1 - ESFORÇOS SOLICITANTES CONSTANTES ($f_x = f_y = m_z = 0$)

Adotada: $M_\omega = 0$

eixos (x, y) principais ($I_{xy} = 0$)

* Eq. equilíbrio

→ $EI_y u^{IV} - N u'' + (M_x - N y_c)\varphi_z'' + \beta_x [u - (y_H - y_c)\varphi_z] = 0$

→ $EI_x v^{IV} - N v'' + (M_y + N x_c)\varphi_z'' + \beta_y [v + (x_H - x_c)\varphi_z] = 0$

→ $EI_\omega \varphi_z^{IV} + (M_x - N y_c) u'' + (M_y + N x_c) v'' + \beta_\varphi \varphi_z - \beta_x (y_H - y_c)$

$+ \beta_y (x_H - x_c) v + [\beta_x (y_H - y_c)^2 + \beta_y (x_H - x_c)^2]\varphi_z -$

$- [G I_t + r_0^2 N + 2 r_{0y} M_x - 2 r_{0x} M_y]\varphi_z'' = 0$

1a) Barras bi-apoiadas ($\beta_x = \beta_y = \beta_\varphi = 0$)

- Condições de contorno:

$u, v, \varphi_z = 0 \qquad z = 0, \ell$

$u'', v'', \varphi_z'' = 0 \qquad z = 0, \ell$

$\therefore u = A \operatorname{sen} \dfrac{n\pi z}{\ell} \; ; \; v = B \operatorname{sen} \dfrac{n\pi z}{\ell} \; ; \; \varphi_z = C \operatorname{sen} \dfrac{n\pi z}{\ell}$

Substituindo-se u, v, φ_z nas eq. equilíbrio ⇒

$(EI_y \chi^2 + N) A - (M_x - N y_c) C = 0$

$(EI_x \chi^2 + N) B - (M_y + N x_c) C = 0$

$-(M_x - N y_c) A - (M_y + N x_c) B + [EI_\omega \chi^2 + (G I_t + r_0^2 N + 2 r_{0y} M_x$

$- 2 r_{0x} M_y)] C = 0$

onde $\mathcal{K}^2 = \dfrac{m^2\pi^2}{\ell^2}$

Soluções $\begin{cases} A = B = C = 0 \\ \text{Determinante dos coeficientes de } (A, B, C) \text{ nulo} \end{cases}$

Para

$$\boxed{\begin{aligned} N_{yE} &= -EI_y\mathcal{K}^2 = -\frac{\pi^2 EI_y}{(\ell/m)^2} \\ N_{xE} &= -EI_x\mathcal{K}^2 = -\frac{\pi^2 EI_x}{(\ell/m)^2} \\ N_\omega &= -\frac{EI_\omega\mathcal{K}^2 + GI_t}{r_0^2} \end{aligned}}$$

$$\rightarrow \begin{bmatrix} N - N_{yE} & 0 & -(M_x - Ny_c) \\ 0 & N - N_{xE} & -(M_y + Nx_c) \\ -(M_x - Ny_c) & -(M_y + Nx_c) & r_0^2\left[N - N_\omega + \frac{1}{r_0^2}(2r_{0y}M_x - 2r_{0x}M_y)\right] \end{bmatrix} = 0$$

* Caso particular 1: compressão axial ($M_x = M_y = 0$)

$$N^3(r_0^2 - x_c^2 - y_c^2) + N^2\left[N_{xE}y_c^2 + N_{yE}x_c^2 - r_0^2(N_{xE} + N_{yE} + N_\omega)\right]$$

$$+ N\left[r_0^2(N_{xE}N_{yE} + N_{xE}N_\omega + N_{yE}N_\omega)\right] - N_{xE}N_{yE}N_\omega r_0^2 = 0$$

ou

$$r_0^2(N - N_{xE})(N - N_{yE})(N - N_\omega) - N^2y_c^2(N - N_{xE}) - N^2x_c^2(N - N_{yE}) = 0$$

