ADDITIONAL TABLES FOR DESIGN OF OPTIMUM LADDER NETWORKS *

BY

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Part II**

IV. MAXIMALLY FLAT TIME DELAY: BESSEL POLYNOMIALS (7, 8, 9)[2]

The preceding two sets of polynomials were used in the approximation of a desired *magnitude* characteristic. However, the nonlinear phase characteristic of both approximations and the resulting variation of the time delay preclude their use where a constant time delay is a paramount requirement. For such time-delay filters an excellent approximation is given by the use of *Bessel polynomials*. This approximation yields a maximally flat time delay along with a low-pass magnitude characteristic.

The Bessel polynomials in the variable $1/s$ are defined by

$$y_n(1/s) = \sum_{k=0}^{n} = \frac{(n+k)!}{(n-k)!\,k!\,(2s)^k}. \tag{7}$$

The polynomials of interest are derived from the above as

$$\begin{aligned} h_n(s) &= s^n y_n(1/s) \\ &= \sum_{k=0}^{n} a_k s^k. \end{aligned} \tag{8}$$

The transfer function is given by

$$Z_{21}(s) = \frac{H}{h_n(s)}. \tag{9}$$

The constant H is equal to a_0 for a ladder terminated in a one-ohm resistance, that is, for this configuration at $s = 0$, $Z_{21}(0)$ is unity. This transfer function has a maximally flat time delay. By this is meant that the time delay t_d is given by a function of the form

$$t_d = -\frac{d\theta}{d\omega} = \frac{t_0(b_0 + b_1\omega^2 + b_2\omega^4 + \cdots + b_{n-1}\omega^{2n-2})}{b_0 + b_1\omega^2 + b_2\omega^4 + \cdots + b_{n-1}\omega^{2n-2} + b_n\omega^{2n}}, \tag{10}$$

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[2] The boldface numbers in parentheses refer to the references appended to Part II of this paper.

where θ is the phase of $Z_{21}(j\omega)$, and t_0 is the zero-frequency time delay. It is noted that the first $(n - 1)$ coefficients of the denominator are equal to the corresponding coefficients of the numerator. Therefore the Maclaurin series for t_d obtained by dividing the numerator by the denominator, namely,

$$-\frac{d\theta}{d\omega} = t_0\left(1 - \frac{b_n}{b_0}\omega^{2n} + \frac{b_1 b_n}{b^2{}_0}\omega^{2n+2} - \cdots\right), \tag{11}$$

will have the first $(n - 1)$ derivatives of t_d (considered as a function of ω^2) at $\omega = 0$ equal to zero. Thus the time delay is as flat as possible in the vicinity of $\omega = 0$; hence the term maximally flat time delay. The delay is very closely equal to t_0, the zero-frequency value, up to a certain frequency (which is an increasing function of n), and then declines smoothly for values greater than this frequency.

To determine the value of n to use for satisfying a specific requirement, it is necessary to have expressions for the magnitude and time delay. We give here the exact expressions in terms of Bessel functions of half an odd integer;[3] but most often the values of the magnitude and time delay for varying n given in Table VIII suffice so that the need

TABLE VIII.—*Significant Values of u for Time Delay and Loss Characteristic of a Maximally Flat Time-Delay Network.*

a) Time-Delay Table: Giving Frequencies (u) at which Time Delay Deviates a Specified Value from its Zero-Frequency Value

n	u for 1% deviation	u for 10% deviation	u for 20% deviation	u for 50% deviation
1	0.10	0.34	0.50	1.00
2	0.56	1.09	1.39	2.20
3	1.21	1.94	2.29	3.40
4	1.93	2.84	3.31	4.60
5	2.71	3.76	4.20	5.78
6	3.52	4.69	5.95	6.97
7	4.36	5.64	6.30	8.15
8	5.22	6.59	7.30	9.33
9	6.08	7.55	8.31	10.50
10	6.96	8.52	9.33	11.67
11	7.85	9.49	10.34	12.84

b) Loss ($L = -20 \log |Z_n(ju)|$, in db) Table: Giving Frequencies (u) at which Loss is a Specified Number of db Down from its Zero-Frequency Value

