

Chapter 1

Statistical and Quantum Physics

Topics

Introduction. The importance of statistical and quantum concepts in physics. Particles undergoing collisions, experiments with air tables, the concept of distributions of velocities and statistical equilibrium. What we aim to explain – the macroscopic properties of gases, Boyle’s law, Charles/Gay-Lussac law, perfect gas law, Avogadro’s number, Boltzmann’s constant, Dalton’s law of partial pressures. Black-body radiation, spectral energy distribution as a function of temperature.

1.1 Introduction

The third part of the first year physics course is called *Statistical and Quantum Physics*. Let us review relevant aspects of what we have learned so far and what we will do in this part of the course.

In the *Mechanics and Relativity* course, the central themes were:

- *Newton’s Laws of Motion* for point particles and then extended to bodies of finite size. The central concepts of *kinetic* and *potential energies* were introduced for these systems, as well as *rotational motion*.
- The elements of the *Special Theory of Relativity*. The Galilean transformations of Newtonian mechanics were replaced by the Lorentz transformations of Special Relativity. In turn, the Lorentz transformations lead to Einstein’s

relation between mass and energy $E = mc^2$. This tells us that mass and energy are the same thing. We will need the concept that there is an *inertial mass* associated with any type of energy we care to think of, in particular, even ‘massless particles’ such as *photons*, the particles of light, have energy E , which is related to the frequency ν of the associated electromagnetic wave by the fundamental relation

$$E = h\nu.$$

where h is Planck’s constant. Photons also have momenta p according to the quantum relation which we will develop in the last part of the course,

$$E = pc.$$

In the last part of the course *Fields, Oscillations and Waves*, wave concepts were extended to derive the *Schrödinger wave equation* and we will use it a great deal in the last part of this course to describe the behaviour of quantum phenomena at a fundamental level.

To generalise rather sweepingly, so far we have been dealing with individual objects, either particles or waves, and understanding their mechanical and dynamical properties. We developed rules about how Newton’s laws can be extended to deal with extended bodies such as tables, chairs and elephants, but these extended objects were taken to be *rigid bodies*.

In the first part of this course, we extend the basic concepts of Newtonian and relativistic mechanics to *very large assemblies of particles*. We have no hope of being able to follow the paths of all the particles and waves and so we need a *statistical description* of these large assemblies and from this we aim to derive the *bulk properties of the system*. This is the beginning of an absolutely vast subject which extends into some of the deepest aspects of physics. These will ultimately provide an understanding of the nature of the two great *Laws of Thermodynamics*, which are probably the most fundamental of all the laws of physics.

The energy-frequency relation for photons

$$E = h\nu.$$

where $h = 6.6261 \times 10^{-34}$ J s is Planck’s constant

The energy-momentum relation for photons

$$E = pc.$$

The second part of this course will concern *quantum physics*, in which we are forced to abandon the classical picture, which has proved so successful so far, and replace it with a wholly new type of physics in which *probabilities* play a central role in determining the outcome of any experiment at the most elementary level. Although quantum phenomena occur at the level of the interaction between particles on the atomic scale, these determine the large scale properties of matter, for example, the forces which hold material objects together.

Thus, there is a strong *statistical and probabilistic flavour* running through this course. Fundamental to these studies is an understanding how to deal with *distributions* of energy, momenta and so on. In this first lecture, we deal with two of the most important distributions in physics:

- the equilibrium distribution of energies, or speeds, of particles which continually undergo collisions with each other;
- the energy distribution of the radiation of a hot body.

Between these, we will review the properties of *ideal gases*, which we will endeavour to explain by processes occurring at the atomic or molecular level.

1.2 Particles Continually Undergoing Collisions

Let us first recall one of the collision calculations carried out in the *Mechanics and Relativity* course. The incoming particle has velocity vector \mathbf{u} and the other particle is stationary. By an *elastic* collision, we mean that there is *no loss of kinetic energy in the collision*. Also, *linear momentum must always be conserved*. Figure 1.1 shows the collision of two identical point masses in the laboratory frame of reference.

