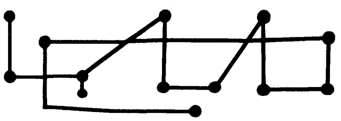
Mathematics and the Twelve-Tone System: Past, Present, and Future



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Introduction

Certainly the first major encounter of non-trivial mathematics and non-trivial music was in the conception and development of the twelve-tone system from the 1920s to the present. Although the twelve-tone system was formulated by Arnold Schoenberg, it was Milton Babbitt whose ample but non-professional background in mathematics made it possible for him to identify the links between the music of the Second Viennese School and a formal treatment of the system. To be sure, there were also important inroads in Europe as well,[1] but these were not often marked by the clarity and rigor introduced by Babbitt in his series of seminal articles from 1955 to 1973 (Babbitt 1955, 1960, 1962, 1974).

This paper has four parts. First, I will sketch a rational reconstruction of the twelve-tone system as composers and researchers applied mathematical terms, concepts, and tools to the composition and analysis of serial music. Second, I will identify some of the major trends in twelve-tone topics that have led up to the present. Third, I will give a very brief account of our present mathematical knowledge of the system and the state of this research. Fourth, I will suggest some future directions as well as provide some open questions and unproven conjectures.

But before I can start, we need to have a working definition of what the twelve-tone system is, if only to make this paper's topic manageable. Research into the system eventually inspired theorists to undertake formal research into other types of music structure; moreover of late, such research has now actually identified twelve-tone music and structure as special cases of much more general musical and mathematical models, so that for instance serial music and Riemannian tonal theory are both aptly modeled by group theory, rather than demarked as fundamentally different or even bi-unique.

Thus I will provisionally define the twelve-tone system as the musical use of ordered sets of pitch classes in the context of the twelve-pitch-class universe (or aggregate) under specified transformations that preserve intervals or other features of ordered sets or partitions of the aggregate. Thus the row, while it once was thought to be the nexus of the system, is only one aspect of the whole—even if the row embraces all of the characteristics I've mentioned: the aggregate, ordered pc sets and, not so obviously, partitions of the aggregate. Thus an object treated by the twelve-tone system can be a series or cycle of any number of pitch classes, with or without repetition or duplication, as well as multi-dimensional constructs such as arrays and networks, or sets of unordered sets that partition the aggregate.

The Introduction of Math into Twelve-Tone Music Research

Schoenberg's phrase, "The unity of musical space," while subject to many interpretations, suggests that he was well aware of the symmetries of the system (Schoenberg 1975). In theoretic word and compositional deed he understood that there was a singular two-dimensional "space" in which his music lived—that is, the space of pitch and time. Indeed, the basic transformations of the row, retrograde and inversion, plus retrograde-inversion for closure (and P as the identity) were eventually shown to form a Klein four-group (see Example 1).

That this space is not destroyed or deformed under these operations

P	R	I	RI
R	P	RI	I
I	RI	P	R
RI	I	R	P

P:	0 1 6 2 7 9 3 4 A B 5 8
R:	8 5 B A 4 3 9 7 2 6 1 0
I:	0 B 6 A 5 3 9 8 2 1 7 4
RI:	4 7 1 2 8 9 3 5 A 6 B 0

EXAMPLE 1: THE SERIAL FOUR-GROUP

gives it unity. Yet, from today's standpoint, the details of this symmetry are quite unclear. What kinds of pitch spaces? Pitch, or pitch-class, or merely contour? Is I mirror inversion or pitch-class inversion? Is RI a more complex operation than I or R alone? What about transposition's interaction with the P, I, R, RI group? And so forth. The lack of clarity, which is actually more equivocal than I've mentioned (because there is no acknowledgment of the different conceptions of intervals between pitch entities), fostered misconceptions about the aural reality of the system on one hand and the justification of its application to structuring other so-called parameters of music on the other. Future research would correct this ambiguity, differentiating it into different musical spaces and entities.

Neither Schoenberg nor his students, nor even the next generation of European serial composers, ever addressed these questions. It was detailed analysis of the music of Schoenberg, Webern, and to some extent Berg, that led to clarity and rigor. The results of such studies beginning circa 1950 revealed that the first generation of twelve-tone composers had principled reasons for deploying rows in music. First, the system itself was shown to preserve musical properties such as interval and interval class; Babbitt (1960) called this twelve-tone invariance.

In Example 2, I have distilled identities from Babbitt 1960 and Martino 1961; these identities can easily be derived from the definition of rows, ordered intervals and the twelve-tone operators T_n, I, and R.

Let the twelve-element array P model a row.
The interval between P_a and P_b is written as the function i:
$\mathrm{i}(P_a, P_b) = P_b - P_a$; $-\mathrm{i}(P_a, P_b)$ is the inversion of $\mathrm{i}(P_a, P_b)$.

$$-\mathrm{i}(P_a, P_b) = \mathrm{i}(P_b, P_a)$$
$$\mathrm{i}(\mathrm{T}_n P_a, \mathrm{T}_n P_b) = \mathrm{i}(P_a, P_b)$$
$$\mathrm{i}(\mathrm{T}_n \mathrm{I} P_a, \mathrm{T}_n \mathrm{I} P_b) = \mathrm{i}(P_b, P_a)$$

Let the array $\mathrm{INT}(P)$ be the interval succession of P;
$\mathrm{INT}(P) = (\mathrm{i}(P_0, P_1), \mathrm{i}(P_1, P_2), \ldots, \mathrm{i}(P_A, P_B))$