* Se $G \equiv C$ $(x_c = y_c = 0)$ → eq. diferenciais desacopladas

$$\left.\begin{aligned} N_{1cr} &= N_{yE} \\ N_{2cr} &= N_{xE} \end{aligned}\right\} \text{flambagem de Euler}$$

$$N_{3cr} = N_{\omega} \; \} \text{ flambagem por torção}$$

* Barra com 1 eixo de simetria (ex. eixo x) → $y_c = 0$

$$r_0^2 (N - N_{xE})(N - N_{yE})(N - N_{\omega}) - N^2 x_c^2 (N - N_{yE}) = 0$$

logo

$$(N - N_{yE})\left[r_0^2 (N - N_{xE})(N - N_{\omega}) - N^2 x_c^2 \right] = 0$$

∴ eq. equilíbrio em u desacoplada

1b) <u>Barras bi-engastadas</u> $(\beta_x = \beta_y = \beta_\varphi = 0)$

- Condições de contorno

$$\begin{aligned} &u, v, \varphi_z = 0 \qquad && z = 0 \; ; z = l \\ &u', v', \varphi_z' = 0 \qquad && z = 0 \; ; z = l \end{aligned} \quad \therefore$$

$$u = A\left[1 - \cos 2\chi z\right] \; ; v = B\left[1 - \cos 2\chi z\right] \; ; \varphi_z = C\left[1 - \cos 2\chi z\right]$$

Substituindo-se u, v, φ_z nas eq. equilíbrio e impondo que o determinante dos coeficientes seja nulo →

$$0 = \begin{bmatrix} N - 4N_{yE} & 0 & -(M_x - Ny_c) \\ 0 & N - 4N_{xE} & -(M_y - Nx_c) \\ -(M_x - Ny_c) & -(M_y - Nx_c) & r_0^2\left[N + \dfrac{GI_t + 4EI_\omega \chi^2}{r_0^2} + \dfrac{1}{r_0^2}(2r_{0y}M_x - 2r_{0x} \ldots \right. \end{bmatrix}$$

Ex. 1

$$I_x = I_y = \frac{8tb^3}{12} = \frac{2tb^3}{3} \qquad I_0^c = I_x^c + I_y^c = \frac{4tb^3}{3}$$

$$I_\omega = 0 \qquad A = 4bt$$

$$I_t = \frac{4bt^3}{3} \qquad r_0^2 = \frac{I_0^c}{A} = \frac{b^2}{3}$$

$$N_{xE} = N_{yE} = -\frac{\pi^2 E I_x}{\ell^2} = -\frac{2}{3}\,\frac{\pi^2 E t b^3}{\ell^2}$$

$$N_\omega = -\frac{G I_t}{r_0^2} = -\frac{4Gt^3}{b} = -\frac{2Et^3}{(1+\nu)\,b}$$

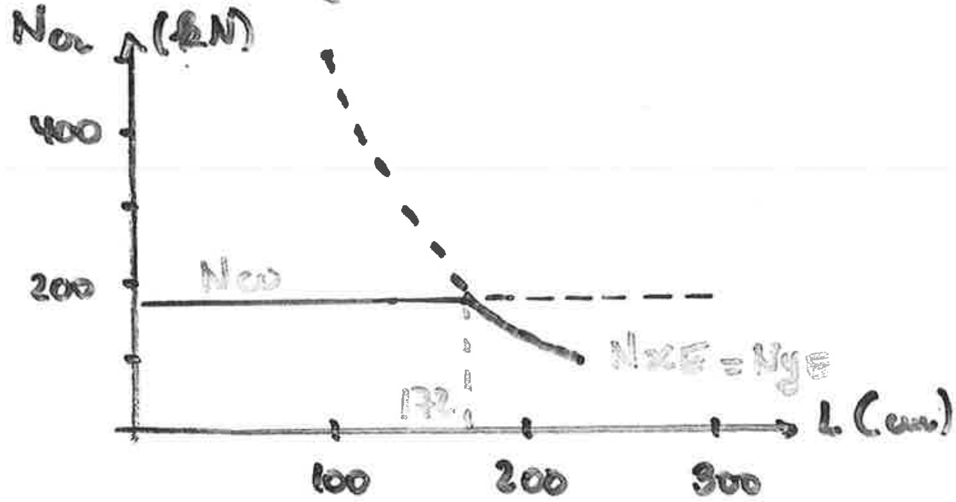
Logo $N_\omega = N_{yE}$ para $\ell^2 = \frac{\pi^2(1+\nu)}{3}\,\frac{b^4}{t^2} = 4{,}28\,\frac{b^4}{t^2}$, ou

$$\underline{\ell^* = 2{,}07\ b^2/t}$$

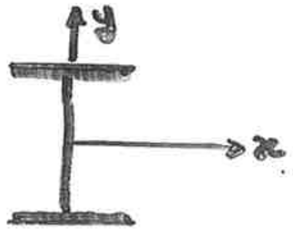
Para $b = 5$ cm e $t = 0{,}3$ cm $\rightarrow$ ($E = 20500$ kN/cm²)

