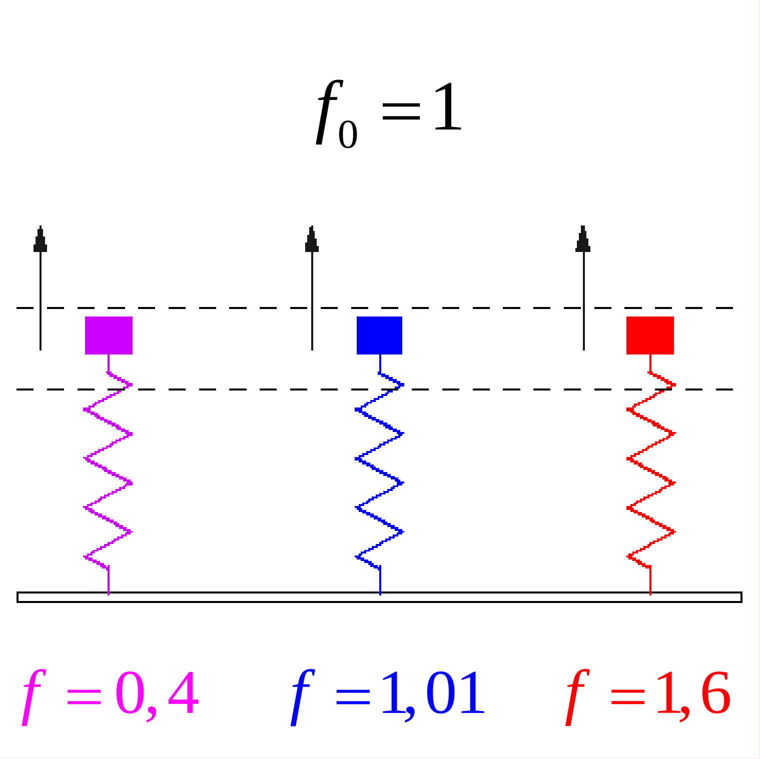
Oscilações Forçadas

Força externa: $$F(t) = F_0 \cos(\omega t) = F_0 \cos(2\pi f t)$$



Sistema oscila com a frequência da força externa (*f*),

mesmo que esta seja diferente da frequência natural do sistema.

Oscilações Forçadas

$$F = m\frac{d^2x}{dt^2} = -kx + F_0\cos(\omega t)$$

$$\frac{d^2x}{dt^2} + \omega_0^2 x = \frac{F_0}{m}\cos(\omega t)$$

Propomos a solução: $x(t) = A\cos(\omega t + \phi)$

$$-A\omega^2\cos(\omega t+\phi)+A\omega_0^2\cos(\omega t+\phi)=\frac{F_o}{m}\cos(\omega t)$$