$\mathrm{INT}(\mathrm{T}_n P) = \mathrm{INT}(P)$	(T_n preserves the interval succession of P, for all n.)
$\mathrm{INT}(\mathrm{T}_n \mathrm{I} P) = \mathrm{I}(\mathrm{INT}(P))$	($\mathrm{T}_n\mathrm{I}$ inverts the interval succession of P, for all n.)
$\mathrm{INT}(\mathrm{RT}_n P) = \mathrm{R}(\mathrm{INT}(\mathrm{I}P))$	(RT_n inverts and retrogrades the interval succession of P, for all n)
$\mathrm{INT}(\mathrm{RIT}_n P) = \mathrm{R}(\mathrm{INT}(P))$	($\mathrm{RT}_n\mathrm{I}$ retrogrades the interval succession of P, for all n)
$\mathrm{i}(P_a, \mathrm{T}_n P_a) = n$	(The interval from pcs in P and $\mathrm{T}_n P$ at order position a is n, for all a and n)
$P_a + \mathrm{T}_n \mathrm{I} P_a = n$	(The sum of pcs at order number a in P and $\mathrm{T}_n\mathrm{I}P$ is n, for all a and n)
$\mathrm{i}(P_a, \mathrm{RT}_n P_a) = -\mathrm{i}(P_{B-a}, \mathrm{RT}_n P_{B-a})$	(The interval from pcs in P to $\mathrm{RT}_n P$ at order position a is the inversion of the interval from pcs in P to $\mathrm{RT}_n P$ at order position $\mathrm{B} - a$, for all a and n.)
$\mathrm{i}(P_a, \mathrm{RT}_n \mathrm{I} P_a) = \mathrm{i}(P_{B-a}, \mathrm{RT}_n \mathrm{I} P_{B-a})$	(The interval from pcs in P to $\mathrm{RT}_n\mathrm{I}P$ at order position a is the interval between pcs in P to $\mathrm{RT}_n\mathrm{I}P$ at order position $\mathrm{B} - a$, for all a and n.)

EXAMPLE 2: TWELVE-TONE INVARIANCE AMONG ORDERED INTERVALS IN ROWS AND PAIRS OF ROWS (AFTER BABBITT 1960 AND MARTINO 1961)

(This example uses an array to model a row but Babbitt (1960) uses a different concept: A row is a set of unordered pairs, each pair consisting of a pc and its order position in the row).

Second, in addition to twelve-tone invariance, Babbitt and others showed that the rows used by Schoenberg, Berg, and Webern were not chosen capriciously, but would depend on features such as shared ordered and unordered sets. Babbitt (1962) called this set-structure invariance (see Example 3). Row succession by complementation or row linking is another aspect of this thinking (see Example 4).

These examples demonstrate that musical objects and relations were supported and cross related from one row to another to build musical continuity, association and form.

Unordered sets shared by related rows.
P is the row of Schoenberg's Violin Concerto, op. 36

P:	$\underline{016279}$	$\underline{34AB58}$
	G	H
RT$_5$IP:	$\underline{906712}$	$\underline{8A3B45}$
	G	H

P:	$\underline{016}$	$\underline{279}$	$\underline{34A}$	$\underline{B58}$
	W	X	Y	Z
T$_4$IP:	$\underline{43A}$	$\underline{297}$	$\underline{106}$	$\underline{5B8}$
	Y	X	W	Z

Ordered sets shared by related rows.

RT$_5$IP:		9	0	6	7	1	2	8	A	3	B	4	5
	J:										B		5
	K:		0	6			2		A				
	L:					1		8		3		4	
	M:	9			7								
T$_B$$P$:		B	0	5	1	6	8	2	3	9	A	4	7
	J:	B		5									
	K:		0			6		2			A		
	L:				1		8		3			4	
	M:									9			7

EXAMPLE 3: ROWS RELATED BY SHARED UNORDERED AND ORDERED SETS

P is the row of Webern's Variations for Orchestra, op. 30.

Row succession by complementation (underlined pcs form aggregates from one row to the next)

P: $/\ \mathrm{RT}_6P$: $/\ \mathrm{T}_5\mathrm{I}P$:

0 1 4 3 2 5 $\underline{\text{6 9 8 7 A B / 5 4 1 2 3 0}}$ $\underline{\text{B 8 9 A 7 6 / 5 4 1 2 3 0}}$

$/\ \mathrm{RT}_\mathrm{B}\mathrm{I}P$: $/\ P$:

$\underline{\text{B 8 9 A 7 6 / 0 1 4 3 2 5}}$ $\underline{\text{6 9 8 7 A B / 0 1 4 3 2 5}}$ etc.

Row succession by linking

Via $\mathrm{T}_n\mathrm{RI}$ succession (a twelve-tone invariant):

P: 0 1 4 3 2 5 6 9 8 7 A B / T_8P: 8 9 0 B A 1 2 5 4 3 6 7

$\mathrm{RT}_9\mathrm{I}P$: A B 2 1 0 3 4 7 6 5 8 9 /

Via T_5 succession (a special invariance of this row):

P: 0 1 4 3 2 5 6 9 8 7 A B

T_5P: 5 6 9 8 7 A B 2 1 0 3 4

$\mathrm{T}_\mathrm{A}P$: A B 2 1 0 3 4 7 6 5 8 9

T_3P: 3 4 7 6 5 8 9 0 B A 1 2

T_8P: 8 9 0 B A 1 2 5 4 3 6 7

EXAMPLE 4: ROW SUCCESSION BY COMPLEMENTATION AND LINKING

Early pre-mathematical research also concerned itself with the relations of the system to tonality. Here are some of the specific questions that arose: Was the first pc of a row a kind of tonic? Or was a row tonal if it contained tonal material such as triads and seventh chords? Did the P and I rows participate in a duality like that of tonic and dominant? In general, tonality was either seen as opposed to the system or both were transcended by a Hegelian sublation into aspects of the same musical and universal laws. But a lack of clarity that conflated reference, quotation, suggestion, analogy, and instantiation made the question impossible to define, much less answer. This obsession with tonality retarded work on the vertical or harmonic combination of rows in counterpoint. Even after the set theories of Howard Hanson (1960), Hubert Howe (1965), and Allen Forte (1964, 1973) had become established, it was not until the 1980s that the problem was generalized to all types of rows and set classes (Morris 1984). By this point in time, clarity about the nature of musical systems and their models helped make the tonality issue manageable. Benjamin Boretz's "Meta-Variations," published serially in *Perspectives of New Music* from 1969 to 1973 is the seminal work on this topic. Understanding tonality as recursive but invariant among levels made it possible to conceive of the multiple-order-number-function rows (Batstone 1972) that implement such properties to various degrees. And it was Babbitt who revealed that Schoenberg's later "American" twelve-tone practice was founded on hierarchic principles so that an entire passage of music would be controlled by a quartet of rows from a row-class preserving partial ordering and hexachordal partitions (Babbitt 1961). David Lewin mathematically elaborated and extended this notion in his early work on row nestings (Lewin 1962).

The early research focused on entities. The row was considered the core idea of the system and specific types of rows, such as order-invariant rows or the all-interval rows (called AIS) were invented (or discovered) and discussed (see Example 5). For example, the all-interval row of Berg's *Lyric Suite* (and also used in other of his works) and its T_6R invariance provides an example. Studies of various types of rows continued up until the 1980s, and I will provide examples below.