n	u for 1/50 db	u for 1/20 db	u for 1/10 db	u for 1/5 db	u for 1/2 db	u for 1 db	u for 3 db
1	0.07	0.11	0.14	0.21	0.35	0.51	1.00
2	0.11	0.18	0.26	0.36	0.57	0.80	1.36
3	0.14	0.23	0.34	0.48	0.75	1.05	1.75
4	0.17	0.28	0.40	0.56	0.89	1.25	2.13
5	0.20	0.32	0.45	0.64	1.01	1.43	2.42
6	0.22	0.36	0.50	0.71	1.12	1.58	2.70
7	0.24	0.39	0.54	0.77	1.22	1.72	2.95
8	0.26	0.41	0.59	0.83	1.31	1.85	3.17
9	0.28	0.44	0.62	0.88	1.40	1.97	3.39
10	0.30	0.47	0.66	0.93	1.48	2.08	3.58
11	0.31	0.49	0.69	0.98	1.55	2.19	3.77

for using the exact analytical form is eliminated. The time delay is given by

$$t_d = t_0\left[1 - \frac{1}{u^2\left(\dfrac{\pi}{2u}\left\{J^2{}_{-n-\frac{1}{2}}(u) + J^2{}_{n+\frac{1}{2}}(u)\right\}\right)}\right] \tag{12}$$

[3] Tables for these functions are given in "Tables of Spherical Bessel Functions," 2 vols., NBS, Math. Tables Project, Columbia University Press, 1947. The particular combination of the spherical Bessel functions that occurs in the magnitude and phase functions is tabulated in Table 13 of "Scattering and Radiation from Circular Cylinders and Spheres, Tables of Amplitude and Phase Angles," Office of Research and Inventions, U. S. Navy Department, July, 1946.

and the magnitude is

$$|Z_{21}(ju)| = \frac{H}{u^{n+1}\left\{\dfrac{\pi}{2u}\left[J^2_{-n-\frac{1}{2}}(u) + J^2_{n+\frac{1}{2}}(u)\right]\right\}^{\frac{1}{2}}}. \tag{13}$$

The loss in db, $L = -\ 20 \log |Z_{21}(ju)|$, tends to the Gaussian form with increasing n,

$$L = \frac{10\, u^2}{(2n - 1) \ln 10}. \tag{14}$$

In the above formulas u is the normalized frequency variable ω/ω_0 and J is a Bessel function.

Use of Eq. 14 gives the 3-db bandwidth as

$$u_{3\text{db}} \cong \sqrt{(2n - 1) \ln 2}, \tag{15}$$

which approximation is good for $n \geq 3$.

In Table VIII are given values of u for four significant points on the time-delay curves and seven significant points on the loss curves. The element values corresponding to the values of n of 1 through 11 are given in Table IX.

Example 4.1. Design a ladder network with a delay of 0.1 μ sec and a constant loss (not greater than 1 db) up to 2.7 mc/s. The network is to be terminated in a load resistance of 2000 ohms and is to be driven by a current source.

Since $t_0 = 1/\omega_0 = 0.1$ sec., then $\omega_0 = 10^7$. For $f = 2.7$ mc/s, $\omega = 5.4\pi \times 10^6$, and $\omega/\omega_0 = 0.54\pi$, which is approximately 1.7.

Using Table VIII(b) for $u = 1.7$, we see that for $n = 7$ the loss is less than 1 db. Now by using Table VIII(a), it is shown that the time delay for $n = 7$ is constant at this frequency.

Consulting Table IX(a) we find the element values for $n = 7$; the unprimed values are used since n is odd and the input is a current source. We remove the normalization by multiplying C's by $1/R\omega_0 = 0.5 \times 10^{-10}$, and L's by $R/\omega_0 = 2 \times 10^{-4}$, and thus obtain the final network given in Fig. 9.

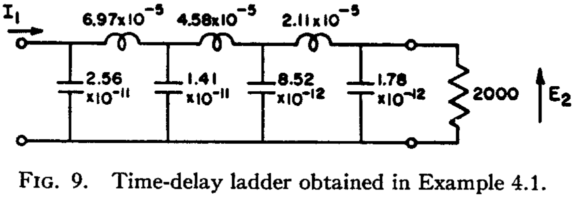

FIG. 9. Time-delay ladder obtained in Example 4.1.