$\phi = 0 \quad (\omega < \omega_0)$ $\phi = -\pi \quad (\omega > \omega_0)$

Em fase **Fora de fase**

com a FORÇA

$$A=\frac{F_0}{m\left|\omega_0^2-\omega^2\right|}$$

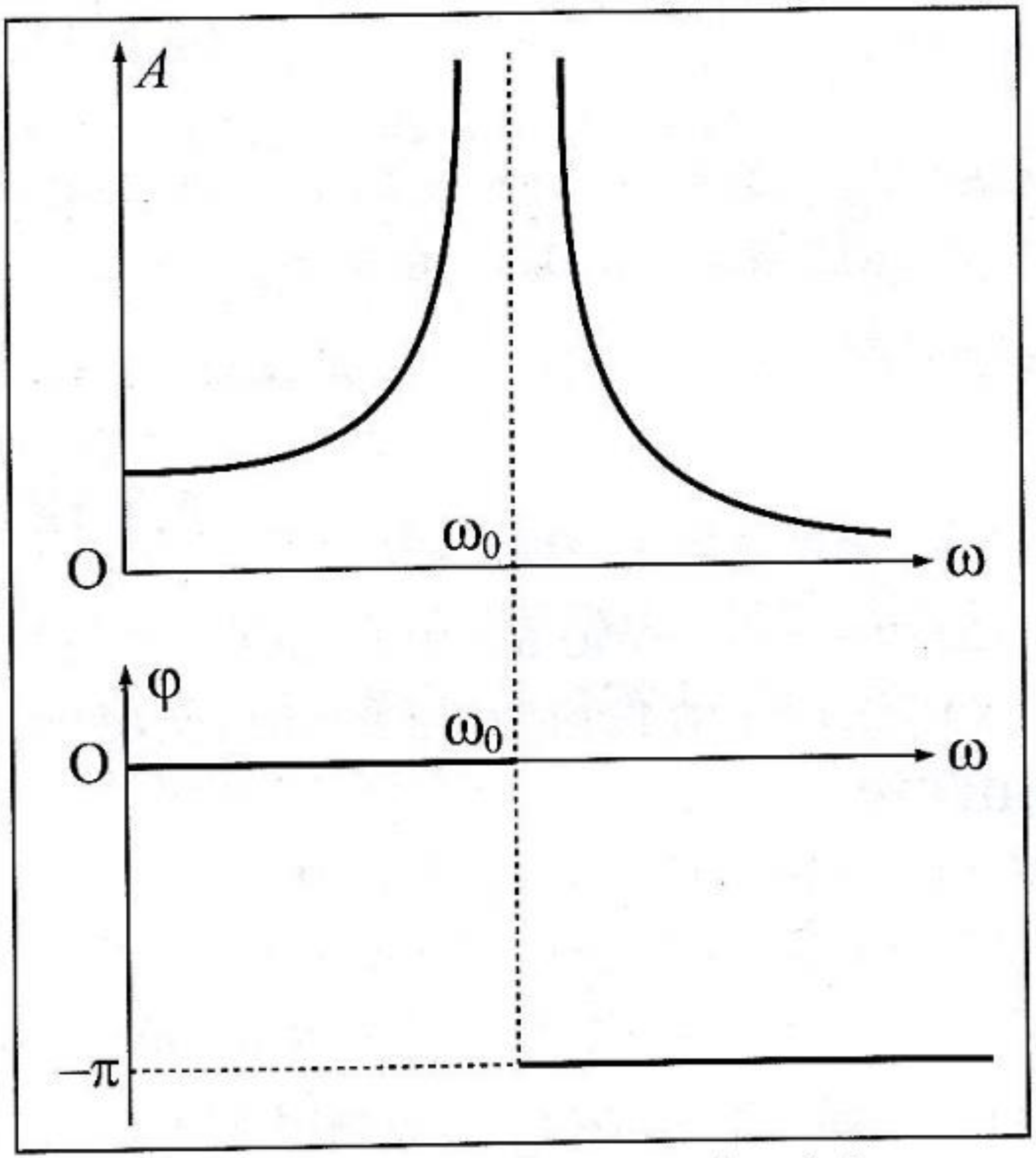


Figura 4.6 — *Solução estacionária*

Oscilações Forçadas

BAIXAS FREQUÊNCIAS: $\omega << \omega_0$

$$\frac{d^2x}{dt^2} + \omega_0^2 x = \frac{F_0}{m}\cos(\omega t)$$

$$-A\omega^2 \cos(\omega t + \phi) + A\omega_0^2 \cos(\omega t + \phi) = \frac{F_o}{m}\cos(\omega t)$$

$$x \approx \frac{F_0}{m\omega_0^2}\cos(\omega t)$$

$$\phi = 0 \quad (\omega < \omega_0)$$

Solução em fase com a Força

Oscilações Forçadas

ALTAS FREQUÊNCIAS: $\omega >> \omega_0$

$$\frac{d^2x}{dt^2} + \omega_0^2 x = \frac{F_0}{m}\cos(\omega t)$$

$$-A\omega^2\cos(\omega t+\phi) + A\omega_0^2\cos(\omega t+\phi) = \frac{F_o}{m}\cos(\omega t)$$

$$x \approx -\frac{F_0}{m\omega^2}\cos(\omega t)$$

$$\phi = -\pi \quad (\omega > \omega_0)$$

Solução fora de fase com a Força

Oscilações Forçadas

$$F = m\frac{d^2x}{dt^2} = -kx + F_0\cos(\omega t)$$

