2. Fundamental steps in structural mechanics

It is easy to recognize that all structures in nature are three-dimensional. However, the reader has surely also encountered structures in simple forms such as an assemblage of geometrically slender members. As we discussed in the previous chapter, these simplified forms lead directly to models that are in accordance with the hierarchical modeling process. In fact, there are a number of structural mechanics mathematical models such as bars, beams, plates and shells, among others, that provide a convenient and efficient way to model structural behavior for design and analysis purposes. These mathematical models will be discussed in the following chapters due to their importance in the structural modeling process.

The objective of this chapter is to address some fundamental conditions that should be met whichever structural mathematical model is used. The discussion will be placed initially in a very general setting – the motion of a deformable body – and we will extract fundamental conditions which lead to "static equilibrium". The simplest structural mathematical model – the truss model – is then studied to exemplify, in a simple setting, the basic steps associated with the formulation and solution of structural mechanics problems. These steps are then systematized leading to the modern matrix approach of analysis.

2.1 General conditions

There exist some fundamental general conditions regarding the analysis of a structure which we present in this section.

2.1.1 Motion of a deformable three-dimensional body

Let us consider the motion of a three-dimensional body in 3-D space. The body can be idealized as a collection of particles which are assumed to have a mass density. At a given time, the set of positions occupied by the body material particles which defines a region in 3-D space is referred to as a configuration. In Figure 2.1 we show two configurations ${}^0V$ and ${}^tV$.

The configuration ${}^0V$ represents the configuration at the onset of the motion, *i.e.*, for time $t = 0$, which is also referred to as the undeformed

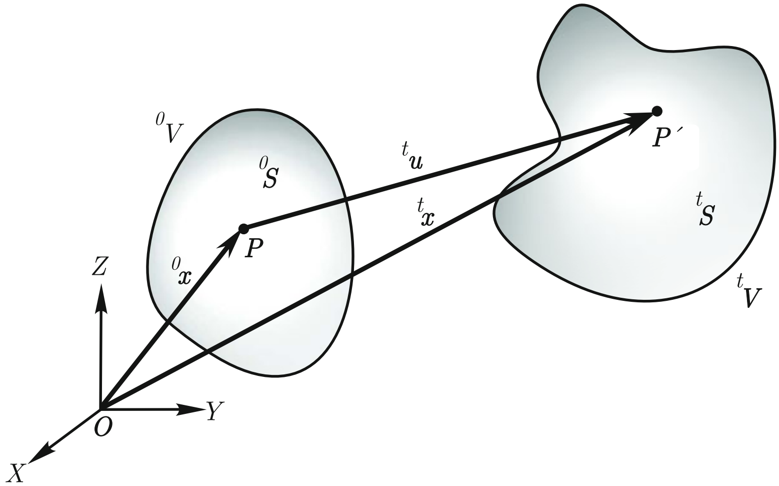

Fig. 2.1. Two configurations of a deformable body in a selected stationary Cartesian coordinate system

configuration, and ${}^tV$ stands for a generic deformed configuration[1], *i.e.*, for time t. We denote by ${}^0S$ and ${}^tS$ the boundary surfaces associated with ${}^0V$ and ${}^tV$.

We adopt a Lagrangian description of the motion, *i.e.*, we "follow" the complete motion of the material particles, from time 0 to time t. The position of a particle at time t is given by[2] ${}^t\mathbf{x} = \mathbf{x}\left({}^0\mathbf{x},\, t\right)$ where ${}^0\mathbf{x}$ is the position vector of this particle at time $t = 0$.

The motion of the body is governed by the action of the "rest of the universe" onto the body. This action is represented by forces which are generically referred to as externally applied forces. These may partly be unknown, namely, when displacements of the body are prescribed, see below. There are two kinds of externally applied forces. There is the field of forces per unit of volume represented by ${}^t\mathbf{f}^B\left({}^t\mathbf{x},\, t\right)$, called body forces. The most common example of such a field is given by gravity acting on the body material particles. And there are externally applied forces on the surface of the body ${}^tS$ represented by a field of tractions – forces per unit of surface area – denoted by ${}^t\mathbf{f}^S\left({}^t\mathbf{x},\, t\right)$. These forces typically arise as a consequence of the contact of

[1] In the 20$^{\text{th}}$ century, continuum mechanics was also presented as a mathematical theory referred to as rational continuum mechanics, see Truesdell, 1977 and references therein. Although we recognize the importance of these works, we keep the mathematics in our presentation as simple as possible

[2] Since t is an argument to our function, we could have simply written $\mathbf{x} = \mathbf{x}\left({}^0\mathbf{x},\, t\right)$ to describe the particle position at time t. However, we choose to use the notation ${}^t\mathbf{x} = \mathbf{x}\left({}^0\mathbf{x},\, t\right)$ with the left superscript t to emphasize that we consider the configuration at the specific time t. This approach is also followed for many other quantities, see also Bathe, 1996

the body with other bodies, that is, the surrounding media, and included are here the effects of the restraints applied to part of the body's surface.

Let us define the resultant of all external forces applied to the body at time t by

$$ {}^t\mathbf{R} = \int_{{}^tV} {}^t\mathbf{f}^B \, d\,{}^tV + \int_{{}^tS} {}^t\mathbf{f}^S \, d\,{}^tS \tag{2.1} $$

and the resultant moment at time t about the system origin $\mathbf{O}$ by

$$ {}^t\mathbf{M}_\mathbf{O} = \int_{{}^tV} {}^t\mathbf{x} \times {}^t\mathbf{f}^B \, d\,{}^tV + \int_{{}^tS} {}^t\mathbf{x} \times {}^t\mathbf{f}^S \, d\,{}^tS. \tag{2.2} $$

Here, ${}^t\mathbf{R}$, ${}^t\mathbf{M}_\mathbf{O}$ are respectively called the external force and moment resultants.

The motion of the body is governed by two principles. The first one is the principle of linear momentum which states that

$$ {}^t\mathbf{R} = \frac{d}{dt} \int_{{}^tV} {}^t\rho \, {}^t\dot{\mathbf{x}} \, d\,{}^tV \tag{2.3} $$

where ${}^t\rho\,({}^t\mathbf{x},t)$ is the mass density function at time t and ${}^t\dot{\mathbf{x}} = d\,{}^t\mathbf{x}/dt$ is the material velocity at ${}^t\mathbf{x}$. The second principle, the principle of angular momentum, is given by

$$ {}^t\mathbf{M}_\mathbf{O} = \frac{d}{dt} \int_{{}^tV} {}^t\mathbf{x} \times {}^t\rho \, {}^t\dot{\mathbf{x}} \, d\,{}^tV. \tag{2.4} $$

These two principles need to be satisfied, in any motion, in an inertial reference system[3].

The principle of linear momentum for the dynamics of a point mass is Newton's 2nd Law, *i.e.*, $\mathbf{R} = \frac{d}{dt}(\mathbf{p})$ where $\mathbf{p}$ is the linear momentum, $\mathbf{p} = m\mathbf{v}$, with m the point mass, $\mathbf{v}$ its velocity and $\mathbf{R}$ the resultant of the forces acting on the point mass. We note that the right-hand side of equation (2.3) gives the time derivative of the vectorial sum of the linear momenta of the material particles of the body.

The principle of angular momentum for a set of point masses is given by $\mathbf{M}_\mathbf{O} = \frac{d}{dt}\sum_i \mathbf{x}_i \times m_i \mathbf{v}_i$ where $\mathbf{M}_\mathbf{O}$ is the resultant moment of all forces acting on the point masses m_i about the origin $\mathbf{O}$ and $\mathbf{x}_i$ and $\mathbf{v}_i$ are the position and velocity of point mass i, respectively. Of course, the right-hand side of equation (2.4) represents the integrated effect for the mass particles in the continuum.

We emphasize that the fields of forces ${}^t\mathbf{f}^B$ and ${}^t\mathbf{f}^S$ represent all the influence of the "rest of the universe" on the motion of the body considered. Note

[3] For structural and solid mechanics applications a reference system which is either at rest or in rectilinear motion with constant velocity with respect to the planet Earth can be taken as inertial

that this general statement includes the very common situation in which the motion of a part of the body's surface is restrained. The physical devices which constrain the motion of the body's surface also belong to the "rest of the universe" and their effect on the body's motion also results into a field of surface tractions. These physical devices are generically referred to as restraints or supports.

Corresponding to these concepts, let us denote by ${}^tS_u$ that part of the body's surface ${}^tS$ which has its motion restrained and by ${}^tS_f$ the complementary part of the body's surface. Therefore, on ${}^tS_f$ there are the interactions with other bodies represented by surface tractions but no restraints. The restraints on ${}^tS_u$ give rise to surface tractions which are referred to as reactive surface tractions, or mostly, simply as reactions. This model situation is schematically summarized in Figure 2.2. The surface tractions on ${}^tS_f$ and the body forces ${}^t\mathbf{f}^B$ are the external loads.

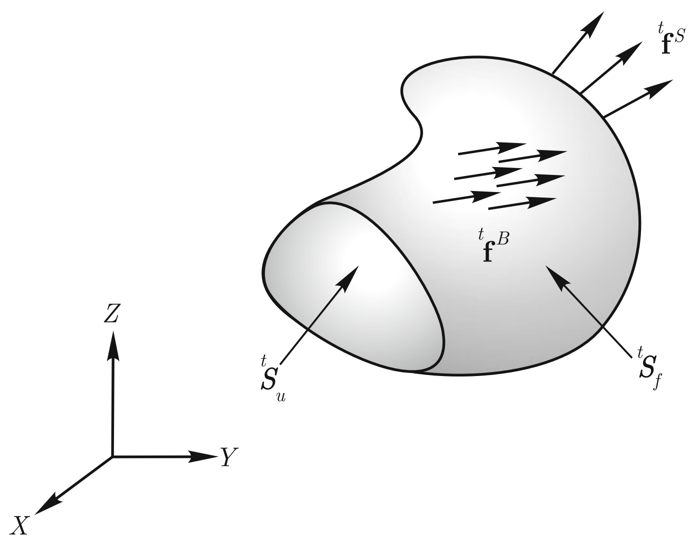

Fig. 2.2. Restrained body under external actions

In the above description we seem to have assumed that on ${}^tS_u$ all displacements (into the X, Y, and Z directions) are restrained. However, in three-dimensional analysis, the particles on the surface ${}^tS_u$ have three independent displacement degrees of freedom and only some of them may be restrained. For example, a particle may be prevented from moving into directions X and Y and free to move into direction Z. Hence, the definition of ${}^tS_u$ and ${}^tS_f$ given above should be generalized and we define ${}^tS_u$ and ${}^tS_f$ for each displacement degree of freedom. That is, we define ${}^tS_u$ and ${}^tS_f$ for the displacement degree of freedom into the direction X and also into the directions Y and Z. Therefore, a given material particle may belong to ${}^tS_f$ for some displacement degree of freedom and to ${}^tS_u$ for another degree of freedom.

2.1.2 Properly supported bodies

A solid[4] is *properly supported* if the supports prevent rigid body motions for any external loading. Hence, in this case the displacements of the material particles can only result from some straining of the material.

According to this definition, a properly supported rigid body can not display motion since rigid body motions are prevented and straining of the material can not occur. This concept can be also used to characterize a properly supported deformable body: indeed, assuming that this deformable body were rigid, if this rigid body can not display motion, then the deformable body is properly supported.

Considering a rigid body which is properly supported, we can conclude directly from equations (2.3) and (2.4) that ${}^t\mathbf{R} = \mathbf{0}$ and ${}^t\mathbf{M_O} = \mathbf{0}$ since the velocity field is always zero (${}^t\dot{\mathbf{x}} = \mathbf{0}$). Here, we note that while a constant velocity field also leads to ${}^t\mathbf{R} = \mathbf{0}$ and ${}^t\mathbf{M_O} = \mathbf{0}$, kinematic restraints that prevent rigid motions represent of course the sufficient condition for a rigid body to display no motion for any externally applied loading.

When assessing whether a deformable body is properly supported or not, we frequently investigate if the associated rigid body is properly supported because this is usually simpler.

In practice, we frequently find that a number of solids are connected by joints. In these cases the above concepts are also directly applicable, but each individual solid — considered just like the single solid above — must then be also properly supported by the rest of the assemblage. If this is not the case, despite the fact that global rigid motions of the whole assemblage are prevented by the supports, the assemblage of solids is said to contain one or more *internal mechanisms.* Corresponding to each internal mechanism there is an independent rigid motion that the solids can undergo while being still connected at the joints.

In all these cases when rigid motions are possible, we sometimes also simply say that the solid or the assemblage of solids is unstable (see Section 8.3 for a further discussion).

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[4] The notion of a body encompasses both solids and fluids. In this book we are interested only in solids and suppose that the reader has an intuitive understanding of the behavior of deformable solids when contrasted with the behavior of a fluid. Frequently we use the term "body" with the implicit understanding that we are actually considering a solid

2.1.3 Internal actions

Consider next a generic part of a properly supported body in the configuration t with volume $\Delta^t V$ and total surface area $\Delta^t S$. Part of that surface area $\Delta^t S$ is the result of having sliced $\Delta^t V$ from the total body (refer to Figure 2.3). When subjected to external loads the deformable body develops internal forces. Let us represent these internal forces, which are given per unit of area, by ${}^t\mathbf{t}$. We postpone a more detailed discussion of these internal forces per unit of area – the stresses – until the next chapter. In Figure 2.3 a typical situation is summarized where the isolated part with all actions on it is shown. The figure showing the isolated part is referred to as a "free body diagram".

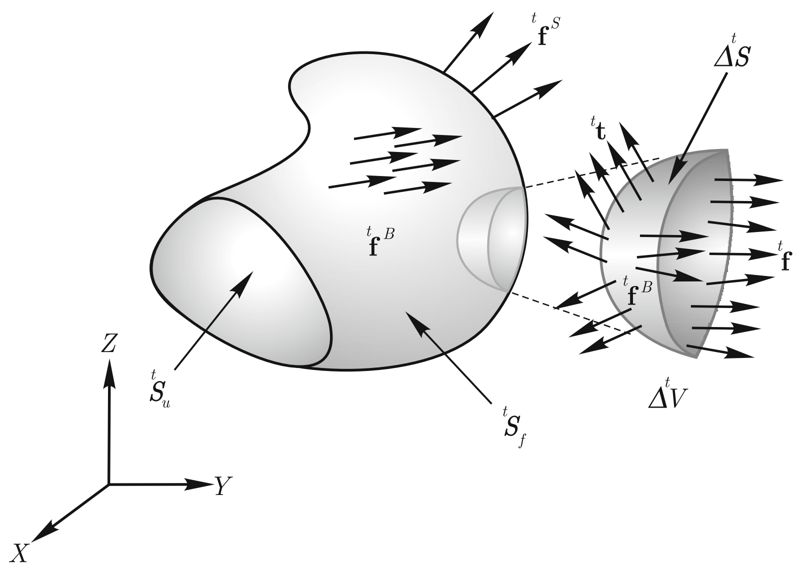


Fig. 2.3. Representation of a generic isolated part of the body showing the internal forces per unit of area: the "free body diagram"

Considering $\Delta^t V$, the principles of linear and angular momenta are, of course, directly applicable as long as we consider all the forces acting on $\Delta^t V$ and $\Delta^t S$ including the field ${}^t\mathbf{t}$. The forces ${}^t\mathbf{t}$ are now part of the "rest of the universe" acting on the body $\Delta^t V$ (and indeed can be thought of as forces ${}^t\mathbf{f}^S$). Hence, the above discussion is directly applicable: the body[5] considered is simply $\Delta^t V$ instead of ${}^t V$.

[5] Actually, we could have introduced the principles of linear and angular momenta for any part of the body, since all that matters to establish (2.3) and (2.4) is to represent all the actions of the "rest of the universe" on whatever body we consider: any body considered will always be a part of the universe

2.1.4 Assumptions for static analysis

Consider a properly supported body which is initially at rest and not subjected to external actions. Assume that the external loads are then applied very "slowly" from zero to their final values such that the induced accelerations ${}^t\ddot{\mathbf{x}}$ and velocities ${}^t\dot{\mathbf{x}}$ can be neglected, that is, the dynamic effects are negligible. These assumptions characterize *static analysis* and, then, equations (2.3) and (2.4) simplify to

$$ {}^t\mathbf{R} = \mathbf{0} \tag{2.5}$$

and

$$ {}^t\mathbf{M}_{\mathbf{O}} = \mathbf{0}. \tag{2.6}$$

Equations (2.5) and (2.6) are the principles of linear and angular momenta for static analysis. In this case, the variable t (used for "time") should be interpreted as a label used only to specify the external loading at time[6] t.

A useful concept is that of a system of forces in *static equilibrium*. A system of forces including body forces and surface tractions is said to be in static equilibrium in the configuration at time t if ${}^t\mathbf{R} = \mathbf{0}$ and ${}^t\mathbf{M}_{\mathbf{O}} = \mathbf{0}$. Hence in static analysis, the force system of all external actions on the body (due to all externally applied loads and all reactions) is to be always in static equilibrium (due to (2.5) and (2.6)).

Finally, we note that if (2.5) is satisfied and (2.6) is satisfied with respect to a point $\mathbf{O}$, then (2.6) is also satisfied when we select any other point $\mathbf{O}'$ instead of $\mathbf{O}$. In fact, since ${}^t\mathbf{R} = \mathbf{0}$ and

$$ {}^t\mathbf{M}_{\mathbf{O}'} = {}^t\mathbf{M}_{\mathbf{O}} - \mathbf{r}_{\mathbf{OO}'} \times {}^t\mathbf{R}$$

where $\mathbf{r}_{\mathbf{OO}'}$ is the vector from $\mathbf{O}$ to $\mathbf{O}'$, we obtain ${}^t\mathbf{M}_{\mathbf{O}} = {}^t\mathbf{M}_{\mathbf{O}'}$. Hence, the resultant moment about $\mathbf{O}'$ is equal to the moment about $\mathbf{O}$. This result implies that if a system of forces is in static equilibrium the moment resultant about any point is zero.

Of course (2.5) and (2.6) also apply to the generic part of the body $\Delta^t V$, but as pointed out above, ${}^t\mathbf{t}$ must then be included as actions from the "rest of the universe" onto this body part. Hence, in static analysis the condition is that the force system given by ${}^t\mathbf{f}^B$, ${}^t\mathbf{f}^S$ and ${}^t\mathbf{t}$ acting onto any part of the body is in static equilibrium. This fact will be used throughout the book.

2.1.5 Assumptions for a linear static analysis

In addition to considering static conditions (and not dynamic effects), most of this book is concerned with situations for which the material internal

[6] Of course, strictly, time is always present and can not be "switched off ", and hence "t" is a convenient label to specify the loading

force-displacement relationships are linear and the loads are such that the body displacements are very small; indeed we can assume in the analysis that the displacements are infinitesimally small. As a result, when we use the linear and angular momentum principles given by equations (2.3) and (2.4), we assume that the deformed configurations are geometrically the same as (identical to) the undeformed configuration. This means that the equilibrium of the body is considered neglecting all body displacements – that is, as if the body did not displace.