TABLE IX.—*Element Values (in ohms, henrys, farads) of a Normalized Maximally Flat Time-Delay Network.*

Value of n	C_1 or L_1'	L_2 or C_2'	C_3 or L_3'	L_4 or C_4'	C_5 or L_5'	L_6 or C_6'	C_7 or L_7'	L_8 or C_8'	C_9 or L_9'	L_{10} or C_{10}'	C_{11} or L_{11}'
a) r = 0											
1	1.0000										
2	0.3333	1.0000									
3	0.1667	0.4800	0.8333								
4	0.1000	0.2899	0.4627	0.7101							
5	0.0667	0.1948	0.3103	0.4215	0.6231						
6	0.0476	0.1400	0.2246	0.3005	0.3821	0.5595					
7	0.0357	0.1055	0.1704	0.2288	0.2827	0.3487	0.5111				
8	0.0278	0.0823	0.1338	0.1806	0.2227	0.2639	0.3212	0.4732			
9	0.0222	0.0660	0.1077	0.1463	0.1811	0.2129	0.2465	0.2986	0.4424		
10	0.0182	0.0541	0.0886	0.1209	0.1549	0.1880	0.2057	0.2209	0.2712	0.4161	
11	0.0152	0.0451	0.0741	0.1016	0.1269	0.1499	0.1708	0.1916	0.2175	0.2639	0.3955
b) r = 1/8											
1	9.0000										
2	8.6533	0.0433									
3	7.1426	0.0615	1.3652								
4	6.0700	0.0589	2.3569	0.0127							
5	5.3229	0.0535	2.5118	0.0246	0.5401						
6	4.7803	0.0484	2.4267	0.0283	1.1309	0.00601					
7	4.3691	0.0442	2.2790	0.0288	1.3738	0.0133	0.2881				
8	4.0462	0.0407	2.1256	0.0280	1.4536	0.0168	0.6627	0.00350			
9	3.7848	0.0378	1.9841	0.0267	1.4558	0.0184	0.8666	0.00830	0.1788		
10	3.5682	0.0354	1.8591	0.0254	1.4215	0.0189	0.9718	0.0111	0.4348	0.00228	
11	3.3850	0.0334	1.7502	0.0240	1.3710	0.0188	1.0191	0.0128	0.6014	0.00589	0.1159
c) r = 1/4											
1	5.0000										
2	4.6409	0.0898									
3	3.7994	0.1258	0.6973								
4	3.2221	0.1198	1.1956	0.0258							
5	2.8247	0.1084	1.2690	0.0498	0.2731						
6	2.5375	0.0980	1.2231	0.0571	0.5703	0.0121					
7	2.3202	0.0893	1.1470	0.0580	0.6915	0.0268	0.1451				
8	2.1496	0.0823	1.0689	0.0563	0.7306	0.0338	0.3333	0.00704			
9	2.0114	0.0764	0.9973	0.0537	0.7310	0.0369	0.4354	0.0167	0.0899		
10	1.8967	0.0716	0.9342	0.0509	0.7132	0.0379	0.4878	0.0224	0.2184	0.00459	
11	1.7999	0.0676	0.8794	0.0482	0.6875	0.0377	0.5112	0.0256	0.2998	0.0115	0.0603
d) r = 1/3											
1	4.0000										
2	3.6330	0.1223									
3	2.9601	0.1700	0.5298								
4	2.5075	0.1613	0.9046	0.0347							
5	2.1981	0.1457	0.9577	0.0669	0.2063						
6	1.9750	0.1316	0.9217	0.0765	0.4300	0.0163					
7	1.8064	0.1199	0.8636	0.0776	0.5207	0.0358	0.1093				
8	1.6740	0.1104	0.8044	0.0753	0.5497	0.0453	0.2509	0.00942			
9	1.5667	0.1026	0.7503	0.0718	0.5496	0.0494	0.3275	0.0223	0.0676		
10	1.4777	0.0962	0.7027	0.0680	0.5360	0.0506	0.3668	0.0299	0.1642	0.00614	
11	1.4024	0.0907	0.6615	0.0644	0.5165	0.0504	0.3842	0.0342	0.2252	0.0153	0.0455
e) r = 1/2											
1	3.0000										
2	2.6180	0.1910									
3	2.1156	0.2613	0.3618								
4	1.7893	0.2461	0.6127	0.0530							
5	1.5686	0.2217	0.6456	0.1015	0.1393						
6	1.4102	0.1999	0.6196	0.1158	0.2894	0.0246					
7	1.2904	0.1821	0.5797	0.1171	0.3497	0.0542	0.0735				
8	1.1964	0.1676	0.5395	0.1135	0.3685	0.0683	0.1684	0.0142			
9	1.1202	0.1558	0.5030	0.1081	0.3680	0.0744	0.2195	0.0336	0.0453		
10	1.0569	0.1460	0.4710	0.1024	0.3586	0.0763	0.2456	0.0450	0.1100	0.00925	
11	1.0033	0.1377	0.4433	0.0970	0.3454	0.0758	0.2570	0.0515	0.1503	0.0228	0.0309
f) r = 1											
1	2.0000										
2	1.5774	0.4226									
3	1.2550	0.5528	0.1922								
4	1.0598	0.5116	0.3181	0.1104							
5	0.9303	0.4577	0.3312	0.2090	0.0718						
6	0.8377	0.4116	0.3158	0.2364	0.1480	0.0505					
7	0.7677	0.3744	0.2944	0.2378	0.1778	0.1104	0.0375				
8	0.7125	0.3446	0.2735	0.2297	0.1867	0.1387	0.0855	0.0289			
9	0.6678	0.3203	0.2547	0.2184	0.1859	0.1506	0.1111	0.0682	0.0230		
10	0.6305	0.3002	0.2384	0.2066	0.1808	0.1539	0.1240	0.0911	0.0557	0.0187	
11	0.5989	0.2834	0.2243	0.1954	0.1739	0.1528	0.1296	0.1039	0.0761	0.0465	0.0154