SOLUÇÃO GERAL =

Solução particular

+

solução da eq. homogênea

$$x(t) = \frac{F_0}{m\left(\omega_0^2 - \omega^2\right)}\cos(\omega t) + B\cos(\omega_0 t + \phi_0)$$

B e ϕ_0 constantes - condições iniciais

Oscilações Forçadas

$$x(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos(\omega t) + B\cos(\omega_0 t + \phi_0)$$

condições iniciais

$$x(0) = 0$$
$$v(0) = 0$$

$$\left.\begin{aligned} x(0) &= \frac{F_0}{m(\omega_0^2 - \omega^2)} + B\cos(\phi_0) = 0 \\ v(0) &= -\omega_0 sen(\phi_0) = 0 \Rightarrow \phi_0 = 0 \end{aligned}\right\} B = -\frac{F_0}{m(\omega_0^2 - \omega^2)}$$

Oscilações Forçadas

$$x(t) = -\frac{F_0}{m(\omega_0+\omega)}\left[\frac{\cos(\omega_0 t)-\cos(\omega t)}{\omega_0-\omega}\right]$$

No limite para: $\omega \to \omega_0$

$$\lim_{\omega\to\omega_0}\left[\frac{\cos(\omega t)-\cos(\omega_0 t)}{\omega-\omega_0}\right] = \left[\frac{d}{d\omega}\cos(\omega t)\right]_{\omega=\omega_0} = -t\,sen(\omega_0 t)$$

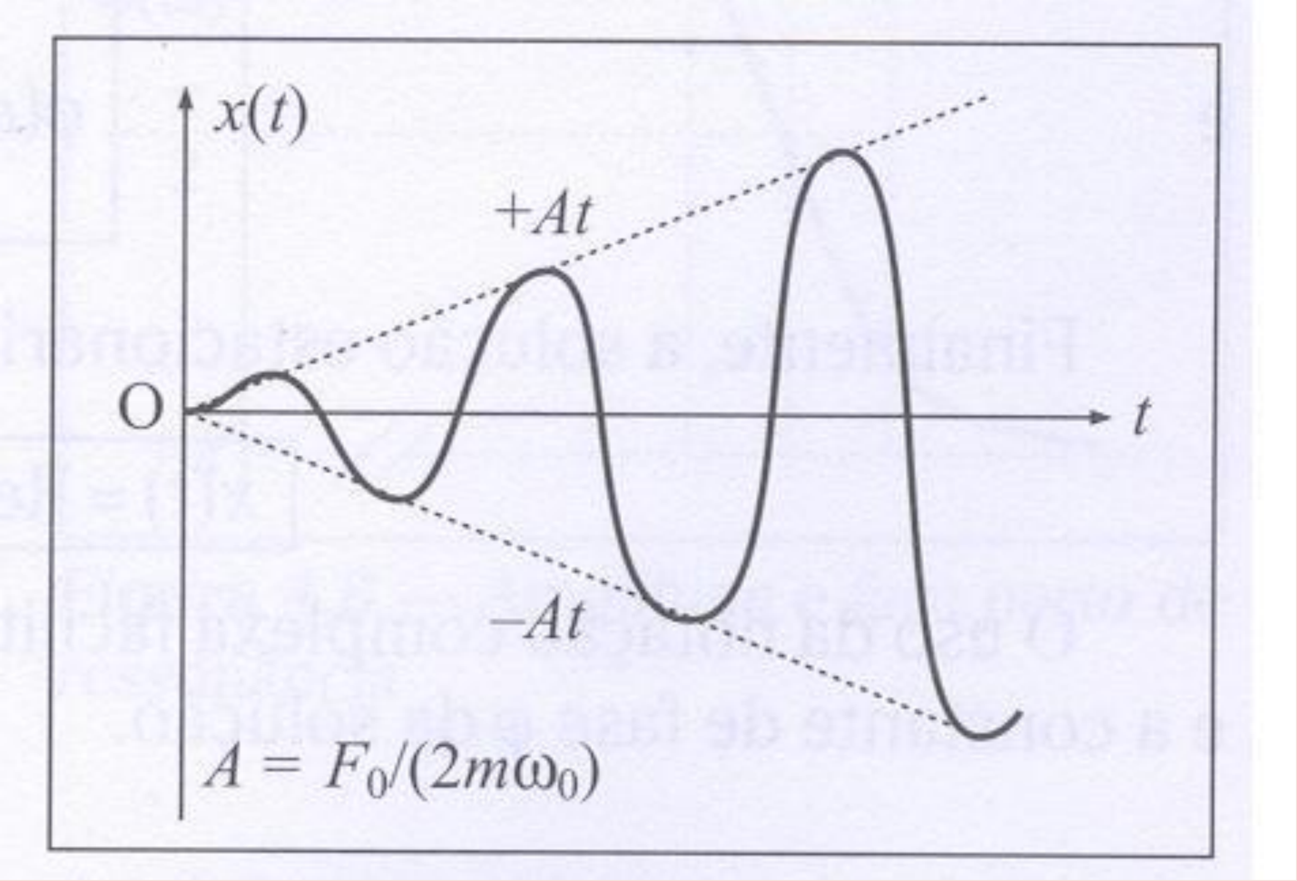
$$x(t) = \frac{F_0}{2m\omega_0}t\,sen(\omega_0 t)$$