With these assumptions

$$
{}^t\mathbf{R} = \int_{{}^0V} {}^t\mathbf{f}^B \, d\,{}^0V + \int_{{}^0S} {}^t\mathbf{f}^S \, d\,{}^0S \tag{2.7}
$$

$$
{}^t\mathbf{M}_{\mathbf{O}} = \int_{{}^0V} {}^0\mathbf{x} \times {}^t\mathbf{f}^B \, d\,{}^0V + \int_{{}^0S} {}^0\mathbf{x} \times {}^t\mathbf{f}^S \, d\,{}^0S. \tag{2.8}
$$

We note that all integrals are using the undeformed configuration.

In general, in linear static analysis we are only interested in the final configuration resulting from the applied loading. In such case, there are only two configurations of interest, the initial configuration – that prior to the application of the loads – and the deformed configuration reached due to the application of the loads. When we assume this situation, we drop the left superscript t associated with time.

2.1.6 Summary

Before we proceed to the next section, we would like to summarize the most important points discussed:

- The motion of a deformable body was characterized in a very general setting, *i.e.*, a body of arbitrary shape was considered undergoing arbitrary motions.
- The interaction of the body with the "rest of the universe" was characterized by fields of forces and displacement restraints. The fields of forces include forces due to the restraints.
- Two general principles, the principle of linear momentum and the principle of angular momentum were stated; they govern every motion of the body (and of any part thereof).
- The concept of a properly supported body was introduced. The concept is based on purely kinematic conditions and reflects the idea that a properly supported deformable body can experience displacements due only to the straining of its material fibers, *i.e.*, an associated rigid body displays no motion for any applied loads.
- The assumptions of linear static analysis were introduced.
- The concept and conditions for static equilibrium were given.

These general concepts and facts are applicable to any structural system. Indeed, they provide the theoretical framework upon which the mathematical models presented in this book will be built.

Until now we mentioned structures and structural systems without a precise definition of "a structure". In fact, we relied on the intuitive understanding of the reader. We can now better characterize a structure or structural system.

We define a structure or a structural system as an assemblage of three-dimensional deformable bodies. Although this definition encompasses all structures, the modeling of bodies as three-dimensional frequently does not lead to the most efficient mathematical models to predict their behavior. Fortunately, in many cases, deformable bodies possess specific characteristics that allow a more effective modeling. These characteristics are linked to the geometries of the bodies, the kind of external loading, the boundary conditions and the connections between bodies. The more effective modeling uses these characteristics to establish displacements and force flow assumptions which lead to the various mathematical models of structural mechanics. These mathematical models are the subject of the forthcoming chapters.

In the next section we study a simple structure – the truss structure. In a truss, the basic "bodies" that together make up the structure are slender bodies, the truss elements connected and only loaded at frictionless hinges. The ability to analyze a truss structure is of course of practical value but in presenting the general framework for truss analyses, we shall also introduce and explore the fundamental conditions which are always part of the formulation of every structural problem, namely: equilibrium, constitutive behavior and compatibility.

2.2 The analysis of truss structures – to exemplify general concepts of analysis

In order to convey the objectives outlined above, we start by characterizing a truss structure and then we apply the fundamental conditions introduced in Section 2.1 to a typical truss structure.

2.2.1 Model assumptions

We define a truss structure as an assemblage of slender prismatic solids of constant transverse cross-section which are called bars, and –

- The bars are connected to each other at frictionless pin joints (detailed later on).
- All external loading is applied as concentrated forces to the joints.
- The truss structure is only restrained at the joints.

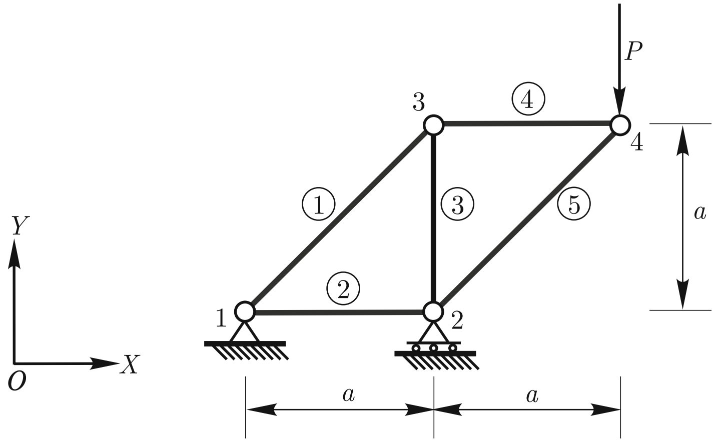


Fig. 2.4. A truss structure in the XY plane

In order to fix ideas let us consider the truss structural model in Figure 2.4. The structure is located in the XY plane and only loaded in that plane. Hence all actions take place in the XY plane (and we can refer to the structure as a “planar truss”). The truss bars are represented by straight lines which meet at the joints represented by the small circles. We shall refer to the joints as nodes; actually, more accurately, each node represents a joint. The supports are connected directly to nodes 1 and 2 and the external load is given by the concentrated force applied (also directly) to node 4.

In Figure 2.5 we show the detail of the connection represented by node 3. We consider not only the model representation used in Figure 2.4 but also a physical representation that gives the reader more insight into the pin-type joint behavior of a truss connection. Although the physical joint representation is still schematic, we can visualize the pin and how the bars are connected through the pin. The kinematics of the bars and the joint are assumed to be such that:

- The frictionless joint does not restrain the rotations of the bars.
- The lines representing the truss bars pass through the axes of the bars and the center of the joint.
- The bars displace with the joint.

2.2.2 Kinematic conditions for a properly supported truss

As we discussed in Section 2.1.2, we can identify whether an assemblage of solids and, hence, a structure is properly supported and does not have an internal mechanism by (first) assuming that the elements of the structure are rigid (where in a truss structure the joints are still assumed to be frictionless). If then the rigid structure and any part thereof can not undergo any motion,

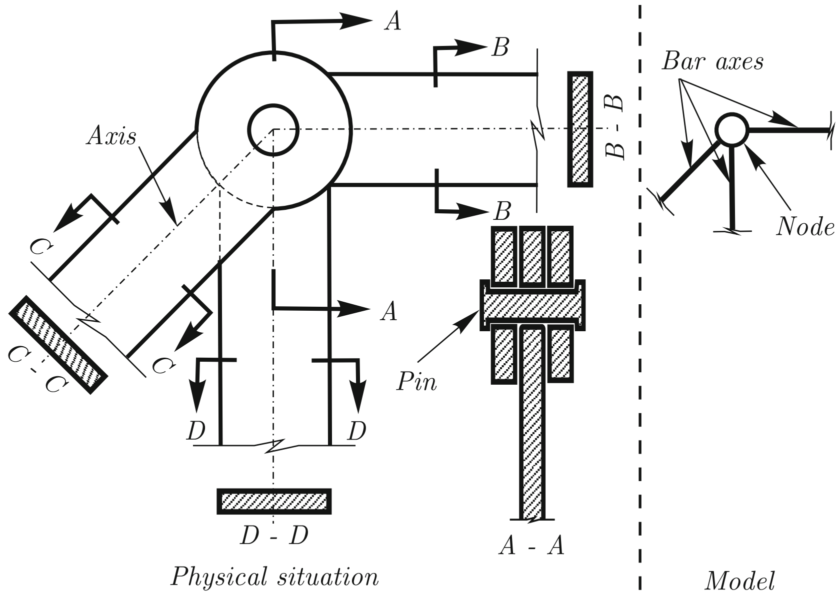


Fig. 2.5. Representation of a truss joint

the actual structure is properly supported and does not possess an internal mechanism.

Considering a truss structure, we should assume that each bar is rigid and verify whether a rigid body motion of the truss as a whole, or of any of its members, or of any of its parts is not possible.

Consider bars 1, 2 and 3 of the truss in Figure 2.4. They form a triangle and since each bar is assumed to be rigid, if bars 1, 2 and 3 were to move, they would have to do so as a rigid triangle. Namely, a triangle with sides of fixed lengths maintains its shape. For bars 3, 4 and 5 the same argument holds. Since bar 3 is common to both "rigid" triangles, the whole assemblage of bars, *i.e.*, bars 1 to 5 would behave as a rigid body.

Now, to examine if the assemblage of bars could have a rigid motion, we need to consider that the structure is supported. Hence, we can immediately conclude that the assemblage considered as rigid can not move since node 1 is fixed and a rotation about node 1 is prevented by the support at node 2.

Therefore, by kinematics alone, we conclude that the truss model of Figure 2.4 is properly supported and does not have an internal mechanism, and structural displacements can only be due to the straining of the bars. Although we consider here a very simple truss structure, this kinematics based approach can be applied to trusses of any complexity to arrive at a correct assessment of whether a truss structure is properly supported and does not have an internal mechanism. Note that this kind of analysis is independent of the external loading acting on the structure.

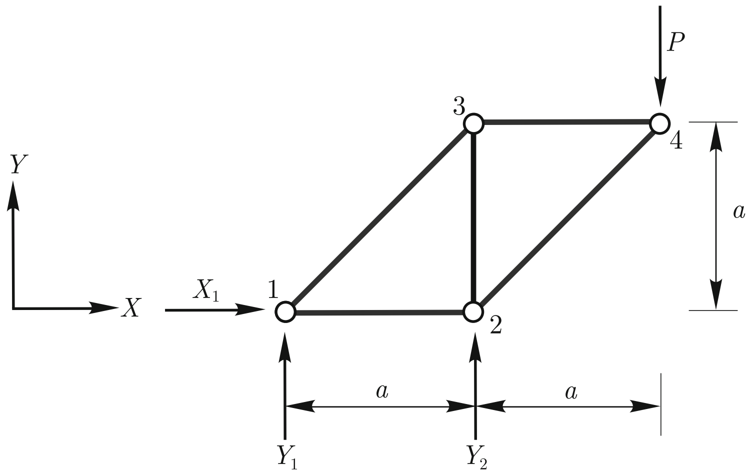


Fig. 2.6. Truss with reactions introduced explicitly

2.2.3 Equilibrium conditions for a truss model

Next, we detail how the "static equilibrium" condition can be applied to the truss structure to obtain the reactions at the supports and the internal forces of the truss bars.

Let us consider the equilibrium condition applied to the whole truss (this would correspond to taking $V \equiv \Delta V$ in the terminology for the solid body considered in Figure 2.3). Introducing the reactions as shown in Figure 2.6 and imposing the equilibrium conditions $\mathbf{R} = \mathbf{0}$ and $\mathbf{M_O} = \mathbf{0}$, we obtain

$$\begin{aligned} \sum F_X &= 0, \qquad & X_1 = 0 \\ \sum F_Y &= 0, \qquad & Y_1 + Y_2 - P = 0 \\ \sum M_1 &= 0, \qquad & Y_2 \cdot a - P \cdot 2a = 0. \end{aligned}$$

In the above equations we are introducing the notation $\sum F_X$ and $\sum F_Y$ to represent the summation of all forces in the X and Y directions respectively, and $\sum M_1$ represents the summation of the moments of the external forces about node 1. We obtain

$$Y_2 = 2P \quad \text{and} \quad Y_1 = -P.$$

Next we impose the equilibrium condition to a generic bar of the truss (in Figure 2.3, this would correspond to taking ΔV as the bar in consideration), as shown in Figure 2.7. Here we also show the internal forces that could possibly arise. We note that no concentrated moment is introduced since we

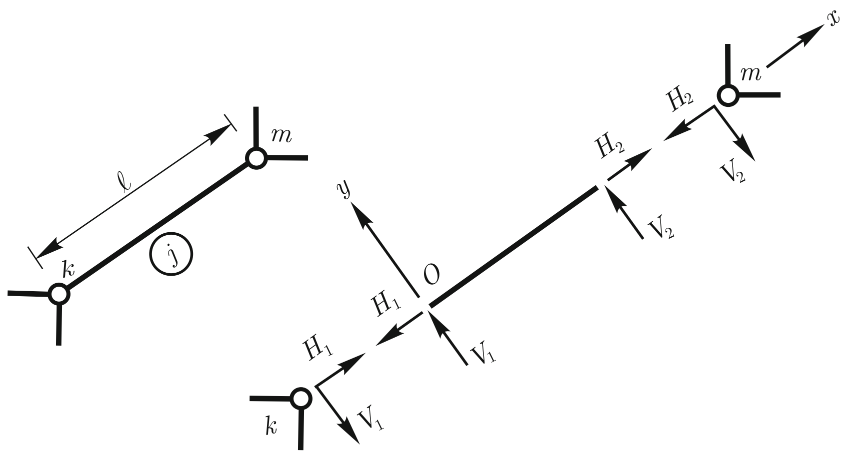


Fig. 2.7. Generic truss element, bar, of a truss. Shown are the magnitudes and directions of the forces acting onto the bar j and onto the nodes k and m (from the bar)

assume that the bars are free to rotate at the nodes. Hence, no such moment can arise.

Imposing $\mathbf{R} = \mathbf{0}$ and $\mathbf{M_O} = \mathbf{0}$ for the bar and using the local axis system shown in Figure 2.7, we arrive at

$$\begin{aligned} \sum F_x &= 0, \qquad H_2 - H_1 = 0, \qquad H_1 = H_2 \\ \sum F_y &= 0, \qquad V_1 + V_2 = 0, \qquad V_1 = -V_2 \\ \sum M_O &= 0, \qquad V_2 \cdot \ell = 0, \qquad V_2 = 0 \end{aligned}$$

and hence $V_1 = 0$ also.

Therefore, each bar can only carry an axial force. We denote this force in bar j by N_j and a positive value is associated with tension. For the generic bar considered $N_j = H_1 = H_2$. This situation is shown in Figure 2.8.

If we next consider the equilibrium of the truss nodes, we can determine the forces in the truss structure of Figure 2.4.

In Figure 2.9 we show all nodes isolated from the rest of the truss structure. As we mentioned earlier, see Figure 2.3, *any* part of the structure must be in equilibrium and so must be each joint, that is each node. Hence, we can suppress the central portions of the truss bars and introduce the axial forces of the bars onto the remaining parts of the structure, namely the joints/nodes. Then, each node shown has to be in equilibrium and, in this example, we can directly solve for all bar forces.

In Figure 2.10, we also include the bars and indicate once more the condition that any part of the structure must be in equilibrium.

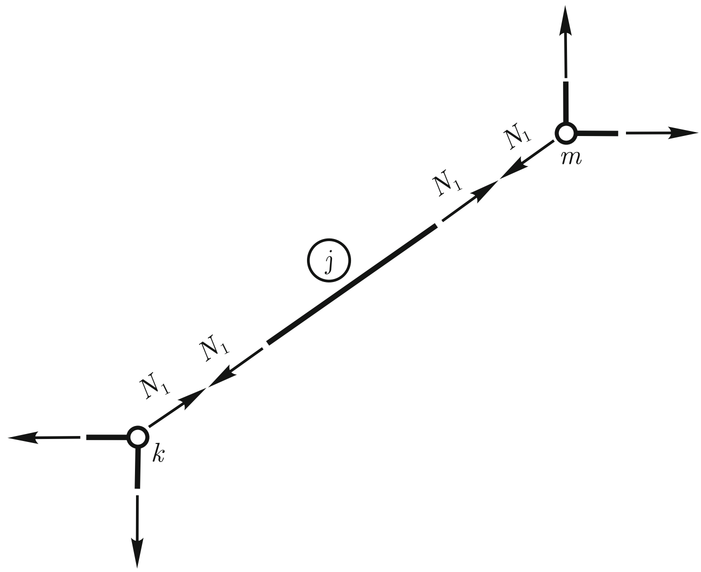


Fig. 2.8. Schematic representation of force in bar j and its action onto the end nodes

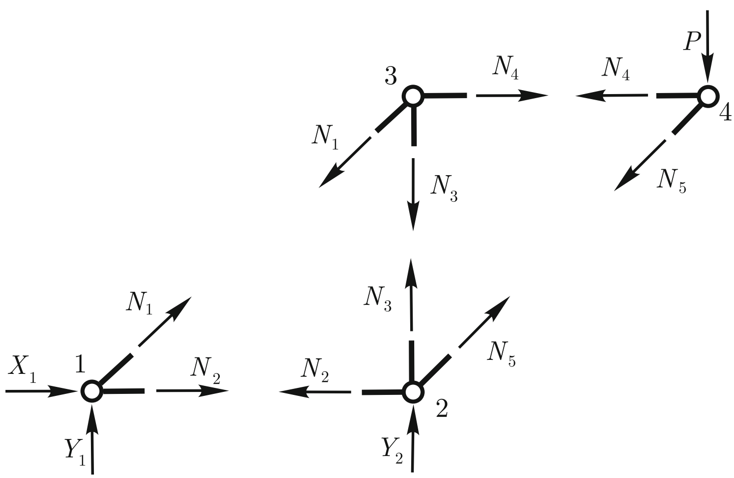


Fig. 2.9. Nodes of the truss structure considered as “free bodies”

Note that moment equilibrium is trivially satisfied for each joint since the lines of action of the forces pass through a point (the node). The condition $\mathbf{R} = \mathbf{0}$ implies for node 4

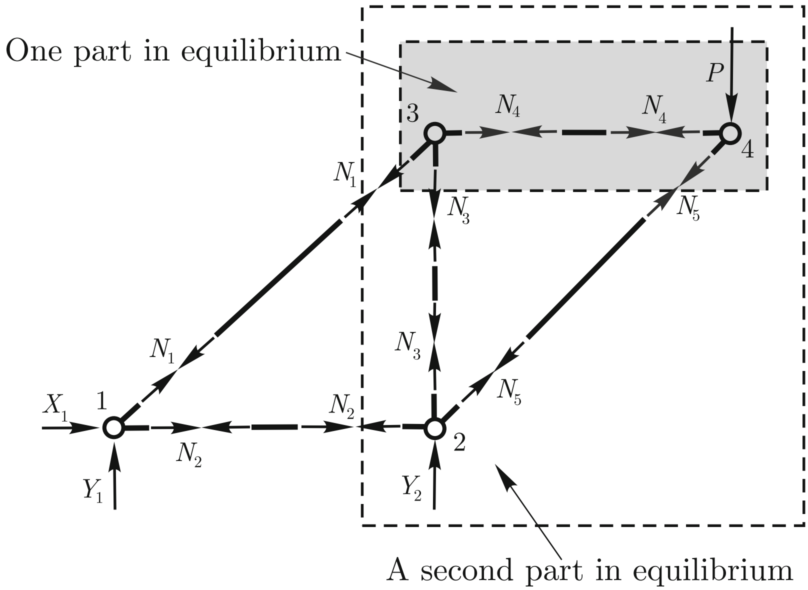


Fig. 2.10. Exploded view of bars and joints of the truss in Figure 2.4, and two typical parts that are in equilibrium

$$\sum F_X = 0, \qquad -N_4 - N_5\frac{\sqrt{2}}{2} = 0$$

$$\sum F_Y = 0, \qquad -P - N_5\frac{\sqrt{2}}{2} = 0$$

and hence

$$N_4 = P \quad \text{and} \quad N_5 = -P\sqrt{2}.$$

Consider next the equilibrium of node 3

$$\sum F_X = 0, \qquad N_4 - N_1\frac{\sqrt{2}}{2} = 0$$

$$\sum F_Y = 0, \qquad -N_3 - N_1\frac{\sqrt{2}}{2} = 0.$$

Using that $N_4 = P$ yields

$$N_1 = P\sqrt{2} \quad \text{and} \quad N_3 = -P.$$

We note that the equilibrium of bar 4 has already been implicitly taken into account. Next, let us impose the equilibrium of node 2

$$\begin{aligned}\sum F_X &= 0, \qquad N_5\frac{\sqrt{2}}{2} - N_2 = 0\\ \sum F_Y &= 0, \qquad Y_2 + N_5\frac{\sqrt{2}}{2} + N_3 = 0\end{aligned}$$

which yields

$$N_2 = -P \quad \text{and} \quad Y_2 = 2P.$$

Finally, considering node 1

$$\begin{aligned}\sum F_X &= 0, \qquad X_1 + N_1\frac{\sqrt{2}}{2} + N_2 = 0\\ \sum F_Y &= 0, \qquad Y_1 + N_1\frac{\sqrt{2}}{2} = 0\end{aligned}$$

which yields

$$X_1 = 0 \quad \text{and} \quad Y_1 = -P.$$

Note that the values of X_1, Y_1 and Y_2 are exactly the reactions already calculated by considering the global equilibrium of the complete structure (see Figure 2.6). This is the consequence of the important fact already mentioned but repeated now:

If equilibrium of each joint and each bar of a truss structure is satisfied, then also global equilibrium of any part the structure and hence of the complete structure is directly satisfied.