V. NORMALIZATION, DUALITY, RECIPROCITY THEOREM; FREQUENCY TRANSFORMATIONS AND TRANSFORMATION OF SYMMETRICAL NETWORKS

Normalization

The element values in the tables are normalized with respect to the load resistance R_1 and the radian frequency. In other words, the value of R_1 is considered as one ohm and that of the cutoff frequency (or $\omega_0 = 1/t_0$ for the time-delay networks) is one radian per second. These frequency and impedance normalizations may be removed simply.

Since the impedance of the three different kinds of elements appearing in a network is given respectively by R, Ls, and $1/Cs$, we note that if the frequency is multiplied by a constant the resistance is unaffected, but that to maintain the impedance of the inductance and capacitance invariant, it is necessary to *divide* L and C by the same constant. This provides the simple rule for removal of the frequency normalization: to raise the radian frequency $\omega = 1$ to $\omega = \omega_c$, divide all L's and C's in the network by ω_c. On the other hand, to raise the impedance level by a factor H we must multiply the impedance of each type of element by this factor, that is, *multiply* every R and L in the network by H, and *divide* every C by H. Thus we see only simple multiplications are involved.

The two rules may be combined into one operation: to raise the radian frequency to ω_c and the impedance level by H, we multiply every resistance by H, every inductance by H/ω_c, and every capacitance by $1/(\omega_c H)$.

Duality

The dual of a ladder network may always be realized simply. The impedance of every series arm is replaced by the admittance of a shunt arm, and vice versa. In simpler terms, this means that every capacitance of C farads is replaced by the dual element which is an inductance of C henrys, every inductance of L henrys is replaced by a capacitance of L farads, and every resistance of R ohms becomes a conductance of R mhos; if the original element is a series arm then the dual element becomes a shunt arm, whereas if the original element is a shunt arm then the dual element is a series arm. For example, the dual of the network in (a) of Fig. 10 is given by the one in (b).

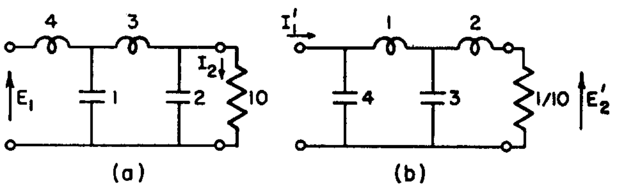

FIG. 10. Ladder network and its dual (values in ohms, henrys, and farads).

What are the characteristics of the dual network with respect to that of a given network? The impedances (admittances) of one network (both transfer and driving-point) become admittances (impedances) of the other. Thus in Fig. 10(a) the input is a voltage source and the output a current so that the transfer function is the admittance $Y_{21} = I_2/E_1$. In the dual given by Fig. 10(b) the transfer impedance $Z'_{21} = E_2'/I_1'$ is the same rational function as Y_{21} of (a).

It is therefore clear that the primed and unprimed values lead to dual networks.