Berg *Lyric Suite* row:

C: 0 B 7 4 2 9 3 8 A 1 5 6 $\qquad C = RT_6C$
INT(C): B 8 9 A 7 6 5 2 3 4 1

Wedge row:

D: 0 1 B 2 A 3 9 4 8 5 7 6 $\qquad D = RT_6D$
INT(D): 1 A 3 8 5 6 7 4 9 2 B

NB: $D = r_6T_9M_5C$

Mallalieu row:

E: 0 1 4 2 9 5 B 3 8 A 7 6 $\qquad E = RT_6E$
INT(E): 1 3 A 7 8 6 4 5 2 9 B

rRM_7-invariant row:

F: 0 5 8 9 3 A 2 4 1 B 7 6 $\qquad F = r_4RT_9M_7F$
INT(F): 5 3 1 6 7 4 2 9 A 8 B

Krenek's AIS row without special invariance:

G: 0 3 A 4 9 B 8 7 5 1 2 6
INT(G): 3 8 6 7 2 9 B A 8 1 4

EXAMPLE 5: BERG'S *LYRIC SUITE* ROW AND OTHER ALL-INTERVAL ROWS (AIS)

The trope system of Hauer (1925), contemporaneous with the invention of the twelve tone system, proposed a system of hexachordal partitions from which music could be made, but the use of tropes was not suitably differentiated from the use of rows, so they seemed to be rival systems rather than different, non-opposed ways of creating music structure. (Up until perhaps 1980 there was similar lack of distinction between Perle's cyclic sets and the twelve-tone row (Perle 1977).)

Questions of enumeration also were raised: How many rows? How many distinct related rows under transposition, inversion and/or retrograde (since some rows are invariant)? How many unordered sets of pitch classes? How many tropes? How many chord types? Not until the 1960s was it understood that answers to such questions were determined by what transformations one included as canonic—as defining equivalence classes. (This involves changes in the cyclic index of Burnside's method of counting equivalence classes.)

As I have pointed out, it was the lack of adequate formal descriptions and models that limited early work on the twelve-tone system. The introduction of mathematical tools changed all that. By the 1970s it became clear that the system was not only about things, but also about the ways in which these things were changed or kept invariant within the system. In 1978 Daniel Starr enunciated the entity/transformational distinction that is so familiar to us today. It took some time however before the difference between a binary group and a transformational group was appreciated—or to put it another way, that the set of transformations that formed a group was distinct from the objects it acted on—and that these objects might be not only pitch classes, but sets, arrays, networks, etc., which in turn might suggest a variety of types of transformation groups (Lewin 1978, Morris 1978). This widened the scope of twelve-tone theory to encompass non-twelve-tone things such as tonal chords, scales, and the like.

The intervention of mathematical tools occurred in three stages. First was the use of mathematical terminology and symbols including the use of numbers to identify pitch classes, order numbers, and transpositional levels. Variable names (with subscripts) such as S_n or P_n, I_n, R_n, and RI_n were used to name rows. However, this practice conflated the difference between a label denoting an entity versus a transformation.

A second stage was the use of mathematical and logical concepts such as equivalence and relation, and the use of mathematical terms borrowed from real math or computer science such as "invariance" or "function." Sometimes, strange terminology from the mathematical point of view resulted, such as the names "set class" or "interval vector," or using the term "complement" to mean "inverse." But at least these ideas and functions were more or less contextually well defined. At this stage, concepts were generally used to describe the properties of musical entities. Perhaps the most important insight was Babbitt's claim that the twelve-tone system was inherently permutational rather than combinational (Babbitt 1960). While this assertion is perhaps too categorical,[2] Babbitt opened the door to the use of group theory in musical research. Researchers also adopted the language of set theory to describe musical properties and relations among sets of musical things. Nevertheless, confusion remained because the same terms were used for different kinds of things. For instance, in the 1970s the term "set" meant row at Princeton and unordered pc set at Yale. Moreover, technical labels did not address all the important differences. The distinction between interval and interval class was not explicitly defined; later, the interval class would finally be understood as the "distance" between pitch classes or pitches, while the term "interval" would define a directed distance between two pitch classes or a transformation of one to another. Sets and set classes were still not adequately distinguished in the literature until around 1975, even after the publication of Allen Forte's important book (1973), which does not explicitly make the distinction.

The third stage involved the use of mathematical reasoning in music theory. At first this reasoning would be alluded to, or presented in words, or in symbols in ad hoc ways. Sometimes this work was done behind the scenes, as in the proof of the complement theorem, which was asserted in the late 1950s but not explicitly proven in the literature until the 1980s (see Example 6).[3]

But it didn't take long before there were ways to do something like professional mathematics in the body of a music theory paper. This led to some consensus about the nature of the terminology and formalisms used in music theory today—but sometimes these do not correspond one to one with mathematical treatment. With the use of real mathematics in music theory, theorists realized that there are branches of mathematics that could be applied to their problems; up to then many theorists constructed the mathematics needed from the ground up. Sometimes this work was original or unique from the mathematical point of view. Indeed, Milton Babbitt has remarked that John Tukey "observed casually that [Babbitt] had produced, inadvertently, some hitherto unknown and, apparently, not uninteresting theorems in group theory" (Babbitt 1976).

The transition from stage two to three was aided by the use of computers to model and/or enumerate aspects of the twelve-tone system. Starting circa 1970, many graduate programs—most notably at Princeton, Yale, and the Eastman School of Music—introduced faculty

1. *Transpositional Common-Tone Theorem:*

$$\#(A \cap \mathrm{T}_n B) = \mathrm{MUL}(A, B, n).$$

The function $\mathrm{MUL}(A, B, n)$ is the multiplicity of $\mathrm{i}(a, b) = n$ for all a and b where A and B are pc sets and $a \in A$, $b \in B$.

2. *Inversional Common-Tone Theorem:*

$$\#(A \cap \mathrm{T}_n \mathrm{I} B) = \mathrm{SUM}(A, B, n).$$

The function $\mathrm{SUM}(A, B, n)$ is the number of *sums* $a + b = n$ for all a and b where A and B are pc sets and $a \in A$, $b \in B$.

3. *Complement Theorem:*

$$\mathrm{MUL}(A', B', n) = 12 - (\#A + \#B) + \mathrm{MUL}(A, B, n).$$

is the cardinality operator; $\#X$ is the cardinality of X.
′ is the complement operator; A' is the complement of A.

