13 Motion under central forces

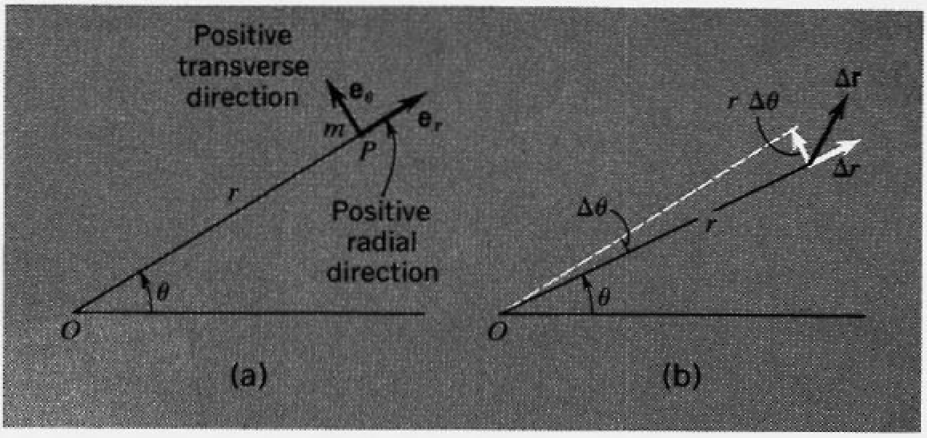
WE HAVE ALREADY seen, especially in Chapter 8, how the motion of objects under the action of forces directed toward some well-defined center is one of the richest areas of study in mechanics. Twice in the history of physics the analysis of such motions has been linked with fundamental advances in our understanding of nature—through the explanation of planetary motions, on the macroscopic scale, and through Rutherford's studies of alpha-particle scattering, which gave man his first clear view of the subatomic world. Up until this point we have limited ourselves to the study of circular orbits, and it is remarkable how much can be learned on that basis. But now we shall begin a more general analysis of motion under the action of central forces.

BASIC FEATURES OF THE PROBLEM

As we saw in Chapter 11 (p. 444), a central force field that is also conservative must be spherically symmetric, and some of the most important fields in nature (notably electrical and gravitational) are precisely of this type. The frequent occurrence of spherically symmetric models to describe physical reality is closely linked to the basic assumption that space is isotropic and is the intuitively natural starting point in building theoretical models of various kinds of dynamical systems.

We shall begin with the specific problem of the motion of a single particle of mass m in a spherically symmetric central field

Fig. 13–1 *(a) Unit vectors associated with radial and transverse directions in a plane polar coordinate system. (b) Radial and transverse components of an elementary vector displacement* $\Delta\mathbf{r}$.



of force. Initially, at least, we shall assume that the object responsible for this central field is so massive that it can be regarded as a fixed center that defines a convenient origin of coordinates for the analysis of the motion.

The first thing to notice is that the path of the moving particle will lie in a fixed plane that passes through the center of force. This plane is defined by the initial velocity vector $\mathbf{v}_0$ of the particle and the initial vector position $\mathbf{r}_0$ of the particle with respect to the center of force. Since the force acting on the particle is in this plane, and since there is no component of initial velocity perpendicular to it, the motion must remain confined to this plane of $\mathbf{r}_0$ and $\mathbf{v}_0$. To analyze the motion we must first pick an appropriate coordinate system. Because the force $\mathbf{F}$ is a function of the scalar distance r only and is along the line of the vector $\mathbf{r}$ (positively or negatively), it is clearly most convenient to work with the plane polar coordinates (r, θ), as indicated in Fig. 13–1(a). This means that we shall be making use of the acceleration vector expressed in these coordinates. In Chapter 3 (p. 108) we calculated this vector for the particular case of circular motion (r = constant). Now we shall develop the more general expression that embraces changes of both r and θ.

Using the unit vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$ as indicated in Fig. 13–1(a) we have

$$\mathbf{r} = r\mathbf{e}_r \tag{13–1}$$

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{dr}{dt}\mathbf{e}_r + r\frac{d\theta}{dt}\mathbf{e}_\theta \tag{13–2}$$

This equation for $\mathbf{v}$ is readily constructed by recognizing that a general infinitesimal change in position, $\Delta\mathbf{r}$, is obtained by combining a radial displacement of length Δr (at constant θ) and a transverse displacement of length $r\,\Delta\theta$ (at constant r), as indicated in Fig. 13–1(b). Alternatively, one can just differentiate both sides of Eq. (13–1) with respect to time, remembering that $d(\mathbf{e}_r)/dt = (d\theta/dt)\mathbf{e}_\theta$ [see Eq. (3–17a)].