First, we find the zero momentum frame of reference. We transform to a frame of reference S' moving at some velocity \mathbf{V} through the laboratory frame of reference. Then, the velocities of the particles in the moving frame S' are $\mathbf{u} - \mathbf{V}$ and $-\mathbf{V}$,

Figure 1.1. Elastic collision of two point masses

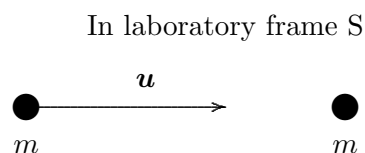


Figure 1.2. The collision in the moving frame S'



so that the total momentum in the frame S' is

$$\mathbf{P}' = m(\mathbf{u} - \mathbf{V}) - m\mathbf{V} \quad (1.1)$$

In the zero momentum frame, this must be zero and so $\mathbf{V} = \mathbf{u}/2$.

We now redraw the vector diagram in the S' frame of reference, which moves at speed $\mathbf{u}/2$ with respect to S in the positive x -direction (Figure 1.2). In S' , the total momentum is zero and, since the collision is elastic, there is no loss of kinetic energy. The velocity vectors must have the same magnitude after the collision and so in S' , after the collision, the velocity vectors must be rotated with respect to the initial direction, as shown in Figure 1.3.

Now, to transform back to the frame S , we need to add vectors $\mathbf{u}/2$ to the rotated vectors, as shown in Figure 1.4, and it is just a piece of geometry to work out the angle between the two resultant vectors. I have completed the parallelograms for the two particles and it can be seen that, for both the upper and lower parallelograms, the sides are all of equal magnitude. Therefore, each of the triangles involving the $|\mathbf{u}|/2$ vectors is an isosceles triangle and so the diagonals intersect at right-angles. Therefore, by geometry, we see that the two masses come out of the collision at right angles to each other.

In the special case, in which the collision is precisely *head-on*, the final result is that the ball which was originally stationary goes off at velocity \mathbf{u} parallel to the initial direction, while the incoming particle is stopped. All the kinetic energy has been transferred from the incoming particle to the other (Figure 1.5).

The opposite extreme is that of a *glancing collision*. In this case, the incoming particle is undeflected and carries on with the same velocity \mathbf{u} that it started with and no kinetic energy is transferred to the stationary particle.

Between these extremes, different amounts of energy are shared between the particles. Thus, elastic collisions provide a means of sharing energy between particles – different amounts of energy are transferred depending upon the angle through which the particles are deflected.

Figure 1.3. In the moving frame S' after the collision

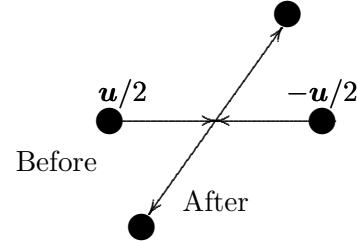


Figure 1.4. In laboratory frame S after the collision

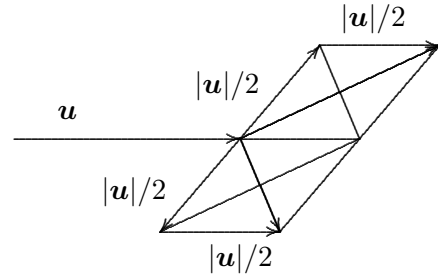
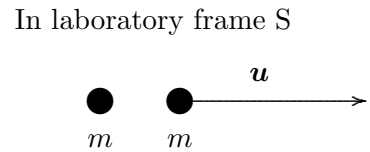


Figure 1.5. The outcome of a head-on collision between two point masses



1.3 Experiments with an Air Table

Suppose we now allow very large numbers of particles to collide. What is the distribution of energies among the particles after a long time? We can illustrate what happens using a simulation of the collisions of many particles using an air table (Figure 1.6). Air flows through tiny holes in the air-table and this keeps the little discs suspended just above its surface. Each of the discs is magnetic and the polarities of the little magnets are all the same so that the discs repel each other. Thus, the collisions between the little discs are essentially perfect, involving no actual contact between the discs – if this were to happen, there would be frictional losses of energy.