Reciprocity Theorem

Often a network designed by the use of the tables does not have the configuration demanded in a particular problem. For example, a shunt capacitance may be desired at the output and a resistance at the input, but the network obtained has the form shown in Fig. 11(a). By the

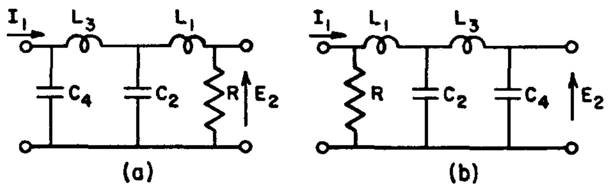

FIG. 11.—Ladder network and one obtained from it by use of reciprocity theorem.

use of the reciprocity theorem the network of Fig. 11(b) with the desired configuration may be obtained.

The reciprocity theorem states that the transfer impedance (or transfer admittance) remains unchanged if the excitation and measuring instrument change places. Thus in Fig. 11(a) we have the transfer impedance

$$Z_{21} = \frac{E_2}{I_1} = \frac{p(s)}{q(s)}, \tag{16}$$

where the excitation is a current source I_1 flowing into the input terminals and the output is a voltage (measured by a voltmeter across R). Now if the current source is placed across R and the voltmeter placed across C_4, then the conditions of the reciprocity theorem have been satisfied. Thus the transfer impedance of Fig. 11(b) is also equal to p/q.

It is therefore clear that by use of reciprocity a whole set of new network configurations may be obtained.

Frequency Transformations (10)

The tables give the element values for *low-pass* filters. However, for the Butterworth and Tschebyscheff cases corresponding characteristics may be obtained for the *high-pass*, *band-pass*, and *band-elimination* filters by the use of transformations of the frequency variable. These transformations do not work for the Bessel-polynomial case because the distortion of the frequency-variable scale makes the phase characteristic nonlinear as a function of the new frequency variable.

High-Pass Filters

A normalized low-pass filter characteristic is shown in Fig. 12(*a*); the corresponding high-pass characteristic is given in Fig. 12(*b*). The

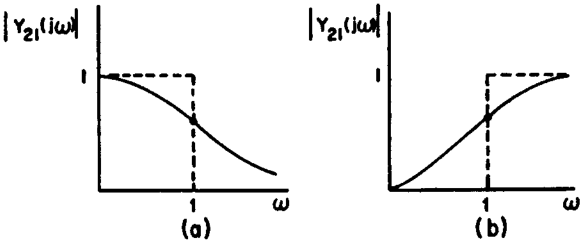

FIG. 12. Low-pass characteristic and the corresponding high-pass one obtained by a frequency transformation.

latter characteristic may be obtained from the former by the use of the transformation $s' = 1/s$. Since by use of this transformation the impedance of an inductance Ls becomes the impedance L/s', the impedance of a capacitance $1/Cs$ becomes s'/C, and the value of a resistance remains unchanged, a simple rule for converting a low-pass ladder network to a high-pass one may be formulated. The rule is: replace every inductance of L henrys by a capacitance of $1/L$ farads; replace every capacitance of C farads by an inductance of $1/C$ henrys; and leave the resistances unchanged. Thus if the network in Fig. 13(*a*) has a low-pass

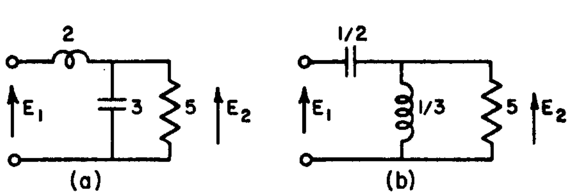

FIG. 13. Low-pass network and its corresponding high-pass network.

characteristic, then the corresponding high-pass network is given in Fig. 13(*b*).

Band-Pass Filters

A low-pass filter of bandwidth ω_c may be converted to a band-pass filter of bandwidth $\omega_c = \omega_b - \omega_a$ by use of the frequency transformation

$$s = \frac{(s')^2 + \omega_0^2}{s'}. \tag{17}$$

Thus the right-hand side of Eq. 17 is substituted for every s in the transfer function. Here ω_b is the upper frequency limit and ω_a is the lower frequency limit of the band, while ω_0 is the center frequency of the band. The band limits have geometric symmetry about the center frequency, that is, $\omega_a\omega_b = \omega_0^2$.

