The two-dimensional consolidation theory of electro-osmosis

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INTRODUCTION
Studies on the application of electro-osmosis in ground improvement include Casagrande (1948, 1983), Bjerrum et al. (1967), Esrig & Gemeinhardt (1967), Fetzer (1967) and Lo et al. (1991). Esrig (1968) presented a one-dimensional consolidation theory of electro-osmosis, which assumes that fluid flows due to an electric field and due to hydraulic gradient may be superimposed to find the total flow. Based on Esrig’s one-dimensional consolidation theory, Wan & Mitchell (1976) also presented a one-dimensional consolidation theory of electro-osmosis, which included the combined effects of electro-osmotic and direct loading consolidation, and proved the effectiveness of the electrode reversal technique. Lewis and Humpheson (1973) provided a numerical analysis that could consider the variation of electric flow in the process of electro-osmosis. However, the effect of electro-osmosis is not uniform, and a one-dimensional consolidation theory cannot thoroughly illustrate the effects of ground improvement. The solutions of two-dimensional consolidation theory with different boundary conditions and initial conditions are provided in this paper.

TWO-DIMENSIONAL CONSOLIDATION THEORY OF ELECTRO-OSMOSIS
The whole ground can be divided into many parts. The average area of each part is shown in Fig. 1(a): the average area of improved ground about a pair of electrodes and the orthogonal coordinate system, x, 0, y.

Equation (1) can be rewritten as

\[
\frac{k_h}{\gamma_w} \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + k_e \left( \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right) = m_v \frac{\partial u}{\partial t}
\]

Assuming uniform soil and orthographic isotropy, the following equations can be obtained, as

\[
k_{hx} = k_{hy} = k_h, k_{ex} = k_{ey} = k_e, \alpha_{ex} = \alpha_{ey} = \alpha_e
\]

Equation (1) can be rewritten as

\[
\frac{k_h}{\gamma_w} \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + \alpha_e \left( \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right) = m_v \frac{\partial u}{\partial t}
\]

Equation (2) can be rewritten as

\[
\alpha_e \left( \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right) = C_p \frac{\partial \phi}{\partial t}
\]

Introducing a variable \( \xi \) (Esrig, 1968; Mitchell, 1976; Banerjee & Mitchell, 1980; Banerjee & Vitayasupakorn, 1984):

\[
\xi = u + \frac{k_e \gamma_w}{k_h} \phi
\]

Substituting equation (5) into equation (3):

\[
\frac{\partial \xi}{\partial t} = a^2 \left( \frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 \xi}{\partial y^2} \right) + f(x, y, t)
\]

where

\[
a^2 = C_h = \frac{k_h}{m_v \gamma_w}, f(x, y, t) = \frac{k_e \gamma_w}{k_h} \frac{\partial \phi}{\partial t}
\]
Equations (4) and (6) are the equations of two-dimensional consolidation of electro-osmosis.

Boundary conditions:
Cathode open: \( x = 0, y = 0, u = 0, \phi = 0, \)
and therefore \( \xi(0, 0, t) = 0 \) \( (7) \)
Cathode closed: \( \xi_x(0, 0, t) + \xi_y(0, 0, t) = 0 \) \( (8) \)
Anode closed: \( \xi_x(L, 0, t) + \xi_y(L, 0, t) = 0 \) \( (9) \)
Anode open: \( x = L, y = 0, u = 0, \phi = \phi_0, \)
and therefore \( \xi(L, 0, t) = -\frac{k_y y_w}{k_h} \phi_0 \) \( (10) \)

When \( y \) is equal to \( \pm p/2 \), it is an impermeable boundary: therefore
\[
\xi(x, \pm p/2, t) = 0 \tag{11}
\]

Initial conditions:
If there is an initial excess pore water pressure \( u(x, y, 0) \), it has the following equation:
\[
\xi(x, y, 0) = u(x, y, 0) + \frac{k_y y_w}{k_h} \phi(x, y, 0) \tag{12}
\]