Let us start the simulation with all the discs at rest except one. We start off that disc with a significant velocity and observe the subsequent behaviour. We can see that, in due course, all the discs are in motion as a result of collisions – the initial kinetic energy is shared among the discs. This shows how the particles share kinetic energy through elastic collisions. But there is much more to observe.

- First, although energy has been shared between the discs, some are going faster and others slower. In other words, there is a *distribution of energy among the discs*. Although they are now sharing the energy, they do not all come to the same energy because different amounts of energy are continually being exchanged randomly in each collision.
- Secondly, if we were to measure the speeds of the discs very carefully at different times, we would find that, although the discs are exchanging energy continually, after a long time, the *distribution of energies among them remains unchanged*. We say that the distribution has reached a state of *statistical equilibrium* – this is a *really important* concept.

The expression for the distribution of particle speeds in such a system was discovered by James Clerk Maxwell, the first Cavendish Professor of Experimental Physics – the velocity distribution is known

Figure 1.6. Discs on an air-table

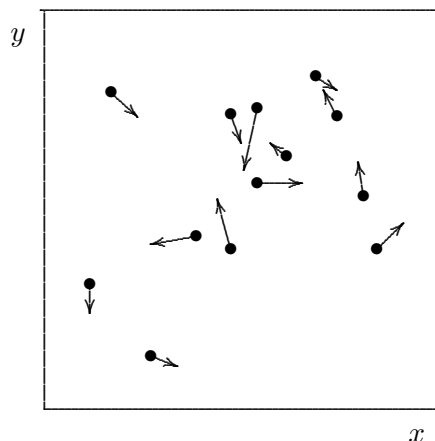
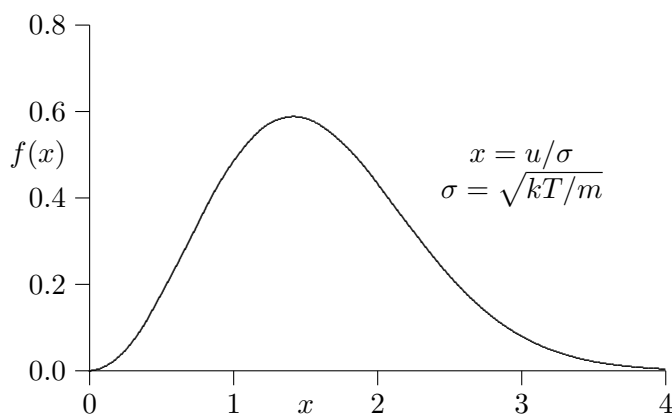


Figure 1.7. Maxwell Distribution

$$f(x) dx = \sqrt{\frac{2}{\pi}} x^2 e^{-x^2/2} dx$$



as the *Maxwell distribution* and has the form shown in Figure 1.7 for the distribution of the *speeds* u of a random distribution of particles *in three dimensions* at temperature T . For the moment, the important concept is that elastic collisions automatically lead to a stationary distribution of speeds.

1.4 The Macroscopic Properties of Gases

We want to use the ideas of the last section to understand the bulk properties of gases. First, we review some of the basic laws of what are known as *perfect* or *ideal gases*.

1.4.1 Boyle's law

In about 1660, Robert Boyle discovered his famous law that, for a fixed amount of gas at a fixed temperature, the product of the pressure p and the volume V is a constant (Figure 1.8),

$$pV = \text{constant} \quad (1.2)$$

The perfect gas law holds good for low density gases at high temperatures. It begins to break down at high densities and close to temperatures at which the gas makes a phase transition to a fluid – the gas condenses.