Oscilações Forçadas

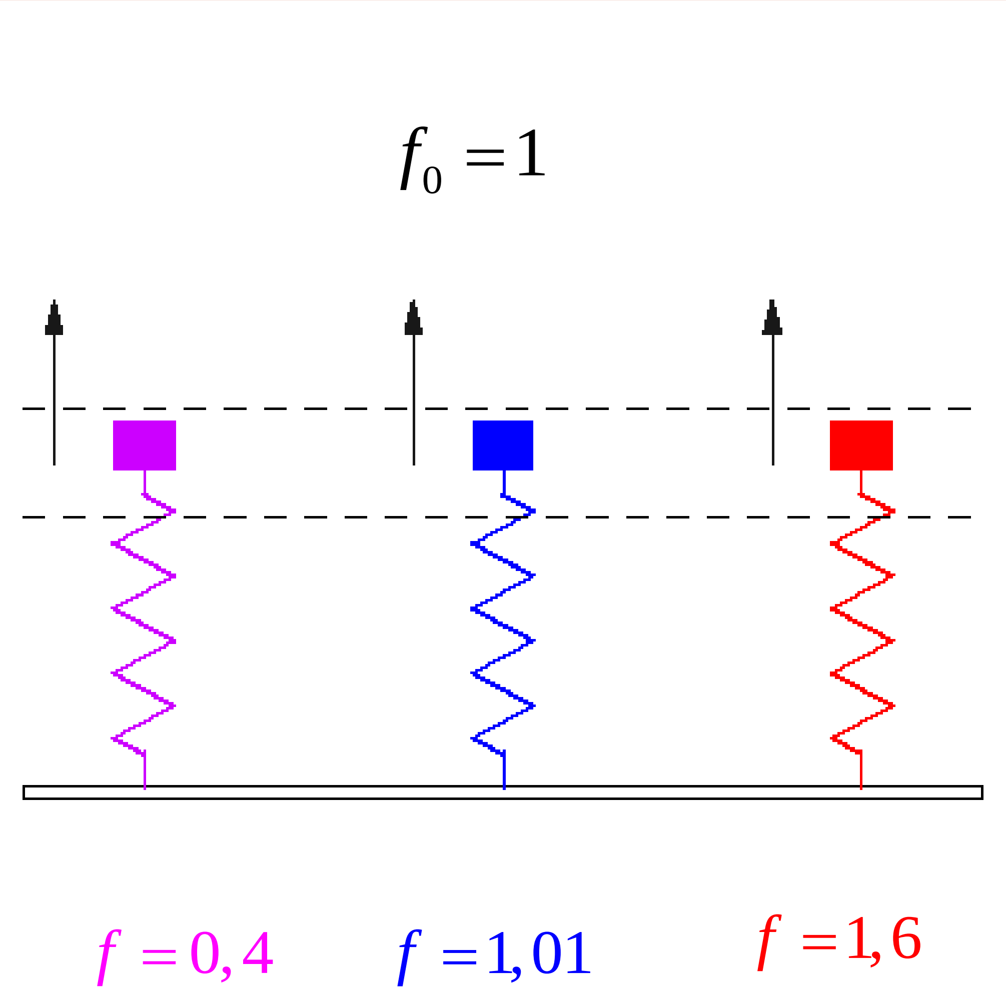
RESSONÂNCIA

$$\omega = \omega_o$$

$$x(t) = \frac{F_0}{2m\omega_0} t \, sen(\omega_0 t)$$



Oscilações Forçadas

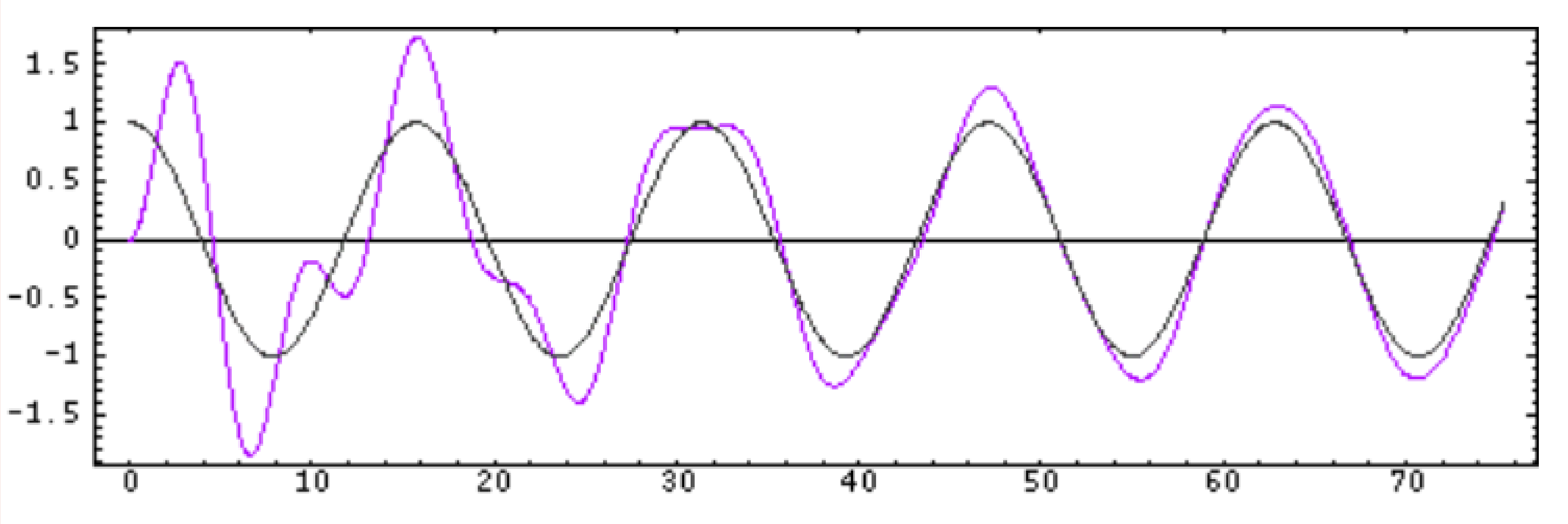


Transient response to an applied force: Three identical damped 1-DOF mass-spring oscillators, all with natural frequency $\mathbf{f_0=1}$, are initially at rest. A time harmonic force $F=F_0\cos(2\ \mathrm{pi}\ f\ t)$ is applied to each of three damped 1-DOF mass-spring oscillators starting at time $t=0$. The driving frequencies ω of the applied forces are (matching colors)

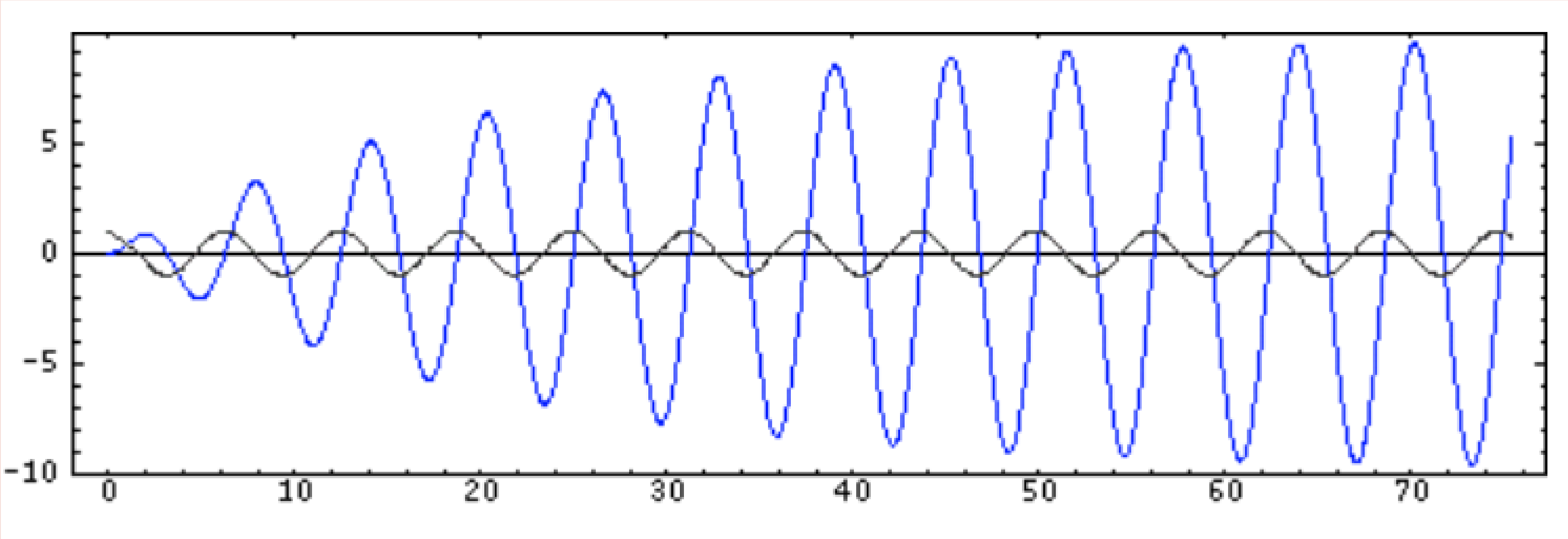
$\mathbf{f_0=0.4}$, $\mathbf{f_0=1.01}$, $\mathbf{f_0=1.6}$

The animation at left shows response of the masses to the applied forces. The direction and magnitude of the applied forces are indicated by the arrows. The dashed horizontal lines provide a reference to compare magnitudes of resulting steady state displacement.