2.2.4 Constitutive behavior for a truss bar

Note that we did not consider so far the material of the truss bars; indeed the bars of the truss in Figure 2.4 could be of steel or wood, and the same forces would be transmitted. However, if – in general – we would like to evaluate the displacements of the nodes of any truss structure – an information of engineering interest – we need to characterize and quantify for each bar the relation between the internal force and the induced deformation. This is achieved by means of the constitutive behavior for each truss bar.

Consider a generic truss bar carrying the axial force N as shown in Figure 2.11.

The relevant quantity to characterize locally internal forces is the stress. For the one-dimensional truss bar the stress is constant over the section and normal to it. Therefore, it is given by $\tau = N/A$ as shown in Figure 2.12.

The material behavior of the truss bar is depicted in Figure 2.13. Here $\tau = E\varepsilon$ where ε is the strain

$$\varepsilon = \frac{\Delta\ell}{\ell}$$

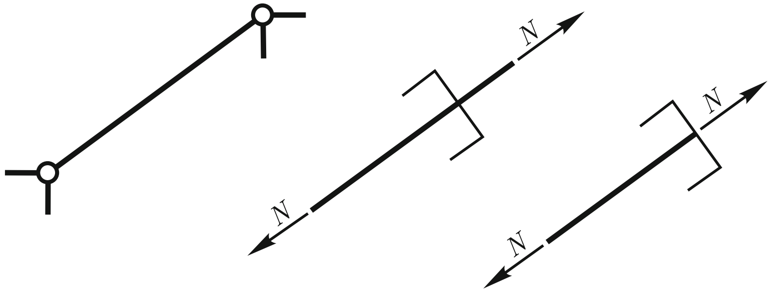


Fig. 2.11. Internal force in a generic truss bar; the force is the same at any section of the bar

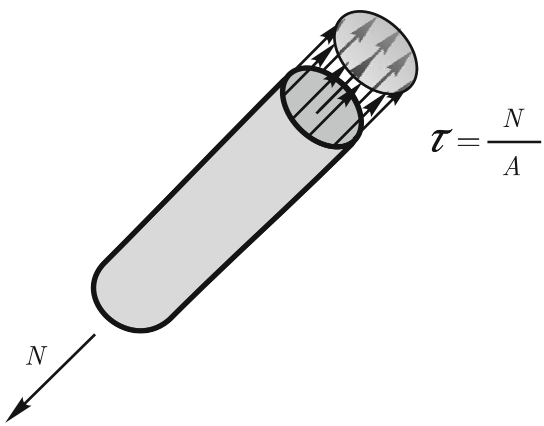


Fig. 2.12. Stress in a truss bar

and E is Young's modulus.

This stress-strain relationship is usually referred to as Hooke's law and materials with this stress-strain property are called "linear elastic". The "linear" refers to the fact that the stress is linearly proportional to the strain. The "elastic" means that the same (τ, ε) curve is followed for any loading or unloading causing an increase or a decrease in the stress/strain values. This property also means that for a given strain the stress value is unique and directly obtained from the (τ, ε) diagram.

Considering our truss model, if we assume that Hooke's law holds we can relate the axial force acting in a bar to the elongation of the bar, *i.e.*,

$$\tau = \frac{N}{A} = E\frac{\Delta\ell}{\ell} \qquad \Rightarrow \qquad N = \frac{EA}{\ell}\Delta\ell \quad \text{or} \quad \Delta\ell = N\frac{\ell}{EA}.$$

2.2.5 Compatibility conditions for a truss

So far we discussed the equilibrium requirements of a truss structure and the constitutive relation of the bars. Considering any truss structure, the

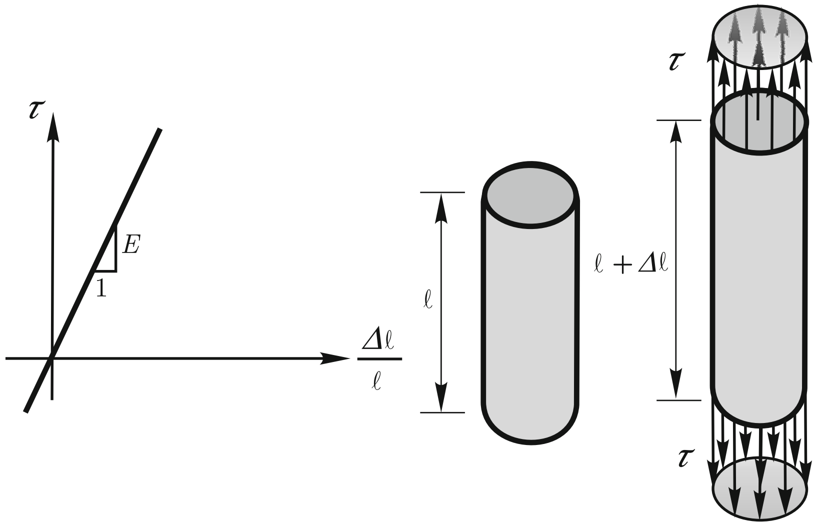


Fig. 2.13. Tension test data for a truss bar. Hooke's law

forces in the truss elements (bars) extend or shorten the bars and yet the bars must remain connected at the frictionless pins and some of the pins are restrained to move. The fact that the bars (the structural members) remain connected and the displacement boundary conditions need to be satisfied – for any externally applied loading – leads to the compatibility conditions: these conditions ensure that in any motion the structure "remains intact" (all elements stay connected) and the displacement boundary conditions are satisfied.

Considering again our truss structure in Figure 2.4, the change of length of each bar is given by

$$\Delta\ell_i = \frac{N_i}{E_i A_i}\ell_i. \tag{2.9}$$

Using the $\Delta\ell_i$, $i = 1, \cdots, 5$, we can now find the final positions of the nodes using the compatibility conditions: that the bars remain connected at the nodes and the structure satisfies the displacement boundary conditions.

We call this complete method of analysis the *elementary method* for analyzing truss structures: the determination of the internal forces of the bars and reactions as accomplished above and the evaluation of the nodal displacements using (2.9) and kinematics. The example below is an application of the elementary method.

Example 2.1

Use the elementary method to solve the truss problem of Figure 2.4.

Table 2.1. Data obtained for the truss in Figure 2.4 considering $a = 2m$, $E_i = 2.1 \times 10^{11} N/m^2$ (steel), $A_i = 1.439 \times 10^{-3} m^2$ for $i = 1, \cdots, 5$ and $P = 60kN$. Forces are given in Newtons and lengths in meters

Bar	N_i	$\Delta\ell_i = \frac{N_i}{E_i A_i}\ell_i$	$\varepsilon = \frac{\Delta\ell_i}{\ell_i}$	$\ell_i + \Delta\ell_i$
1	84853	7.942×10^{-4}	2.808×10^{-4}	2.8292
2	−60000	-3.971×10^{-4}	-1.986×10^{-4}	1.9996
3	−60000	-3.971×10^{-4}	-1.986×10^{-4}	1.9996
4	60000	3.971×10^{-4}	1.986×10^{-4}	2.0004
5	−84853	-7.942×10^{-4}	-2.808×10^{-4}	2.8276

Solution

For the evaluation of the forces of the bars and the reactions we refer to Section 2.2.3.

For the evaluation of the displacements, we summarize in Table 2.1 the data obtained by applying equation (2.9). This data is used for the calculation of the nodal positions of the deformed truss structure.

In Figure 2.14, we describe in four steps the determination of the deformed configuration of the truss. The change of length of the bars is magnified 300 times for visualization purposes. Note that the bars extend/shorten and freely rotate.

In Figure 2.14a, we show the final position of bar 2, which is obtained by introducing its shortening and taking into account the restraints. Hence nodes 1 and 2 are already in their final positions. We also show the stretching of bar 1.

In Figure 2.14b, we show the shortening of bar 3, which is shown in an intermediate position, considering the displacement of node 2 but not the rotation of bar 3. The final position of node 3 is also indicated, and it is obtained by the rotation of bars 1 and 3 around nodes 1 and 2, respectively.

In Figure 2.14c bars 1, 2 and 3 are in their final positions and bars 4 and 5 are shown in intermediate positions considering their extension/shortening and the displacements of nodes 2 and 3, but not the rotations of bars 4 and 5. We indicate how the final position of node 4 is obtained by the rotations of bars 4 and 5 around nodes 3 and 2, respectively.

Finally, in Figure 2.14d the deformed configuration of the complete truss structure is shown. With the steps detailed in Figure 2.14 and using the data of Table 2.1, it is possible to evaluate all nodal displacements of the truss.

□

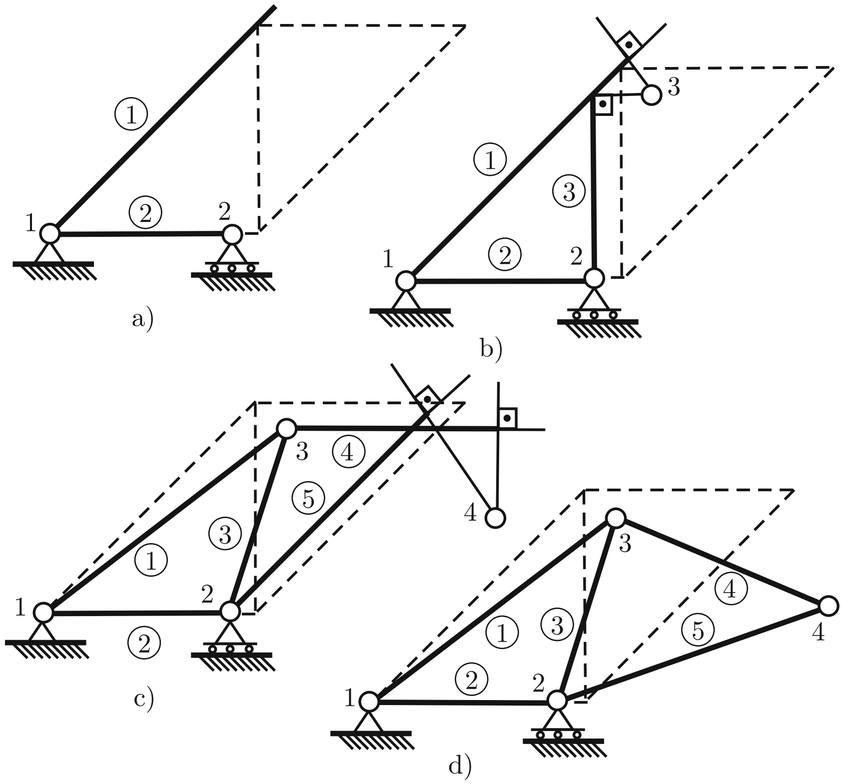


Fig. 2.14. Determination of the deformed configuration of a truss structure

Note that we used in Example 2.1 the important assumption of linear analysis: the displacements of the bars and their rotations are infinitesimally small.

In Figure 2.15a we schematically show a generic bar rotating about A for two conditions: large and infinitesimally small rotations. In Figure 2.15b, we detail the assumption of an infinitesimally small rotation. With θ assumed infinitesimally small, the displacement δ due to the rotation is assumed to take place at the 90 degree (right) angle to the bar. Also, the length of the deformed bar ℓ_d and the magnitude of the displacement δ are given by

$$\ell_d = \frac{\ell}{\cos\theta} \quad \text{and} \quad \delta = \ell_d \sin\theta.$$

which when θ is infinitesimally small (using $\cos\theta = 1$ and $\sin\theta = \theta$) leads to

$$\ell_d = \ell \quad \text{and} \quad \delta = \ell\theta.$$

Note that the bar does not change its length due to the rotation (and hence any force carried by the bar is not changed due to the rotation). For example,

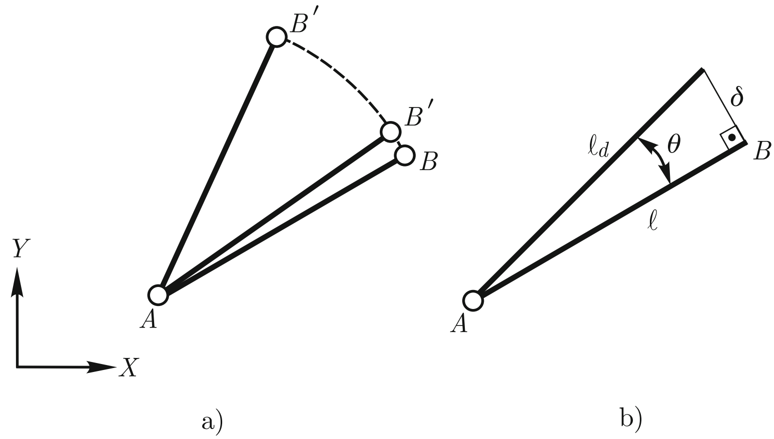


Fig. 2.15. Rotation of a generic truss bar. (a) Large rotation and (b) Infinitesimally small rotation

bars 1 and 3 when rotated about nodes 1 and 2, respectively, to meet at node 3 in the deformed configuration, see Figure 2.14b, do not change their lengths due to the rotations. We use this assumption throughout the book, except in Chapter 8.

Another important assumption due to considering that the displacements are infinitesimally small – already mentioned (see Section 2.1.5) – which we want to recall here once more is that the equilibrium conditions (for the bars, the joints, and any part of the truss) are established and satisfied in the original configuration of the structure. Hence, although the truss nodes and bars moved (see Figure 2.14 for the truss in Figure 2.4) the equilibrium conditions assume that these displacements are so small that they can be entirely neglected.

We finally note that as we use the linear model assumptions and solve a truss problem, as in Example 2.1, the calculated nodal displacements and bar rotations may not come out to be infinitesimally small. This fact is revealing that the solution of the linear model is only an approximation to the response of the actual physical problem – as the hierarchical modeling process emphasizes.

However, for actual engineering truss structures, the nodal displacements and bar rotations predicted by the linear model are mostly small as can be verified examining the numerical solution values and, hence, in most cases, the linear model is adequate for design purposes. If the linear model predicts large nodal displacements and bar rotations, then a nonlinear analysis may be necessary, see Chapter 8.

The objective of Example 2.1 was to present the elementary method for solving truss structures and to give insight into the use of constitutive relations and compatibility conditions to calculate the displacements of a truss structure. Of course, as the number of the bars increases – and there may be many bars in truss structures – the application of the above methodology becomes very cumbersome. We will see in Section 2.3, that the use of matrix methods leads to a much more efficient solution.

2.2.6 Statically determinate and indeterminate trusses

For the truss of Figure 2.4, the equilibrium conditions alone allowed us to determine the reactions and the forces in all bars. This kind of structure is termed statically determinate since the equilibrium conditions alone are sufficient to determine all bar forces and reactions.

However, this is not always the case. To understand when we can obtain the axial forces from equilibrium only and when not, we take a step back and consider two very simple truss structures.

Let us consider the truss structure shown in Figure 2.16a. Of course, this structure is properly supported.

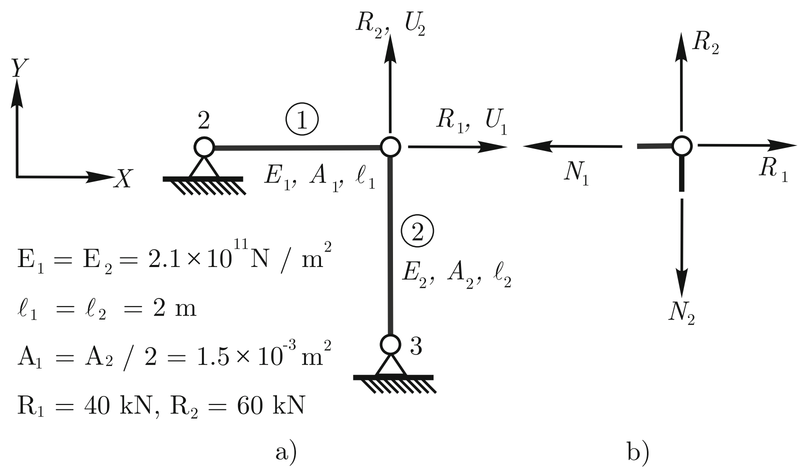


Fig. 2.16. a) Two-bar truss structure. R_1 and R_2 are concentrated applied loads and U_1 and U_2 are the node 1 displacements. Nodes and elements are numbered; b) Equilibrium of node 1 for the two bar truss

The structure can be solved by considering the equilibrium of node 1, as shown in Figure 2.16b, which leads to

$$\sum F_X = 0, \qquad R_1 - N_1 = 0 \qquad \Rightarrow \qquad N_1 = R_1$$

$$\sum F_Y = 0, \qquad R_2 - N_2 = 0 \qquad \Rightarrow \qquad N_2 = R_2.$$

The bar elongations are given by

$$\Delta\ell_1 = \frac{N_1}{E_1 A_1}\ell_1 = 2.53968\times 10^{-4} \text{ m}, \qquad \Delta\ell_2 = \frac{N_2}{E_2 A_2}\ell_2 = 1.90476\times 10^{-4} \text{ m}.$$

and

$$U_1 = \Delta\ell_1, \qquad U_2 = \Delta\ell_2.$$

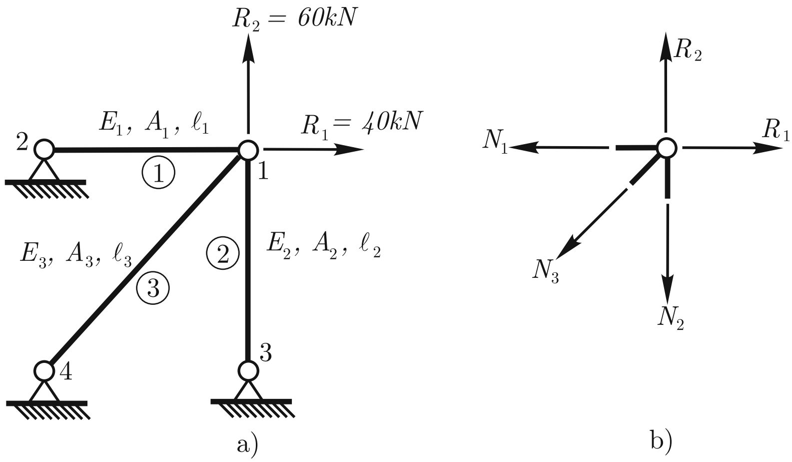


Fig. 2.17. a) Three-bar truss structure. The properties of bars 1 and 2 are as in Figure 2.16 and $E_3 = 2.1\times 10^{11}$ N/m^2, $A_3 = 3A_1$; b) Equilibrium of node 1 for the three-bar truss

Let us now add to the structure of Figure 2.16 an inclined bar as shown in Figure 2.17a. Obviously, this new structure is still properly supported. The equilibrium of node 1, as shown in Figure 2.17b, now yields

$$\sum F_X = 0, \qquad -N_1 - N_3\frac{\sqrt{2}}{2} + R_1 = 0$$

$$\sum F_Y = 0, \qquad -N_2 - N_3\frac{\sqrt{2}}{2} + R_2 = 0$$

which leads to

$$N_1 + N_3\frac{\sqrt{2}}{2} = R_1, \qquad N_2 + N_3\frac{\sqrt{2}}{2} = R_2. \tag{2.10}$$

Therefore there are infinitely many values of N_1, N_2 and N_3 that satisfy the equilibrium conditions. However, if we consider the actual structure (the physical problem) – which admits as a mathematical model the truss model of Figure 2.17a – we would, of course, be able to experimentally measure unique forces in the truss bars for given loads R_1 and R_2. It is easy to conclude that the equilibrium conditions in equation (2.10) alone are not sufficient to determine the truss internal forces and the reactions. The truss of Figure 2.17a is a simple example of a statically indeterminate structure: for such structures we also need to consider the constitutive relations of the materials used and the compatibility conditions to solve for the internal forces.