$N_\omega = 170$ kN

$N_{xE} = N_{yE} = 5.058.170 / L^2$ kN



Ex 2



a) CS 250x52 ($b_f = h = 250$ mm ; $t_f = 9{,}5$ mm ; $t_w = 8{,}0$ mm)

$I_x = 7694\ cm^4$; $I_y = 2475\ cm^4$; $I_\omega = I_y \dfrac{d^2}{4} = 356.400\ cm^6$

$I_0^c = I_x + I_y = 10.169\ cm^4$; $A = 66\ cm^2$; $r_0 = 12{,}41\ cm$; $I_t = 18{,}2$ c

$N_{xE} = -\pi^2 E I_x / L^2 = -1{,}56 \times 10^9 / L^2 > N_{yE}$ (não condicionante

$N_{yE} = -\pi^2 E I_y / L^2 = -0{,}50 \times 10^9 / L^2$

$N_\omega = -\dfrac{G I_t + E I_\omega \pi^2 / L^2}{r_0^2} = -\left(923 + \dfrac{0{,}47 \times 10^9}{L^2}\right)$

$N_{yE} = N_\omega$ para $L = 18$
($\bar{N}_{yE} = 15400$ kN

Para $L = 100$ cm $\rightarrow \bar{N}_\omega = 47.750$ kN $< \bar{N}_{yE} = 50.076$ kN

$L = 400$ cm $\rightarrow \bar{N}_\omega = 3850$ kN $> \bar{N}_{yE} = 3130$ kN

b) CVS 350x73 ($b_f = 250$ mm ; $h = 350$ mm ; $t_f = 12{,}5$ mm ; $t_w = 9{,}5$ m

$I_x = 20524\ cm^4$; $I_y = 3258\ cm^4$; $I_\omega = I_y d^2/4 = 942\,000\ cm^6$

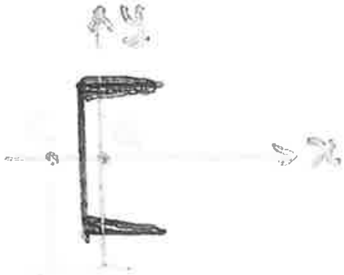
$I_0^c = 23782\ cm^4$; $A = 93{,}4\ cm^2$; $r_0 = 15{,}97\ cm$

$N_{yE} = -\pi^2 E I_y / L^2 = -0{,}66 \times 10^9 / L^2$

$N_\omega = -\dfrac{G I_t + E I_\omega \pi^2 / L^2}{r_0^2} = -\left(1301 + \dfrac{0{,}75 \times 10^9}{L^2}\right)$

Logo $\bar{N}_\omega > \bar{N}_{yE}\ \ \forall L$ (não ocorre flambagem por torção).

Ex 3



C 102×8,0

$y_c = 0$ $\qquad x_c = -2{,}66\ cm$

$I_x = 160\ cm^4 \qquad I_y = 13\ cm^4 \qquad I_t = 1{,}66\ cm^4$

$I_\omega = 248\ cm^6 \qquad A = 10\ cm^2$

$I_0^G = I_x^G + I_y^G = 173\ cm^4$

$I_0^C = I_0^G + A(x_c^2 + y_c^2) \rightarrow r_0^2 = \dfrac{I_0^G}{A} + x_c^2 + y_c^2 = 2$