1.4.2 Charles' or Gay-Lussac's Law and the Perfect Gas Temperature Scale

The French physicists Charles (1787) and Gay-Lussac (1802) established the expansion law for gases at constant pressure (Figure 1.9). For a given mass of gas maintained at constant pressure, the volume increases linearly with the temperature t ,

$$V = V_0(1 + \alpha t), \quad (1.3)$$

where t is measured in degrees Celsius and V_0 is the volume of the gas at some reference temperature. This is the behaviour of a *perfect gas*. For such a gas, we can define the zero of the temperature scale as that corresponding to zero volume.

Nowadays, the reference temperature is taken to correspond to the unique value at which the three

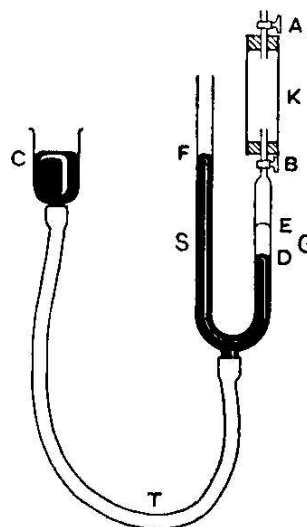


Figure 1.8 Demonstration of Boyle's law. The fluid shown in black was mercury in old versions of Boyle's experiments.

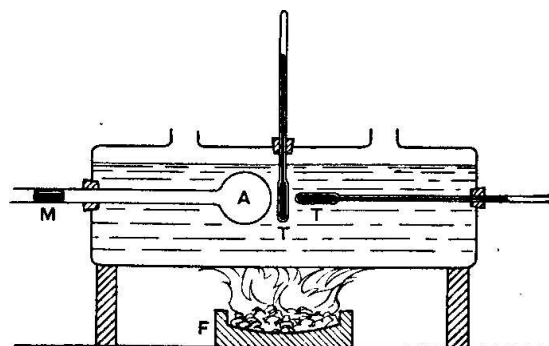


Figure 1.9. Gay-Lussac's experiment to find the dependence of the volume of a gas upon temperature. M is a mercury plug which could move along the tube as the gas expanded at constant pressure.

phases of water – solid, liquid and gas – can co-exist. It is known as the *triple point of water*. Using the Celsius temperature scale, the constant α is $1/273.16$ and so the zero of the perfect gas scale of temperature is at -273.16°C . Temperatures $T = t + 273.16$ are measured from this zero of temperature and are called kelvins (K). We can rewrite (1.3) as

$$V/V_0 = T/T_0, \quad (1.4)$$

where V_0 and T_0 are some reference values and the temperature is measured in kelvins.

Strictly, we call this temperature scale the *perfect gas temperature scale* since it is based upon the physical properties of perfect gases, which don't really exist in nature. When we look more deeply into classical thermodynamics, we will find that the perfect gas temperature scale is identical to the *thermodynamic temperature scale* (see later).

1.4.3 The Perfect Gas Law

Experimentally we find that, if the volume of a gas is kept constant, the pressure increases linearly with the temperature and the constant describing the proportionality is exactly the same as that in Charles' law. Thus,

$$p/p_0 = T/T_0, \quad (1.5)$$

where p_0 and T_0 are suitable reference values and the temperature is measured in kelvins. We can therefore combine all three results (1.2), (1.4) and (1.5) to form the relation

$$pV = (\text{constant}) T. \quad (1.6)$$

The value of the constant depends upon the quantity of gas used but, for a given gas, the constant is proportional to the mass of gas present. Even more important, Avogadro established that the constant is the same for one *mole* of any gas.

Let us recall the definition of the *mole*.

One mole is the mass in grams of a substance which is equal to its molecular weight.

Note: In the SI system, the carbon-12 atom is used as the standard for atomic weights. Therefore, we can define the mole as being the mass of a substance which contains the same number of chemical units (atoms or molecules) as exactly 12 grams of carbon-12. Since the molecular weight of oxygen is 31.9988, one mole of oxygen has mass 31.9988 grams.

