CHAPTER 10

Dynamic Programming

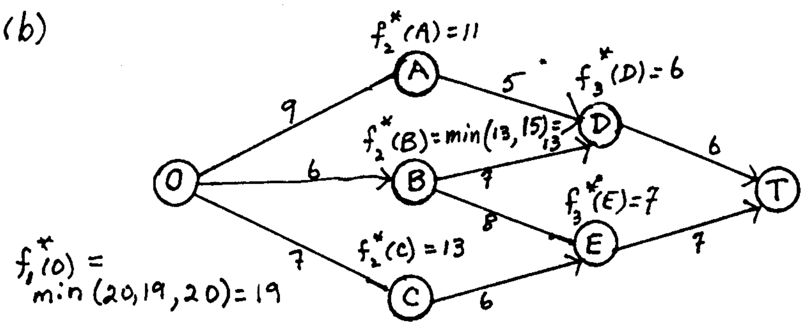
10.2-1 (a) Notice that the nodes of the network may be divided in "layers": the nodes in the nth layer are accessible from the origin only through nodes in the $(n-1)$st layer (i.e., the columns of nodes in the figure constitute the layers).

Stage n — nth column, $n = 1, 2, 3, 4$
States — nodes of the network
x_n (decision on stage n) — immediate destination on stage n
c_{ij} — distance between nodes i and j

$$f_n^*(s) = \min_{x_n} (c_{s x_n} + f_{n+1}^*(x_n))$$

$$f_4^*(T) = 0$$

(b)



Optimal route is O-B-D-T.

(c) Number of Stages = 3

Calculations:

	S3	f3*(S3)	X3*
Node D	1	6	0
Node E	2	7	0

	S2 \ X2	f2(S2, X2): 1	2	f2*(S2)	X2*
Node A	3	11	---	11	1
Node B	4	13	15	13	1
Node C	5	---	13	13	2

	S1 \ X1	f1(S1, X1): 3	4	5	f1*(S1)	X1*
Node O	6	20	19	20	19	4

Optimal solution(s):

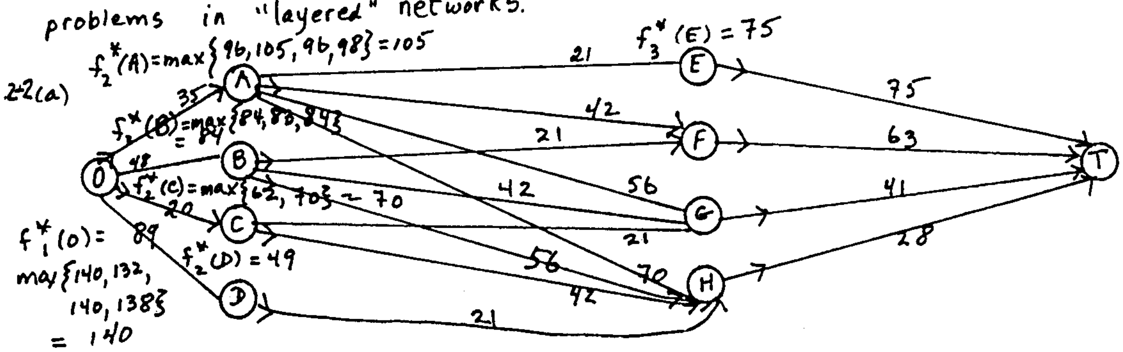
Optimal solution	X1*	X2*	X3*
1	4	1	0

10.2-1(d) Applying the shortest route algorithm

n	Solved nodes directly connected to unsolved nodes	Closest connected unsolved node	total distance	nth nearest node	Distance to nth nearest node	Last connection
1	O	B	6	B	6	OB
2	O B	C D	7 6+7=13	C	7	OC
3	O B C	A D E	9 6+7=13 7+6=13	A	9	OA
4	A B C	D D E	9+5=14 6+7=13 7+6=13	D E	13	BD CE
5	D E	T T	13+6=19 13+7=20	T	20	DT

Using the shortest route algorithm we had to do 8 sums and 6 comparisons, while using dynamic programming we did only 7 sums and 3 comparisons. The latter approach seems to be more efficient for this type of shortest route problems, i.e., shortest route problems in "layered" networks.