THE ANALYTIC SOLUTION OF TWO-DIMENSIONAL CONSOLIDATION THEORY
Case 1. Anode closed and cathode open
From equations (5), (6), (7), (9), (11) and (12), equation (13) can be obtained by separation of variables:
\[
u(x, y, t) = -\frac{k_y y_w}{k_h} \phi(x, y, t) + \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \left[ B_{mn} + \int_0^{p/2} f_{mn}(\tau) e^{-\left(\frac{(m\pi)^2+(n\pi)^2}{L^2}\right)\tau} d\tau \right] \times \sin \left(\frac{(m\pi)^2+(n\pi)^2}{2L}\right) \times \frac{2n\pi}{P} y e^{-\left(\frac{(m\pi)^2+(n\pi)^2}{L^2}\right)\tau} \tag{13}
\]

where
\[
B_{mn} = \frac{8}{P} \int_0^{p/2} \left[ u(x, y, 0) + \frac{k_y y_w}{k_h} \phi(x, y, 0) \right] \times \sin \left(\frac{(m\pi)^2+(n\pi)^2}{2L}\right) \times \frac{2n\pi}{P} y dx dy \quad n \geq 1
\]
\[
B_{mn} = \frac{4}{P} \int_0^{p/2} \left[ u(x, y, 0) + \frac{k_y y_w}{k_h} \phi(x, y, 0) \right] \times \sin \left(\frac{(m\pi)^2+(n\pi)^2}{2L}\right) \times dx dy \quad n = 0
\]
\[
f_{mn}(\tau) = \frac{8}{P} \int_0^{p/2} \left[ f(x, y, \tau) \sin \left(\frac{(m\pi)^2+(n\pi)^2}{2L}\right) x \times \cos \left(\frac{2n\pi}{P} y \right) dx dy \quad n \geq 1
\]
\[
f_{mn}(\tau) = \frac{4}{P} \int_0^{p/2} \left[ f(x, y, \tau) \sin \left(\frac{(m\pi)^2+(n\pi)^2}{2L}\right) x \times dx dy \quad n = 0
\]
\[
T_H = \frac{C_h t}{L}, \text{ time factor.}
\]

Discussion:
(a) For the case where there is no variation of electric flow during the whole process of ground improvement, \( \partial \phi / \partial t = 0 \), and therefore \( f_{mn}(t) = 0 \). Equation (13) can be rewritten as
\[
u(x, y, t) = -\frac{k_y y_w}{k_h} \phi(x, y) + \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} B_{mn} \sin \left(\frac{(m\pi)^2+(n\pi)^2}{2L}\right) \times \cos \left(\frac{2n\pi}{P} y \right) e^{-\left(\frac{(m\pi)^2+(n\pi)^2}{L^2}\right)\tau} \tag{14}
\]

(b) For the case where there is no variation of electric flow and no direct loading during the whole process of ground improvement, \( \partial \phi / \partial t = 0 \), \( u(x, y, 0) = 0 \), and the \( B_{mn} \) in equation (14) can be written as
\[
B_{mn} = \frac{8k_y y_w}{P} \int_0^{p/2} \left[ f(x, y, \tau) \sin \left(\frac{(m\pi)^2+(n\pi)^2}{2L}\right) x \times \cos \left(\frac{2n\pi}{P} y \right) \right] dx dy \quad n \geq 1
\]
\[
B_{mn} = \frac{4k_y y_w}{P} \int_0^{p/2} \left[ f(x, y, \tau) \sin \left(\frac{(m\pi)^2+(n\pi)^2}{2L}\right) x \times \left(\frac{2n\pi}{P} y \right) \right] dx dy \quad n = 0
\]

Assuming that \( \phi(x, y) = (x/L)(1-2|y|/p)\phi_0 \), the following equations can be obtained:
\[
B_{mn} = \frac{16k_y y_w \phi_0}{k_h \pi^2} \left(\frac{(-1)^n-1}{(m\pi)^2+(n\pi)^2} \right) \quad n \geq 1
\]
\[
B_{mn} = \frac{4k_y y_w \phi_0}{k_h \pi^2} \left(\frac{(-1)^n}{(m\pi)^2} \right) \quad n = 0
\]