Suppose we choose values for N_1, N_2 and N_3 which satisfy (2.10). These values would then satisfy the equilibrium conditions. Of course, these values could then be used to evaluate the bar elongations $\Delta\ell_1$, $\Delta\ell_2$ and $\Delta\ell_3$. However, in general these elongations will not lead to a valid deformed configuration, *i.e.*, the ends of bars 1, 2, and 3 would not connect to a single point, the supposedly new position of node 1. Therefore, the compatibility condition that the bars remain connected at node 1 would be violated.

In Figure 2.18a, we show bar elongations that lead to a kinematically admissible configuration, *i.e.*, a compatible deformed configuration. Of course, there is a relation that should be satisfied by $\Delta\ell_1$, $\Delta\ell_2$ and $\Delta\ell_3$, namely the compatibility condition. In Figure 2.18b, we show the region around node 1 magnified and we can write

$$\tan\alpha = \frac{\Delta\ell_1 - \Delta\ell_3\cos\alpha}{\Delta\ell_3\sin\alpha - \Delta\ell_2}$$

which for $\alpha = 45^\circ$ leads to the compatibility condition

$$\Delta\ell_1 + \Delta\ell_2 = \Delta\ell_3\sqrt{2}. \tag{2.11}$$

Using the constitutive relations, equation (2.11) can be written in terms of the axial forces

$$\frac{N_1}{E_1A_1}\ell_1 + \frac{N_2}{E_2A_2}\ell_2 = \frac{N_3}{E_3A_3}\ell_3\sqrt{2}.$$

Introducing the data given in Figure 2.17a

$$N_1 + \frac{N_2}{2} - \frac{2N_3}{3} = 0. \tag{2.12}$$

Equations (2.10) and (2.12) contain the requirements of equilibrium, compatibility and material behavior. We can solve and obtain

$$N_1 = 11.34 \text{ kN}, \qquad N_2 = 31.34 \text{ kN}, \qquad N_3 = 40.52 \text{ kN}.$$

The nodal displacements are now given by

$$\begin{aligned} U_1 &= \Delta\ell_1 = \frac{N_1}{E_1A_1}\ell_1 = 7.203\times10^{-5}\text{ m} \\ U_2 &= \Delta\ell_2 = \frac{N_2}{E_2A_2}\ell_2 = 9.951\times10^{-5}\text{ m} \end{aligned}$$

Summarizing, we recognize that, since the three-bar truss structure is a statically indeterminate structure, we had to use the following conditions

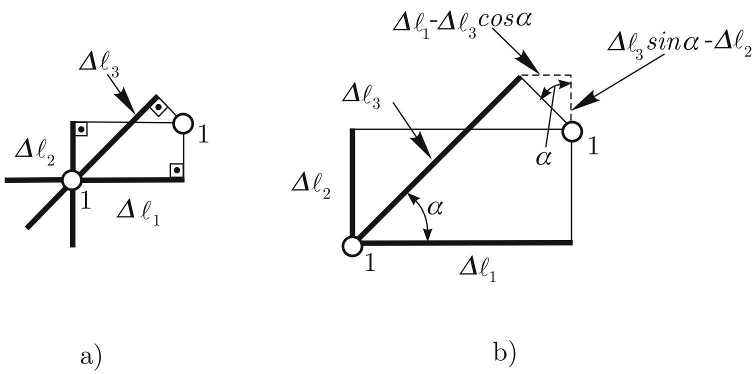


Fig. 2.18. Compatibility of displacements for node 1

- equilibrium
- compatibility
- constitutive

in order to solve for the internal forces in the structure. Once the internal forces have been calculated, the nodal displacement can be obtained as in the analysis of a statically determinate truss structure. These three conditions are the *fundamental conditions* that govern the behavior of every problem in structural mechanics.

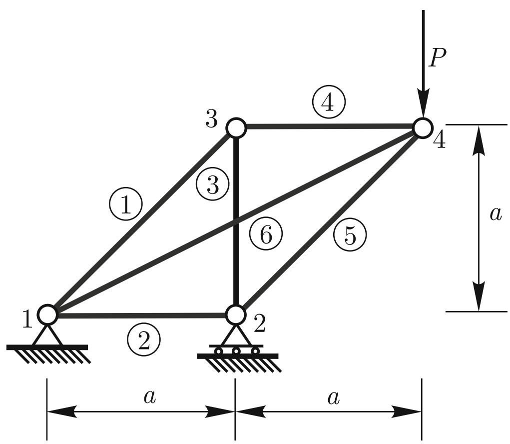


Fig. 2.19. New truss structure obtained by adding bar 6 to truss in Figure 2.4

In order to further elaborate on statically indeterminate structures, we show in Figure 2.19 the truss of Figure 2.4 with an extra bar linking nodes 1 and 4. The truss of Figure 2.19 is no longer statically determinate. Conceptually, the change from the statically determinate to the statically indeterminate structure is very similar to the change from the two-bar structure (Figure 2.16a) to the three-bar truss structure (Figure 2.17a). In fact, we can study the truss of Figure 2.19 as shown in Figure 2.20: the displacement of node 4 will depend on the value of N_6 and this bar force extends/shortens bar 6. The compatibility condition (that the bar 6 must fit into the distance between nodes 1 and 4 in the deformed geometry) can only be enforced by also using the constitutive relations of the bars.

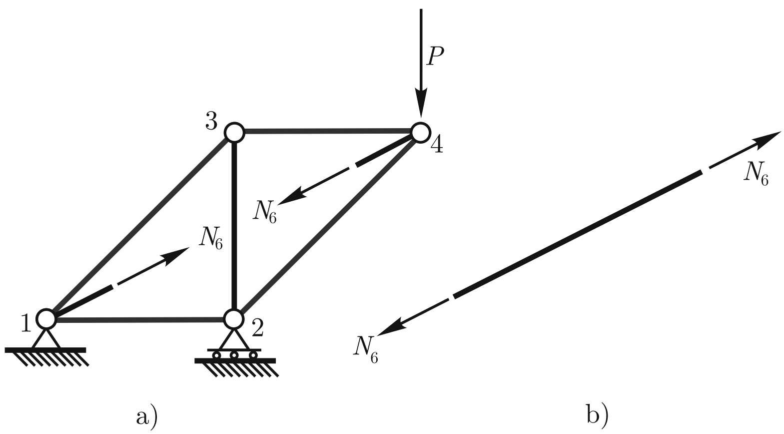


Fig. 2.20. Another representation of the previous truss

As for the three-bar truss in Figure 2.17, nodal equilibrium alone would not give the bar forces of the truss of Figure 2.19. We would again obtain a system of equations with one degree of indeterminacy.

This problem is solved later on in this chapter using the matrix method of analysis which provides a much more efficient solution procedure.

2.3 Matrix displacement method for trusses

In this section we introduce the matrix displacement method for planar truss structures. It is important to note that the concepts discussed in the context of trusses are also directly applicable to more complex structural analyses, like when considering frame structures. Therefore, the objective of this section is not only to present an efficient method for solving truss structures of arbitrary complexity but also to introduce the main concepts of matrix structural analysis.

We recall that the fundamental conditions of equilibrium, compatibility and constitutive behavior translate in the case of trusses, subjected to joint forces only, to:

- Every node should be in equilibrium considering the forces of the truss bars that connect to that node and, possibly, external forces applied directly to the node. Each bar is automatically in equilibrium as it only carries an axial (constant) force.
- The axial deformations of the truss bars must lead to a compatible deformation of the complete structure taking into account how the bars are linked to each other and to the supports. The joints (nodes) do not deform.

In the matrix formulation, the above conditions of equilibrium, constitutive behavior and compatibility are directly – and in a very elegant manner – enforced.

2.3.1 Truss bar stiffness matrix in its local system

We begin by establishing a relation between the end displacements and forces of a truss bar.

Let us use the convention given in Figure 2.21 for the end forces and displacements. The symbol $\sim$ over the quantities is used to show that a local coordinate system, aligned with the bar axis, is adopted. Then

$$\begin{aligned} \Delta\ell &= \tilde{u}_2 - \tilde{u}_1, \qquad N = EA\frac{\Delta\ell}{\ell} = \frac{EA}{\ell}(\tilde{u}_2 - \tilde{u}_1) \\ \tilde{f}_2 &= N, \qquad \tilde{f}_1 = -N \end{aligned}$$

with N positive when the bar is in tension.

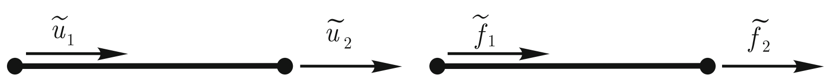


Fig. 2.21. Local end-displacements and forces acting onto a truss bar

The equations above can be written in matrix form as follows

$$\begin{bmatrix} \frac{EA}{\ell} & -\frac{EA}{\ell} \\ -\frac{EA}{\ell} & \frac{EA}{\ell} \end{bmatrix} \begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{bmatrix} = \begin{bmatrix} \tilde{f}_1 \\ \tilde{f}_2 \end{bmatrix}.$$

Let us define

$$\begin{aligned} \tilde{\mathbf{u}} &= \begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{bmatrix}, \qquad \tilde{\mathbf{f}} = \begin{bmatrix} \tilde{f}_1 \\ \tilde{f}_2 \end{bmatrix} \\ \tilde{\mathbf{k}} &= \begin{bmatrix} \tilde{k}_{11} & \tilde{k}_{12} \\ \tilde{k}_{21} & \tilde{k}_{22} \end{bmatrix} = \begin{bmatrix} \frac{EA}{\ell} & -\frac{EA}{\ell} \\ -\frac{EA}{\ell} & \frac{EA}{\ell} \end{bmatrix} \end{aligned}$$

where $\tilde{\mathbf{u}}$ is referred to as the column matrix of element nodal point displacements, $\tilde{\mathbf{f}}$ as the column matrix of element nodal point forces (acting onto the bar) and $\tilde{\mathbf{k}}$ as the element stiffness matrix. All quantities are referred to the local coordinate system. The term "element" is representing a truss bar and is also used in later chapters to represent other structural members.

It is instructive to interpret the physical meaning of the coefficients in the stiffness matrix. For that purpose, let us impose a unit displacement at the left end and restrain the displacement to be zero at the right end, *i.e.*, $\tilde{u}_1 = 1$ and $\tilde{u}_2 = 0$. Then

$$\begin{bmatrix} \tilde{k}_{11} & \tilde{k}_{12} \\ \tilde{k}_{21} & \tilde{k}_{22} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \tilde{f}_1 \\ \tilde{f}_2 \end{bmatrix}$$

leading to $\tilde{k}_{11} = \tilde{f}_1$ and $\tilde{k}_{21} = \tilde{f}_2$. In other words, the stiffness coefficient $\tilde{k}_{11} = \frac{EA}{\ell}$ is the force that must be applied in the displacement degree of freedom $\tilde{u}_1$ onto the bar to impose a unit displacement when $\tilde{u}_2 = 0$. The coefficient $\tilde{k}_{21} = -\frac{EA}{\ell}$ is the force (reaction) at the right end onto the bar, *i.e.*, the force of the restraint onto the bar. Of course, the interpretation of $\tilde{k}_{11}$ as a stiffness coefficient is now evident since it gives the magnitude of the force necessary to produce a unit displacement. An analogous interpretation can be given for $\tilde{k}_{12}$ and $\tilde{k}_{22}$ associated with imposing a unit displacement at the right end and fixing the left end. These results are summarized in Figure 2.22.

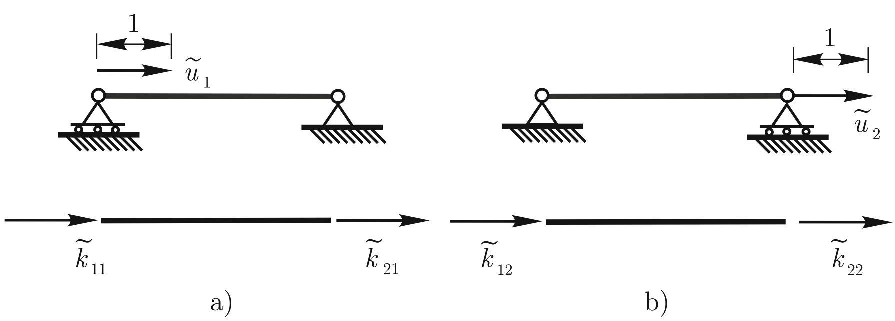


Fig. 2.22. Interpretation of the stiffness coefficients as forces applied *onto* the bar

Suppose now that we would like to solve the problem depicted in Figure 2.16 with the aid of the truss element stiffness matrix. This is a simple problem to demonstrate the matrix method of analysis.

Considering bar 1, we note that at its right end the node can displace not only along the axial direction but also along the transverse direction (the Y direction). In Figure 2.21, a nodal transverse displacement was not considered as a degree of freedom of the truss bar because there is no stiffness provided by the bar to such displacement. As a consequence, for example, if in Figure 2.22b there were no support at the right end, which prevents the transverse displacement, the bar could rigidly rotate about the left support. Of course, in the problem of Figure 2.16 the stiffness for the transverse displacement at the right end of bar 1 is provided by bar 2, which shares the node with bar 1, because for bar 2, such nodal displacement is along its axial direction.

Considering that a truss structure always consists of an assemblage of truss bars, we add, as shown in Figure 2.23, the transverse degrees of freedom for a generic truss element. Then the truss element stiffness matrix is given by

$$\tilde{\mathbf{k}} = \begin{bmatrix} \frac{EA}{\ell} & 0 & -\frac{EA}{\ell} & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{EA}{\ell} & 0 & \frac{EA}{\ell} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{2.13}$$

Fig. 2.23. Degrees of freedom of a truss bar

The physical meaning of the second and fourth rows and columns being zero is that there is no stiffness associated with the degrees of freedom given by $\tilde{u}_2$ and $\tilde{u}_4$. In fact, the equation

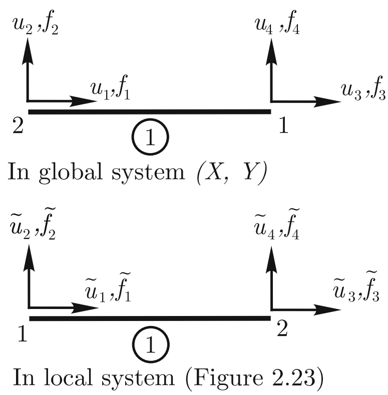
$$\tilde{\mathbf{k}}\tilde{\mathbf{u}} = \tilde{\mathbf{f}} \tag{2.14}$$

tells that for any values of $\tilde{u}_2$ and $\tilde{u}_4$ (which would amount to vertical displacement and rotation of the bar) there is no induced internal bar force and there are no induced nodal forces (see also Figure 2.15 and the corresponding discussion).

Of course, the stiffness matrix given in equation (2.13) is useful as long as the truss bar is part of an assemblage and the stiffness for the transverse degrees of freedom $\tilde{u}_2$ and $\tilde{u}_4$ is provided by other bars.

2.3.2 Solution of a two-bar truss structure using the matrix method

Returning to the problem of Figure 2.16, if we obtain the displacements of node 1, which is the only node with free degrees of freedom, the problem is solved. Let U_1 and U_2 be such displacements as shown in Figure 2.24 (the same notation for these displacements has been used before). Note that we use capital letters to denote that these displacements are degrees of freedom defined for the whole structural assemblage. In this way, we distinguish such degrees of freedom from the individual bar degrees of freedom for which we use lower case letters. Note also that U_1 and U_2 (the structure degrees of freedom) are here referred to the global coordinate system X, Y.



Bar	Global node	Local node
1	2	1
	1	2
2	3	1
	1	2

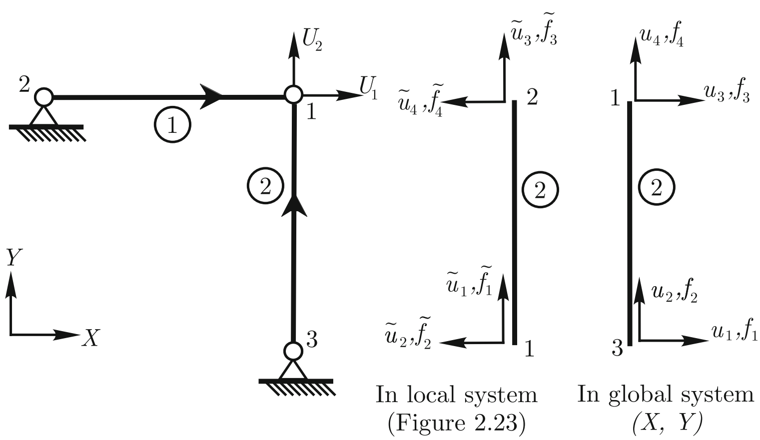


Fig. 2.24. Definitions for the two bar truss structure

In Figure 2.24, the degrees of freedom of bars 1 and 2 are also shown. The arrow on a bar axis defines the orientation of the bar and establishes a local (bar attached) numbering for the end nodes of the bar. The table included in Figure 2.24 shows, for each bar, the relation between the global node numbering (for the structure) and the local node numbering of the bar. We observe that the numbering of the displacements and forces of the bar starts always from the local node 1.