We now proceed to differentiate both sides of Eq. (13–2) with respect to t:

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2r}{dt^2}\mathbf{e}_r + \frac{dr}{dt}\frac{d}{dt}\mathbf{e}_r + \frac{dr}{dt}\frac{d\theta}{dt}\mathbf{e}_\theta + r\frac{d^2\theta}{dt^2}\mathbf{e}_\theta + r\frac{d\theta}{dt}\frac{d}{dt}\mathbf{e}_\theta$$

Substituting $d(\mathbf{e}_r)/dt = (d\theta/dt)\mathbf{e}_\theta$, and $d(\mathbf{e}_\theta)/dt = -(d\theta/dt)\mathbf{e}_r$, the expression for $\mathbf{a}$ can be rewritten as follows:

$$\mathbf{a} = \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\mathbf{e}_r + \left[r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right]\mathbf{e}_\theta \qquad (13\text{–}3)$$

It will be convenient to extract from this the separate radial and transverse components of the total acceleration:

$$a_r = \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 \qquad (13\text{–}4)$$

$$a_\theta = r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt} \qquad (13\text{–}5)$$

The statement of Newton's law in plane polar coordinates can then be made in terms of these separate acceleration components:

$$F_r = m\left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right] \qquad (13\text{–}6)$$

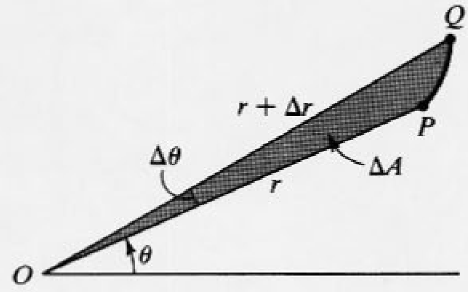
$$F_\theta = m\left[r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right] \qquad (13\text{–}7)$$

The above two equations provide a basis for the solution of any problem of motion in a plane, referred to an origin of polar coordinates. We shall, however, consider their application to central forces in particular.

THE LAW OF EQUAL AREAS

In the case of any kind of conservative central force, we have $F_r = F(r)$ simply, and $F_\theta = 0$. The second of these immediately implies that $a_\theta = 0$. Substituting the specific expression for a_θ

Fig. 13–2 *Illustrating the basis of calculating areal velocity (area swept out per unit time by the radius vector).*



from Eq. (13–5), we have

$$r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt} = 0 \qquad (13\text{–}8)$$

This equation contains a somewhat veiled statement of a simple geometrical result—that the vector **r** sweeps out area at a constant rate. One way of seeing this is to multiply Eq. (13–8) throughout by r:

$$r^2\frac{d^2\theta}{dt^2} + 2r\frac{dr}{dt}\frac{d\theta}{dt} = 0$$

The left-hand side may then be recognized as the derivative with respect to t of the product $r^2\, d\theta/dt$:

$$\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = r^2\frac{d^2\theta}{dt^2} + 2r\frac{dr}{dt}\frac{d\theta}{dt}$$

If we integrate this expression, we therefore have

$$r^2\frac{d\theta}{dt} = \text{const.} \qquad (13\text{–}9)$$

Now in Fig. 13–2 we show the area ΔA (shaded) swept out by **r** in a short time Δt. It is the triangle POQ (we take PQ to be indistinguishable from a straight line if it is short enough) and we have

$$\Delta A = \tfrac{1}{2}r(r + \Delta r)\sin\Delta\theta$$

The rate at which area is being swept out, instantaneously, is the limit of $\Delta A/\Delta t$ for $\Delta t \to 0$. Since, as we approach this limit, $\Delta r/r \to 0$ and $\sin\Delta\theta \to \Delta\theta$, we arrive at the result

$$\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt} \qquad (13\text{–}10a)$$

Thus we recognize the constant on the right-hand side of Eq.