However, it is not necessary to actually carry out the functional transformation, since there is a simple rule for converting the low-pass network to a band-pass one: for each inductance in the network of L henrys add a capacitance in series with it of value $1/(\omega_0^2L)$ farads; for each capacitance in the network of C farads add an inductance in parallel with it of $1/(\omega_0^2C)$ henrys (that is, the added element always resonates with the original element at the center frequency ω_0); leave the resistances unchanged.

The complete process for converting a normalized low-pass filter to a desired band-pass one may be given as the following:

1. Determine the desired bandwidth $\omega_c = \omega_b - \omega_a$ and the desired center frequency $\omega_0^2 = \omega_a\omega_b$ from the given data.
2. Change the bandwidth of the low-pass filter to ω_c.
3. Perform the low-pass to band-pass transformation on the network.
4. Remove the level normalization from the resulting band-pass filter.

Example 5.1. Design an equal-ripple band-pass filter with the following characteristics:

(*a*) The ripple in the pass band is 1 db.
(*b*) The center frequency is $f_0 = 1000$ cps.
(*c*) The bandwidth f_c measured at 1-db points is 100 cps.
(*d*) At the frequencies corresponding to three times f_c the response is to be down approximately 50 db.
(*e*) The network is driven by a current source and should have a load resistance of 1000 ohms.

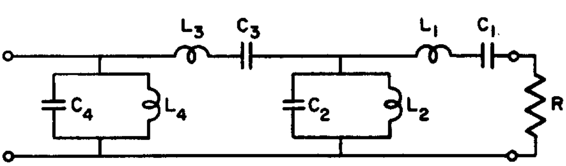

FIG. 14. Band-pass filter for Example 5.1.

In order to design this filter it is not necessary to find the actual frequencies at which the response is down 1 db and 50 db, but if we wished to find them we could use the formulas $f_af_b = f_a(f_a + 100) = 10^6$ and $f_{50}(f_{50} + 300) = 10^6$, where f_a is the lower 1-db frequency and f_{50} is the lower 50-db frequency.

From Table V we find that the 1-db ripple corresponds to $\epsilon = 0.5088$. We now calculate n and find that $n = 4$ yields approximately 49-db attenuation at $\omega = 3$. Therefore using $n = 4$ and the primed values of Table V(a), we find the element values:

$$L_1' = 1.0495 \qquad L_3' = 1.9093$$
$$C_2' = 1.4126 \qquad C_4' = 1.2817$$

The bandwidth is now changed to $\omega_c = 2\pi \times 100$ by dividing the above values by ω_c. The network is then converted to the band-pass form and the impedance level raised to 1000 ohms. The final network given in Fig. 14 has the element values (in ohms, henrys, and farads):

$$R = 1000 \qquad L_3 = 3.04$$
$$L_1 = 1.67 \qquad C_3 = 8.33 \times 10^{-9}$$
$$C_1 = 1.52 \times 10^{-8} \qquad L_4 = 1.15 \times 10^{-2}$$
$$L_2 = 1.41 \times 10^{-2} \qquad C_4 = 2.20 \times 10^{-6}$$
$$C_2 = 2.25 \times 10^{-6}$$

Band-Elimination Filters

The transformation from a low-pass to a band-elimination characteristic is given by

$$s = \frac{s'}{(s')^2 + \omega_0^2}. \tag{18}$$

As for the band-pass filter the transformation can be achieved by direct operation on the low-pass network. The rule follows:

(a) Add a capacitance in parallel with each inductance in the low-pass network; the value of the capacitance is $1/(\omega_0^2 L)$, where L is the value of the original inductance.

(b) Add an inductance in series with each capacitance of the network; the value of the inductance is $1/(\omega_0^2 C)$, where C is the value of the original capacitance.

(c) Since the resistances are unaffected by the transformation, their values are not changed.

Transformation of Symmetrical Networks

It has been pointed out that the Butterworth and Tschebyscheff networks obtained for $r = 1$ and n odd are symmetrical. This symmetry allows any specified resistance ratio to be obtained simply; the method used transforms the symmetrical network to an unsymmetrical one with the desired resistance ratio.