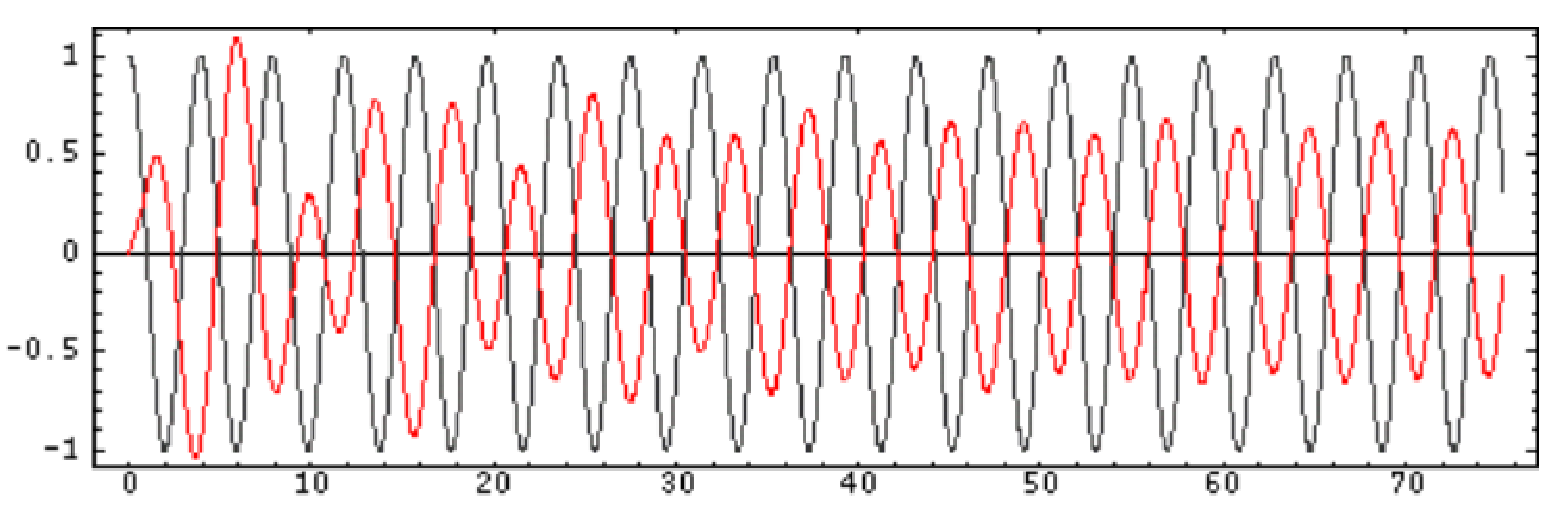
Mass 1: Below Resonance In the plot below the forcing frequency is f=0.4, so that the first oscillator is being driven **below resonance**. The grey curve shows the applied force (positive is upwards), and the purple curve shows the displacement of the mass in response to the applied force. After the transient motion decays and the oscillator settles into steady state motion, the displacement is in phase with force. Notice that the frequency of the steady state motion of the mass is the driving (forcing) frequency, not the natural frequency of the mass-spring system.



Mass 2: At Resonance In the plot below the forcing frequency is f=1.01, so that the second oscillator is being driven very **near resonance**. The grey curve shows the applied force (positive is upwards), and the blue curve shows the displacement of the mass in response to the applied force. Since the oscillator is being driven near resonance the amplitude quickly grows to a maximum. After the transient motion decays and the oscillator settles into steady state motion, the displacement 90° out of phase with force (displacement lags the force). Notice, again, that the frequency of the steady state motion of the mass is the driving (forcing) frequency, not the natural frequency of the mass-spring system.



Mass 3: Above Resonance In the plot below the forcing frequency is f=1.6, so that the third oscillator is being driven **above resonance**. The grey curve shows the applied force (positive is upwards), and the red curve shows the displacement of the mass in response to the applied force. Since the oscillator is being driven near resonance the amplitude quickly grows to a maximum. After the transient motion decays and the oscillator settles into steady state motion, the displacement 180° out of phase with force. Notice, again, that the frequency of the steady state motion of the mass is the driving (forcing) frequency, not the natural frequency of the mass-spring system. Also notice that the amplitude of motion is less than when the mass was driven below resonance.



Oscilações forçadas amortecidas

$$m\ddot{x} = -\rho\dot{x} - kx + F_0 \cos(\omega t)$$

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = \frac{F_0}{m}\cos(\omega t)$$

Equação diferencial Linear de segunda ordem não homogênea

Solução Geral:

$$x(t) = x_h(t) + x_p(t)$$

Solução da homogênea

Solução da particular

Por exemplo, no caso sub-crítico ($\omega_0 > \gamma/2$)

$$x_h(t) = Be^{-\frac{\gamma t}{2}} \cos(\Omega t + \phi)$$

Onde:

$$\Omega = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}$$

B e ϕ são constantes definidas a partir das condições iniciais.

Solução da particular

Considerando a equação complexa:

$$\ddot{z} + \gamma \dot{z} + \omega_0^2 z = \frac{F_0}{m} e^{i\omega t}$$

A parte real da equação acima reproduz a equação do oscilador forçado

Uma solução tentativa do tipo: $z(t) = z_0 e^{i\omega t}$

$$\left(-\omega^2 + i\gamma\omega + \omega_0^2\right) z_0 \exp(i\omega t) = \frac{F_0}{m} \exp(i\omega t)$$

A equação acima é satisfeita para todo *t* quando:

$$z_0 = \frac{F_0}{m}\frac{1}{\left(\omega_0^2 - \omega^2 + i\gamma\omega\right)}$$

z_0 é complexo e para ser escrito na forma: $z_0 = a + bi$

Multiplicamos e dividimos por:

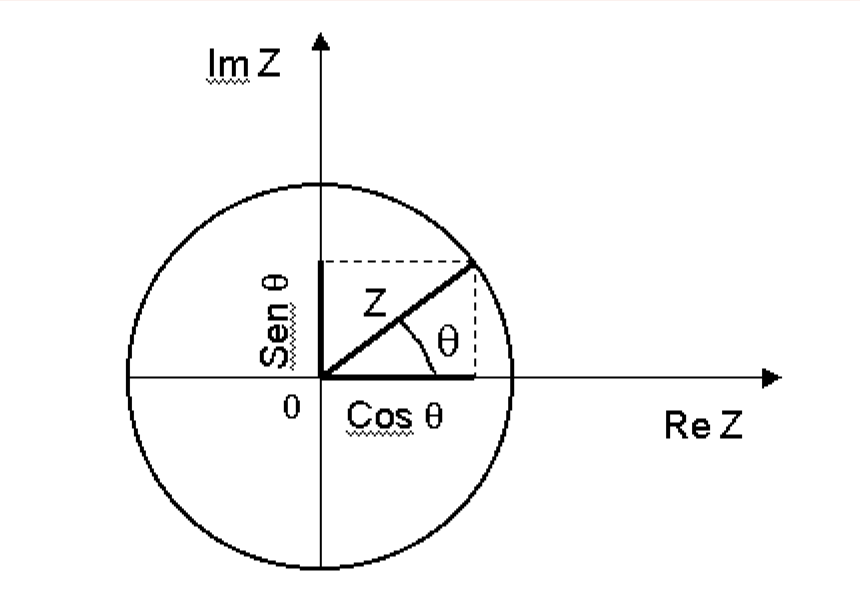
$$z_0 = \frac{F_0}{m}\frac{1}{\left(\omega_0^2 - \omega^2 + i\gamma\omega\right)} \cdot \frac{\left(\omega_0^2 - \omega^2 - i\gamma\omega\right)}{\left(\omega_0^2 - \omega^2 - i\gamma\omega\right)}$$

$$z_0 = \frac{F_0}{m}\frac{\left(\omega_0^2 - \omega^2 - i\gamma\omega\right)}{\left[\left(\omega_0^2 - \omega^2\right)^2 + \gamma^2\omega^2\right]}$$

$$a = \frac{F_0}{m}\frac{\left(\omega_0^2 - \omega^2\right)}{\left[\left(\omega_0^2 - \omega^2\right)^2 + \gamma^2\omega^2\right]}$$

$$b = -\frac{F_0}{m}\frac{\gamma\omega}{\left[\left(\omega_0^2 - \omega^2\right)^2 + \gamma^2\omega^2\right]}$$

$$z_0 = a + bi = Ae^{i\varphi}$$



$$z = |z|e^{i\theta}$$

$$z = x + yi = |z|\cos\theta + |z|\sin\theta i$$

$$A = \sqrt{a^2 + b^2} \qquad \tan\varphi = \frac{b}{a}$$

$$A(\omega) = \left(\frac{F_0}{m}\right) \frac{1}{\left[\left(\omega_0^2 - \omega^2\right)^2 + \gamma^2\omega^2\right]^{1/2}}$$

$$\varphi(\omega) = \arctan\left(-\frac{\gamma\omega}{\omega_0^2 - \omega^2}\right) = -\arctan\left(\frac{\gamma\omega}{\omega_0^2 - \omega^2}\right)$$

$$z(t) = z_0 \exp(i\omega t) = A\exp(i\varphi)\exp(i\omega t) = A\exp\left[i(\omega t + \varphi)\right]$$

$$x_p(t) = \mathrm{Re}\left[z(t)\right] = A\cos(\omega t + \varphi)$$

Oscilador harmônico de frequência ω, amplitude A(ω) e fase inicial φ(ω)

Ressonância da amplitude

Amplitude A(ω) da onda estacionária é máxima quando denominador for mínimo:

$$\frac{d}{d\omega}\left[\left(\omega_0^2-\omega^2\right)^2+\gamma^2\omega^2\right]=0 \quad \Rightarrow \quad \omega=\sqrt{\omega_0^2-\frac{\gamma^2}{2}}$$

Efeito da ressonância:

Amortecimento fraco: $\gamma << \omega_0$

Próximo da ressonância: $\left|\omega-\omega_0\right| << \omega_0$

$$\omega_0^2-\omega^2=\left(\omega_0+\omega\right)\left(\omega_0-\omega\right)=\left[2\omega_0+\left(\omega-\omega_0\right)\right]\left(\omega_0-\omega\right)$$

$$\omega_0^2-\omega^2=2\omega_0\left(\omega_0-\omega\right)-\left(\omega_0-\omega\right)^2\simeq 2\omega_0\left(\omega_0-\omega\right)$$

$$A(\omega) \cong \left(\frac{F_0}{m}\right) \frac{1}{\left[4\omega_0^2(\omega_0-\omega)^2+\gamma^2\omega_0^2\right]^{1/2}} \cong \left(\frac{F_0}{2m\omega_0}\right) \frac{1}{\left[(\omega_0-\omega)^2+\frac{\gamma^2}{4}\right]^{1/2}}$$

$$\varphi(\omega) \cong -\arctan\left[\frac{\gamma\omega_0}{2\omega_0(\omega_0-\omega)}\right] \cong -\arctan\left[\frac{\gamma}{2(\omega_0-\omega)}\right]$$