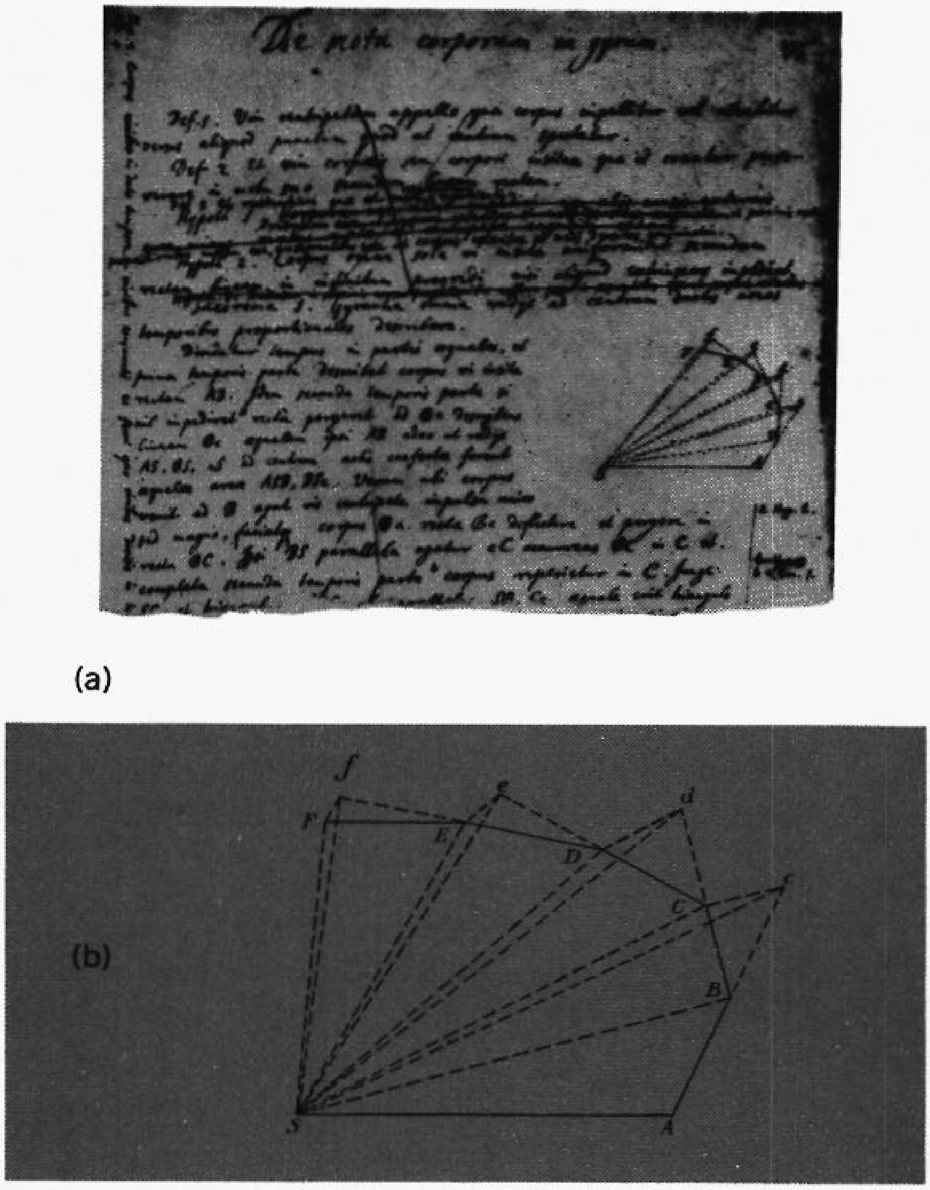


Fig. 13–3 (a) A portion of Newton's manuscript, De Motu, *showing the basis of his proof of the law of equal areas for a central force. (b) Enlarged copy of Newton's diagram. (From J. Herivel,* The Background to Newton's Principia, *Oxford University Press, London, 1965.)*

(13–9) as twice the rate (a *constant* rate) at which the radius vector **r** sweeps out area, and we therefore have

$$\text{(Any central force)} \qquad \frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt} = \text{const.} \qquad (13\text{–}10b)$$

The result expressed by Eq. (13–10b) was first discovered by Kepler in his analysis of planetary motions (of which more later). It was stated by him in what is known as his second law (although it was actually the first chronologically). Newton understood it on the same dynamical grounds as we have discussed above, i.e., as a feature of the motion of an object acted on by any kind of force that is directed always to the same point. He visualized the action of such a force as a succession of small kicks or impulses, which in the limit would go over into a continuously applied influence. He set out this view of the process in a tract written in 1684 (about 2 years before the *Principia*).[1] Figure 13–3(a) is a reproduction of a small fragment of the work, indicating Newton's approach to the problem. With the help of an enlarged version of his sketch [Figure 13–3(b)] we can more readily follow Newton's argument, which as usual was geometrical.

Newton imagines an object traveling along AB and then receiving an impulse directed toward the point S. As a result it now travels along the line BC instead of Bc. Similar impulses carry it to D, E, and F. To make things quantitative, Newton visualizes the displacement BC as being, in effect, the combination of the displacement Bc, equal to AB, that the object would have undergone if it had continued for an equal length of time with its original velocity, together with the displacement cC parallel to the line BS along which the impulse was applied. This at once yields the law of areas by a simple argument: The triangles SAB and SBc are equal, having equal bases (AB and Bc) and the same altitude. The triangles SBc and SBC are equal, having a common base (SB) and lying between the same parallels. Hence $\triangle SAB = \triangle SBC$.

THE CONSERVATION OF ANGULAR MOMENTUM

We give a more modern and more fundamental slant to the law of areas by expressing it in terms of the conservation of *orbital angular momentum*. If a particle at P [Fig. 13–4(a)] is acted on by a force $\mathbf{F}$, we have

$$\mathbf{F} = m\mathbf{a} = m\frac{d\mathbf{v}}{dt}$$

Let us now form the vector (cross) product of the position vector $\mathbf{r}$ with both sides of this equation:

$$\mathbf{r} \times \mathbf{F} = \mathbf{r} \times m\frac{d\mathbf{v}}{dt} \tag{13–11}$$

The left-hand side is the torque $\mathbf{M}$ due to $\mathbf{F}$ about O.

[1]This tract, called *De Motu* (Concerning the Motion of Bodies), contains many of the important results that were later incorporated in the *Principia*.