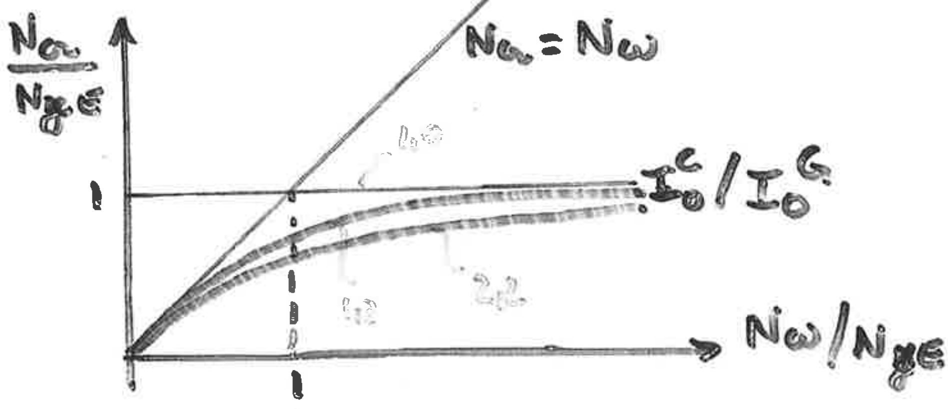
$$N_{cr1} = -\frac{\pi^2 E I_y}{L^2} = -\frac{2{,}63\times10^6}{L^2} = N_{yE}$$

$$N_{xE} = -\frac{\pi^2 E I_x}{L^2} = -\frac{32{,}4\times10^6}{L^2}$$

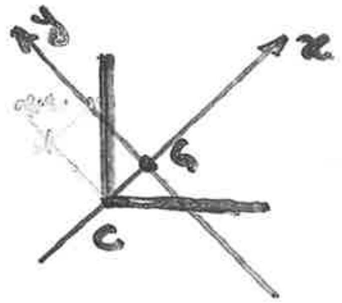
$$N_\omega = -\frac{E I_\omega \pi^2/L^2 + G I_t}{r_0^2} = -\left(537 + \frac{2{,}05\times10^6}{L^2}\right)$$

$$(r_0^2 - x_c^2)N^2 - r_0^2(N_\omega + N_{xE})N + r_0^2 N_{xE} N_\omega = 0 < \begin{matrix} N_{cr2} \\ N_{cr3} \end{matrix}$$

L	$\bar{N}_{xE}$	$\bar{N}_{yE}$	$\bar{N}_\omega$	$\bar{N}_{cr2}$	$\bar{N}_{cr3}$
25	51840	4208	3817	3734	74765
50	12960	1052	1357	1314	18879
100	3240	263	742	688	4928



Ex 4



L 102 × 6,4 ; L = 100 cm

$y_c = 0$; $x_c = -3{,}5$ cm

$A = 12{,}51\ cm^2$; $I_x = 200\ cm^4$; $I_y = 50\ cm^4$

$I_t = 1{,}75\ cm^4$; $I_\omega = 0$; $I_0^G = 250\ cm^4$

$I_0^C = I_0^G + A x_c^2 = 400\ cm^4 \rightarrow I_0^C / I_0^G = 1{,}6$; $r_0^2 = ...$

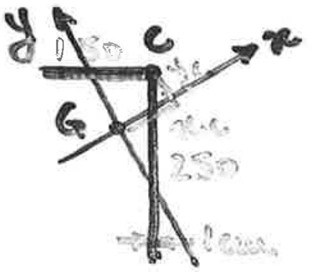
$$N_{cr,1} = N_{yE} = -\frac{\pi^2 E I_y}{L^2} = -\frac{10{,}1 \times 10^6}{L^2} = -1010\ kN$$

$$N_{xE} = -\frac{\pi^2 E I_x}{L^2} = -\frac{40{,}4 \times 10^6}{L^2} = -4040\ kN$$

$$N_\omega = -\frac{G I_t}{r_0^2} = -432\ kN$$

$$(r_0^2 - x_c^2) N^2 - r_0^2 (N_\omega + N_{xE}) N + r_0^2 N_\omega N_{xE} = 0 \rightarrow \begin{cases} N_{cr,2} = -414 \\ N_{cr,3} = -6838 \end{cases}$$

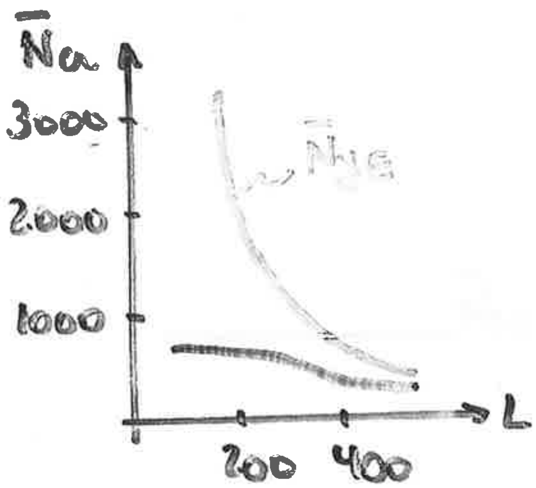
Ex 5



$x_c = 5{,}2$ cm ; $y_c = 6{,}3$ cm ; $I_\omega = 0$; $A = 39\ cm^2$

$I_x = 2898\ cm^4$; $I_y = 434\ cm^4$; $I_t = 13\ cm^4$

$$r_0^2 = \frac{I_x + I_y}{A} + x_c^2 + y_c^2 = 152\ cm^2 \rightarrow N_\omega = -\frac{G I_t}{r_0^2} = -7...$$