A amplitude máxima é dada por:

$$A_{max} = A(\omega_0) = \frac{F_0}{m\omega_0\gamma}$$

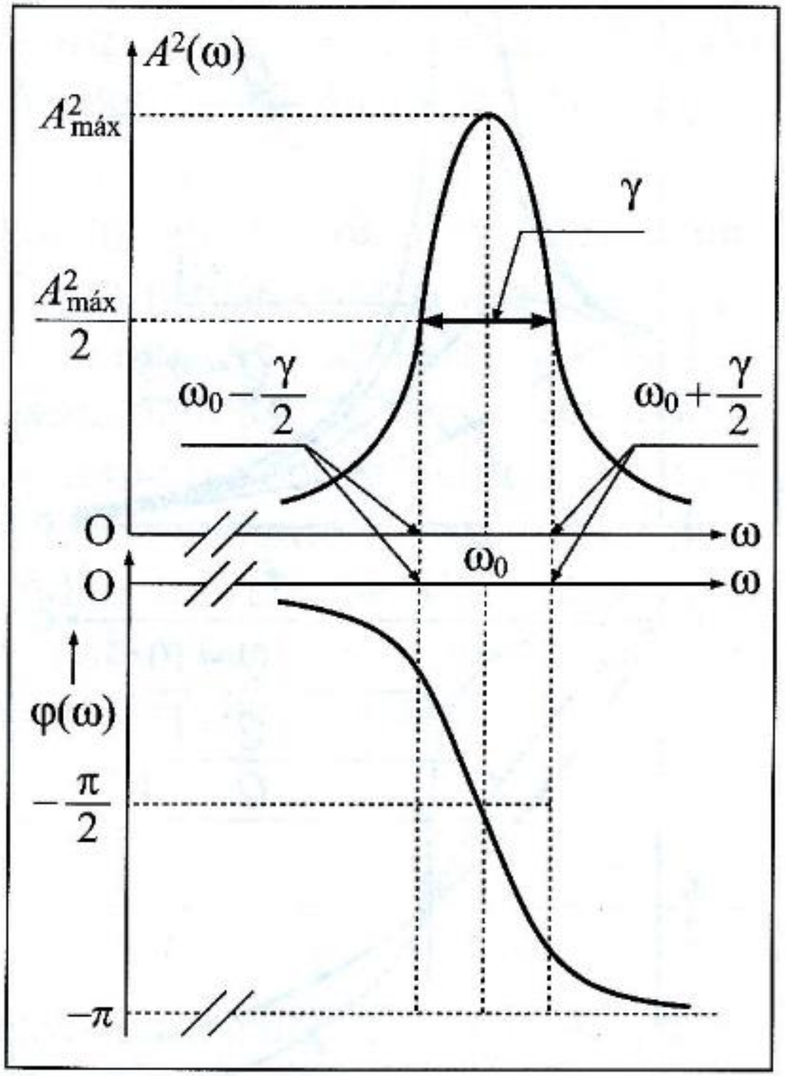
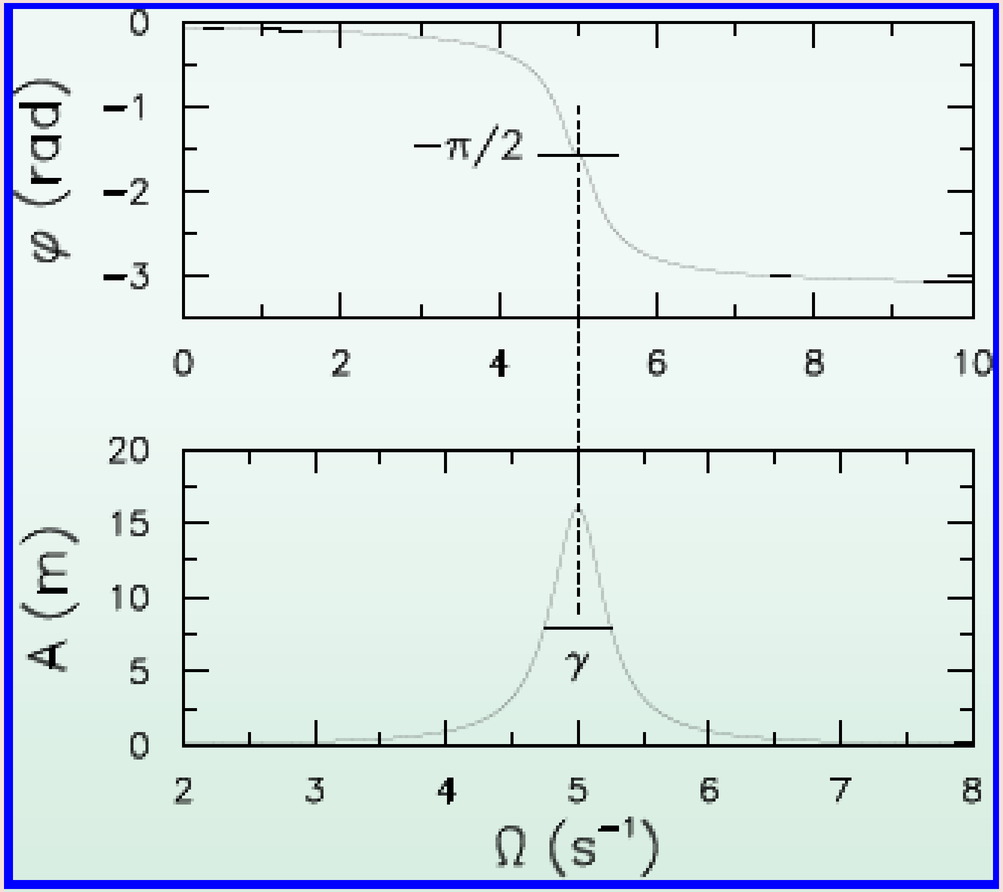


