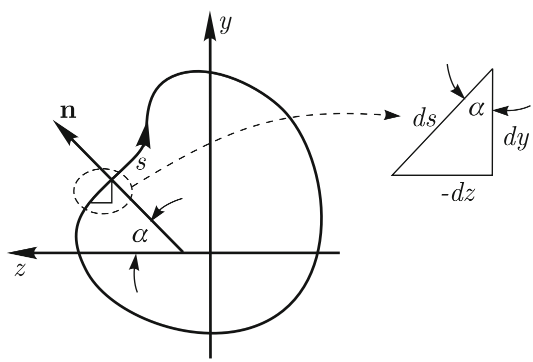

Fig. 3.64. Generic cross section

$$\mathbf{Tn} = \begin{bmatrix} 0 & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & 0 & 0 \\ \tau_{xz} & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ n_y \\ n_z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or

$$\tau_{xy} n_y + \tau_{xz} n_z = 0. \tag{3.134}$$

Let s be the arc length coordinate along the section boundary as shown in Figure 3.64. Of course, for a given cross-section, the section boundary is characterized by $y = y(s)$ and $z = z(s)$. Defining α as the angle that the surface outward's unit normal makes with the z axis, we can write

$$n_y = \sin\alpha, \; n_z = \cos\alpha$$

$$ds\cos\alpha = dy, \quad ds\sin\alpha = -dz$$

and introducing these relations into equation (3.134)

$$\tau_{xz}\frac{dy}{ds} - \tau_{xy}\frac{dz}{ds} = 0. \tag{3.135}$$

This equation can be written in terms of the warping function

$$G\theta'\left(\frac{\partial\psi}{\partial z} + y\right)\frac{dy}{ds} - G\theta'\left(\frac{\partial\psi}{\partial y} - z\right)\frac{dz}{ds} = 0$$

which finally leads to

$$\left(\frac{\partial\psi}{\partial z} + y\right)\frac{dy}{ds} - \left(\frac{\partial\psi}{\partial y} - z\right)\frac{dz}{ds} = 0. \tag{3.136}$$

This condition is to be satisfied for every point on the section boundary.

The field equation (3.133) subjected to the boundary condition (3.136) should determine the warping function $\psi(y,z)$. We note that the boundary condition (3.136) depends only on the geometry of the cross-section. Therefore, given a cross-section, the warping function can be determined and depends only on the shape of the cross-section.

Although the warping function can be determined as described above, Prandtl (see Timoshenko and Goodier, 1970) has introduced a stress function $\phi(y,z)$ to propose an alternative, somewhat simpler, formulation of the torsion problem. In order to introduce the stress function, let us consider the equilibrium equation (3.130) and re-write it as

$$\frac{\partial\tau_{xy}}{\partial y} = -\frac{\partial\tau_{xz}}{\partial z}. \tag{3.137}$$

Therefore, we can define a function $\phi(y,z)$ such that

$$\frac{\partial\phi}{\partial z} = \tau_{xy} \quad \text{and} \quad \frac{\partial\phi}{\partial y} = -\tau_{xz} \tag{3.138}$$

and (3.137) is automatically satisfied.

Using (3.127) and (3.128) we also have

$$\frac{\partial\phi}{\partial z} = G\theta'\left(\frac{\partial\psi}{\partial y} - z\right) \tag{3.139}$$

$$\frac{\partial\phi}{\partial y} = -G\theta'\left(\frac{\partial\psi}{\partial z} + y\right). \tag{3.140}$$

To eliminate ψ and obtain the governing differential equation in terms of ϕ we take derivatives of equations (3.139) and (3.140) with respect to z and y respectively, and then add to obtain

$$\frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = -2G\theta'. \tag{3.141}$$

Expressing the boundary condition (3.135) in terms of ϕ, we obtain

$$\frac{\partial \phi}{\partial y}\frac{dy}{ds} + \frac{\partial \phi}{\partial z}\frac{dz}{ds} = 0$$

which leads to

$$\frac{d\phi}{ds} = 0 \;\Rightarrow\; \phi \equiv \text{constant over the boundary.}$$

Since the stresses are obtained from ϕ by taking derivatives we can choose

$$\phi = 0 \text{ on the boundary of the cross-section.} \tag{3.142}$$

Let us summarize what we have obtained so far.