To solve the problem, we need to enforce equilibrium of the global node 1. We see that the end displacements and forces of the bars 1 and 2 (the quantities identified by a curl) at the global node 1 are not referred to a single coordinate system. Therefore, to facilitate the enforcement of nodal equilibrium, it is convenient to define the bar end displacements and forces in a common coordinate system which is chosen to be the global one. These nodal displacements and forces for bars 1 and 2 in the global system are also shown in Figure 2.24. The $\sim$ symbol over the lower case letters is dropped since we are considering a global system for these quantities.

The element stiffness matrices in the global system are, in general, different from those in the local system. Later on, in this section, we will derive a general expression which relates these stiffness matrices. However, for bars 1 and 2 in this problem we can easily obtain the global matrices as shown below. In fact, for bar 1 since the local and global systems are the same, we directly write

$$\mathbf{k}^{(1)} = \tilde{\mathbf{k}}^{(1)}$$

where again the $\mathbf{k}^{(1)}$ (without the $\sim$ symbol) indicates that we are using the global system, and therefore

$$\mathbf{k}^{(1)}\mathbf{u}^{(1)} = \mathbf{f}^{(1)}.$$

For bar 2, we have $\tilde{u}_1 = u_2$, $\tilde{u}_2 = -u_1$, $\tilde{u}_3 = u_4$ and $\tilde{u}_4 = -u_3$ with analogous relations for the forces. Using

$$\tilde{\mathbf{k}}^{(2)}\tilde{\mathbf{u}}^{(2)} = \tilde{\mathbf{f}}^{(2)}.$$

and the relations between the global and local quantities, we obtain

$$\begin{bmatrix} \frac{E_2A_2}{\ell_2} & 0 & -\frac{E_2A_2}{\ell_2} & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{E_2A_2}{\ell_2} & 0 & \frac{E_2A_2}{\ell_2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_2 \\ -u_1 \\ u_4 \\ -u_3 \end{bmatrix} = \begin{bmatrix} f_2 \\ -f_1 \\ f_4 \\ -f_3 \end{bmatrix}. \tag{2.15}$$

Re-ordering the equations leads to

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{E_2A_2}{\ell_2} & 0 & -\frac{E_2A_2}{\ell_2} \\ 0 & 0 & 0 & 0 \\ 0 & -\frac{E_2A_2}{\ell_2} & 0 & \frac{E_2A_2}{\ell_2} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix} \tag{2.16}$$

and, therefore, since $\mathbf{k}^{(2)}\mathbf{u}^{(2)} = \mathbf{f}^{(2)}$,

$$\mathbf{k}^{(2)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{E_2 A_2}{\ell_2} & 0 & -\frac{E_2 A_2}{\ell_2} \\ 0 & 0 & 0 & 0 \\ 0 & -\frac{E_2 A_2}{\ell_2} & 0 & \frac{E_2 A_2}{\ell_2} \end{bmatrix}.$$

It is important to note that the nodal forces are always defined in the same coordinate system as the nodal displacements and that the f_i are the forces acting *onto* the bar elements (just like the displacements are imposed *onto* the bar element).

We are now ready to enforce equilibrium at the global node 1. Referring to Figure 2.25, equilibrium in the X direction leads to

$$R_1 - \left(f_3^{(1)} + f_3^{(2)}\right) = 0 \qquad \Rightarrow \qquad R_1 = f_3^{(1)} + f_3^{(2)} \tag{2.17}$$

where we use the superscripts to identify the contributions from bar 1 and bar 2. However, the superscripts are used only when necessary. For example, in equation (2.15) the superscripts for u_i, f_i were not used since it is implicitly understood that we are working with bar 2. Equilibrium in the Y direction gives

$$R_2 - \left(f_4^{(1)} + f_4^{(2)}\right) = 0 \qquad \Rightarrow \qquad R_2 = f_4^{(1)} + f_4^{(2)}. \tag{2.18}$$

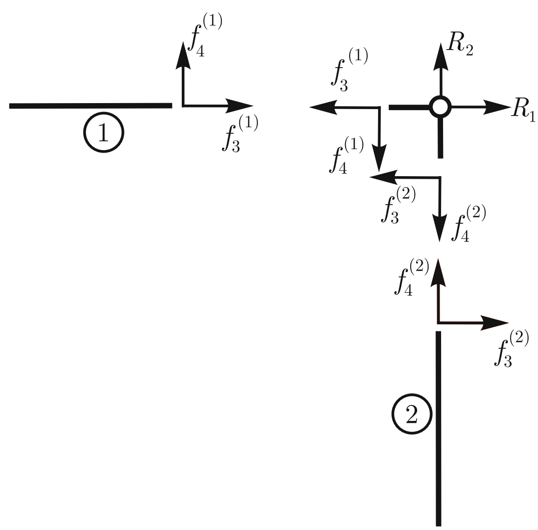


Fig. 2.25. Equilibrium of node 1, forces acting onto the bars and onto the node

Of course, since we have only an axial force in a truss bar

$$f_4^{(1)} = f_3^{(2)} = 0. \tag{2.19}$$

Introducing the stiffness relations and using equation (2.19), we can write (2.17) as

$$R_1 = k_{31}^{(1)} u_1^{(1)} + k_{33}^{(1)} u_3^{(1)}. \tag{2.20}$$

Considering that $u_1^{(1)} = 0$ (global node 2 is fixed) and the compatibility relation

$$U_1 = u_3^{(1)}$$

we arrive at

$$k_{33}^{(1)} U_1 = R_1. \tag{2.21}$$

Analogously, using (2.19) , equation (2.18) can be written as

$$R_2 = k_{42}^{(2)} u_2^{(2)} + k_{44}^{(2)} u_4^{(2)}. \tag{2.22}$$

Since node 3 is fixed $u_2^{(2)} = 0$ and using the compatibility relation

$$U_2 = u_4^{(2)}$$

we obtain

$$k_{44}^{(2)} U_2 = R_2. \tag{2.23}$$

Introducing the values of the stiffness coefficients, equations (2.21) and (2.23) can be written in matrix form as

$$\begin{bmatrix} \frac{E_1 A_1}{\ell_1} & 0 \\ 0 & \frac{E_2 A_2}{\ell_2} \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} R_1 \\ R_2 \end{bmatrix}. \tag{2.24}$$

Let

$$\mathbf{U} = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}, \qquad \mathbf{R} = \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} \tag{2.25}$$

where $\mathbf{U}$ is the column matrix of the free nodal degrees of freedom of the structure and $\mathbf{R}$ is the column matrix of the external nodal forces acting on the free degrees of freedom. We can write

$$\mathbf{KU} = \mathbf{R} \tag{2.26}$$

where $\mathbf{K}$, implicitly defined by (2.24) and (2.26), is the global stiffness matrix of the structure associated with the free degrees of freedom.

Note that the physical interpretations given earlier for stiffness coefficients also hold for those of the global **K** matrix. That is, the first column in **K** gives the external forces necessary to impose $U_1 = 1$ with $U_2 = 0$, and analogously the second column gives the external forces associated with $U_2 = 1$ with $U_1 = 0$. Due to the very simple nature of this problem, the stiffness matrix **K** could have been simply obtained in this way.

Solving (2.24) using the mechanical and geometrical properties of the bars leads to

$$U_1 = 2.53968 \times 10^{-4} \text{ m}, \qquad U_2 = 1.90476 \times 10^{-4} \text{ m} \tag{2.27}$$

which are the values obtained earlier.

Of course, since the bars are orthogonal there is no coupling between the vertical and horizontal displacements. This fact is reflected by the zero off-diagonal elements in the stiffness matrix.

Let us next consider the solution of the problem defined in Figure 2.17; that is, when an inclined bar is added to the structure. If we had the stiffness matrix of element 3 in the global coordinate system we could directly enter its contributions to the equilibrium of node 1. Therefore, we need to derive the stiffness matrix of a bar arbitrarily oriented in the global coordinate system.

2.3.3 Stiffness matrix of an arbitrarily oriented truss element

The degrees of freedom of an arbitrarily oriented truss element are summarized in Figure 2.26.

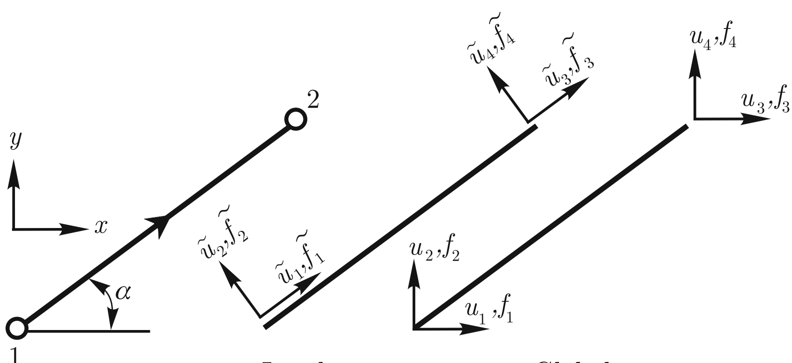


Fig. 2.26. Local and global degrees of freedom of an arbitrarily oriented truss element

We would like to obtain the matrix **k** such that

$$\mathbf{ku} = \mathbf{f}$$

where $\mathbf{u}$ and $\mathbf{f}$ are as in (2.16). Of course, we have already derived $\tilde{\mathbf{k}}$ such that, see (2.13),

$$\tilde{\mathbf{k}}\tilde{\mathbf{u}} = \tilde{\mathbf{f}}. \tag{2.28}$$

Before deriving $\mathbf{k}$ based on a transformation matrix, let us show how $\mathbf{k}$ could be constructed column by column imposing unit displacements.

As an example, we obtain the first column by imposing a unit displacement $u_1 = 1$ and fixing the remaining degrees of freedom, *i.e.*, $u_2 = u_3 = u_4 = 0$. We know that under such conditions $k_{11} = f_1$, $k_{21} = f_2$, $k_{31} = f_3$ and $k_{41} = f_4$.

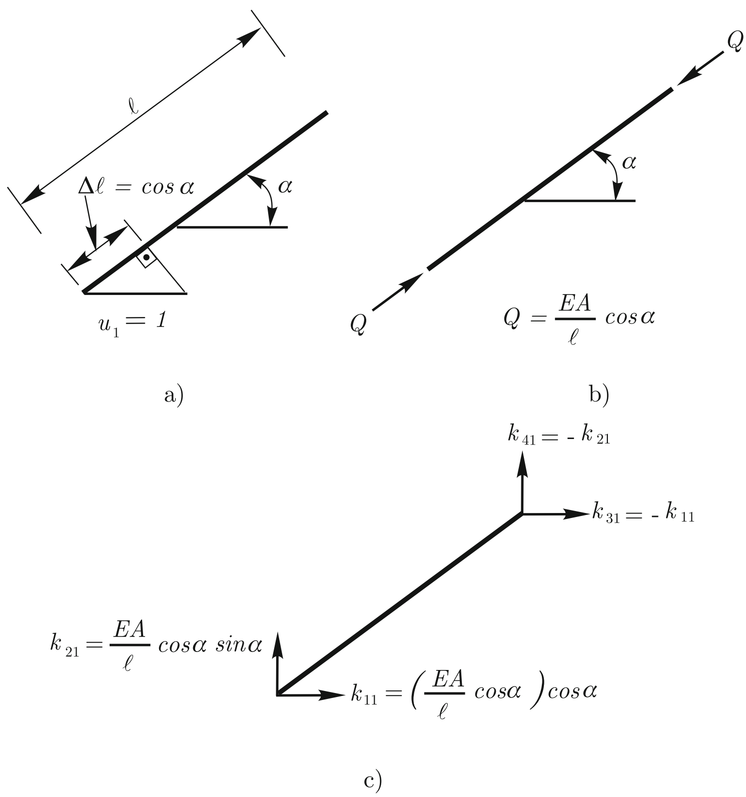


Fig. 2.27. Imposed horizontal unit displacement and corresponding forces. a) Imposed displacement and corresponding shortening of bar; b) Resulting force Q acting onto the bar; c) Nodal forces (stiffness coefficients) corresponding to (replacing) force Q

Referring to Figure 2.27, we have

$$
\begin{aligned}
\Delta\ell &= 1 \cdot \cos\alpha \\
Q &= EA\frac{\Delta\ell}{\ell} = \frac{EA}{\ell}\cos\alpha .
\end{aligned}
$$

Then, because the stiffness coefficients are forces into directions u_1, u_2, u_3 and u_4, and applied *onto* the element

$$
\begin{aligned}
k_{11} &= f_1 = Q\cos\alpha = \frac{EA}{\ell}\cos^2\alpha \\
k_{21} &= f_2 = Q\sin\alpha = \frac{EA}{\ell}\cos\alpha\sin\alpha \\
k_{31} &= f_3 = -Q\cos\alpha = -\frac{EA}{\ell}\cos^2\alpha \\
k_{41} &= f_4 = -Q\sin\alpha = -\frac{EA}{\ell}\cos\alpha\sin\alpha .
\end{aligned}
\tag{2.29}
$$

Proceeding in an analogous way, we could construct the remaining columns. Of course, the bar axial force $N = -Q$.

However, a more effective procedure to obtain $\mathbf{k}$ is to use transformation matrices, where $\tilde{\mathbf{u}}$ and $\tilde{\mathbf{f}}$ in (2.28) are expressed in terms of $\mathbf{u}$ and $\mathbf{f}$, respectively.

The kinematic relation between the displacements at node 1 measured in the local $(\tilde{x}, \tilde{y})$ and global (x, y) systems is, see Figures 2.28 and 2.29,

$$
\begin{aligned}
\tilde{u}_1 &= u_1\cos\alpha + u_2\sin\alpha \\
\tilde{u}_2 &= -u_1\sin\alpha + u_2\cos\alpha .
\end{aligned}
$$

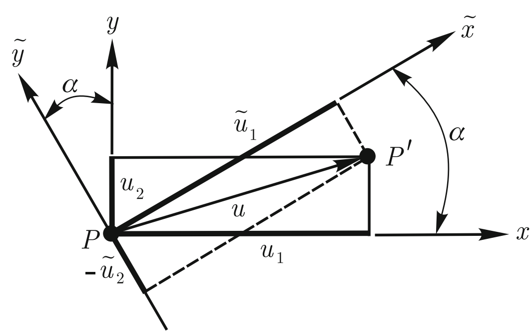


Fig. 2.28. Nodal point displacement vector $\mathbf{u}$ of local node 1 expressed in (x, y) and $(\tilde{x}, \tilde{y})$ coordinate systems

These relations can be written in matrix form

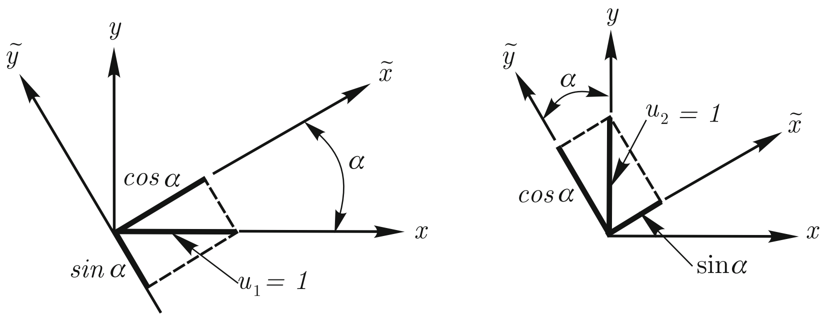


Fig. 2.29. Displacements $u_1 = 1$ and $u_2 = 1$ expressed in $(\tilde{x}, \tilde{y})$ coordinate system

$$\begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

For the displacements of the local node 2, the same kind of relationship holds

$$\begin{bmatrix} \tilde{u}_3 \\ \tilde{u}_4 \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} u_3 \\ u_4 \end{bmatrix}$$

which allows us to write

$$\tilde{\mathbf{u}} = \mathbf{T}\mathbf{u} \tag{2.30}$$

where

$$\mathbf{T} = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 & 0 \\ -\sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & \cos\alpha & \sin\alpha \\ 0 & 0 & -\sin\alpha & \cos\alpha \end{bmatrix}.$$

Since we transformed vector components, the same relation holds for the forces

$$\tilde{\mathbf{f}} = \mathbf{T}\mathbf{f}. \tag{2.31}$$

It is easy to verify that $\mathbf{T}$ is an orthogonal matrix, *i.e.*,

$$\mathbf{T}^{-1} = \mathbf{T}^T, \qquad \mathbf{T}^T\mathbf{T} = \mathbf{I}.$$

Now substituting for $\tilde{\mathbf{u}}$ and $\tilde{\mathbf{f}}$ in (2.28) yields

$$\mathbf{T}\mathbf{f} = \tilde{\mathbf{k}}\mathbf{T}\mathbf{u}$$

and left multiplying both sides by $\mathbf{T}^T$

$$\mathbf{T}^T\mathbf{T}\mathbf{f} = \mathbf{T}^\mathbf{T}\tilde{\mathbf{k}}\mathbf{T}\mathbf{u}$$

we obtain

$$\mathbf{f} = \left(\mathbf{T}^\mathbf{T}\tilde{\mathbf{k}}\mathbf{T}\right)\mathbf{u}$$

yielding

$$\mathbf{k} = \mathbf{T}^\mathbf{T}\tilde{\mathbf{k}}\mathbf{T}. \tag{2.32}$$

Performing the matrix multiplications we arrive at

$$\mathbf{k} = \frac{EA}{\ell}\begin{bmatrix} \cos^2\alpha & \sin\alpha\cos\alpha & -\cos^2\alpha & -\sin\alpha\cos\alpha \\ \sin\alpha\cos\alpha & \sin^2\alpha & -\sin\alpha\cos\alpha & -\sin^2\alpha \\ -\cos^2\alpha & -\sin\alpha\cos\alpha & \cos^2\alpha & \sin\alpha\cos\alpha \\ -\sin\alpha\cos\alpha & -\sin^2\alpha & \sin\alpha\cos\alpha & \sin^2\alpha \end{bmatrix}. \tag{2.33}$$

Note that the first column in (2.33) corresponds to the results given in (2.29).

2.3.4 Solution of the three-bar truss structure using the matrix method

We can now efficiently solve the problem described in Figure 2.17. Using relation (2.32) we evaluate the stiffness matrix of element 3, $\mathbf{k}^{(3)}$, choosing node 4 as its initial node and impose the equilibrium of node 1. Equilibrium in the X direction gives

$$R_1 - \left(f_3^{(1)} + f_3^{(3)}\right) = 0 \tag{2.34}$$

where we used that $f_3^{(2)} = 0$. Equilibrium in the Y direction gives

$$R_2 - \left(f_4^{(2)} + f_4^{(3)}\right) = 0 \tag{2.35}$$

where we used that $f_4^{(1)} = 0$. Introducing the stiffness relations into (2.34) and (2.35) leads to

$$k_{33}^{(1)}u_3^{(1)} + k_{33}^{(3)}u_3^{(3)} + k_{34}^{(3)}u_4^{(3)} = R_1 \tag{2.36}$$

$$k_{44}^{(2)}u_4^{(2)} + k_{43}^{(3)}u_3^{(3)} + k_{44}^{(3)}u_4^{(3)} = R_2. \tag{2.37}$$

We note that in the above equations the stiffness coefficients $k_{34}^{(1)}$, $k_{43}^{(2)}$ are not included since, as bar 1 is horizontal and bar 2 is vertical, these coefficients are zero. Of course, for bar 3 these terms are not zero.