L	$\bar{N}_{xE}$	$\bar{N}_{yE}$	$\bar{N}_\omega$	$\bar{N}_{cr}$
100	60070	9000	720	700
300	6690	1000	720	540
500	2400	360	720	340

* Caso particular 2 : Força normal de extremidade excêntrica

$$M_x = + N e_y \qquad M_y = - N e_x$$

* Eq. equilíbrio:

$$EI_y u^{IV} - N u'' - N(y_c - e_y)\varphi_z'' = 0$$

$$EI_x v^{IV} - N v'' + N(x_c - e_x)\varphi_z'' = 0$$

$$EI_\omega \varphi_z^{IV} - N(y_c - e_y)u'' + N(x_c - e_x)v'' - \Big[GI_t + N\underbrace{(r_o^2 + 2r_{oy}e_y + 2r_{ox}e_x)}_{r_o^{*2}}\Big]\varphi_z''$$

* Barra bi-apoiada ($u = A \operatorname{sen} \chi z$, $v = B \operatorname{sen} \chi z$, $\varphi_z = C \operatorname{sen} \chi z$):

$$\begin{bmatrix} N - N_{yE} & 0 & N(y_c - e_y) \\ 0 & N - N_{xE} & -N(x_c - e_x) \\ N(y_c - e_y) & -N(x_c - e_x) & N r_o^{*2} - N_\omega r_o^2 \end{bmatrix} = 0$$

a) Força normal em C ($e_x = x_c$; $e_y = y_c$) → 3 equações desacopladas ∴

$$N_{1cr} = N_{xE} \quad ; N_{2cr} = N_{yE} \quad ; N_{3cr} = N_\omega^* = - \frac{EI_\omega \chi^2 + GI_t}{r_o^{*2}}$$

b) *Barra com um eixo de simetria (ex. eixo y) → $x_c = 0$; r_{ox}

*Força normal no plano de simetria (yz) → $e_x = 0$

A eq. equilíbrio em v fica desacoplada →

$$N_{1cr} = N_{xE} = -\pi^2 EI_x / \ell^2$$

Das demais, obtém-se:

$$(N-N_{yE})\left[N\, r_0^{*2} - N_\omega r_0^2\right] - N^2 (y_c - e_y)^2 = 0$$

Se $e_y = y_c \rightarrow N_{2cr} = N_{yE}$ e $N_{3cr} = N_\omega^* = N_\omega \left(\dfrac{r_0}{r_0^*}\right)^2$

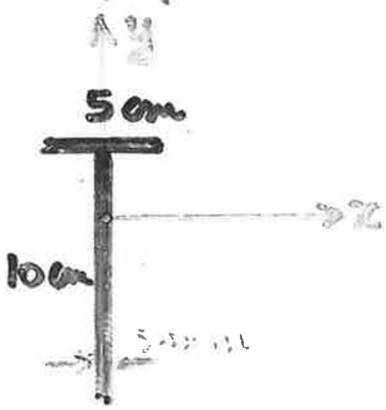
Se $e_y = 0 \rightarrow r_0^2 (N-N_{yE})(N-N_\omega) - N^2 y_c^2 = 0$ (ver caso 1)

c) Barra com dois eixos de simetria ($C \equiv G$; $x_c = y_c = 0$; $r_{0x} = r_{0y} = 0$) e $e_x = 0$

$$r_0^2 (N-N_{yE})(N-N_\omega) - N^2 e_y^2 = 0 \rightarrow$$

$$N^2 (r_0^2 - e_y^2) - N (N_\omega + N_{yE})\, r_0^2 + r_0^2 N_\omega N_{yE} = 0$$