10.2-2(a)



Optimal routes: O-A-F-T and 140 is the corresponding sales income.
O-C-H-T

Optimal Solution:

	Region 1	2	3
number of salespeople	1	2	3
	3	2	1

(b) Stage n — region n
state s_n — number of salespeople remaining to be allocated at stage n
x_n — number of salespeople allocated to region n
$c_n(x_n)$ — increase in sales in region n if x_n salespeople are allocated to it

INTERACTIVE DETERMINISTIC DYNAMIC PROGRAMMING ALGORITHM SOLUTION

Number of Stages = 3

Calculations:

S3	f3*(S3)	X3*
1	28	1
2	41	2
3	63	3
4	75	4

S2 \ X2	f2(S2, X2) 1	2	3	4	f2*(S2)	X2*
2	49	---	---	---	49	1
3	62	70	---	---	70	2
4	84	83	84	---	84	1,3
5	96	105	97	98	105	2

10.2-2 (b) (Continued)

S1 \ X1	1	2	3	4	f1*(S1)	X1*
6	140	132	140	138	140	1,3

Optimal solution(s):

Optimal solution	X1*	X2*	X3*
1	1	2	3
2	3	2	1

10.2-3

a) stages: 5 "columns" in network

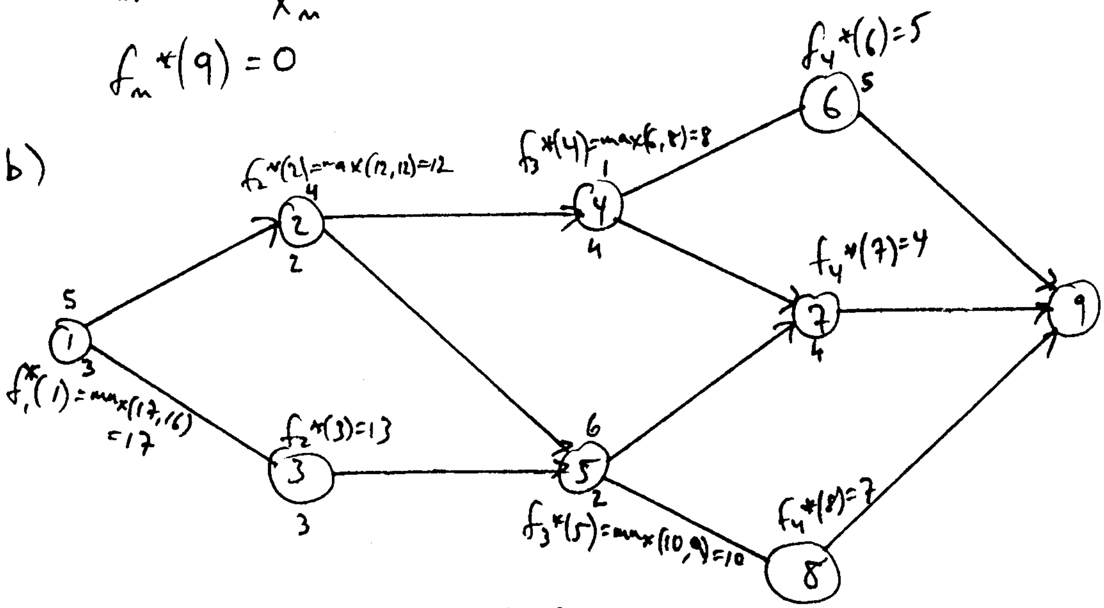
states: nodes of the network

let t_{ij} be times for activities, given:

$$f_n^*(s) = \min_{x_n} \left(t_{s,x_n} + f_{n+1}^*(x_n)\right)$$

$$f_n^*(9) = 0$$

b)



optimal path: $1 \to 2 \to 4 \to 7 \to 9$

or $1 \to 2 \to 5 \to 7 \to 9$

c) Interactive Deterministic Dynamic Programming Algorithm:

s_4	$f_4^*(s_4)$	x_4^*
1	5	1
2	4	1
3	7	1

s_3 \ x_3	$f_3(s_3,x_3)$ 1	2	$f_3^*(s_3)$	x_3^*
1	6	8	8	2
2	10	9	10	1

s_2 \ x_2	$f_2(s_2,x_2)$ 1	2	$f_2^*(s_2)$	x_2^*
1	12	12	12	1,2
2	13	---	13	1

s_1 \ x_1	$f_1(s_1,x_1)$ 1	2	$f_1^*(s_1)$	x_1^*
1	17	16	17	1

Optimal Sol.	x_1^*	x_2^*	x_3^*	x_4^*
1	1	1	2	1
2	1	2	1	1

10.2-4

a) False. "preceding" should read "following": feature No. 7, section 11-2.

b) False. The optimal decision for that stage onward was found by the solution procedure. How you got there is no longer important. (feature No. 5, section 10.2, principle of optimality)

c) False. This is the principle of optimality again; need only know current state, not history.