The formulation based on the warping function can be written as:

Find $\psi(y,z)$ defined on the cross-section domain such that

$$\begin{cases} \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} = 0 & \text{for every point in the cross-section domain} \\ \left(\frac{\partial \psi}{\partial z} + y\right)\frac{dy}{ds} - \left(\frac{\partial \psi}{\partial y} - z\right)\frac{dz}{ds} = 0 & \text{for every point on the cross-section boundary.} \end{cases}$$

Having determined $\psi(y,z)$, the displacements can be obtained from equations (3.123) – (3.125) and the stresses from (3.126) – (3.129) . We note that θ' is still to be found as will be discussed shortly. The formulation based on the stress function can be summarized as:

Find $\phi(y,z)$ defined on the cross-section domain such that

$$\begin{cases} \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = -2G\theta' & \text{for every point in the cross-section domain} \\ \phi = 0 & \text{for every point on the cross-section boundary.} \end{cases}$$

Having determined $\phi(y,z)$, we can obtain $\psi(y,z)$ by integration of equations (3.139)–(3.140) and hence the remaining quantities can be found as described above. The same observation regarding θ' applies.

To complete the formulation of the torsion problem and to obtain θ' we consider the tractions on the boundary surfaces at $x = L$ and $x = 0$. Specifically

$$\mathbf{f}^S = \mathbf{T}\mathbf{e}_x \text{ at } x = L$$

or

$$\begin{bmatrix} f_x^S \\ f_y^S \\ f_z^S \end{bmatrix} = \begin{bmatrix} 0 & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & 0 & 0 \\ \tau_{xz} & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \tau_{xy} \\ \tau_{xz} \end{bmatrix}$$

where τ_{xy} and τ_{xz} are given by equation (3.127) and (3.128) or alternatively by (3.138). These stress components are shown in Figure 3.65.

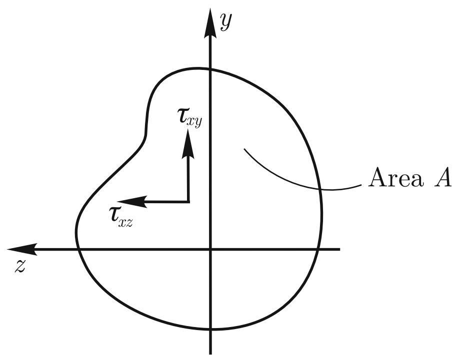

Fig. 3.65. Stress components at section $x = L$

Let us evaluate the stress resultants at the end section $x = L$. The normal force N and the bending moments $\mathbf{M}_y = M_y \mathbf{e}_y$ and $\mathbf{M}_z = M_z \mathbf{e}_z$ are trivially found to be zero since

$$N = \int_A \tau_{xx}\, dA = 0, \quad M_y = \int_A \tau_{xx}\, z\, dA = 0, \quad M_z = \int_A -\tau_{xx}\, y\, dA = 0$$

when we take into account that $\tau_{xx} = 0$. The shear forces $\mathbf{V}_y = V_y \mathbf{e}_y$ and $\mathbf{V}_z = V_z \mathbf{e}_z$ are given by

$$V_y = \int_A \tau_{xy}\, dA, \quad V_z = \int_A \tau_{xz}\, dA$$

which can be shown to be zero by expressing τ_{xy} and τ_{xz} in terms of ϕ (see equation (3.138)) and performing the above integrations.

Therefore, we can conclude that the stresses at the end section $x = L$ are mechanically equivalent to a torsional moment M_t only, since the resultants $N = M_y = M_z = V_y = V_z = 0$ and we have

$$\begin{aligned} M_t &= \int_A (\tau_{xz} y - \tau_{xy} z)\, dA \\ &= \int_A \tau_{xz}\, y\, dA - \int_A \tau_{xy}\, z\, dA. \end{aligned} \tag{3.143}$$