EXAMPLE 6: COMMON-TONE AND COMPLEMENT THEOREMS

and students to computer programming via seminars and courses. The result was an appreciation of the need for correct and apt formalization of music theoretic concepts and reasoning. This paved the way for researchers to go directly into the math that underlay the design and implementation of the computer programs. Moreover, the output of programs posed new puzzles. What was the structure underlying the output data?

These three stages actually overlapped in the literature depending on the mathematical sophistication of both authors and readers. Some mathematical treatments of serial topics remained virtually unread until music theory as a whole caught up.

For instance, Walter O'Connell (1968) wrote a mathematically interesting and prescient article in *Die Reihe* 8; however, theorists and composers have generally overlooked it even though it is the first published account of the multiplicative pitch-class operations, the order-number/pitch-number exchange operator, and networks of pitch classes and transformations in multiple dimensions. Sometimes such work was not even published or, if published, criticized as irrelevant to music study—as unwanted applied mathematics. The prime example involves the classical papers by David Lewin on the interval function. Lewin's sketch of the mathematical derivation of the function via Fourier analysis —published in *JMT* in 1959 and 1960—was not appreciated and developed until recently by young theorists such as Ian Quinn (2006).

Important Results and Trends

Perhaps the most important development in twelve-tone theory was the invention of invariance matrices by Bo Alphonce at Yale in 1974. Here T- and I-matrices were shown to display properties of pairs of ordered or unordered sets. In addition, Alphonce used them to analyze one passage of music in terms of another. Since the row table (probably invented by Babbitt in the 1950s) is a special case of the T-matrix, the complex of rows was shown to be related to its generating row in ways supplementing those already formalized by earlier research such as the common tone and hexachord theorem. Example 7 shows the T- and I-matrices for a row and a hexachord/trichord pair.

T-matrix E: $E_{ij} = P_i + \mathrm{I}P_j$

$P = 0\ 1\ 6\ 2\ 7\ 9\ 3\ 4\ \mathrm{A}\ \mathrm{B}\ 5\ 8$

	0	1	6	2	7	9	3	4	A	B	5	8
0	0	1	6	2	7	9	3	4	A	B	5	8
B	B	0	5	1	6	8	2	3	9	A	4	7
6	6	7	0	8	1	3	9	A	4	5	B	2
A	A	B	4	0	5	7	1	2	8	9	3	6
5	5	6	B	7	0	2	8	9	3	4	A	1
3	3	4	9	5	A	0	6	7	1	2	8	B
9	9	A	3	B	4	6	0	1	7	8	2	5
8	8	9	2	A	3	5	B	0	6	7	1	4
2	2	3	8	4	9	B	5	6	0	1	7	A
1	1	2	7	3	8	A	4	5	B	0	6	9
7	7	8	1	9	2	4	A	B	5	6	0	3
4	4	5	A	6	B	1	7	8	2	3	9	0

I-matrix F: $F_{ij} = P_i + P_j$

$P = 0\ 1\ 6\ 2\ 7\ 9\ 3\ 4\ \mathrm{A}\ \mathrm{B}\ 5\ 8$

	0	1	6	2	7	9	3	4	A	B	5	8
0	0	1	6	2	7	9	3	4	A	B	5	8
1	1	2	7	3	8	A	4	5	B	0	6	9
6	6	7	0	8	1	3	9	A	4	5	B	2
2	2	3	8	4	9	B	5	6	0	1	7	A
7	7	8	1	9	2	4	A	B	5	6	0	3
9	9	A	3	B	4	6	0	1	7	8	2	5
3	3	4	9	5	A	0	6	7	1	2	8	B
4	4	5	A	6	B	1	7	8	2	3	9	0
A	A	B	4	0	5	7	1	2	8	9	3	6
B	B	0	5	1	6	8	2	3	9	A	4	7
5	5	6	B	7	0	2	8	9	3	4	A	1
8	8	9	2	A	3	5	B	0	6	7	1	4

T-matrix G: $G_{ij} = X_i + \mathrm{I}\Upsilon_j$

$X = \{0\ 1\ 2\ 4\ 7\ 8\}$; $\Upsilon = \{3\ 4\ 8\}$

	0	1	2	4	7	8
9	9	A	B	1	4	5
8	8	9	A	0	3	4
4	4	5	6	8	B	0

I-matrix H: $H_{ij} = X_i + \Upsilon_j$

$X = \{0\ 1\ 2\ 4\ 7\ 8\}$; $\Upsilon = \{3\ 4\ 8\}$

	0	1	2	4	7	8
3	3	4	5	7	A	B
4	4	5	6	8	B	A
8	8	9	A	0	3	4

Example 7: T- and I- matrices for a row and a hexachord/trichord pair

From one point of view, the T-matrix is a complete list of the directed intervals between the entities that generate it. Thus, as illustrated in Example 8, T- and I-matrices generate Lewin's interval function. Such matrices have many other functions and uses, such as spelling out the verticals in Stravinsky's rotational arrays, since those array's columns are the diagonals of the T-matrix. Example 9 shows that the *Tonnetz* is a T-matrix and illustrates the derivation of a rotational array from a T-matrix, and the transpositional combination of two unordered sets. Example 10 displays permutations between two rows and their interaction to form a determinate contour.

In my 1987 book, invariance matrices underlie and unify many different aspects of serial theory including the relations of sets of transformations and mathematical groups. This is because a T-matrix is a group table or a part thereof.

T-matrix G: $G_{ij} = X_i + \mathrm{I}\Upsilon_j$ I-matrix H: $H_{ij} = X_i + \Upsilon_j$

$X = \{0\ 1\ 2\ 4\ 7\ 8\}$; $\Upsilon = \{3\ 4\ 8\}$ $X = \{0\ 1\ 2\ 4\ 7\ 8\}$; $\Upsilon = \{3\ 4\ 8\}$

	0	1	2	3	7	8
9	9	A	B	1	4	5
8	8	9	A	0	3	4
4	4	5	6	8	B	0

	0	1	2	3	7	8
3	3	4	5	7	A	B
4	4	5	6	8	B	A
8	8	9	A	0	3	4

IFUNC(X, Υ) = {210132102222} IFUNC(X, IΥ) = {200232112122}

IFUNC$_n$(X, Υ) is the number of *n*s in the T-matrix
IFUNC$_n$(X, IΥ) is the number of *n*s in the I-matrix

$$\mathrm{IFUNC}_n(X, \Upsilon) = \mathrm{MUL}(X, \Upsilon, n)$$
$$\mathrm{IFUNC}_n(X, \mathrm{I}\Upsilon) = \mathrm{SUM}(X, \Upsilon, n)$$

Corollary of Transpositional Common-Tone Theorem:
$$\#(X \cap \mathrm{T}_n\Upsilon) = \mathrm{IFUNC}_n(X, \Upsilon)$$

Corollary of Inversional Common-Tone Theorem:
$$\#(X \cap \mathrm{T}_n\mathrm{I}\Upsilon) = \mathrm{IFUNC}_n(X, \mathrm{I}\Upsilon)$$

(# is the cardinality operator; $\#X$ is the cardinality of X.)