10.3-1

Let x_n be the number of crates allocated to store n.
Let $p_n(x_n)$ be the expected profit from allocating x_n crates to store n.
Let s_n be the number of crates remaining.
Then, $f_n^*(s_n) = \max_{0 \le x_n \le s_n} [p_n(x_n) + f_{n+1}^*(s_n - x_n)]$

Number of Stages = 3

Calculations:

S3	f3*(S3)	X3*
0	0	0
1	4	1
2	9	2
3	13	3
4	18	4
5	20	5

S2 \ X2	f2(S2, X2) 0	1	2	3	4	5	f2*(S2)	X2*
0	0	---	---	---	---	---	0	0
1	4	6	---	---	---	---	6	1
2	9	10	11	---	---	---	11	2
3	13	15	15	15	---	---	15	1,2,3
4	18	19	20	19	19	---	20	2
5	20	24	24	24	23	22	24	1,2,3

S1 \ X1	f1(S1, X1) 0	1	2	3	4	5	f1*(S1)	X1*
5	24	25	24	25	23	21	25	1,3

Optimal solution(s):

Optimal solution	X1*	X2*	X3*
1	1	2	2
2	3	2	0

10.3-2

Let x_n be the number of study days allocated to course n.
Let $p_n(x_n)$ be the number of grade points expected when x_n days are spent on course n.
Let s_n be the number of study days not yet allocated.
Then, $f_n^*(s_n) = \max_{1 \le x_n \le \min(s_n,4)} [p_n(x_n) + f_{n+1}^*(s_n - x_n)]$

Number of Stages = 4

Calculations:

S4	f4*(S4)	X4*
1	6	1
2	7	2
3	9	3
4	9	4

S3 \ X3	f3(S3, X3) 1	2	3	4	f3*(S3)	X3*
2	8	---	---	---	8	1
3	9	10	---	---	10	2
4	11	11	13	---	13	3
5	11	13	14	14	14	3,4

10.3-2 (CONTINUED)

S2 \ X2	f2(S2, X2) 1	2	3	4	f2*(S2)	X2*
3	13	---	---	---	13	1
4	15	13	---	---	15	1
5	18	15	14	---	18	1
6	19	18	16	17	19	1

S1 \ X1	f1(S1, X1) 1	2	3	4	f1*(S1)	X1*
7	22	23	21	20	23	2

Optimal solution(s):

Optimal solution	X1*	X2*	X3*	X4*
1	2	1	3	1

10.3-3 Let x_n = number of commercials run in area n.
Let $p_n(x_n)$ = number of votes garnered when x_n commercials are run in area n
Let s_n = number of commercials remaining
Then $f_n^*(s_n) = \max_{0 \le x_n \le s_n} \{p_n(x_n) + f_{n+1}^*(s_n - x_n)\}$

INTERACTIVE DETERMINISTIC DYNAMIC PROGRAMMING ALGORITHM SOLUTION

Number of Stages = 4

Calculations:

S4	f4*(S4)	X4*
0	0	0
1	3	1
2	7	2
3	12	3
4	14	4
5	16	5

10.3-3 (CONTINUED)

S3 \ X3	0	1	2	3	4	5	f3*(S3)	X3*
	f3(S3, X3)							
0	0	---	---	---	---	---	0	0
1	3	5	---	---	---	---	5	1
2	7	8	9	---	---	---	9	2
3	12	12	12	11	---	---	12	0,1,2
4	14	17	16	14	10	---	17	1
5	16	19	21	18	13	9	21	2

S2 \ X2	0	1	2	3	4	5	f2*(S2)	X2*
	f2(S2, X2)							
0	0	---	---	---	---	---	0	0
1	5	6	---	---	---	---	6	1
2	9	11	8	---	---	---	11	1
3	12	15	13	10	---	---	15	1
4	17	18	17	15	11	---	18	1
5	21	23	20	19	16	12	23	1

S1 \ X1	0	1	2	3	4	5	f1*(S1)	X1*
	f1(S1, X1)							
5	23	22	22	20	18	15	23	0

Optimal solution(s):

Optimal solution	X1*	X2*	X3*	X4*
1	0	1	1	3

10.3-4

Let x_n be the number of workers allocated to precinct n.
Let $p_n(x_n)$ be the increase in the number of votes if x_n workers are assigned to precinct n.
Let s_n be the number of workers remaining.
Then $f_n^*(s_n) = \max\limits_{1 \le x_n \le s_n} [p_n(x_n) + f_{n+1}^*(s_n - x_n)]$.