Introducing the compatibility relations

$$U_1 = u_3^{(1)} = u_3^{(3)}, \qquad U_2 = u_4^{(2)} = u_4^{(3)}$$

we can re-write equations (2.36) and (2.37) as

$$\begin{aligned}
\left(k_{33}^{(1)} + k_{33}^{(3)}\right) U_1 + k_{34}^{(3)} U_2 &= R_1 \\
k_{43}^{(3)} U_1 + \left(k_{44}^{(2)} + k_{44}^{(3)}\right) U_2 &= R_2.
\end{aligned}$$

The global stiffness coefficients are implicitly defined in the above equations and they are given by

$$\begin{aligned}
K_{11} &= k_{33}^{(1)} + k_{33}^{(3)}, \qquad K_{12} = k_{34}^{(3)} \\
K_{22} &= k_{44}^{(2)} + k_{44}^{(3)}, \qquad K_{21} = k_{43}^{(3)}.
\end{aligned} \tag{2.38}$$

Therefore the matrix equation for the structural assemblage is

$$\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} \tag{2.39}$$

and its numerical solution is given by

$$U_1 = 7.203 \times 10^{-5} \text{ m}, \qquad U_2 = 9.951 \times 10^{-5} \text{ m}.$$

We note that:

- The off-diagonal stiffness coefficients are now different from zero due to the inclined bar which couples the horizontal and vertical displacements U_1 and U_2.
- The structure stiffness matrix $\mathbf{K}$ is obtained by summing the appropriate stiffness coefficients of the bar elements. This assemblage process is a direct consequence of imposing the equilibrium and compatibility conditions at the nodes (joints).
- The equilibrium, compatibility and constitutive conditions for the bar elements are satisfied by use of the (correct) element stiffness matrices.
- The structure stiffness matrix is established for the bars in their original configuration, *i.e.*, the joint displacements which are caused by the applied loading do not enter $\mathbf{K}$.
- We have obtained U_1 and U_2 from (2.39) which completely characterize the solution of this statically indeterminate structure. Therefore the matrix method of solution gives directly the solution of statically indeterminate (and statically determinate) structures by enforcing all equilibrium, compatibility and constitutive conditions simultaneously.

- The displacements U_1 and U_2 have decreased due to adding the diagonal bar. This, of course, makes sense physically.

Let us further explore the three-bar truss structure problem. We formulated this problem only in terms of the free degrees of freedom U_1 and U_2. We could have also included the degrees of freedom at the supports which are shown in Figure 2.30.

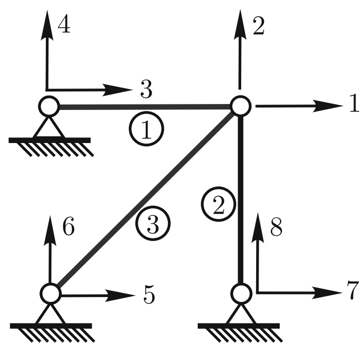


Fig. 2.30. Three-bar truss structure with all degrees of freedom shown

We observe that we have numbered all degrees of freedom. It is implied that for each degree of freedom i there is a displacement U_i and a force R_i. For the free degrees of freedom, the force is specified and the displacement is to be determined. For a restrained degree of freedom, the displacement is specified and the reaction force associated with such restraint is to be determined. The fixed conditions at the supports in this case imply, of course, that

$$U_3 = U_4 = U_5 = U_6 = U_7 = U_8 = 0.$$

Therefore, once the free degrees of freedom have been calculated, *i.e.*, when equation (2.39) has been solved, all the end bar displacements are known and all the bar end forces can be readily obtained by using the element stiffness matrices; that is, for bar (m)

$$\mathbf{f}^{(m)} = \mathbf{k}^{(m)}\mathbf{u}^{(m)}. \tag{2.40}$$

Consider bar 1, we have

$$U_1 = u_3^{(1)}, \qquad U_2 = u_4^{(1)}, \qquad U_3 = u_1^{(1)}, \qquad U_4 = u_2^{(1)}.$$

Equation (2.40) applied to bar 1 leads to

$$\mathbf{f}^{(1)} = \frac{E_1A_1}{\ell_1}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ U_1 \\ U_2 \end{bmatrix}$$

which yields

$$\mathbf{f}^{(1)} = \begin{bmatrix} -11344 \\ 0 \\ 11344 \\ 0 \end{bmatrix}.$$

The reactions R_3 and R_4 can be evaluated using that node 2 is in equilibrium. Hence

$$R_3 = f_1^{(1)} = -11344\ \text{N}, \qquad R_4 = f_2^{(1)} = 0.$$

In an analogous manner, for bar 2

$$u_3^{(2)} = U_1, \qquad u_4^{(2)} = U_2, \qquad u_1^{(2)} = U_7, \qquad u_2^{(2)} = U_8.$$

$$\mathbf{f}^{(2)} = \frac{E_2A_2}{\ell_2}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ U_1 \\ U_2 \end{bmatrix}, \qquad \mathbf{f}^{(2)} = \begin{bmatrix} 0 \\ -31344 \\ 0 \\ 31344 \end{bmatrix}.$$

Hence

$$R_7 = f_1^{(2)} = 0, \qquad R_8 = f_2^{(2)} = -31344\ \text{N}.$$

And for bar 3

$$u_3^{(3)} = U_1, \qquad u_4^{(3)} = U_2, \qquad u_1^{(3)} = U_5, \qquad u_2^{(3)} = U_6.$$

$$\mathbf{f}^{(3)} = \frac{E_3A_3}{\ell_3}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ U_1 \\ U_2 \end{bmatrix}, \qquad \mathbf{f}^{(3)} = \begin{bmatrix} -28655 \\ -28655 \\ 28655 \\ 28655 \end{bmatrix}.$$

Hence

$$R_5 = f_1^{(3)} = -28655 \text{ N}, \qquad R_6 = f_2^{(3)} = -28655 \text{ N}.$$

Considering the three-bar truss structure, we observe that, in general, the displacements at the supports could be prescribed to have values different from zero. To consider this coupling explicitly, we evaluate the stiffness matrix corresponding to all degrees of freedom of the structure.

$$\begin{bmatrix} k_{33}^{(1)} + k_{33}^{(3)} & k_{34}^{(3)} & k_{31}^{(1)} & 0 & k_{31}^{(3)} & k_{32}^{(3)} & 0 & 0 \\ k_{43}^{(3)} & k_{44}^{(2)} + k_{44}^{(3)} & 0 & 0 & k_{41}^{(3)} & k_{42}^{(3)} & 0 & k_{42}^{(2)} \\ k_{13}^{(1)} & 0 & k_{11}^{(1)} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ k_{13}^{(3)} & k_{14}^{(3)} & 0 & 0 & k_{11}^{(3)} & k_{12}^{(3)} & 0 & 0 \\ k_{23}^{(3)} & k_{24}^{(3)} & 0 & 0 & k_{21}^{(3)} & k_{22}^{(3)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & k_{24}^{(2)} & 0 & 0 & 0 & 0 & 0 & k_{22}^{(2)} \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \\ U_7 \\ U_8 \end{bmatrix} = \begin{bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \\ R_5 \\ R_6 \\ R_7 \\ R_8 \end{bmatrix} \tag{2.41}$$

The additional stiffness coefficients – besides K_{11}, K_{12}, K_{21}, K_{22} which are already given in (2.38) – can be obtained by also considering the equilibrium and compatibility conditions of nodes 2 to 4, in the same way as given above for node 1. These considerations lead to adding the element stiffness matrices into the global structure stiffness matrix following the correspondence between the numbering of structure global and element local degrees of freedom. The non-zero contributions from each element can be identified in the matrix of (2.41).

The equations in (2.41) are then solved by specifying all known nodal displacements and the known externally applied nodal forces. The reactions can then be directly obtained by simply evaluating the left-hand side of (2.41).

We next develop this procedure in detail for general truss structures – and indeed for any other structural element assemblage.

2.3.5 Systematization of the matrix formulation for truss structures

We show in Figure 2.31 a part of a generic truss structure. In the matrix formulation of a truss problem, all bars, all nodes and all degrees of freedom are numbered, and the bar orientations are chosen. We select node g as a representative truss node to which we impose equilibrium.

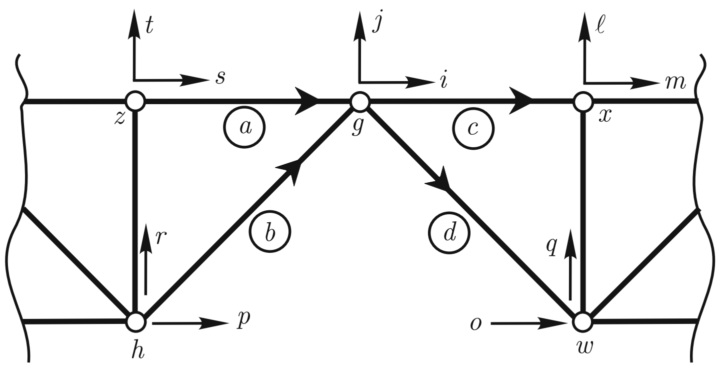


Fig. 2.31. Part of a truss structure

Referring to Figure 2.32, we can write for the node g

$$R_i = f_3^{(a)} + f_3^{(b)} + f_1^{(c)} + f_1^{(d)}, \qquad R_j = f_4^{(a)} + f_4^{(b)} + f_2^{(c)} + f_2^{(d)} \tag{2.42}$$

where we recall that the element end forces are applied onto the elements.

Hence, equations (2.42) reflect the fact that the external loads acting on a node are equilibrated by the sum of the bar end forces that connect to this node.

In order to facilitate the accounting of bar local and structure global numbering of degrees of freedom and the force summation process, we define for a generic bar (m) a $N \times 1$ column matrix $\mathbf{F}^{(m)}$ where N is the *total* number of degrees of freedom of the structure. The nodal forces of bar (m) are placed in $\mathbf{F}^{(m)}$ at the positions corresponding to the global numbering of the bar degrees of freedom. The remaining positions in $\mathbf{F}^{(m)}$ are each filled with 0. For example, for bar (b)

$$\mathbf{F}^{(b)^T} = [0 \cdots 0 \; \overset{i}{f_3^{(b)}} \; \overset{j}{f_4^{(b)}} \; 0 \cdots \overset{p}{f_1^{(b)}} \; 0 \cdots 0 \; \overset{r}{f_2^{(b)}} \; 0 \cdots 0]_{1\times N} \, .$$

With $\mathbf{F}^{(m)}$ given for every bar of the bar assemblage we can write the equilibrium equations for *every* degree of freedom as

$$\mathbf{R} = \sum_{m=1}^{n_e} \mathbf{F}^{(m)} \tag{2.43}$$

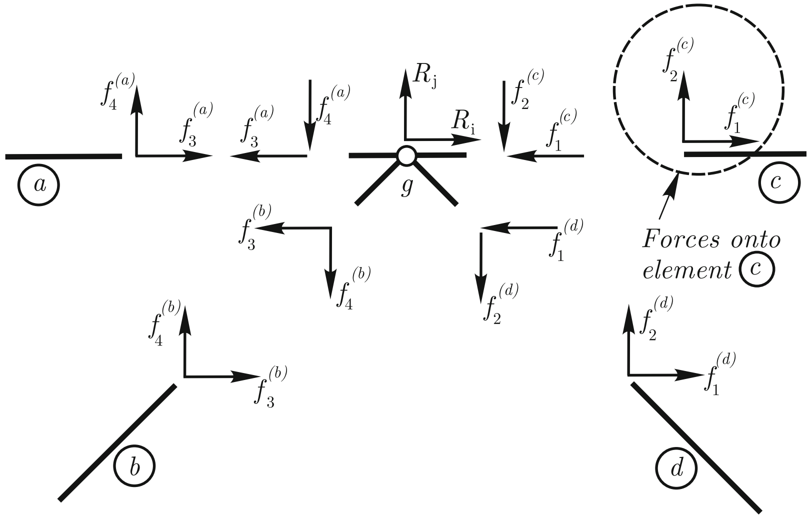


Fig. 2.32. Pictorial representation of equilibrium of node g

where $\mathbf{R}$ is the column matrix of external forces applied to the nodes of the truss corresponding to the global structural degrees of freedom and n_e is the total number of bar elements of the structure. Considering, for example, the degree of freedom i, we have

$$R_i = F_i^{(a)} + F_i^{(b)} + F_i^{(c)} + F_i^{(d)} \tag{2.44}$$

since (a), (b), (c) and (d) are the only bars which have end forces corresponding to the global degree of freedom i (that is, the i^{th} entries of $\mathbf{F}^{(m)}$ are zero for all other bars $(m \neq a, b, c, d)$). Referring to Figure 2.32 we can write

$$F_i^{(a)} = f_3^{(a)}, \qquad F_i^{(b)} = f_3^{(b)}, \qquad F_i^{(c)} = f_1^{(c)}, \qquad F_i^{(d)} = f_1^{(d)}$$

and, hence, equation (2.44) is just the first equation of (2.42).

Now let us define a $N \times N$ matrix denoted by $\mathbf{K}^{(m)}$ such that

$$\mathbf{K}^{(m)}\mathbf{U} = \mathbf{F}^{(m)} \tag{2.45}$$

where $\mathbf{U}$ is the column matrix of the global displacement degrees of freedom and $\mathbf{K}^{(m)}$ has non-zero entries only at the positions associated with the nodal displacements of bar (m) when these nodal displacements are numbered according to the global ordering. Equation (2.45) contains, considering the relations between global and local numbering of degrees of freedom, the equations given by

$$\mathbf{k}^{(m)}\mathbf{u}^{(m)} = \mathbf{f}^{(m)} \tag{2.46}$$

and all remaining equations in (2.45) result into $0 = 0$ identities. Hence, all non-zero coefficients of $\mathbf{K}^{(m)}$ can be obtained from the coefficients of $\mathbf{k}^{(m)}$. For example, for bar (b), we have

$$K_{pp}^{(b)} = k_{11}^{(b)}, \quad K_{rr}^{(b)} = k_{22}^{(b)}, \quad K_{pr}^{(b)} = k_{12}^{(b)}, \quad K_{pi}^{(b)} = k_{13}^{(b)}, \quad K_{pj}^{(b)} = k_{14}^{(b)}$$

and so on.

Note that as we use equation (2.45) to reproduce (2.46) , we are implicitly enforcing compatibility since the bar end displacements are taken to be the global displacements.

Now we are ready to present the following important derivation. Substituting (2.45) into (2.43) yields

$$\mathbf{R} = \left(\sum_{m=1}^{n_e} \mathbf{K}^{(m)} \right) \mathbf{U} \tag{2.47}$$

leading to

$$\mathbf{KU} = \mathbf{R} \tag{2.48}$$

and, of course,

$$\mathbf{K} = \sum_{m=1}^{n_e} \mathbf{K}^{(m)} \tag{2.49}$$

is the stiffness matrix of the total structure obtained from the element stiffness matrices.

We observe that equation (2.48) represents the matrix formulation for a generic truss structure and this equation contains all three fundamental requirements:

- Equilibrium
- Compatibility
- Constitutive

"Equilibrium" because each truss element is always in equilibrium for any force it carries and (2.43) enforces the equilibrium of the nodes. "Compatibility" because the bars are connected to the joints which are undergoing unique displacements (some of which are imposed as displacement boundary conditions). "Constitutive" because the correct Young's modulus E is used for each element. Hence, once $\mathbf{U}$ has been calculated from (2.48) , the truss problem has been solved.

The matrices $\mathbf{K}^{(m)}$, $\mathbf{F}^{(m)}$ were defined because they are very useful to present the above theoretical derivations in a rigorous and elegant manner. However, most entries of the matrices $\mathbf{K}^{(m)}$and $\mathbf{F}^{(m)}$ are zero and, in actual computations, we need to take advantage of this fact.

Let us briefly describe an efficient computational procedure to obtain $\mathbf{K}$. Since all non-zero entries of $\mathbf{K}^{(m)}$ are in $\mathbf{k}^{(m)}$, $\mathbf{K}$ can be obtained without constructing the $\mathbf{K}^{(m)}$. We define for every bar element of the truss the following row matrix

$$\mathbf{LM}^{(m)} = \begin{array}{c} \begin{matrix} u_1 & u_2 & u_3 & u_4 \end{matrix} \\ \begin{bmatrix} \ell & p & q & r \end{bmatrix} \end{array} \tag{2.50}$$

which is referred to as the element connectivity array. The number ℓ in the first entry is the number of the degree of freedom of the *structure* which corresponds to the displacement u_1 of bar (m). Analogously, p, q and r correspond to u_2, u_3 and u_4. For example, the element arrays for bars (a), (b), (c) and (d) defined in Figure 2.31 are

$$\mathbf{LM}^{(a)} = \begin{bmatrix} s & t & i & j \end{bmatrix}, \qquad \mathbf{LM}^{(b)} = \begin{bmatrix} p & r & i & j \end{bmatrix}$$

$$\mathbf{LM}^{(c)} = \begin{bmatrix} i & j & m & \ell \end{bmatrix}, \qquad \mathbf{LM}^{(d)} = \begin{bmatrix} i & j & o & q \end{bmatrix}.$$

The assemblage process implied by the summation sign in equation (2.49) can be effectively performed starting with an array of an empty $N \times N$ matrix (each entry in the matrix is initially zero) which eventually will contain $\mathbf{K}$. For every bar element in the structural assemblage, $m = 1, \cdots, n_e$, we then add the element stiffness matrix into this array. Considering bar (m) for which $\mathbf{LM}^{(m)}$ is given in (2.50) we add

$k_{11}^{(m)}$ to the entry $\ell\ell$ of the array

$k_{12}^{(m)}$ to the entry ℓp

$k_{13}^{(m)}$ to the entry ℓq

$k_{14}^{(m)}$ to the entry ℓr

$k_{22}^{(m)}$ to the entry pp

$k_{23}^{(m)}$ to the entry pq

$k_{24}^{(m)}$ to the entry pr

$k_{33}^{(m)}$ to the entry qq

$k_{34}^{(m)}$ to the entry qr

$k_{44}^{(m)}$ to the entry rr.

We note that since each bar stiffness matrix is symmetric (see equation (2.33)), the structure stiffness matrix $\mathbf{K}$ is also symmetric, see equation

(2.49). For this reason we construct in the assemblage procedure only the upper diagonal part of $\mathbf{K}$.

To demonstrate the matrix procedure we consider the following examples.

Example 2.2

Formulate and solve the problem described in Figure 2.4 and considered in Example 2.1 using the matrix method.