EXAMPLE 8: T- AND I-MATRICES GENERATE LEWIN'S IFUNC

T-matrix H:

$H_{i,j} = X_i + \mathrm{I}X_j$ Rotational array derived from T-matrix, H.

(Diagonals of H become columns of rotational array)

X = 0 A 7 9 8* Columns 0 and 3 are I-invariant;
columns 1 and 5 and 2 and 4 are I-related

0	A	7	9	2	8
2	0	9	B	4	A
5	3	0	2	7	1
3	1	A	0	5	B
A	8	5	7	0	6
4	2	B	1	6	0

0	A	7	9	2	8
0	9	B	4	A	2
0	2	7	1	5	3
0	5	B	3	1	A
0	6	A	8	5	7
0	4	2	B	1	6

Transpositional combination of {025} and {289A}
{023456789A} = {025} ∗ {289A}
set-class 10-3[012345679A] = 3-5[025] ∗ 4-5[0126]

X = {0 2 5}; Υ = {2 8 9 A}

	0	2	5
2	A	0	3
8	4	6	9
9	3	5	8
A	2	4	7

The *Tonnetz* is the T-matrix for X = {0369} and Υ = {048}

	0	3	6	9
0	0	3	6	9
4	4	7	A	1
8	8	B	2	5

* X is the first hexachord of Stravinsky's "A Sermon, a Narrative, and a Prayer."

EXAMPLE 9: T-MATRIX, DERIVED ROTATIONAL ARRAYS, TRANSPOSITIONAL COMBINATION, AND *TONNETZ*

T-matrix K: $K_{i,j} = P_i + \mathrm{I}P_j$ $P = 0143256987AB$

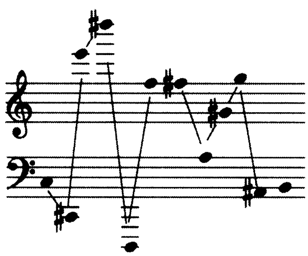
	0	1	4	3	2	5	6	9	8	7	A	B
0	0	1	4	3	2	5	6	9	8	7	A	B
B	B	0	3	2	1	4	5	8	7	6	9	A
8	8	9	0	B	A	1	2	5	4	3	6	7
9	9	A	1	0	B	2	3	6	5	4	7	8
A	A	B	2	1	0	3	4	7	6	5	8	9
7	7	8	B	A	9	0	1	4	3	2	5	6
6	6	7	A	9	8	B	0	3	2	1	4	5
3	3	4	7	6	5	8	9	0	B	A	1	2
4	4	5	8	7	6	9	A	1	0	B	2	3
5	5	6	9	8	7	A	B	2	1	0	3	4
2	2	3	6	5	4	7	8	B	A	9	0	1
1	1	2	5	4	3	6	7	A	9	8	B	0

Submatrices show permutations of P with T_3P and T_5P

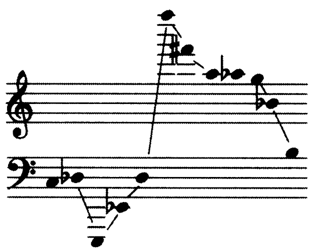
	0	1	4	3	2	5	6	9	8	7	A	B
3				3								
4			3									
7										3		
6							3					
5						3						
8									3			
9								3				
0	3											
B												3
A											3	
1		3										
2					3							

	0	1	4	3	2	5	6	9	8	7	A	B
5						5						
6							5					
9								5				
8									5			
7										5		
A											5	
B												5
2					5							
1		5										
0	5											
3				5								
4			5									

Permutations realized as musical contours:



Vertical (top to bottom) = T_3P;
Horizontal (left to right) = P



Vertical (top to bottom) = T_5P;
Horizontal (left to right) = P

EXAMPLE 10: PERMUTATIONS BETWEEN TWO ROWS AND THEIR INTERACTION TO FORM A DETERMINATE CONTOUR

There are many other important landmarks on the way to the present, and I will mention some of them as part of an identification of the important trends starting from about 1950. I believe the books and articles I shall cite are still of interest to us today and should be read by students who wish to go further into mathematical music theory—even if they are not studying anything directly connected to what I call "composition with pitch-classes."

I've already mentioned Babbitt's important articles on serial music. His earliest work, as documented in his 1947 Princeton dissertation (but not recommended for acceptance until 1992) was to develop the compositional practices of Schoenberg and Webern into a system of combinatoriality—that is, aggregate preservation among contrapuntal combinations of rows. (It is quite an achievement, for Babbitt was able to make much progress without the explicit distinction of pitch and pitch-class, operator and entity, and set and set-class, and without any explicit invocation of group theory.) Babbitt (1961, 1974), Donald Martino (1961), and Starr and myself (Starr and Morris 1978) continued to develop the theory of combinatoriality. This work, generalized as two-dimensional arrays with rows or other types of pitch-class entities in the array rows and with aggregates or other pitch-class "norms" in the columns, de-emphasized the row. Example 11 provides some combinatorial arrays.

It was established that while small combinatorial arrays depended on the properties on the generating row, larger and more elaborate arrays depended on more global principles, as in the rotational array of Stravinsky's serial music. Consequently, the emphasis shifted from the row to the array so that the array might be considered the more basic musical unit (Winham 1964; Morris 1984, 1987). This was inherent in Babbitt's serial music, which, while unnoticed for quite a time in the literature, had been composed from pairs of combinatorial rows rather than rows alone. This meant, as with the arrays, that musical realization was a matter of choosing an ordering out of the partial ordering given by the array columns.