Exemplo: Perfil T



$A = 7{,}5\ cm^2$; $x_c = 0$; $y_c = 3{,}33\ cm$

$I_x = 83{,}3\ cm^4$; $I_y = 5{,}2\ cm^4$; $I_t = 0{,}63\ cm$

$I_\omega = 0$; $I_0^C = I_x + I_y + A y_c^2 = 172\ cm^4$;

$I_{xy} = 0$; $r_0^2 = 22{,}9\ cm^2$

$$H_x = \int_A y(x^2+y^2)\, dA = \int_{-6,67}^{+3,33} t y^3 dy + t y_c \int_{-2,5}^{+2,5} x^2 dx + t y_c^3 \int_{-2,5}^{+2,5} dx = -122\ c$$

$$2 r_{0y} = \frac{H_x}{I_x} - 2 y_c = -8{,}13\ cm$$

$$r_0^{*2} = r_0^2 + 2 r_{0y} e_y = 22{,}9 - 8{,}13\, e_y$$

Para $L = 167$ cm, tem-se:

$N_{xE} = -\pi^2 E I_x / \ell^2 = -604$ kN $= N_{1cr}$

$N_{yE} = -\pi^2 E I_y / \ell^2 = -37{,}7$ kN

$N_{\omega} = G I_t / r_0^2 = -219$ kN

Eq. : $N^2 [r_0^{*2} - (y_c - e_y)^2] - N [N_{\omega} r_0^2 + N_{yE} r_0^{*2}] + r_0^2 N_{\omega} N_{yE} =$

a) Força normal em C ($e_y = y_c = +3{,}33$ cm)

$N_{2cr} = N_{yE} = \underline{-37{,}7 \text{ kN}}$

$N_{3cr} = N_{\omega}^* = N_{\omega} r_0^2 / r_0^{*2}$; $r_0^{*2} = -4{,}17$ cm $\rightarrow N_{3cr} = \underline{+1200 \text{ k}}$

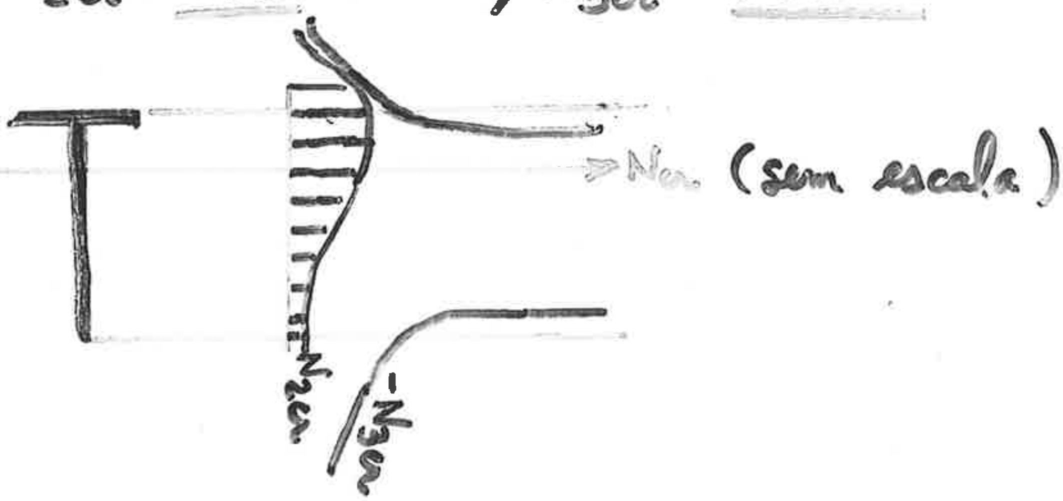
b) Força normal em G ($e_y = 0$)

$r_0^2 (N - N_{yE})(N - N_{\omega}) - N^2 y_c^2 = 0 \rightarrow N_{2cr} = \underline{-34{,}5 \text{ kN}}$

$N_{3cr} = -452$ kN

c) $e_y = -6{,}67$ cm , analogamente

$N_{2cr} = \underline{-22 \text{ kN}}$; $N_{3cr} = \underline{+378 \text{ kN}}$



* Caso particular 3 : Flexão pura simples

$M_\omega = N = 0$; $f_x = f_y = 0$

* Eq. equilíbrio

$$EI_y\, u^{IV} + M_x \varphi_z'' = 0$$

$$EI_x\, v^{IV} + M_y \varphi_z'' = 0$$

$$EI_\omega \varphi_z^{IV} + M_x u'' + M_y v'' - [GI_t + 2r_{0y} M_x - 2r_{0x} M_y]\varphi_z'' = 0$$