Solution

We adopt the nodal and bar numbering already given in Figure 2.4 and define in Figure 2.33 the numbering of the degrees of freedom as well as the bar orientations.

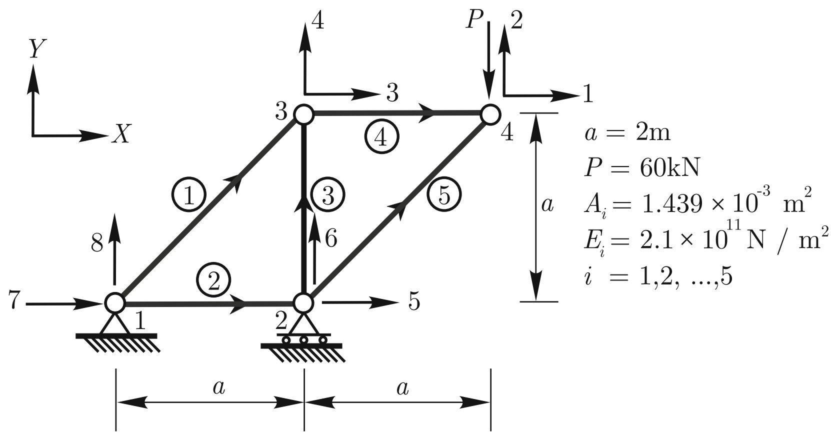


Fig. 2.33. Definitions for the matrix formulation of the problem in Figure 2.4

The next step is to obtain the stiffness matrices of the bar elements in the global coordinate system. We note that for bars 2 and 4, the local and global coordinate systems (of displacements and nodal forces) are the same. Therefore, we can write

$$\mathbf{k}^{(2)} = \mathbf{k}^{(4)} = \frac{EA}{a} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} .$$

The stiffness matrices of bars 1 and 5 are the same and can be obtained, corresponding to the global coordinate system, using equation (2.33) with $\alpha = 45°$ leading to

$$\mathbf{k}^{(1)} = \mathbf{k}^{(5)} = \frac{EA\sqrt{2}}{4a} \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix}$$

and for bar 3 we use again equation (2.33) with $\alpha = 90^\circ$ arriving at

$$\mathbf{k}^{(3)} = \frac{EA}{a} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}.$$

To assemble the global stiffness matrix, we need the element connectivity arrays

$$\begin{aligned} \mathbf{LM}^{(1)} &= \begin{bmatrix} 7 & 8 & 3 & 4 \end{bmatrix}, \quad \mathbf{LM}^{(2)} = \begin{bmatrix} 7 & 8 & 5 & 6 \end{bmatrix} \\ \mathbf{LM}^{(3)} &= \begin{bmatrix} 5 & 6 & 3 & 4 \end{bmatrix}, \quad \mathbf{LM}^{(4)} = \begin{bmatrix} 3 & 4 & 1 & 2 \end{bmatrix} \\ \mathbf{LM}^{(5)} &= \begin{bmatrix} 5 & 6 & 1 & 2 \end{bmatrix}. \end{aligned}$$

Then performing the assemblage procedure leads to

$$\mathbf{K} = \begin{bmatrix} k_{33}^{(4)} + k_{33}^{(5)} & k_{34}^{(4)} + k_{34}^{(5)} & k_{13}^{(4)} & k_{23}^{(4)} & k_{13}^{(5)} & k_{23}^{(5)} & 0 & 0 \\ & k_{44}^{(4)} + k_{44}^{(5)} & k_{14}^{(4)} & k_{24}^{(4)} & k_{14}^{(5)} & k_{24}^{(5)} & 0 & 0 \\ & & k_{33}^{(1)} + k_{33}^{(3)} + k_{11}^{(4)} & k_{34}^{(1)} + k_{34}^{(3)} + k_{12}^{(4)} & k_{13}^{(3)} & k_{23}^{(3)} & k_{13}^{(1)} & k_{23}^{(1)} \\ & & & k_{44}^{(1)} + k_{44}^{(3)} + k_{22}^{(4)} & k_{14}^{(3)} & k_{24}^{(3)} & k_{14}^{(1)} & k_{24}^{(1)} \\ & & & & k_{33}^{(2)} + k_{11}^{(3)} + k_{11}^{(5)} & k_{34}^{(2)} + k_{12}^{(3)} + k_{12}^{(5)} & k_{13}^{(2)} & k_{23}^{(2)} \\ & symmetric & & & & k_{44}^{(2)} + k_{22}^{(3)} + k_{22}^{(5)} & k_{14}^{(2)} & k_{24}^{(2)} \\ & & & & & & k_{11}^{(1)} + k_{11}^{(2)} & k_{12}^{(1)} + k_{12}^{(2)} \\ & & & & & & & k_{22}^{(1)} + k_{22}^{(2)} \end{bmatrix}$$

and introducing the numerical values we obtain

$$
\mathbf{K} = \frac{EA}{a} \begin{bmatrix}
\frac{\sqrt{2}}{4}+1 & \frac{\sqrt{2}}{4} & -1 & 0 & -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & 0 & 0 \\
\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & 0 & 0 & -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & 0 & 0 \\
-1 & 0 & \frac{\sqrt{2}}{4}+1 & \frac{\sqrt{2}}{4} & 0 & 0 & -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} \\
0 & 0 & \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4}+1 & 0 & -1 & -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} \\
-\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & 0 & 0 & \frac{\sqrt{2}}{4}+1 & \frac{\sqrt{2}}{4} & -1 & 0 \\
-\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & 0 & -1 & \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4}+1 & 0 & 0 \\
0 & 0 & -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & -1 & 0 & \frac{\sqrt{2}}{4}+1 & \frac{\sqrt{2}}{4} \\
0 & 0 & -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & 0 & 0 & \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4}
\end{bmatrix}.
$$

Hence the complete set of equilibrium equations with the applied nodal force P and imposed displacement restraints is

$$
\frac{EA}{a} \left[\begin{array}{ccccc|ccc}
\frac{\sqrt{2}}{4}+1 & \frac{\sqrt{2}}{4} & -1 & 0 & -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & 0 & 0 \\
\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & 0 & 0 & -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & 0 & 0 \\
-1 & 0 & \frac{\sqrt{2}}{4}+1 & \frac{\sqrt{2}}{4} & 0 & 0 & -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} \\
0 & 0 & \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4}+1 & 0 & -1 & -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} \\
-\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & 0 & 0 & \frac{\sqrt{2}}{4}+1 & \frac{\sqrt{2}}{4} & -1 & 0 \\
\hline
-\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & 0 & -1 & \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4}+1 & 0 & 0 \\
0 & 0 & -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & -1 & 0 & \frac{\sqrt{2}}{4}+1 & \frac{\sqrt{2}}{4} \\
0 & 0 & -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & 0 & 0 & \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4}
\end{array}\right]
$$

$$
\left[\begin{array}{c}
U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ \hline 0 \\ 0 \\ 0
\end{array}\right] = \left[\begin{array}{c}
0 \\ -P \\ 0 \\ 0 \\ 0 \\ \hline R_6 \\ R_7 \\ R_8
\end{array}\right]. \tag{2.51}
$$

Hence we can identify natural partitions for the load and displacement column matrices. The displacement partitioning is obtained according to whether the displacement degrees of freedom are free or restrained. Denoting by $\mathbf{U}_a$ the free displacement degrees of freedom and by $\mathbf{U}_b$ the restrained degrees of freedom, we can write

$$\mathbf{U}^T = \begin{bmatrix} \mathbf{U}_a^T & \mathbf{U}_b^T \end{bmatrix}$$

where for this particular case

$$\mathbf{U}_a^T = \begin{bmatrix} U_1 & U_2 & U_3 & U_4 & U_5 \end{bmatrix}$$

and

$$\mathbf{U}_b^T = \begin{bmatrix} U_6 & U_7 & U_8 \end{bmatrix}.$$

Analogously, for the load column matrix

$$\mathbf{R}^T = \begin{bmatrix} \mathbf{R}_a^T & \mathbf{R}_b^T \end{bmatrix}$$

where $\mathbf{R}_a$ collects the external loads for the free degrees of freedom and for this case is given by

$$\mathbf{R}_a^T = \begin{bmatrix} R_1 & R_2 & R_3 & R_4 & R_5 \end{bmatrix}$$

with $R_1 = R_3 = R_4 = R_5 = 0$ and $R_2 = -P$. The column matrix $\mathbf{R}_b$ collects the reactions and is given by

$$\mathbf{R}_b^T = \begin{bmatrix} R_6 & R_7 & R_8 \end{bmatrix}.$$

Furthermore, the partitions of the load and displacement column matrices also induce the following partitioning for the stiffness matrix

$$\begin{bmatrix} \mathbf{K}_{aa} & \mathbf{K}_{ab} \\ \mathbf{K}_{ba} & \mathbf{K}_{bb} \end{bmatrix} \begin{bmatrix} \mathbf{U}_a \\ \mathbf{U}_b \end{bmatrix} = \begin{bmatrix} \mathbf{R}_a \\ \mathbf{R}_b \end{bmatrix}. \tag{2.52}$$

Here $\mathbf{R}_a$ and $\mathbf{U}_b$ contain always known values whereas $\mathbf{U}_a$ and $\mathbf{R}_b$ contain always unknown values. In order to solve the system in (2.52) we use

$$\mathbf{K}_{aa}\mathbf{U}_a + \mathbf{K}_{ab}\mathbf{U}_b = \mathbf{R}_a \tag{2.53}$$

$$\mathbf{K}_{ba}\mathbf{U}_a + \mathbf{K}_{bb}\mathbf{U}_b = \mathbf{R}_b \tag{2.54}$$

and obtain

$$\mathbf{K}_{aa}\mathbf{U}_a = \mathbf{R}_a - \mathbf{K}_{ab}\mathbf{U}_b \tag{2.55}$$

which can be solved for $\mathbf{U}_a$. Having obtained $\mathbf{U}_a$, the reactions $\mathbf{R}_b$ can be evaluated from (2.54).

In this case, (2.55) reads

$$\frac{EA}{a}\begin{bmatrix} \frac{\sqrt{2}}{4}+1 & \frac{\sqrt{2}}{4} & -1 & 0 & -\frac{\sqrt{2}}{4} \\ \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & 0 & 0 & -\frac{\sqrt{2}}{4} \\ -1 & 0 & \frac{\sqrt{2}}{4}+1 & \frac{\sqrt{2}}{4} & 0 \\ 0 & 0 & \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4}+1 & 0 \\ -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & 0 & 0 & \frac{\sqrt{2}}{4}+1 \end{bmatrix}\begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{bmatrix} = \begin{bmatrix} 0 \\ -P \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

since $\mathbf{U}_b = \mathbf{0}$. Introducing the numerical values for E, A, a and P and solving yields

$$\begin{aligned} U_1 &= 1.91737 \times 10^{-3}\ \text{m}, \qquad U_2 = -3.43765 \times 10^{-3}\ \text{m} \\ U_3 &= 1.52027 \times 10^{-3}\ \text{m}, \qquad U_4 = -3.97101 \times 10^{-4}\ \text{m} \\ U_5 &= -3.97101 \times 10^{-4}\ \text{m}. \end{aligned}$$

The reactions are evaluated using (2.54)

$$\frac{EA}{a}\begin{bmatrix} -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & 0 & -1 & \frac{\sqrt{2}}{4} \\ 0 & 0 & -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & -1 \\ 0 & 0 & -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & 0 \end{bmatrix}\begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{bmatrix} = \begin{bmatrix} R_6 \\ R_7 \\ R_8 \end{bmatrix}$$

which leads to

$$R_6 = 120000\ \text{N}, \qquad R_7 = 0, \qquad R_8 = -60000\ \text{N}.$$

To complete the solution, we need to evaluate the internal forces in the truss bars. Considering bar 1 and using the nodal displacements, we obtain the end displacements of bar 1

$$\begin{aligned} u_1^{(1)} &= U_7 = 0, \qquad u_2^{(1)} = U_8 = 0 \\ u_3^{(1)} &= U_3 = 1.52027 \times 10^{-3}\ \text{m}, \qquad u_4^{(1)} = U_4 = -3.97101 \times 10^{-4}\ \text{m}. \end{aligned}$$

Therefore, the global nodal forces acting onto the bar are

$$\mathbf{f}^{(1)} = \mathbf{k}^{(1)}\mathbf{u}^{(1)} = \begin{bmatrix} -60000 \\ -60000 \\ 60000 \\ 60000 \end{bmatrix}$$

where the values are given in the units used throughout the solution (Newtons).
We could extract the axial force from $\mathbf{f}^{(1)}$ by projecting the components into the axial direction. Equivalently, we can evaluate $\tilde{\mathbf{f}}^{(1)}$ using equation (2.31) with $\alpha = 45°$, which leads to

$$\begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 0 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 0 & 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} -60000 \\ -60000 \\ 60000 \\ 60000 \end{bmatrix} = \begin{bmatrix} -84853 \\ 0 \\ 84853 \\ 0 \end{bmatrix}$$

and hence, of course,

$$N_1 = -\tilde{f}_1^{(1)} = \tilde{f}_3^{(1)} = 84853 \text{ N}.$$

In an analogous manner we can evaluate the remaining internal bar forces to obtain

$$N_2 = -60000 \text{ N}, \quad N_3 = -60000 \text{ N}, \quad N_4 = 60000 \text{ N}, \quad N_5 = -84853 \text{ N}.$$

□

Of course, the solution results given in Example 2.2 are those we obtained earlier (see Example 2.1).

It is instructive to compare the two solution methods. We solved the problem described in Figure 2.4, firstly, by using an elementary method in which the bar forces are determined by equilibrium of the joints and then the displacements are found and then, secondly, by the matrix method. We note that even in this case in which this elementary method allows to find the nodal forces by nodal equilibrium only – with good insight into the deformation of the structure – the determination of the nodal displacements is relatively complex and difficult to systematize. Hence, the matrix method has great appeal since the whole procedure can be easily systematized and implemented in computer programs leading to the solution of very large problems in a straight forward and fast manner.

Furthermore, the solution of statically indeterminate trusses requires no additional considerations.

Example 2.3

Consider the truss described in Figure 2.19, which is obtained by adding bar 6 to the truss structure of Figure 2.4. Assuming that bar 6 has the same Young's modulus and cross-sectional area as the other bars, modify the matrix formulation presented in Example 2.2 to solve this problem.

Solution

The only modification necessary is to add the contribution of bar 6 to the stiffness matrix of the structure. Considering that for bar 6 we have $\ell = a\sqrt{5}$ and $\alpha = arctg(1/2) = 26.565°$, its stiffness matrix is given by

$$\mathbf{k}^{(6)} = \frac{EA\sqrt{5}}{25a}\begin{bmatrix} 4 & 2 & -4 & -2 \\ 2 & 1 & -2 & -1 \\ -4 & -2 & 4 & 2 \\ -2 & -1 & 2 & 1 \end{bmatrix}$$

where we assume the bar orientation from node 1 to 4. The coefficients of the upper diagonal part of the global stiffness matrix which should be updated are

$$\begin{aligned} K_{11}^6 &= K_{11}^5 + k_{33}^{(6)}, \qquad & K_{12}^6 &= K_{12}^5 + k_{34}^{(6)} \\ K_{22}^6 &= K_{22}^5 + k_{44}^{(6)}, \qquad & K_{17}^6 &= 0 + k_{31}^{(6)} \\ K_{18}^6 &= 0 + k_{32}^{(6)}, \qquad & K_{27}^6 &= 0 + k_{41}^{(6)} \\ K_{28}^6 &= 0 + k_{42}^{(6)}, \qquad & K_{77}^6 &= K_{77}^5 + k_{11}^{(6)} \\ K_{78}^6 &= K_{78}^5 + k_{12}^{(6)}, \qquad & K_{88}^6 &= K_{88}^5 + k_{22}^{(6)} \end{aligned}$$

where we have used K_{ij}^5 and K_{ij}^6 to represent the stiffness coefficient K_{ij} of the truss structure with 5 and 6 bars respectively. The solution is then obtained, as for Example 2.2, by considering the updated stiffness matrix.

□

We emphasize that since equilibrium and compatibility are enforced simultaneously in the matrix method, there is no need to consider in the solution procedure whether the truss is a statically determinate or a statically indeterminate structure. Namely, adding bar 6 in Example 2.3, which makes the structure statically indeterminate, has very little impact on the complete solution effort. In fact, we only need to add the contribution of bar 6 to the structure stiffness matrix and there is no increase in the order of the system of linear algebraic equations to be solved. On the other hand, when we try to use the elementary method, the addition of bar 6 significantly increases the effort of solution because, since the structure becomes statically indeterminate, we can no longer determine the bar forces by nodal equilibrium only.

These observations reinforce the earlier conclusion that the matrix method is a very efficient method for the computerized analysis of complex truss structures.

2.3.6 Principle of superposition

We note that equation (2.48) establishes a linear relation between the applied forces and the resulting displacements. Of course, this linear relation is a direct consequence of the assumptions used in the formulation of the truss mathematical model, which all together result into a constant stiffness matrix $\mathbf{K}$ (that is independent of the nodal displacements). Such models are called "linear (elastic) models" and for such models the *principle of superposition* holds. Suppose that the total load $\mathbf{R}$ acting on a truss structure is decomposed into n_ℓ load sets given by $\mathbf{R}_i$, that is

$$\mathbf{R} = \sum_{i=1}^{n_\ell} \mathbf{R}_i.$$

If we solve for each load set

$$\mathbf{K}\mathbf{U}_i = \mathbf{R}_i$$

then the solution for the total load $\mathbf{R}$ is

$$\mathbf{U} = \sum_{i=1}^{n_\ell} \mathbf{U}_i.$$

Namely, we have

$$\mathbf{K}\mathbf{U} = \mathbf{K}\left(\sum_{i=1}^{n_\ell} \mathbf{U}_i\right) = \sum_{i=1}^{n_\ell} (\mathbf{K}\mathbf{U}_i) = \sum_{i=1}^{n_\ell} \mathbf{R}_i = \mathbf{R}.$$

and these relations hold true because $\mathbf{K}$ is constant.

The principle of superposition is valid for all linear mathematical models studied in this book. In practice, a structure may be analyzed for many different load cases (gravity, wind loading, snow loading, settlement of supports, etc.) and the analyst/designer needs to seek the worst valid combination of loads to identify the highest internal forces that the structure may possibly experience. Then, of course, for each load combination the principle of superposition is used to obtain the structural response.

To give a simple example, we mention that the three-bar structure described in Figure 2.17 could have been solved considering only R_1 and then R_2. Of course, the total response would be obtained by superimposing the two resulting solutions. However, the maximum force in bar 2 is reached when R_2 acts alone leading to $N_2 = 47.71$ kN (when R_1 acts alone $N_2 = -16.37$ kN).

Also, to obtain insight into the structural behavior, the principle of superposition is sometimes used to break up the structural response for complex loading, allowing the analyst/designer to examine the contribution of each load case to the total response.