Performing the integrations it can be shown that

$$M_t = 2\int_A \phi\, dA \tag{3.144}$$

which relates the stress function to the torsional moment applied. We can also relate the torsional moment to the warping function. Starting from (3.143) and introducing (3.127) and (3.128), we obtain

$$\begin{aligned} M_t &= G\theta' \int_A \left[\left(\frac{\partial \psi}{\partial z} + y \right) y - \left(\frac{\partial \psi}{\partial y} - z \right) z \right] dA \\ &= G\theta' \int_A \left[\left(\frac{\partial \psi}{\partial z} y - \frac{\partial \psi}{\partial y} z \right) + \left(y^2 + z^2 \right) \right] dA. \end{aligned}$$

Recalling that the warping function ψ depends only on the geometry of the cross-section, we can define the torsional moment of inertia of the cross-section

$$I_t = \int_A \left[\left(\frac{\partial \psi}{\partial z} y - \frac{\partial \psi}{\partial y} z \right) + \left(y^2 + z^2 \right) \right] dA. \tag{3.145}$$

and hence

$$\frac{M_t}{GI_t} = \theta' \tag{3.146}$$

establishing the link between the torsional moment and θ'. Analogous derivations hold for the end section given by $x = 0$.

The formulation of the torsion problem is now complete and its solutions is an exact solution to the 3–D elasticity mathematical model as long as M_t is applied by means of the surface tractions $\mathbf{f}^S = \mathbf{T}\mathbf{e}_x$ at $x = L$ and $\mathbf{f}^S = \mathbf{T}\left(-\mathbf{e}_x\right)$ at $x = 0$.

As an example, we present below the solution of the torsion problem for an elliptical cross-section.

Example 3.8

Consider a bar with the elliptical cross-section in Figure 3.66. The bar is subjected to self–equilibrated torsional moments $\mathbf{M}_t = M_t \mathbf{e}_x$ at $x = L$ and $-\mathbf{M}_t$ at $x = 0$. Solve for the stress and displacement fields.

Solution

We consider the solution in terms of the stress function ϕ. Since the boundary equation is

$$\frac{y^2}{a^2} + \frac{z^2}{b^2} - 1 = 0$$

we can investigate solutions of the form

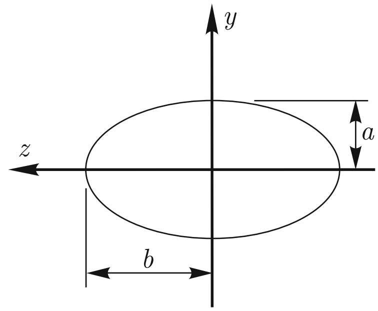

Fig. 3.66. Definitions for elliptic cross section

$$\phi = C\left(\frac{y^2}{a^2} + \frac{z^2}{b^2} - 1\right) \tag{3.147}$$

where C is a real constant. Of course, (3.147) satisfies the condition $\phi = 0$ at the boundary and substituting (3.147) in (3.141) yields

$$\frac{2C}{a^2} + \frac{2C}{b^2} = -2G\theta'$$

which leads to

$$C = -\frac{a^2 b^2 G\theta'}{a^2 + b^2}.$$

Therefore

$$\phi = -\frac{a^2 b^2}{a^2 + b^2} G\theta' \left(\frac{y^2}{a^2} + \frac{z^2}{b^2} - 1\right)$$

and the stresses

$$\tau_{xy} = \frac{\partial \phi}{\partial z} = -\frac{2a^2}{a^2 + b^2} G\theta' z, \quad \tau_{xz} = -\frac{\partial \phi}{\partial y} = \frac{2b^2}{a^2 + b^2} G\theta' y.$$

Using relations (3.139) and (3.140) we obtain

$$\left(\frac{\partial \psi}{\partial y} - z\right) = -\frac{2a^2}{a^2 + b^2} z, \quad \left(\frac{\partial \psi}{\partial z} + y\right) = \frac{2b^2}{a^2 + b^2} y \tag{3.148}$$

which lead to $\psi(y, z)$ by integration. In fact, integrating (3.148) with respect to y and z respectively leads to

$$\psi = yz - \frac{2a^2}{a^2 + b^2} yz + f(z), \quad \psi = -yz + \frac{2b^2}{a^2 + b^2} yz + f(y).$$