INTERACTIVE DETERMINISTIC DYNAMIC PROGRAMMING ALGORITHM SOLUTION

Number of Stages = 4

Calculations:

S4	f4*(S4)	X4*
0	0	0
1	6	1
2	11	2
3	14	3
4	15	4
5	17	5
6	18	6

S3 \ X3	0	1	2	3	4	5	6	f3*(S3)	X3*
	f3(S3, X3)								
0	0	---	---	---	---	---	---	0	0
1	6	5	---	---	---	---	---	6	0
2	11	11	10	---	---	---	---	11	0,1
3	14	16	16	15	---	---	---	16	1,2
4	16	19	21	21	18	---	---	21	2,3
5	17	21	24	26	24	21	---	26	3
6	18	22	26	29	29	27	22	29	3,4

S2 \ X2	0	1	2	3	4	5	6	f2*(S2)	X2*
	f2(S2, X2)								
0	0	---	---	---	---	---	---	0	0
1	6	7	---	---	---	---	---	7	1
2	11	13	11	---	---	---	---	13	1
3	16	18	17	16	---	---	---	18	1
4	21	23	22	22	18	---	---	23	1
5	26	28	27	27	24	20	---	28	1
6	29	33	32	32	29	26	21	33	1

10.3-4 (CONTINUED)

X1 \ S1	0	1	2	3	4	5	6	f1*(S1)	X1*
6	33	32	32	33	31	29	24	33	0,3

Optimal solution(s):

	Precinct 1	2	3	4	
Optimal solution	X1*	X2*	X3*	X4*	
1	0	1	3	2	number
2	3	1	0	2	of
3	3	1	1	1	workers

10.3-5

Let x_n be the number produced in month n.
Let s_n be the inventory.
Then

$$f_n^*(s_n) = \min_{\max[r_n - s_n, 0] \le x_n \le m_n} [c_n x_n + d_n \max[s_n + x_n - r_n, 0] + f_{n+1}^*(\max[s_n + x_n - r_n, 0])]$$

where

Month	r_n	m_n	c_n	d_n
1	2	5	5.40	.075
2	3	7	5.55	.075
3	5	6	5.50	.075
4	4	2	5.65	.075

$n = 4$:

s_4	$f_4^*(s_4)$	x_4^*
2	11.30	2
3	5.65	1
4	0.00	0

$n = 3$:

s_3 \ x_3	$f_3(s_3, x_3)$ 0	1	2	3	4	5	6	$f_3^*(s_3)$	x_3^*
1	—	—	—	—	—	—	44.45	44.45	6
2	—	—	—	—	—	38.95	38.875	38.875	6
3	—	—	—	—	33.45	33.375	33.30	33.30	6
4	—	—	—	27.95	27.875	27.80	—	27.80	5
5	—	—	22.45	22.375	22.30	—	—	22.30	4
6	—	16.95	16.875	16.80	—	—	—	16.80	3
7	11.45	11.375	11.30	—	—	—	—	11.30	2

$n = 2$:

s_2 \ x_2	$f_2(s_2, x_2)$ 0	1	2	3	4	5	6	7	$f_2^*(s_2)$	x_2^*
0	—	—	—	—	66.725	66.775	66.825	66.95	66.725	4
1	—	—	—	61.175	61.225	61.275	61.40	61.175	61.175	3
2	—	—	55.625	55.675	55.725	55.85	55.975	55.605	55.625	2
3	—	50.075	50.125	50.175	50.30	50.425	50.55	50.675	50.075	1

$n = 1$:

s_1 \ x_1	$f_1(s_1, x_1)$ 0	1	2	3	4	5	$f_1^*(s_1)$	x_1^*
0	—	—	77.525	77.45	77.375	77.30	77.30	5

Hence, produce five in period 1, one in period 2 six in period 3 and two in period 4.