Thus various types of posets of the aggregate and their possible realization as rows became the focus of this research. Lewin (1976) and Starr (1984) were the first to specify and formalize the use of posets in twelve-tone theory (Morris 1984, 1987, 1995a). An order lattice, posets, and order matrix derived from an array column are given in Example 12.

Eventually the array concept became detached from aggregates and rows so that it could model the preservation of harmonic relations among simultaneous linear presentations of any kinds of pitch or

P is the row of Schoenberg's op. 36.
Each array column is aggregate.
Each array row is a transformation of *P*.
Below each column is a multiset, identifying the 12-partition of the column.

P	0 1 6 2 7 9	3 4 A B 5 8
RP	8 5 B A 4 3	9 7 2 6 1 0
	6^2	6^2

P	0 1 6	2 7 9	3 4 A	B 5 8
RP	8 5 B	A 4 3	9 7 2	6 1 0
T_3IP	3 2 9	1 8 6	0 B 5	4 A 7
RT_3IP	7 A 4	5 B 0	6 8 1	9 2 3
	3^4	3^4	3^4	3^4

P	0 1 6 2 7	9 3 4 A B 5 8	
T_3P	3 4 9 5 A	0 6	7 1 2 8 B
RT_3P	B 8	2 1 7	6 0 A 5 9 4 3
	5^22	732	75

P	0 1 6	2 7 9		3 4 A	B 5 8
T_4IP	4 3 A 2 9 7	1 0 6 5 B 8	($T_{11}P$)	B 0 5 1 6 8	2 3 9 A 4 7
RP	8 5 B	A 4 3		9 7 2	6 1 0
	63^2	63^2		63^2	63^2

P	0 1 6			2 7 9 3	4 A B 5 8
T_4P	4 5 A	6 B 1	7 8 2 3 9		0
T_9IP	9 8 3	7 2 0	6	5 B A 4	1
RT_3IP	7	A 4 5	B	0 6 8 1	9 2 3
RT_7IP	B 2	8 9 3	4 A 0 5 1		6 7
	3^321	3^4	5^21^2	4^3	5321^2

EXAMPLE 11: SOME COMBINATORIAL ARRAYS

pitch-class entities. Such non-aggregate combinatoriality was useful in formalizing and extending aspects of the music of Carter and others. The topic extends into set-type saturated rows, two-partition graphs, and the complement-union property; see Morris 1985, 1987. A simple example of a non-aggregate combinatorial array occurs in Example 13.

Research on partitions of the aggregate form a related trend to combinatoriality. Babbitt was the first composer to use all 77 partitions of the number 12 in his music by inventing the all-partition array (Babbitt 1961, 1974). See Example 14. The earliest emphasis on partitions is that of Hauer, whose tropes are collections of 6/6 partitions grouped by transposition. There are 44 such tropes, but if we use transposition and inversion to group the partitions there are 35 tropes—the number of pairs of complementary hexachordal set-classes.

Array column

2	7	8	3
5	B	A	4
0	6	8	1

Lattice derived from array column

$$\begin{tikzcd}
 & 2 \arrow[r] & 7 \arrow[r] & 9 \arrow[r] & 3 \arrow[dr] & \\
* \arrow[ur] \arrow[r] \arrow[dr] & 5 \arrow[r] & B \arrow[r] & A \arrow[r] & 4 \arrow[r] & * \\
 & 0 \arrow[r] & 6 \arrow[r] & 8 \arrow[r] & 1 \arrow[ur] &
\end{tikzcd}$$

Poset derived from lattice

{ (2, 7) (2, 9) (2, 3) (7, 9) (7, 3) (9, 3) (5, B) (5, A) (5, 4) (B, A) (B, 4) (A, 4) (0, 6) (0, 8) (0, 1) (6, 8) (6, 1) (8, 1) }

Order matrix derived from poset

	0	1	2	3	4	5	6	7	8	9	A	B
0		1					1		1			
1												
2				1				1		1		
3												
4												
5					1						1	1
6		1							1			
7				1						1		
8		1										
9				1								
A					1							
B					1						1	

EXAMPLE 12: LATTICE, POSET, AND ORDER MATRIX DERIVED FROM AN ARRAY COLUMN

0	245	79	
578			03A
A	037	25	
9		46	18B

Array rows are members of set-class 6-32[024579] (C all-combinatorial hexachord)

Array columns are members of set-class 6-8[023457] (B all-combinatorial hexachord)

EXAMPLE 13: NON-AGGREGATE COMBINATORIAL ARRAY

(Self-conjugates are in boldface.)

max. length of part ↓ / number of parts →	1	2	3	4	5	6	7	8	9	10	11	12
1												1^{12}
2						2^6	$2^5 1^2$	$2^4 1^4$	$2^3 1^6$	$2^2 1^8$	$2 1^{10}$	
3				3^4	$3^2 2^3$ $3^3 2 1$	$3 2^4 1$ $3^2 2^2 1^2$ $3^3 1^3$	$3 2^3 1^3$ $3^2 2 1^4$	$3 2^2 1^5$ $3^2 1^6$	$3 2 1^7$	$3 1^9$		
4			4^3	$4 3^2 2$ $\mathbf{4^2 2^2}$ $4^2 3 1$	$4 2^4$ $4 3 2^2 1$ $4 3^2 1^2$ $4^2 2 1^2$	$4 2^3 1^2$ $4 3 2 1^3$ $4^2 1^4$	$4 2^2 1^4$ $4 3 1^5$	$4 2 1^6$	$4 1^8$			
5			$5^2 2$ $5 4 3$	$5^2 1^2$ $5 4 2 1$ $5 3^2 1$ $5 3 2^2$	$5 4 1^3$ $\mathbf{5 3 2 1^2}$ $5 2^3 1$	$5 2^2 1^3$ $5 3 1^4$	$5 2 1^5$	$5 1^7$				
6		6^2	$6 5 1$ $6 4 2$ $6 3^2$	$6 4 1^2$ $6 3 2 1$ $6 2^3$	$6 3 1^3$ $6 2^2 1^2$	$\mathbf{6 2 1^4}$	$6 1^6$					
7		$7 5$	$7 4 1$ $7 3 2$	$7 3 1^2$ $7 2^2 1$	$7 2 1^3$	$7 1^5$						
8		$8 4$	$8 3 1$ $8 2^2$	$8 2 1^2$	$8 1^4$							
9		$9 3$	$9 2 1$	$9 1^3$								
10		$10\ 2$	$10\ 1^2$									
11		$11\ 1$										
12	12											

EXAMPLE 14: THE 77 PARTITIONS OF THE AGGREGATE

Martino's article of 1961 is an early development of the partitions of the aggregate followed more than 25 years later by Andrew Mead (1988), Harald Fripertinger (1993), Brian Alegant (1993), and Alegant and Lofthouse (2002).