It can be seen from equation (14) that the ultimate excess pore water pressure is not zero, but a negative value, which depends on the soil property and electric potential distribution. It is necessary to modify the definition of the average degree of consolidation to
\[
U = \int_0^{\infty} \int_0^{p/2} \left[ u(x, y, 0) - u(x, y, t) \right] dx dy + \int_0^{\infty} \int_0^{p/2} \left[ u(x, y, 0) - u(x, y, \infty) \right] dx dy \tag{15}
\]

The initial excess pore water pressure is zero, and the ultimate excess pore water pressure is equal to \( -(k_y y_w / k_h) \phi(x, y) \). From the assumption that \( \phi(x, y) = (x/L)(1-2|y|/p)\phi_0 \), the isolines of the ultimate excess pore water pressure are shown in Fig. 2, which are zero at the cathode and \( -1 \) at the anode. The value of ultimate excess pore water pressure is decreasing gradually from the anode to the cathode. If \( y \) is equal to zero, we can see from Fig. 2 that the linear distribution of the ultimate excess pore water pressure is the same as Esrig's one-dimensional consolidation. We can also see that the ultimate excess pore water pressure is determined by the difference of electrical potential: the distribution depends on the distribution of electrical potential and the boundary conditions, independent of initial conditions. The average degree of consolidation is
\[
U = 1 - \frac{4}{\pi} \sum_{m=1}^{\infty} \left(\frac{1}{m-1/2}\right) e^{-\left(\frac{(m\pi)^2+(n\pi)^2}{L^2}\right)\tau} \tag{16}
\]
which is the same as that of Esrig’s one-dimensional consolidation.

(c) For the case where there is no variation of electric flow and the uniformly direct loading is applied during the whole process of ground improvement, \( \partial \phi / \partial t = 0 \), \( u(x, y, 0) = u_0 \), and the \( B_{mn} \) in equation (14) can be written as

\[
B_{mn} = \frac{4}{Lp} \int_0^L \frac{y}{2} \int_0^L \left[ u_0 + \frac{k_e y_w}{k_0} \phi(x, y, 0) \right] \times \sin \frac{(2m-1)\pi x \cos 2\pi y}{2L} \, dx \, dy \quad n \geq 1
\]

\[
B_{m0} = \frac{4}{Lp} \int_0^L \frac{y}{2} \int_0^L \left[ u_0 + \frac{k_e y_w}{k_0} \phi(x, y, 0) \right] \times \sin \frac{(2m-1)\pi x \cos 2\pi y}{2L} \, dx \, dy \quad n = 0
\]

Assuming that \( \phi(x, y) = (x/L)(1 - |y|/p)\phi_0 \), the following equations can be obtained:

\[
B_{mn} = \frac{16k_e y_w\phi_0}{k_0\pi^2} \left\{ \frac{(-1)^{m-1}(-1)^{n-1}+1}{(2m-1)^2 n^2} \right\} \quad n \geq 1
\]

\[
B_{m0} = \frac{4k_e y_w\phi_0}{k_0\pi^2} \left( \frac{(-1)^{m-1}}{(2m-1)^2} \right) \quad n = 0
\]

The initial excess pore water pressure is \( u_0 \), and the ultimate excess pore water pressure is equal to \( -(k_e y_w/k_0)\phi(x, y) \). The average degree of consolidation is

\[
\bar{U} = \frac{1}{\pi^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\pi u_0}{(2m-1)^2} \frac{k_e y_w\phi_0}{k_0} \left( \frac{(-1)^{m-1}}{(2m-1)^2 n^2} \right) e^{-\left(m-\frac{j}{2}\right)2r T_n}
\]

\[
= \frac{4}{\pi^2} \sum_{m=1}^{\infty} \int_0^\infty \frac{\pi u_0}{(2m-1)^2} \frac{k_e y_w\phi_0}{k_0} \left( \frac{(-1)^{m-1}}{(2m-1)^2} \right) e^{-\left(m-\frac{j}{2}\right)2r T_n} \, dr
\]

If \( u_0 \) is equal to zero, equation (17) is changed into equation (16).