Thus, we can write (1.6)

$$pV = RT \quad R = 8.31 \text{ J K}^{-1} \text{ mole}^{-1} \quad (1.7)$$

where R is known as the *gas constant* or *molar gas constant*. If there are N moles of gas present, then $pV = NRT$.

The expression (1.7) is known as the *equation of state* of the perfect gas. Equations of state are defined to be the relation which determines the pressure of any substance as a function of its volume (or density) and temperature. Equations of state can become very complex and are very important indeed in all physics.

Let us now introduce *Avogadro's constant* N_A , the number of atoms or molecules per mole,

$$N_A = 6.02 \times 10^{23} \text{ mol}^{-1} \quad (1.8)$$

This enables us to write the perfect gas law (1.7) in an important and suggestive form. We define *Boltzmann's constant* k to be the gas constant *per molecule*. In other words,

$$k = \frac{R}{N_A} = 1.38 \times 10^{-23} \text{ J K}^{-1} \quad (1.9)$$

We can therefore write the perfect gas law in the form

$$p = \frac{RT}{V} = \frac{N_A k T}{V} = nkT \quad (1.10)$$

where $n = N_A/V$ is the *number density* of molecules, that is, the number of molecules of the gas per unit volume.

Boltzmann's constant k is one of the key fundamental constants of physics and it will dominate much of the development of the first part of this course. Whenever you see k in a formula, you immediately think 'thermal physics'.

1.4.4 Dalton's Law of Partial Pressures

Among other important laws is *Dalton's law of partial pressures* (Figure 1.10). Dalton found that, at a fixed temperature, the pressure of a mixture of gases is equal to the sum of the pressures which they would exerted separately, if the other constituents were not there.

Equation of State for one mole of a Perfect gas

$$pV = RT \quad R = 8.31 \text{ J K}^{-1} \text{ mole}^{-1}$$

Avogadro's constant

$$N_A = 6.02 \times 10^{23} \text{ mol}^{-1}$$

Boltzmann's constant

$$k = \frac{R}{N_A} = 1.38 \times 10^{-23} \text{ J K}^{-1}$$

Equation of State of a Gas in Molecular Terms

$$p = \frac{RT}{V} = \frac{N_A k T}{V} = nkT$$

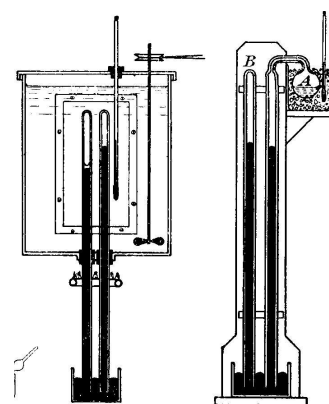


Figure 1.10. The apparatus used by Dalton to establish his law of partial pressures.

1.5 Black-body Radiation

In the second example of distributions in physics, we study the *distribution of radiant energy emitted by a hot body*. We will study this radiation in much more detail in the quantum physics part of the course. We are familiar with the spectrum of white light from the appearance of rainbows, in which the raindrops act as tiny ‘prisms’ which disperse the light of the Sun into its different colours. More precisely, we observe that light possesses a *spectrum* of radiation and a *spectral energy distribution*. Let us perform an experiment which demonstrates the same phenomenon. As the source of light, we use a tungsten light bulb, and allow the current through the filament to be variable. We can estimate the temperature of the filament from the fact that its resistance is a strong function of temperature, as illustrated in Figure 1.11. Also shown is the layout of the experiment. I have already calibrated the temperature scale as a function of the resistance of the filament (Table 1.1). The light from the lamp is collimated and passed through a prism to disperse the light into its spectral components.