Hence

$$\psi = \frac{b^2 - a^2}{a^2 + b^2} yz + K$$

where K is a real constant. Assuming that $u = 0$ for $y = z = 0$ yields

$$\psi(y, z) = \frac{b^2 - a^2}{a^2 + b^2} yz.$$

Having the warping function $\psi(y, z)$, we can obtain I_t from (3.145) which gives

$$I_t = \frac{\pi a^3 b^3}{a^2 + b^2}$$

and we can relate M_t to θ', *i.e.*,

$$\theta' = \frac{M_t}{GI_t} = \frac{M_t}{G} \frac{a^2 + b^2}{\pi a^3 b^3}. \tag{3.149}$$

Alternatively we could have used

$$M_t = 2 \int_A \phi dA$$

leading also to the result given in (3.149). The displacements are

$$\begin{aligned} u &= \theta' \psi(y, z) = \frac{M_t}{G} \frac{b^2 - a^2}{\pi a^3 b^3} yz \\ v &= -\theta' xz = -\frac{M_t}{G} \frac{a^2 + b^2}{\pi a^3 b^3} xz \\ w &= \theta' xy = \frac{M_t}{G} \frac{a^2 + b^2}{\pi a^3 b^3} xy \end{aligned}$$

and the stresses can be written in terms of M_t by

$$\tau_{xy} = -\frac{2M_t a^2}{\pi a^3 b^3} z, \quad \tau_{xz} = \frac{2M_t b^2}{\pi a^3 b^3} y.$$

The stress variation along the selected lines is schematically represented in Figure 3.67a and the out-of-plane displacements $u(y, z)$ – the warping displacements – are shown in Figure 3.67b. We note that the above solution includes the classical torsion solution of a prismatic bar of circular cross-section when $a = b = r$, r being the radius of the circular cross-section. In this case, there is no warping of the cross-section and since the solution derived above is an exact solution of the 3-D elasticity problem, the classical solution of a circular prismatic bar is also exact.

□

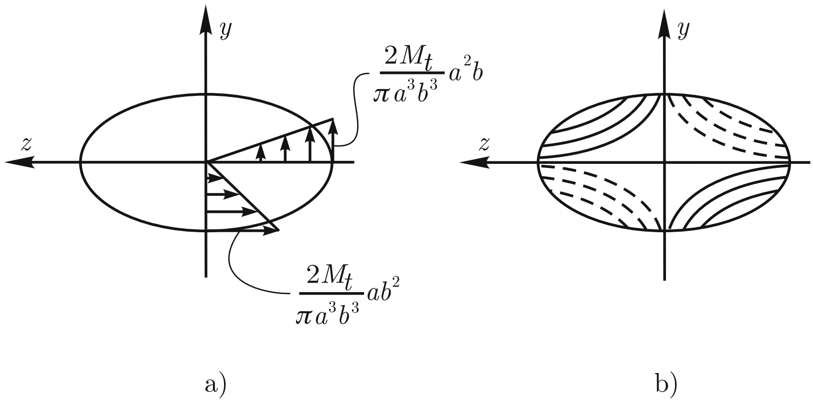

Fig. 3.67. a) Some shear stresses for the solution of a prismatic bar of elliptic cross section; b) Lines connecting points of equal values of warping displacements. The solid lines represent positive displacements and the dashed lines represent negative displacements

The membrane analogy

A valuable physical interpretation of the solution reached in (3.141) is obtained by the Prandtl membrane analogy. Namely, a membrane prestressed with uniform tension T and subjected to transverse pressure p undergoes the transverse displacement $u(y, z)$ calculated from (see *e.g.* Bathe, 1996)

$$\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = -\frac{p}{T} \tag{3.150}$$

with $u = 0$ on the boundary. Hence from (3.141) , we obtain

$$\phi = c\, u(y, z) \text{ with } c = \frac{2G\theta' T}{p}. \tag{3.151}$$

Therefore, the transverse displacement of the membrane is proportional to ϕ with c the constant of proportionality.