2.3.7 Remarks about the structure stiffness matrix

We gave already a physical explanation of the stiffness coefficients of a bar element, see Figure 2.22. The same discussion also applies to the stiffness coefficients in $\mathbf{K}$. The element K_{ii} gives the force that should be applied to the degree of freedom i to impose a unit displacement at this same degree of freedom, when all other structural degrees of freedom are fixed. The stiffness coefficients K_{ji} for $j \neq i$ give the reaction forces associated with the degrees of freedom that have been fixed. Based on this property, the stiffness matrix can be constructed column by column. For example, referring to Example 2.2, the first column of the global stiffness matrix can be obtained by imposing a unit displacement at the degree of freedom 1, *i.e.*, for the horizontal displacement of node 4, and fixing all other degrees of freedom.

We can also interpret the contributions of the bar stiffness coefficients to the global stiffness matrix coefficients for such column. In fact, to impose $U_1 = 1$ we need to impose a unit displacement at the end sections of bars 4 and 5 which couple into node 4. Therefore

$$K_{11} = k_{33}^{(4)} + k_{33}^{(5)}$$

since $k_{33}^{(4)}$ and $k_{33}^{(5)}$ give the horizontal forces that should be imposed at the end sections of bars 4 and 5, respectively, for a unit end displacement when all the remaining bar degrees of freedom are fixed. By an analogous reasoning, the reactions at the fixed degrees of freedom can also be obtained as

$$\begin{aligned} K_{21} &= k_{43}^{(4)} + k_{43}^{(5)}, \qquad K_{31} = k_{13}^{(4)} \\ K_{41} &= k_{23}^{(4)}, \qquad K_{51} = k_{13}^{(5)}, \qquad K_{61} = k_{23}^{(5)}. \end{aligned}$$

In addition, $K_{71} = K_{81} = 0$ since there are no bars connecting nodes 1 and 4.

The above discussion also shows that the equation $\mathbf{KU} = \mathbf{R}$ represents a linear system of N algebraic equilibrium equations. The i^{th} equation of the system given by

$$\sum_{j=1}^{N} K_{ij} U_j = R_i$$

reflects the equilibrium at the i^{th} degree of freedom. In other words, $K_{ij}U_j$ gives the internal force contribution associated with the displacement U_j to the equilibrium at the i^{th} degree of freedom.

2.3.8 Strain energy of a truss structure

In this section we introduce the strain energy concept for truss structures.

Elastic solids when subjected to external loading deform and store energy associated with the deformation like a spring which is called *strain energy*. We detail below this concept for a truss bar and then for truss structures.

The truss bar of linear elastic material shown in Figure 2.34a is subjected to a slowly increasing external load up to the value R. The final configuration is shown in Figure 2.34b.

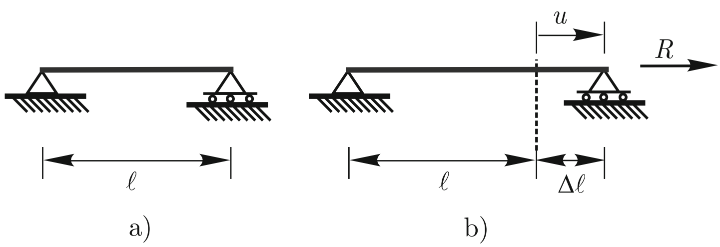


Fig. 2.34. Deformation of single bar structure

Let W_e be the external work done by the applied load. The differential external work dW_e associated with an induced differential displacement is given by the shaded area in Figure 2.35a, *i.e.*,

$$dW_e = R_u \, du_u$$

and, therefore,

$$W_e = \int_0^R R_u \, du_u = \frac{1}{2} R u.$$

Let W_i be the internal work. The differential increment of internal work associated with an induced increment in strain is given by the shaded area in Figure 2.35b, multiplied by the differential volume element $dxdA$, *i.e.*,

$$dW_i = \tau_u d\varepsilon_u dx \, dA$$

and, therefore[7],

$$W_i = \int_V \left(\int_0^\tau \tau_u \, d\varepsilon_u \right) dxdA = \frac{1}{2} \tau \varepsilon A \ell.$$

Since

$$\tau = \frac{R}{A}, \qquad \varepsilon = \frac{\Delta \ell}{\ell} = \frac{u}{\ell}$$

[7] Note that although the bar will change its thickness (cross-sectional area) as indicated in Figure 2.35b, we are integrating over the original volume, in correspondence with the linear analysis assumptions (see Section 2.1.5)

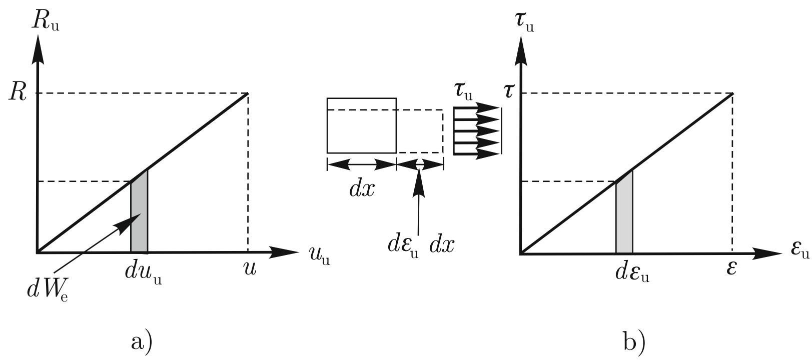


Fig. 2.35. a) Load displacement diagram for a truss bar; b) Stress-strain diagram for a generic point of a truss bar and side view of a differential volume element of the truss bar

we obviously have

$$W_i = W_e.$$

This is an important result that shows that the work done by the external force is equal to the work done by the internal forces/stresses.

Using Hooke's law, we can write

$$W_i = \frac{\varepsilon^2}{2} EA\ell.$$

Note that W_i depends only on the current strain and gives the work stored in the truss bar as elastic deformation. Therefore, we define

$$\mathcal{U}(\varepsilon) = W_i(\varepsilon) = \frac{\varepsilon^2}{2} EA\ell$$

as the strain energy of the truss bar. Of course, the strain energy per unit of volume is given by

$$W(\varepsilon) = \frac{E\varepsilon^2}{2} = \frac{1}{2}\tau\varepsilon.$$

It is usual to express the strain energy of a bar in terms of the axial force carried, then

$$\mathcal{U} = \frac{N^2\ell}{2EA}.$$

Since the strain energy is a scalar, we can evaluate the strain energy of an assemblage of bars by adding up the contribution of every bar. For a generic bar m of the assemblage, since $\mathbf{u}^{(m)}$ and $\mathbf{f}^{(m)}$ are end displacements and forces and the external work $W_e^{(m)}$ done by the forces $\mathbf{f}^{(m)}$ is equal to the bar strain energy $\mathcal{U}^{(m)}$, we have

$$\mathcal{U}^{(m)} = \frac{1}{2}\mathbf{u}^{(m)^T}\mathbf{f}^{(m)} = \frac{1}{2}\mathbf{u}^{(m)^T}\mathbf{k}^{(m)}\mathbf{u}^{(m)} = \frac{1}{2}\mathbf{U}^T\mathbf{K}^{(m)}\mathbf{U}.$$

Hence, for a complete truss structure the strain energy is

$$\mathcal{U} = \sum_{m=1}^{n_e}\mathcal{U}^{(m)} = \frac{1}{2}\mathbf{U}^T\left(\sum_{m=1}^{n_e}\mathbf{K}^{(m)}\right)\mathbf{U} = \frac{1}{2}\mathbf{U}^T\mathbf{K}\mathbf{U}.$$

Note that since $\mathbf{R} = \mathbf{K}\mathbf{U}$, we also have

$$\mathcal{U} = \frac{1}{2}\mathbf{U}^T\mathbf{R}.$$

2.3.9 Properly supported truss structures in the context of the matrix method

In Section 2.1.2 we discussed the concept of a properly supported deformable body and later we applied this concept to a truss structure. Recall that when a truss is properly supported and without an internal mechanism, any motion of its bars requires some bar to shorten or to extend.

Therefore, the strain energy associated with any motion of a properly supported truss structure, that is, corresponding to any non-trivial nodal displacements $\mathbf{U}$, will be positive

$$\mathcal{U}(\mathbf{U}) = \frac{1}{2}\mathbf{U}^T\mathbf{K}\mathbf{U} > 0 \qquad \text{for any } \mathbf{U} \neq \mathbf{0}. \tag{2.56}$$

Here "non-trivial $\mathbf{U}$" means $\mathbf{U} \neq \mathbf{0}$.

Mathematically, condition (2.56) defines $\mathbf{K}$ to be a positive definite matrix.

It is a mathematical property that a positive definite matrix has always an inverse, *i.e.*, it is not singular (Bathe, 1996). This leads to a very important result: for any given nodal load $\mathbf{R}$ acting on a truss structure which is properly supported and without an internal mechanism we can always find a unique nodal displacement $\mathbf{U}$ such that

$$\mathbf{K}\mathbf{U} = \mathbf{R}.$$

Now we would like to show that when a truss structure is not properly supported and/or has an internal mechanism the $\mathbf{K}$ matrix is singular and, therefore, there is no unique solution $\mathbf{U}$ for any $\mathbf{R}$. Before we do so, we note that:

- If the truss structure is not properly supported a global rigid motion of the complete structure is possible (see Figure 2.36).

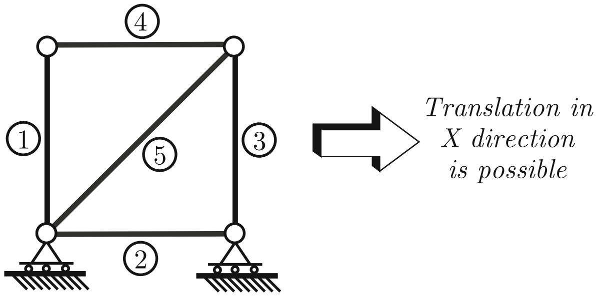


Fig. 2.36. Not properly supported truss structure: Rigid body motion of complete structure is possible

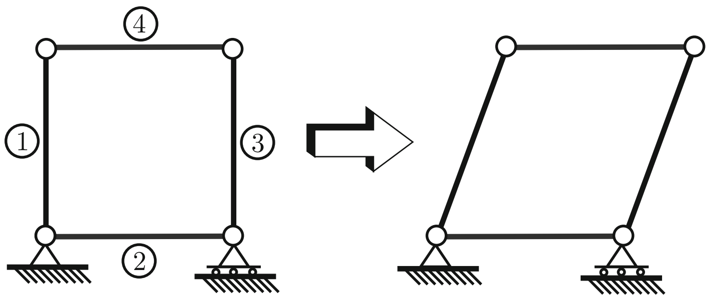


Fig. 2.37. Truss structure with an internal mechanism. Rigid body motion of some bars is possible, *i.e.*, there is a mechanism. Here bars 1, 3, 4 undergo a rigid body motion

- If the truss structure has an internal mechanism, a rigid motion of one or more of its parts without any motion of the remaining parts is possible (see Figure 2.37).

Note that the bars 1, 3 and 4 of the truss of Figure 2.37 display, individually, rigid motions but bar 2 displays no motion and hence does not strain. On the other hand, Figure 2.38 illustrates that for a properly supported truss structure without an internal mechanism parts of the structure may undergo rigid motion but then always cause straining in (some) other bars.

Consider a generic truss structure which either is not properly supported or has an internal mechanism (or both). Let us choose $\mathbf{U} \neq \mathbf{0}$ for which

$$\mathcal{U}(\mathbf{U}) = \frac{1}{2}\mathbf{U}^T\mathbf{K}\mathbf{U} = 0.$$

Of course, this choice of $\mathbf{U}$ is always possible since there is always a motion for which each bar (m) either displays a rigid motion or does not move. Therefore, the stiffness matrix is positive semidefinite and, hence, singular. Since the $\mathbf{K}$ matrix is singular we can not find a unique nodal displacement solution $\mathbf{U}$ for any given load $\mathbf{R}$.

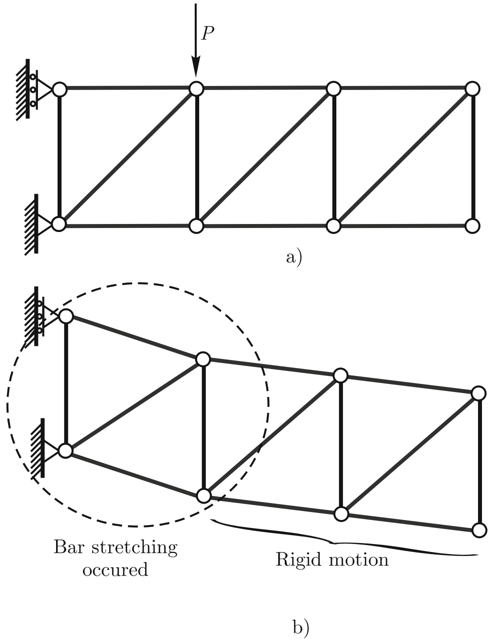


Fig. 2.38. a) Problem definition; b) Schematic and magnified deformed configuration

Summarizing, through the use of the strain energy concept, we arrived at two important results:

- For a truss structure which is properly supported and does not have an internal mechanism, given any loading $\mathbf{R}$, there exists a unique nodal displacement solution $\mathbf{U}$.
- If the truss structure is not properly supported and/or has an internal mechanism, then for any $\mathbf{R}$ there is no unique solution $\mathbf{U}$.

Let us consider an example.

Example 2.4

Show using a purely kinematic approach that the stiffness matrix of a bar is singular whenever the bar can display rigid motions.

Solution

Let us consider a structure composed of a single truss bar as described in Figure 2.39.

Since there are no supports, this bar can display rigid body motions. In fact, there are three linearly independent rigid body motions: a translation along the X direction, a translation along the Y direction and a rigid body

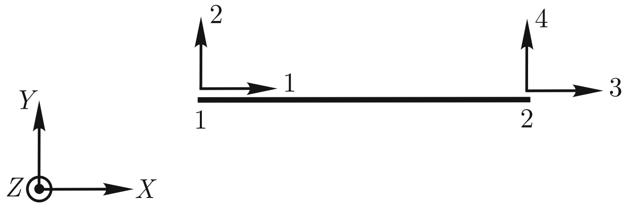


Fig. 2.39. One bar truss structure in X-Y plane

rotation in the XY plane. Any combination of these three rigid body motions also constitutes a rigid body motion.

Let us show that for each set of nodal displacements corresponding to a rigid body motion, we have

$$\mathbf{ku} = \mathbf{0}. \tag{2.57}$$

For a rigid body mode translation along the X direction

$$\mathbf{u}^T = \mathbf{u}_x^T = \begin{bmatrix} C_x & 0 & C_x & 0 \end{bmatrix}$$

where C_x is an arbitrary constant. Clearly (2.57) is satisfied when we use the stiffness matrix given by equation (2.13). An analogous result is found for a rigid body mode translation along the Y direction, which can be defined by

$$\mathbf{u}^T = \mathbf{u}_y^T = \begin{bmatrix} 0 & C_y & 0 & C_y \end{bmatrix}$$

where C_y is also an arbitrary constant.

Next we find the bar nodal displacements for an infinitesimally small rigid body rotation about the Z axis, as shown in Figure 2.40.

Referring to Figure 2.40 we can write

$$\begin{aligned} -u_1 &= r_1 d\varphi \sin\theta_1 \\ u_2 &= r_1 d\varphi \cos\theta_1 \end{aligned}$$

and since

$$\begin{aligned} X_1 &= r_1 \cos\theta_1 \\ Y_1 &= r_1 \sin\theta_1 \end{aligned}$$

we obtain

$$\begin{aligned} u_1 &= -d\varphi Y_1 \\ u_2 &= d\varphi X_1 \end{aligned}$$

Considering a similar derivation for node 2, we obtain

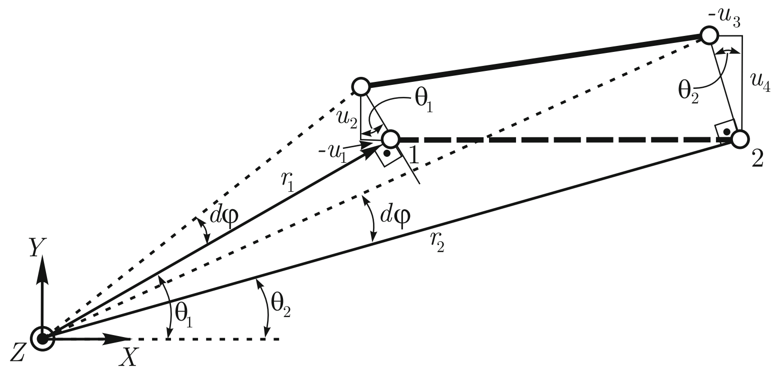


Fig. 2.40. Rigid rotation of the bar of an infinitesimally small angle $d\varphi$. Nodal coordinates are X_1, Y_1 and X_2, Y_2

$$\mathbf{u}_{\varphi}^{T} = \left[\begin{array}{cccc} -d\varphi Y_1 & d\varphi X_1 & -d\varphi Y_2 & d\varphi X_2 \end{array} \right].$$

Since $Y_1 = Y_2$, we also obtain (2.57) . Furthermore, we note that for a displacement $\mathbf{u}$ which is any linear combination of $\mathbf{u}_x$, $\mathbf{u}_y$ and $\mathbf{u}_{\varphi}$, the relation (2.57) also holds.

Of course, when equation (2.57) holds, $\mathbf{k}$ is singular. Equation (2.57) also means that there are no bar end forces associated with a rigid motion.

We note that the choice of a bar which is aligned with the X axis does not imply lack of generality, since, if (2.57) holds for a given coordinate system, it is also satisfied for any other coordinate system.

Note that for the one bar structure of Figure 2.39 three restraints are necessary to kinematically suppress the three rigid body motions. For example, the restraints shown in Figure 2.22.

□

2.4 Modeling considerations for truss structures

While reading this chapter, the reader might have thought of the real physical truss structures that are part of the every day environment, for example, the structures that are frequently encountered in bridges and roofs. Possibly the joints of these "real" trusses are different from the joint described in Figure 2.5, in particular, there might not be an actual pin. In fact, in most truss structures, there are no pins and different joint options are used to connect the bars.

We need to resort to the hierarchical modeling approach to properly address this apparent inconsistency. The truss model with the pin joints is an adequate low-order mathematical model to represent the behavior of bar structures which are not pinned, as long as:

- The structure can not display either local or global rigid motions when its joints are considered as pin-type joints (the truss model is properly supported and does not have an internal mechanism).
- All external loads can be assumed to be introduced at the nodes as point forces only (no moments).
- The bars are long (typically the thickness/length ratio is smaller than 1/10) and have similar magnitudes of cross-sectional areas.
- The bar axes always intersect at unique geometric points (at the joints).

In such cases, the truss model provides a good description of the structure and, in general, it represents a reliable model for most analysis purposes.

The analysis and design of truss structures represent a very broad subject. While we did not discuss issues related to the design of engineering trusses, the analysis of truss structures is important in mechanics and, in our discussion, provided a convenient setting to introduce and explore the fundamental facts of structural mechanics.

In the next chapter we extend the ideas studied in this chapter to model 2-D and 3-D deformable solids.