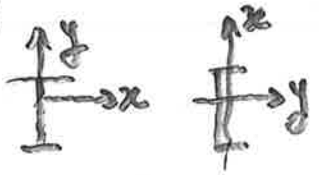
* Barra bi-apoiada

$u = A\,\text{sen}\,\chi z \qquad v = B\,\text{sen}\,\chi z \qquad \varphi_z = C\,\text{sen}\,\chi z$

$$\begin{bmatrix} N_{yE} & 0 & -M_x \\ 0 & N_{xE} & -M_y \\ -M_x & -M_y & r_0^2 N_\omega + 2r_{0y} M_x - 2r_{0x} M_y \end{bmatrix} = 0$$

a) Barras com um eixo de simetria (eixo y) $\rightarrow x_c = r_{0x} = 0$

a.1) M no plano de simetria yz ($M_y = 0$)



$$M_{x\,cr} = N_{yE}\left[r_{0y} \pm \sqrt{r_{0y}^2 + r_0^2 \frac{N_\omega}{N_{yE}}}\right]$$

a.2) M no plano normal ao de simetria ($M_x = 0$)

$$M_{y\,cr} = \pm r_0 \sqrt{N_{xE} N_\omega}$$

Obs: (a1) vale para $M_y = 0$ mesmo sem simetria

b) Barras com 2 eixos de simetria - $M_y = 0$ ($x_c = y_c = 0$; $r_{ox} = r_{oy}$)

$$M_{cr} = r_o \sqrt{N_{yE} N_\omega} = \sqrt{\frac{\pi^2 E I_y}{\ell^2}\left(G I_t + \frac{\pi^2 E I_\omega}{\ell^2}\right)} =$$

$$= \frac{\pi}{L} \sqrt{E I_y G I_t} \sqrt{1 + \frac{\pi^2}{L^2} \frac{E I_\omega}{G I_t}}$$

ex: seção retangular

$$I_\omega = 0 \; ; \; I_t = h t^3/3 \; ; \; I_y = h t^3/12$$

$$M_{cr} = \frac{\pi}{L} \sqrt{E G} \, \frac{h t^3}{6}$$

ex: seção I

$$I_\omega = I_y d^2/4 \; ; \; I_y = t_f b_f^3/6 \; ; \; I_t \cong 2/3 \, b_f t_f^3$$

$$I_x = 2 b_f t_f \frac{(d - t_f)^2}{4} + \frac{t_w (d - 2t_f)^3}{12} \rightarrow W_x = \frac{I_x}{d/2} \cong d\left(A_f + \frac{A_w}{6}\right)$$

$$M_{cr} = \sqrt{\frac{\pi^2}{L^2} E I_y G I_t + \left(\frac{\pi^2}{L^2}\right)^2 E^2 I_y I_\omega} = \sqrt{M_{cr1}^2 + M_{cr2}^2}$$

onde

$$M_{cr1} = \frac{\pi E}{L} \sqrt{\frac{I_y I_t}{2(1+\nu)}} = \frac{\pi E}{L} \frac{t_f^2 b_f^2}{3\sqrt{2(1+\nu)}} \cong \overbrace{\frac{\pi}{3\sqrt{2(1+\nu)}}}^{\cong 0{,}65} \frac{E W_x}{(L d/A_f)}$$

$$M_{cr2} = \frac{\pi^2 E}{L^2} \sqrt{I_y I_\omega} = \frac{\pi^2 E I_y d}{2 L^2} \cong \frac{\pi^2 E}{L^2} \overbrace{\frac{I_y/2}{(A_f + A_w/6)}}^{= r_T^2} W_x = \frac{\pi^2 E}{\lambda_T^2} W_x$$

c) Barra ponto-simétrica (perfil Z) - $r_{ox} = r_{oy} = 0$

$$\rightarrow M_{xcr} = \pm \sqrt{r_o^2 N_{yE} N_\omega} \; ; \; M_{ya} = \pm \sqrt{r_o^2 N_{xE} N_\omega}$$

IV - FLAMBAGEM LATERAL DE VIGAS

$I_{xy}=0$; $\beta_x=\beta_y=\beta_\varphi=0$; $M_\omega=0$; $M_y=0$ (logo $f_x=0$, $V_x=0$)