As I've mentioned, studies of kinds of rows have led to generalities beyond rows. The next example provides a brief survey of some of these special rows. These types shown in Example 15 are not mutually exclusive so that a row might reside in all of these categories. (Example 15 does not show derived, all-combinatorial, and all-interval rows because I've already given examples of these types.) Example 16 illustrates the use of a self-deriving array with one of my short piano compositions. This kind of array allows each aggregate to be ordered according to the rows in the row class of the generating row.

These examples point out that research on derived, all-combinatorial

Derived rows

All-combinatorial rows

All-interval rows

Order-invariant rows

$P = r_9RT_4IP$: 0 3 7 B 2 5 9 1 4 6 8 A
(Berg, *Violin Concerto*)

All-trichord rows

Q: 0 1 B 3 8 A 4 9 7 6 2 5

Trichords: 3–1 3–9 3–7 3–3

3–6 3–8 3–2

3–11 3–5 3–4

Multiple-order-function rows

S: $\underset{X}{\underline{0\ 1\ B}}$ $\underset{Y}{\underline{4\ 8\ 5\ 9\ A}}$ $\underset{Z}{\underline{7\ 3\ 2\ 6}}$

RT_AIS:	4	8	7	3	0	1	5	2	6	B	9	A
Y:	4	8					5				9	A
Z:			7	3				2	6			
X:					0	1				B		

S embedded in successions of RT_AIS:
$4873\underline{01}526\underline{B}9A\underline{48}7301\underline{5}26B\underline{9A}48\underline{73}0152\underline{6}$...

Set-type saturated rows

T: 0 1 4 7 8 A B 2 5 6 9 3 (0 1 4 7)

Hexachords: 6–28 6–49

6–49 6–28

6–28 6–49

(Morris, *Concerto for Piano and Winds*)

EXAMPLE 15: TYPES OF ROWS

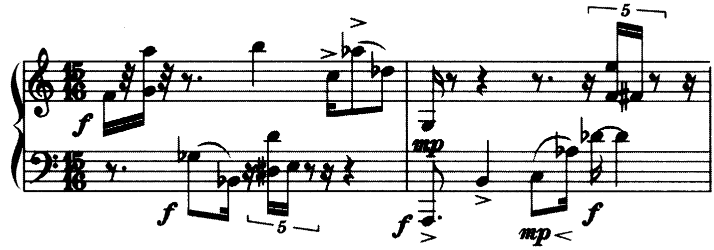
Z: 0 5 1 9 A B 2 7 3 4 6 8

Array:

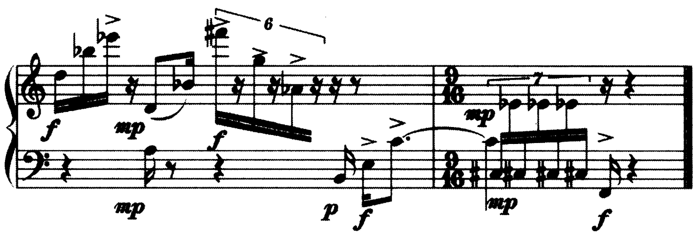
RT_5IZ:		9B 1 2A3	678 40 5
RT_3IZ:	79 B 081	456	2A 3
T_5Z:	5 A6 234	7 08	9 B 1

Linear rows:	579A6B234081	79B0814562A3	92A678B40135
	RT_1IZ	RT_3IZ	T_9Z

Piano:



RT_5IZ:	9 B 1
RT_3IZ:	79 B 0 8 1 456
T_5Z:	5 A 6 234 7 0 8



2 A 3 6 7 8 4 0 5
2 A 3
9 B 1

EXAMPLE 16: SELF-DERIVING ROW AND ARRAY

rows, all-interval rows, order-invariant rows, all-trichord rows, multiple-order-function rows (Batstone 1972; Morris 1976, 1977; Mead 1988, 1989; Scotto 1995), set-type saturated rows (Morris 1985) and self-deriving array rows (Starr 1984, Kowalski 1985) reflected new orientations to the use and function of the twelve-tone system, which developed, in turn, into considerations of various kinds of saturation in addition to aggregate completion, the embedding of one musical thing in itself or another, the preservation of properties among like entities such as ordering, transformations, and set structure. These topics are grounded in the cycles of transformations considered as permutations and the orbits of the permutation groups. These questions of preservation often hinge on whether pairs of transformations commute, and if their orbits and cycles are invariant under interval-preserving transformations. Mead's (1988) elaboration on the pitch-class/order-number isomorphism introduced by Babbitt and O'Connell is another signal contribution to this topic for it allows any subset of an ordered pc entity to be characterized as batches of pcs at batches of order numbers or vice versa; in this way, all partitions of the aggregate are available in each and every row and the difference between rows is based on the distributions of these partitions over the class of all rows.

The development of ways to extend adequately the relationships among pitch classes to time and other musical dimensions was an unsolved problem until the advent of Stockhausen's article ". . . how time passes . . ." (1959) and Babbitt's (1962) time-point system. Such elaborations were further developed by Rahn (1975), Morris (1987) and especially David Lewin (1987), who constructed non-commutative temporal GISs that do not preserve simultaneity, succession, or duration.

Another line of research concerns the construction of networks of pitches or other musical entities connected by succession, intersection or transformation. Perle's (1977) elaboration of his cyclic sets together with Lansky's (1973) formalization via matrix algebra, and the further generalizations to K-nets (Lewin 1990) represents one strand in network theory. Another strand is the use of networks of protocol pairs to create poset lattices for generalizing order relationships in serial music (Lewin 1976, Starr 1984). Yet another strand begins with similarity graphs among pc sets and set classes (Morris 1980), two-partition graphs (Morris 1987), transformation networks (Lewin 1987) and some types of compositional spaces (Morris 1995a).

In the interest of time and space, I've left out a great deal of important research including the application of pc theory to musical contour and time.