If \( r = u_x/u_0 \) and \( u_a = -k_e y_w\phi_0/k_0 \), equation (17) can be rewritten as

\[
\bar{U} = \frac{1}{\pi^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\pi u_0}{(2m-1)^2} \frac{k_e y_w\phi_0}{k_0} \left( \frac{(-1)^{m-1}}{(2m-1)^2 n^2} \right) e^{-\left(m-\frac{j}{2}\right)2r T_n}
\]

\[
= \frac{4}{\pi^2} \sum_{m=1}^{\infty} \int_0^\infty \frac{\pi u_0}{(2m-1)^2} \frac{k_e y_w\phi_0}{k_0} \left( \frac{(-1)^{m-1}}{(2m-1)^2} \right) e^{-\left(m-\frac{j}{2}\right)2r T_n} \, dr
\]

When \( r \) is equal to negative infinity it is electro-osmosis consolidation, and when \( r \) is equal to zero, it is Terzaghi consolidation, which is shown in Fig. 3.

(d) The initial excess pore water pressure after electrode reversal, \( u(x, y, 0) \), is the ultimate excess pore water pressure before electrode reversal, \( u(x, y, \infty) \). By changing the coordinate system, the following equations are obtained:

\[
u(x, y, 0) = \frac{k_e y_w\phi_0}{k_0} \left( \frac{1 - \frac{x}{L}}{1 - \frac{2}{p} |y|} \right)
\]

\[
\phi(x, y, t) = \frac{x}{L} \left( \frac{1 - \frac{2}{p} |y|}{1 - \frac{x}{L}} \phi_0 \right)
\]

The \( B_{mn} \) in equation (14) can be written as

\[
B_{mn} = -\frac{8k_e y_w\phi_0}{k_0\pi^2} \left( \frac{(-1)^{m-1}(-1)^{n-1}+1}{(2m-1)^2 n^2} \right)
\]

\[n \geq 1\]

\[
B_{m0} = -\frac{2k_e y_w\phi_0}{k_0\pi(2m-1)} + \frac{4k_e y_w\phi_0}{k_0\pi^2} \left( \frac{(-1)^{m-1}}{(2m-1)^2} \right)
\]

\[n = 0\]

Case 2. Cathode closed and anode open

From equations (5), (6), (8), (10), (11) and (12), equation (21) also can be obtained:

\[
u(x, y, t) = \frac{k_e y_w}{k_0} \left[ \phi_0 - \phi(x, y, t) \right] + \sum \int_{m=1}^{\infty} \int_{n=1}^{\infty} B_{mn} \left( \int_0^\infty f_{mn}(t) e^{-\left(m-\frac{j}{2}\right)2r T_n} \, dr \right)
\]

\[
\times \cos \frac{(2m-1)\pi x \cos 2\pi y}{2L} \cos \frac{2n\pi y}{p} ye^{-\left(m-\frac{j}{2}\right)2r T_n}
\]

\[n \geq 1\]

If \( u_0 \) is equal to zero, equation (17) is changed into equation (16). If \( r = u_x/u_0 \) and \( u_a = -k_e y_w\phi_0/k_0 \) equation (17) can be rewritten as
where

\[ B_{mn} = \frac{8}{L} \int_0^{\pi/2} \left\{ u(x, y, 0) + \frac{k_e \gamma_w}{k_h} \left[ \phi(x, y, 0) - \phi_0 \right] \right\} \]
\[ \times \cos \left( \frac{(2m-1)\pi}{2L} \right) x \cos \frac{2n\pi}{p} y \, dx \, dy \quad n \geq 1 \]
\[ B_{m0} = \frac{4}{L} \int_0^{\pi/2} \left\{ u(x, y, 0) + \frac{k_e \gamma_w}{k_h} \left[ \phi(x, y, 0) - \phi_0 \right] \right\} \]
\[ \times \cos \left( \frac{(2m-1)\pi}{2L} \right) x \, dx \, dy \quad n = 0 \]
\[ f_{mn}(\tau) = \frac{8}{L} \int_0^{\pi/2} f(x, y, \tau) \cos \left( \frac{(2m-1)\pi}{2L} \right) x \, dx \, dy \quad n \geq 1 \]
\[ f_{m0}(\tau) = \frac{4}{L} \int_0^{\pi/2} f(x, y, \tau) \cos \left( \frac{(2m-1)\pi}{2L} \right) x \, dx \, dy \quad n = 0 \]