Eq. equilíbrio

$$EI_y\, u^{IV} + (M_x \varphi_z)'' = 0$$

$$EI_x\, v^{IV} = 0$$

$$EI_\omega \varphi_z^{IV} - \left[(2r_{oy}M_x + GI_t)\varphi_z'\right]' + M_x u'' + f_y(e_y - y_c)\varphi_z = 0$$

Energia potencial total

$$U = \frac{1}{2}\int_0^l \left[EI_y u''^2 + EI_\omega \varphi_z''^2 + (GI_t + 2r_{oy}M_x)\varphi_z'^2 - 2(M_x\varphi_z' + V_y\varphi_z)u'\right]dz$$

$$+ \frac{1}{2}\int_0^l f_y(e_y - y_c)\varphi_z^2\, dz$$

__Obs__:

$$EI_y u'' + M_x\varphi_z = -\bar{V}_{0x}z - \bar{M}_{oy}$$

__Vigas bi-apoiadas__:

$z=0 \rightarrow u''=0$; $\varphi_z=0$ $\therefore$ $\bar{M}_{oy}=0$

$z=l \rightarrow u''=0$; $\varphi_z=0$ $\therefore$ $0=\bar{V}_{0x}\cdot l \rightarrow \bar{V}_{0x}=0$

$$\therefore \boxed{u'' = -\frac{M_x}{EI_y}\varphi_z}$$

na expressão de U; para $V_y = M_x'$ $\rightarrow$

$$-\int_0^\ell (M_x \varphi_z' + M_x' \varphi_z)\mu' dz = -\int_0^\ell (M_x \varphi_z)' \mu' dz = -\int_0^\ell (-\mu'' E I_y)' \mu' dz$$

$$= + E I_y \int_0^\ell \mu' \mu''' dz = E I_y \left[\mu' \mu''\right]_0^\ell - \int_0^\ell E I_y \mu''^2 dz =$$

$$= \underbrace{E I_y \left[\mu' \mu''\right]_0^\ell}_{} - \int_0^\ell \frac{(M_x \varphi_z)^2}{EI} dz$$

nova condição de contorno (=0 para vigas bi-apoiadas)...

Tomando-se ainda $f_y = -M_x''$ tem-se, p/ vigas bi-apoiadas,

$$U = \frac{1}{2}\int_0^\ell \left[\frac{M_x^2 \varphi_z^2}{EI_y} + E I_\omega \varphi_z''^2 + (G I_t + 2 r_{oy} M_x)\varphi_z'^2 - 2\frac{M_x^2 \varphi_z^2}{EI_y} - M_x''(e_y - y_c)\varphi_z^2\right] dz$$

$$= \frac{1}{2}\int_0^\ell \left[-\frac{(M_x \varphi_z)^2}{EI_y} + E I_\omega \varphi_z''^2 + (G I_t + 2 r_{oy} M_x)\varphi_z'^2 - M_x''(e_y - y_c)\varphi_z^2\right] dz$$

Impondo-se $U = 0$ e sendo $M_x = M_o\, m(z)$, onde $M_o = M_{máx}$ $\rightarrow$

$$\left[-\frac{1}{2}\int_0^\ell \frac{(m \varphi_z)^2}{EI_y} dz\right] M_o^2 + \left[r_{oy}\int_0^\ell m \varphi_z'^2 dz - \frac{(e_y - y_c)}{2}\int_0^\ell m'' \varphi_z^2 dz\right] M_o +$$

$$\left[\frac{EI_\omega}{2}\int_0^\ell \varphi_z''^2 dz + \frac{GI_t}{2}\int_0^\ell \varphi_z'^2 dz\right] = 0$$

$\therefore$ eq. 2º grau em M_o $\rightarrow$ M_{ocr}. Para $\varphi_z = \sum \varphi_i \operatorname{sen} \frac{i\pi z}{\ell}$ $\rightarrow$

$$M_{ocr} = C_b \frac{\pi^2 E I_y}{(K\ell)^2}\left\{\left[C_p(e_y - y_c) + C_r r_{oy}\right] + \right.$$
$$\left. + \sqrt{\left[C_p(e_y - y_c) + C_r r_{oy}\right]^2 + \frac{I_\omega}{I_y}\left[K^2 + \frac{GI_t}{EI_\omega}\left(\frac{K\ell}{\pi}\right)^2\right]}\right\}$$