If the symmetrical network is divided as shown in Fig. 15, then the over-all transfer impedance is given in terms of the impedances of the component networks by (11)

$$Z_{21} = \frac{Z_{21a}Z_{21b}}{Z_a + Z_b}. \tag{19}$$

The subscripts a and b have been used to designate the networks on the left and right, respectively. But because of the symmetry, the component networks are the same and consequently $Z_{21b} = Z_{21a}$ and $Z_b = Z_a$. Now suppose it is desired to increase the resistance ratio by r. If the impedance level of N_a is multiplied by r, the desired effect will have been accomplished. But this change also increases Z_{21a} and Z_a by r. Because $Z_b = Z_a$, however, the Z_{21} of the whole network is not changed

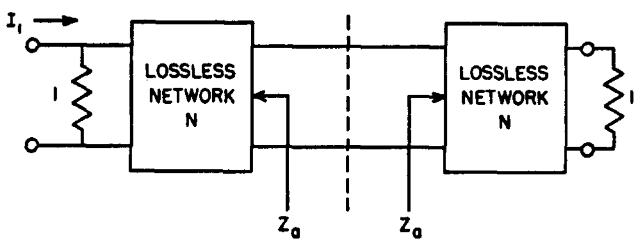

FIG. 15. Decomposition of a symmetrical network into two network halves.

except by a constant multiplier. For example, if $r = 10$ then the transfer impedance before the level change is

$$Z_{21} = \frac{(Z_{21a})^2}{2Z_a}, \tag{20}$$

whereas after the change it is

$$Z'_{21} = \frac{10(Z_{21a})^2}{11Z_a}, \tag{21}$$

which differs from Eq. 20 only by a constant multiplier.

An analogous situation of course holds for transfer admittances.

Thus it is possible to obtain two different networks with the same value of r; one is derived from the table for the desired value of r, and the second, as indicated above, by means of a transformation of a symmetrical network, the symmetrical network being obtained from the table for $r = 1$. For example, for the transfer impedance of a Tschebyscheff network with $n = 3$, $r = 1/2$, and a 1/10-db ripple, the network shown in Fig. 16 is obtained from Table II(e). However, if Table II(f) is used the symmetrical network in Fig. 17(a) results; multiplying the impedance level of the left half of this network by 1/2 yields the network of Fig. 17(*b*), for which *r* is now 1/2. Inspection of the networks in Figs. 16 and 17(*b*) shows that they differ, even though their transfer

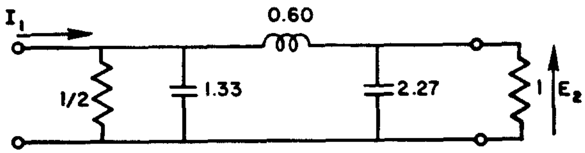

FIG. 16. Normalized Tschebyscheff network with $r = 1/2$, $n = 3$, and 1/10-db ripple.

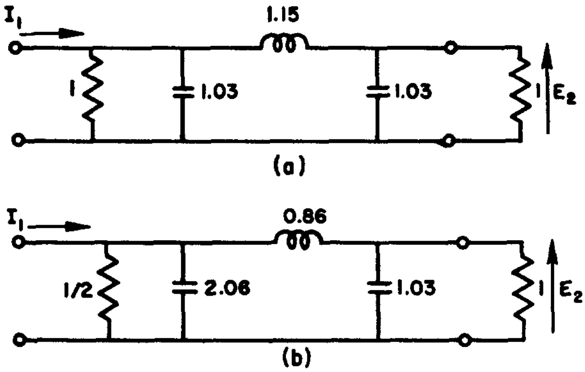

FIG. 17. Normalized Tschebyscheff network with $n = 3$, and 1/10-db ripple; (*a*) $r = 1$; (*b*) $r = 1/2$ achieved by an impedance level change on half the network.

impedances are *identical*. The reason for this is that the tables are derived for a network reflection coefficient all of whose zeros lie in *only one* half-plane, whereas the network obtained by transformation of the symmetrical network has the zeros of its reflection coefficient alternating in the left and right half-planes. This phenomenon has important implications and is discussed elsewhere (12).

CONCLUSION

The design of three classes of practical networks with resistance terminations at both ends becomes simple by use of the tables presented in this paper. The tables give the element values for the normalized low-pass network with a Butterworth, Tschebyscheff, or Bessel-polynomial characteristic. The low-pass networks that are realized in the Butterworth and Tschebyscheff cases can also be easily transformed to serve high-pass, band-pass, or band-elimination functions.

In the future tables will be presented for networks with uniform dissipation and for networks whose reflection coefficients possess zeros that alternate in the left and right half-planes.

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