10.3-8

(a) Let x_n (in \$1,000,000) be amount spent in phase n.
Let s_n (in \$1,000,000) be amount yet to be spent.
Let $p_n(x_n)$ be: ① the initial share of the market attained in phase 1 when x_1 is spent in phase 1 or ② the fraction of this market share retained in phase n when x_n is spent in phase n ($n=2$ or 3).

INTERACTIVE DETERMINISTIC DYNAMIC PROGRAMMING ALGORITHM SOLUTION

Number of Stages = 3

Calculations:

S3	f3*(S3)	X3*
0	0.3	0
1	0.5	1
2	0.6	2
3	0.7	3

S2 \ X2	f2(S2, X2): 0	1	2	3	f2*(S2)	X2*
0	0.06	---	---	---	0.06	0
1	0.1	0.12	---	---	0.12	1
2	0.12	0.2	0.15	---	0.2	1
3	0.14	0.24	0.25	0.18	0.25	2

S1 \ X1	f1(S1, X1): 1	2	3	4	f1*(S1)	X1*
4	5	6	4.8	3	6	2

Optimal solution(s):

Optimal solution	X1*	X2*	X3*
1	2	1	1

Hence, the optimal plan is to spend \$2,000,000 in Phase 1 and \$1,000,000 in each of Phases 2 and 3, which will result in a final market share of [ ]%.

10.3-6

1) Phase 3:

s	$f_3^*(s)$	x_3^*
$0 \le s \le 4$	$.6 + .07s$	s

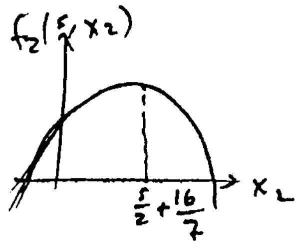
Phase 2: $f_2(s, x_2) = (.4 + .1x_2)(.6 + .07(s - x_2))$

$$= -.007x_2^2 + (.007s + .032)x_2 + (.24 + .028s)$$

for $0 \le x_2 \le s$

$$\frac{\partial f(s, x_2)}{\partial x_2} = -.014x_2 + .007s + .032 = 0$$

$$\Rightarrow x_2 = \frac{.007s + .032}{.014} = \frac{s}{2} + \frac{16}{7}$$



$\therefore$ if $s \le \frac{s}{2} + \frac{16}{7}$ then $x_2^* = s$ because $f_2(s, x_2)$ is strictly increasing on $[0, \frac{s}{2} + \frac{16}{7}]$ and hence is strictly increasing on $[0, s]$. If $s > \frac{s}{2} + \frac{16}{7}$ then $x_2^* = \frac{s}{2} + \frac{16}{7}$ because the unconstrained maximum is then feasible.

$$\Rightarrow x_2^* = \min\left\{\frac{s}{2} + \frac{16}{7}, s\right\}$$

But $0 \le s \le \frac{32}{7} \Rightarrow s \le \frac{s}{2} + \frac{16}{7}$, and $s \le 4 \le \frac{32}{7}$

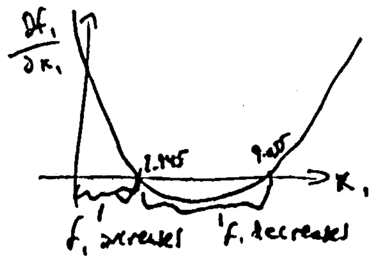
$\Rightarrow x_2^* = s$ and $f_2^*(s) = .06s + .24$

Phase 1: $f_1(4, x_1) = (10x_1 - x_1^2)(.06(4 - x_1) + .24)$

$$= .06x_1^3 - 1.08x_1^2 + 4.8x_1$$

$$\frac{\partial f_1(4, x_1)}{\partial x_1} = .18x_1^2 - 2.16x_1 + 4.8 = 0$$

$$\Rightarrow x_1 = \frac{2.16 \pm \sqrt{2.16^2 - 4(.18)(4.8)}}{2(.18)} = 2.945 \text{ or } 9.055$$



Thus, $f_1(4, x)$ achieves its maximum in the interval $[0, 4]$ at $x_1^* = 2.945$ with $f_1^*(4) = 6.302$. Thus, optimally \$2,945,000 is spent in Phase 1, \$1,055,000 in Phase 2 and none in Phase 3, which gives a market share of 6.302%.