Present State of Research

Today, the nature of the twelve-tone system is well understood. In a few words, the field is supported by an application of mathematical group theory, where various kinds of groups act on pcs, sets, arrays, etc. The most important group is the affine group including the T_n and M_m

operations acting on $\mathbf{Z}_{12}$ or simply $\mathbf{Z}$. Other subgroups of the background group S_{12} have been used to relate musical entities; these fall into two categories: the so-called context-sensitive groups, some of which are simply-transitive, and groups that are normalized by operations in the affine group. Other branches of math having strong connections with group theory such as semi-groups and fields, number theory, combinatorial analysis, and graph theory are often implicated in twelve-tone research.

In addition to the unification provided by group theory, topics in different branches of pitch-class theory have been connected and revealed to be instances of the same concept or model. For instance, I've shown that Forte's K and Kh-relations are related via two-partitions to hexachordal combinatoriality (Morris 1987) or that Forte's genera are actually the result of operations on his set-complexes (Morris 1997). In any case, our understanding of the twelve-tone system is so general so that to divide the topic into set theory (à la Forte), cycles (à la Perle) and transformations (à la Lewin) is no longer viable. If there are any differences between these three orientations to the system it is due to the applications.

What is more, when it became obvious that serial theory was actually an application of group theory, research shifted over from modeling serial composition and analysis to other aspects of music that involved symmetry. David Lewin's (1987) work on general interval systems (GIS) and transformation networks represents this change of orientation. Thus, the development of the twelve-tone system has been so extended and ramified that there is no longer a need to distinguish this line of work from other mathematically informed branches of theory. Neo-Riemannian, scale theory, networks, and compositional spaces unify and interconnect music theory in hitherto unexpected ways. Thus the distinction between tonal and atonal may no longer very meaningful; rather, distinctions between types and styles of music are much more context-sensitive and nuanced thanks to the influence of mathematics.

Future

While the twelve-tone system is no longer isolated from other aspects of music theory such as models for tonality, there are many research projects that can be identified to carry on previous work.

One obvious direction is to ask what happens when we change the "twelve" in the twelve-tone system. Carlton Gamer (1967a and b) was one of the first theorists to raise such issues. He showed that equal-tempered systems of other moduli not only have different structures; they allow different types of combinatorial entities to be built within them. Another aspect that individuates mod-n systems is that their (multiplicative) units need not be their own inverses as they are in the twelve-tone system. Moreover, when n is a prime, all integers mod n are units. Jumping out of any modular system into the pitch space, there are other ways of conceptualizing and hearing pitch relations, as in spectral composition.

Here I list a few more specific research issues. What are the ranges for models of similarity between and among ordered sets (including rows)? A few models have been introduced: order inversions (Babbitt 1961), BIPs (Forte 1973), and the correlation coefficient (Morris 1987). At the time of this writing, Tuukka Ilomaki is working on a dissertation on row similarity.

Generating functions and algorithms have been useful in enumerating the number of entities or equivalence classes such as rows, set classes, partition classes, and the like. Are there mathematical ways of generating entities of certain types such as all-interval series or multiple-order-function rows? Some preliminary results are found in Fripertinger (1993). Babbitt has pointed out that the famous multiple-order-function Mallalieu row can be generated by the enumeration of imprimative roots. Can most or all multiple-order-function rows be similarly generated? Caleb Morgan has been working on this question and will soon publish the results.

There is much work to be done on the generation of combinatorial and other arrays. For instance, it is an unproven conjecture that any row can generate a twelve-row, all-77-partition array, but only special rows can generate a four-row, all-34-partition array. However, in the later case, even the necessary criteria are not known. Bazelow and Brickle carried out an initial probe into this problem in 1976. A host of other similar problems surround the creation and transformation of arrays.

In twelve-tone partition theory, the Z-relation is generally understood, but what about in systems of other moduli? David Lewin (1982) showed that there were Z-triples in the 16-tone system. Does the Z-phenomenon have one root cause or many?

Multisets are of use for modeling doubling and repetition in voice-leading and weighted arrays. Even the most basic questions of enumeration and transformation of pc multisets have yet to be investigated.

Existence proofs have been lacking to explain—for instance—why there are no all-interval rows that are also all-trichordal.[4] Another open question: do 50-pc rings that imbricate an instance of each of the 50 hexachordal set classes exist?

Conclusion

The introduction of math into music theoretic research has had a number of important consequences. At first, the work simply became more rigorous and pointed in the questions that it could ask and in the generality of the answers. On one hand, this led to the identification of different types of twelve-tone music and the models for each type within the twelve-tone system. On the other hand, group theory eventually unified what seemed to be different aspects of music so that the twelve-tone system could no longer completely be conceptually differentiated from tonality, modality, and even aspects of non-Western music.

I say "not completely differentiated," for there are other mathematical bases for music besides group theory. For instance, Schenkerian tonal theory is not modeled by groups of transformations; here we have tree structures. Thus there might be a distinction between serial and tonal music by transformation graphs that have no loops or cycles and those that do. Of course, graph theory would model these two ways of organizing transformations and entities accordingly—as different types of graphs. But since every graph whatsoever has at least one shortest spanning tree, graphs that do not have loops and cycles are involved with ones that do. It can be easily be shown that "motivic" association among structural levels in Schenkerian theory is not hierarchical—i.e., tree-like—and that relationships among serial entities like rows can be recursive and/or hierarchically configured. The moral here is not that graph theory might be a better general model for music than group theory, but that no one mathematical theory is going to partition musical structures into closed categories or completely unify it. Thus, links between mathematical categories will have relevance for modeling the diversity of music, as Mazzola (2002) and his co-workers have begun to demonstrate.

In any case, while there can be no doubt that the way we regard music has been transfigured by the use of math in music theory, the music we study remains or remains to be written. But math is not music, and one must remember to distinguish between a task and the tools used to accomplish it. Nevertheless, a traditional verity has been falsified: the way we explore a given composition's particularity is no longer different in kind from the way we associate and/or group different pieces, genres, and styles.

Notes

This paper was read at the first international conference of the Society for Mathematics and Computation in Music held at the Staatliches Institut für Musikforschung, Berlin, May 2007.

1. Luigi Verdi has documented some of this early European research (Verdi 2007).

2. Tonality and theories of chords involve permutation and aspects of the twelve-tone system involve unordered sets.

3. The theorem was enunciated as early as Hanson 1960, and sketches for a proof were given in Regener 1974 and Starr 1978. An elegant proof appears in Lewin 1987.

4. Here we mean all-trichordal in Babbitt sense of the term: a row that imbricates an instance of each of ten different trichordal set classes, leaving out [036] and [048].

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