The ultimate excess pore water pressure is equal to \( k_e \gamma_w / k_h [\phi_0 - \phi(x, y, \tau)] \). From the assumption that \( \phi(x, y) = (x/L)(1 - |y|/p)\phi_0 \), the isoclines of the ultimate excess pore water pressure are shown in Fig. 4, which are zero at the anode and 1 at the cathode. The value of ultimate excess pore water pressure increases gradually from the anode to the cathode. If \( y \) is equal to zero, we can see from Fig. 4 that the linear distribution of the ultimate excess pore water pressure is the same as Esrig’s one-dimensional consolidation.

**Case 3. Anode and cathode open**

From equations (5), (6), (7), (10), (11) and (12), equation (22) can also be obtained:

\[ u(x, y, \tau) = \frac{k_e \gamma_w}{k_h} \left[ \frac{x}{L} \phi_0 - \phi(x, y, \tau) \right] \]
\[ + \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \left\{ B_{mn} + \int_0^{\pi/2} f_{mn}(\tau) e^{-\left(\frac{m-1}{2}\right)^2 \tau \tau_0} \, d\tau \right\} \]
\[ \times \sin \frac{m\pi}{L} x \cos \frac{2n\pi}{p} y e^{-\left(\frac{m-1}{2}\right)^2 \tau \tau_0} \]  \hspace{2cm} (22)

From equation (22), the ultimate excess pore water pressure is \( k_e \gamma_w / k_h [\phi_0 - \phi(x, y, \tau)] \). From the assumption that \( \phi(x, y) = (x/L)(1 - |y|/p)\phi_0 \), the isoclines of the ultimate excess pore water pressure are shown in Fig. 5. If \( y \) is equal to zero, we can see from Fig. 5 that the zero distribution of the ultimate excess pore water pressure is the same as Esrig’s one-dimensional consolidation.

**CONCLUSIONS**

The solutions of two-dimensional consolidation theory due to electro-osmosis are provided in this paper based on three kinds of boundary condition: anode closed and cathode open; cathode closed and anode open; and both anode and cathode open. The solutions are presented graphically as isoclines of ultimate excess pore water pressure. These show that the ultimate excess pore water pressure produced by electro-osmosis may be positive or negative, whose maximum is determined by the difference of electrical potential between cathode and anode, and whose distribution is dependent on the electrical potential distribution and boundary conditions, independent of the initial conditions. When \( y \) is equal to zero, the solution is the same as Esrig’s one-dimensional consolidation.
**NOTATION**

\( a \) coefficient of equations

\( C_h \) hybrid coefficient of consolidation for vertical compression due to horizontal drainage

\( C_p \) capacitance per unit volume of soil

\( i_e \) electrical potential gradient

\( i_h \) hydraulic gradient

\( k_e \) electro-osmotic permeability

\( k_{e_x} \) coefficient of electro-osmotic permeability in \( x \) direction

\( k_{e_y} \) coefficient of electro-osmotic permeability in \( y \) direction

\( k_h \) coefficient of horizontal hydraulic permeability

\( k_{h_x} \) coefficient of horizontal hydraulic permeability in \( x \) direction

\( k_{h_y} \) coefficient of horizontal hydraulic permeability in \( y \) direction

\( m_v \) compressibility of soil

\( r \) ratio between ultimate excess pore water pressure and initial pore water pressure

\( t \) time

\( u \) excess pore water pressure

\( u_u \) ultimate excess pore water pressure

\( \bar{U} \) average degree of consolidation

\( x \) \( x \) coordinate

\( y \) \( y \) coordinate

\( \gamma_w \) unit weight of water

\( \sigma_{e_x} \) electrical conductivity in \( x \) direction

\( \sigma_{e_y} \) electrical conductivity in \( y \) direction

\( t \) a variable of integral

\( \phi \) electrical potential

\( \phi_0 \) electrical potential at anode, assuming that electric potential at cathode is zero

\( \phi_0^e \) electrical potential at new anode after electrode reversal

\( \psi \) a variable

**REFERENCES**