10.3-7

Let x_n be the number of parallel units of component n that are installed.
Let $p_n(x_n)$ be the probability that the component will function if it contains x_n parallel units.
Let $c_n(x_n)$ be the cost of installing x_n units of component n.
Let s_n be the number of dollars (in hundreds of \$) remaining to be spent.
Then $f_n^*(s_n) = \max_{x_n = 0,\ldots,\min\{3,\alpha_{s_n}\}} [p_n(x_n) f_{n+1}^*(s_n - c_n(x_n))]$

where $\alpha_{s_n} = \max\{\alpha : c_n(\alpha) \le s_n, \alpha \in \mathbb{Z}\}$

$n=4$:

s_4	$f_4^*(s_4)$	x_4^*
0, 1	0	0
2	.5	1
3	.7	2
$4 \le s_4 \le 10$	.9	3

$n=3$: $f_3(s_3, x_3) = p_3(x_3) f_4^*(s_3 - c_3(x_3))$

s_3 \ x_3	0	1	2	3	$f_3^*(s_3)$	x_3^*
0	0	—	—	—	0	0
1, 2	0	0	—	—	0	0, 1
3	0	.35	0	—	.35	1
4	0	.49	0	0	.49	1
5	0	.63	.4	0	.63	1
6	0	.63	.56	.45	.63	1
7	0	.63	.72	.63	.72	2
$8 \le s_3 \le 10$	0	.63	.72	.81	.81	3

$n=2$: $f_2(s_2, x_2) = p_2(x_2) f_3^*(s_2 - c_2(x_2))$

s_2 \ x_2	0	1	2	3	$f_2^*(s_2)$	x_2^*
0, 1	0	—	—	—	0	0
2, 3	0	0	—	—	0	0, 1
4	0	0	0	—	0	0, 1, 2
5	0	.21	0	0	.21	1
6	0	.294	0	0	.294	1
7	0	.378	.245	0	.378	1
8	0	.378	.343	.28	.378	1
9	0	.432	.441	.392	.441	2
10	0	.486	.441	.504	.504	3

(cont.)

10.3-7 cont.

$n=1$:

s_1 \ x_1	$f_1(s_1,x_1)=p_1(x_1)f_2^*(s_1-c_1(x_1))$: 0	1	2	3	$f_1^*(s_1)$	x_1^*
10	0	.22	.227	.302	.302	3

Thus, the optimal solution is $x_1^*=3$, $x_2^*=1$, $x_3^*=1$, $x_4^*=3$ yielding a system reliability of .3024.

10.3-8 Interactive Deterministic Dynamic Programming Algorithm:

s_2	$f_2^*(s_2)$	x_2^*
0	0	0
1	0	0
2	4	1
3	4	1
4	12	2

stages are x_1, x_2
state is amount of slack remaining in constraint
find $f_1^*(4)$:

s_1 \ x_1	$f_1(s_1,x_1)$: 0	1	2	3	4	$f_1^*(s_1)$	x_1^*
4	12	6	8	0	-16	12	0

Optimal Sol.	x_1^*	x_2^*
1	0	2

10.3-9 stages x_1, x_2, x_3
state as in 10.3-11; find $f_1^*(11)$

s_3	$f_3^*(s_3)$	x_3^*
0	0	0
1	0	0
2	0	0
3	10	1
4	10	1
5	10	1
6	20	2
7	20	2
8	20	2
9	30	3
10	30	3
11	30	3

s_2 \ x_2	$f_2(s_2,x_2)$: 0	1	2	$f_2^*(s_2)$	x_2^*
0	0	—	—	0	0
1	0	—	—	0	0
2	0	—	—	0	0
3	10	—	—	10	0
4	10	20	—	20	1
5	10	20	—	20	1
6	20	20	—	20	0,1
7	20	30	—	30	1
8	20	30	40	40	2
9	30	30	40	40	2
10	30	40	40	40	1,2
11	30	40	50	50	2

s_1 \ x_1	$f_1(s_1,x_1)$: 0	1	2	3	4	5	$f_1^*(s_1)$	x_1^*
11	50	57	62	65	66	65	66	4

optimal solution: $x_1^*=4$, $x_2^*=0$, $x_3^*=1$
$Z=66$

10.3-10

Let s_n denote the slack remaining in the constraint $x_1+x_2 \le 3$.