***Figura 4.8** — Amplitude e fase perto de ressonância*

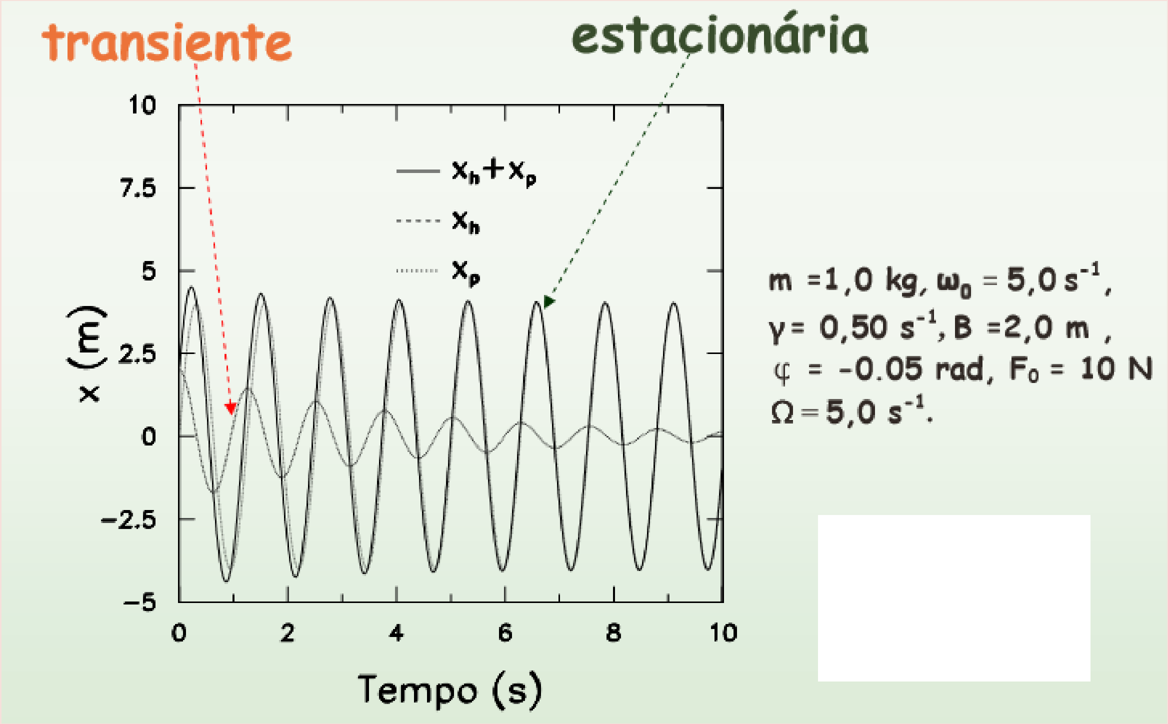


Solução Geral:

$$x(t) = x_h(t) + x_p(t)$$

Exemplo: Solução da homogênea para $\omega_0 > \frac{\gamma}{2}$

$$x(t) = Be^{-\frac{\gamma}{2}t}\cos(\omega_0 t + \phi) + A(\omega)\cos[\omega t + \varphi(\omega)]$$



Para ω<< ω_0

$$\frac{A(\omega_0)}{A(0)} = \frac{\omega_0}{\gamma} = Q$$

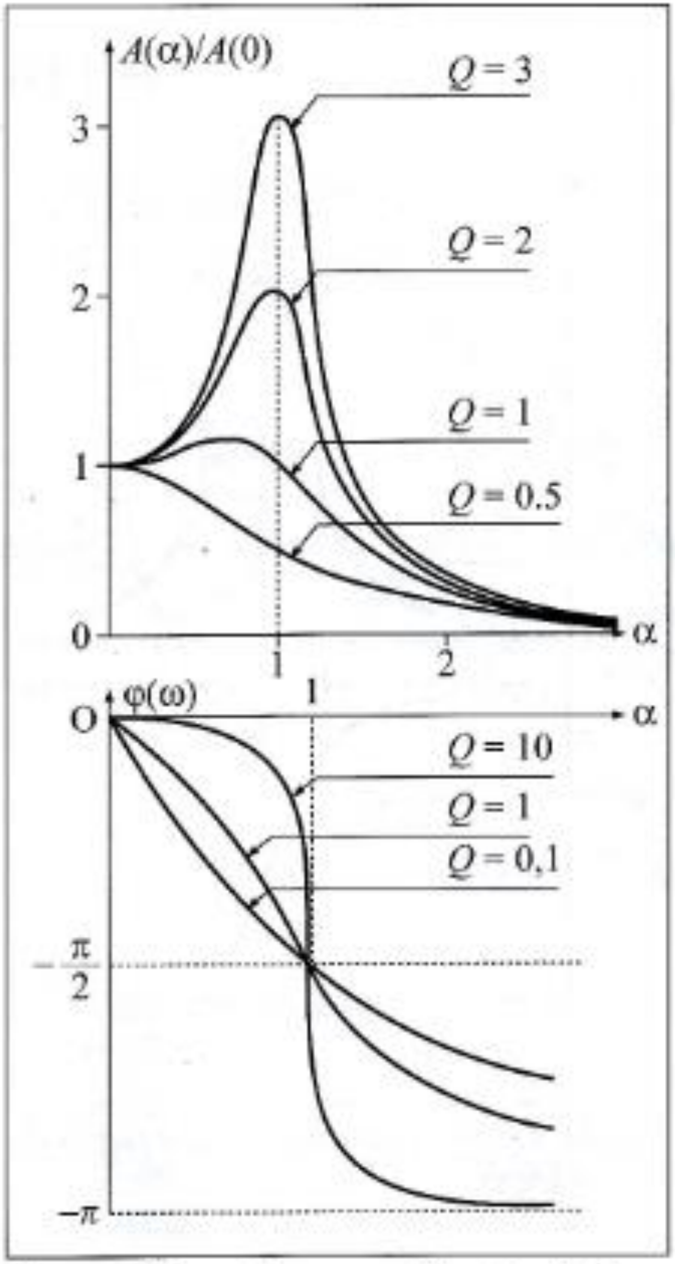


Figura 4.9 — *Curvas em função de Q*