There are two properties linked to the fact that the shear stresses are obtained as derivatives of ϕ which are of interest. In Figure 3.68a, we show a generic cross-section and a curve for which ϕ is constant. It can be shown (Timoshenko and Goodier, 1970) that

$$\mathbf{t}_s = t_s \mathbf{s}$$

that is, $\mathbf{t}_s$ is always tangent to the curve of constant ϕ since $\mathbf{s}$ is the tangent unit vector to this curve, and that

$$t_s = -\frac{d\phi}{dn} \tag{3.152}$$

that is, the derivative of ϕ with respect to the normal direction to the curve of constant ϕ gives the magnitude of the shear stress t_s. In Figure 3.68b a typical distribution is shown. Due to the membrane analogy, a curve which gives equal values of ϕ corresponds to a level curve of the deformed membrane and hence we can qualitatively anticipate the shear stress distributions.

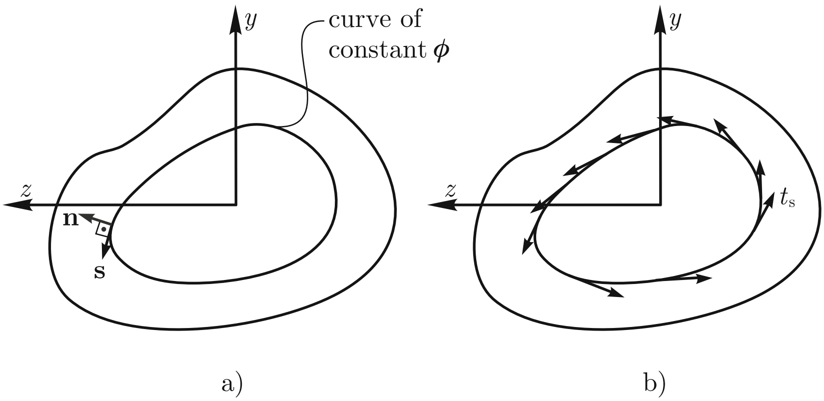

Fig. 3.68. a) A generic curve of constant ϕ; b) A typical shear stress distribution for the points on a curve of constant ϕ

We demonstrate these observations in the following example.

Example 3.9

Derive the shear stress distribution and the torsional moment of inertia for the torsion problem for a thin rectangular cross-section as shown in Figure 3.69a.

Solution

In Figure 3.69b, the deformation of the associated membrane is shown. We can see that, except in the small end regions, the membrane displacements are such that

$$\frac{\partial u}{\partial y} = 0.$$

Therefore, equation (3.150) simplifies to

$$\frac{\partial^2 u}{\partial z^2} = -\frac{p}{T}$$

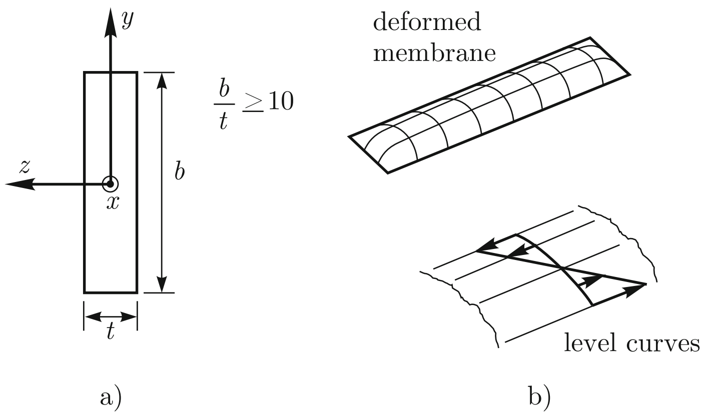

Fig. 3.69. a) Thin rectangular cross section; b) Associated deformed membrane

which upon integration leads to

$$u = \frac{p}{8T}\left(t^2 - 4z^2\right).$$

Using relation (3.151), we have

$$\phi = \frac{G\theta'}{4}\left(t^2 - 4z^2\right).$$

The shear stresses can be evaluated from (3.138) which leads to

$$\tau_{xz} = -\frac{\partial \phi}{\partial y} = 0, \quad \tau_{xy} = \frac{\partial \phi}{\partial z} = -2G\theta' z.$$

Note the shear stresses are tangent to the level curves of the membrane as shown in Figure 3.69b. The torsional moment of inertia can be calculated using (3.144) and (3.146)

$$I_t = \frac{bt^3}{3}.$$

Of course, the derived quantities are approximate since we assumed that $\frac{\partial u}{\partial y} = 0$ also in the end regions.

□