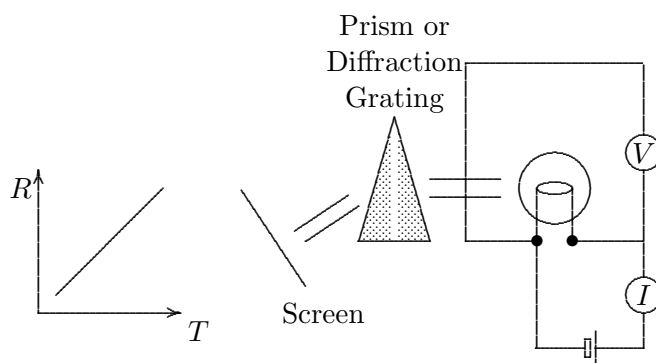
You will observe the following points about this experiment:

- The spectrum of the radiation from the lamp is a continuous function of the frequency. Thus, we need to find some suitable way of describing the variation of its spectral energy distribution with frequency, specifically, we need to define the power per unit frequency (or wavelength) interval.
- As the temperature is raised, there is a greater intensity in the green and blue region of the spectrum. Indeed, if we were to measure the intensity spectrum of the radiation, we would find that it has a maximum frequency and that the frequency of this maximum is proportional to the absolute temperature of the filament,

$$\nu_{\max} \propto T.$$

This is one aspect of *Wien’s Displacement Law*, which we will derive in the quantum physics section of the course.

Figure 1.11. The Tungsten Light Bulb



Voltage V	Current I	Resistance $R = V/I$	R/R_0	Temp. (K)
-	-	0.36	1	293
3.0	1.06	2.83	7.92	1622
6.0	1.45	4.14	11.59	2265
8.0	1.68	4.76	13.33	2572
10.0	1.88	5.32	14.89	2846
12.0	2.05	5.85	16.39	3109

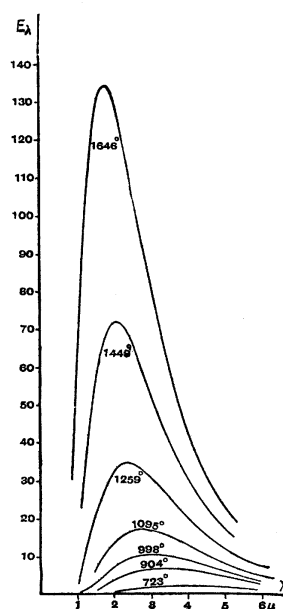


Figure 1.12. Illustrating the experimentally determined change of the spectrum of black-body radiation with temperature. Note the abscissa is in wavelength units.

If we took sufficient care in the design of the source of radiation, we would find that *the intensity is a universal function of frequency*. By this, we mean that we would maintain an enclosure at a fixed temperature *for a very long time* until all the matter and radiation were at the same single temperature. Radiation of this form is known as *black-body radiation*. The term black-body is derived from the concept that a black object is a perfect absorber of radiation of all frequencies, and hence thermodynamically must also be a perfect emitter.

Notice a key point. In the case of the elastic collisions between identical particles, energy was exchanged among all the particles and, in the equilibrium state, a unique energy distribution was found, the Maxwell distribution. In the same way, there are elementary physical processes which enable the frequency, or energy, of any given set of waves to be changed and so energy can be exchanged between different frequencies. If we were to wait long enough, these processes result in the black-body equilibrium spectrum. The form of the blackbody curve is shown in Figure 1.13 and will be derived in the quantum physics part of the course.

1.6 Summary

We have considered only two of many possible examples of physical phenomena involving very large numbers of particles and waves. These are:

- The velocity, or energy, distribution of the particles in a gas at some fixed temperature T , the Maxwell distribution.
- The intensity distribution of radiation in thermal equilibrium at some fixed temperature T , the spectrum of black-body radiation.

The next step is to understand how to describe such distributions so that we can uncover the physics lying behind the properties of perfect gases and radiation fields.

Figure 1.13. Spectrum of black body radiation