$f_2^*(s_2) = \max_{0 \le x_2 \le s_2} [36x_2 - 3x_2^3]$

$\dfrac{\partial f_2(s_2,x_2)}{\partial x_2} = 36 - 9x_2^2 \begin{cases} >0 & \text{for } 0 \le x_2 < 2 \\ =0 & \text{for } x_2 = 2 \\ <0 & \text{for } x_2 > 2 \end{cases} \Longrightarrow x_2^* \begin{cases} s_2 & \text{for } 0 \le s_2 \le 2 \\ 2 & \text{for } 2 \le s_2 \le 3. \end{cases}$

$$f_1^*(3) = \max_{0 \le x_1 \le 3} [36x_1 + 9x_1^2 - 6x_1^3 + f_2^*(3-x_1)]$$

$$= \max\left[\max_{0 \le x_1 \le 1} [36x_1 + 3x_1^2 - 6x_1^3 + 48] + \max_{1 \le x_1 \le 3} [36x_1 + 9x_1^2 - 6x_1^3 + 36(3-x_1) - 3(3-x_1)^3]\right)$$

for $0 \le x_1 \le 1$ $\quad \dfrac{\partial f_1(3,x_1)}{\partial x_1} = -18(x_1^2 - x_1 - 2) > 0 \Rightarrow x_{1,max} = 1$

for $1 \le x_1 \le 3$ $\quad \dfrac{\partial f_1(3,x_1)}{\partial x_1} = -9(x_1^2 + 4x_1 - 9) \begin{cases} >0 & \text{for } 1 \le x_1 < -2+\sqrt{13} \\ =0 & \text{for } x_1 = -2+\sqrt{13} \\ <0 & \text{for } x_1 > -2+\sqrt{13} \end{cases}$

$\therefore x_{1,max} = -2 + \sqrt{13}$

Hence, $f_1(3,x_1)$ attains its maximum at $x_1 = -2+\sqrt{13}$

summarizing:

s_1	$f_1^*(3)$	x_1^*
3	98.233	$-2+\sqrt{13}=1.606$

; $(x_1^*, x_2^*) = (-2+\sqrt{13}, 5-\sqrt{13}) = (1.606, 1.394)$; $Z^* = 98.233$

10.3-11

$$f_n^*(s_n) = \min_{r_n \le x_n \le 255} [100(x_n - s_n)^2 + 2000(x_n - r_n) + f_{n+1}^*(x_n)]$$

$\underline{n=4}$:

s_4	$f_4^*(s_4)$	x_4^*
$200 \le s_4 \le 255$	$100(255-s_4)^2$	255

(Cont.)

10.3-11 (CONT'D)

$\underline{n=3}$: $f_3(s_3, x_3) = 100(x_3 - s_3)^2 + 2000(x_3 - 200) + 100(255 - x_3)^2$

$$\frac{\partial f_3(s_3, x_3)}{\partial x_3} = 200(x_3 - s_3) + 2000 - 200(255 - x_3) = 200(2x_3 - (s_3 + 245)) = 0$$

$$\Rightarrow x_3 = \frac{s_3 + 245}{2}$$

$$200 \le \frac{s_3 + 245}{2} \Rightarrow s_3 \ge 155 \text{ and } \frac{s_3 + 245}{2} \le 255 \Rightarrow s_3 \le 265$$

Hence, $x_3 = \frac{s_3 + 245}{2}$ is feasible for $240 \le s_3 \le 255$ so

$$f_3^*(s_3) = 100((s_3 + 245)/2 - s_3)^2 + 2000((s_3 + 245)/2 - 200) + 100(255 - (s_3 + 245)/2)$$

$$= 25(245 - s_3)^2 + 25(265 - 3)^2 + 1000(s_3 - 155) \quad \text{or}$$

s_3	$f_3^*(s_3)$	x_3^*
$240 \le s_3 \le 255$	$25(245 - s_3)^2 + 25(265 - s_3)^2 + 1000(s_3 - 155)$	$\frac{s_3 + 245}{2}$

$\underline{n=2}$: $f_2(s_2, x_2) = 100(x_2 - s_2)^2 + 2000(x_2 - 240) + f_3^*(x_2)$

$$\frac{\partial f_2(s_2, x_2)}{\partial x_2} = 200(x_2 - s_2) + 2000 - 50(245 - x_2) - 50(265 - x_2) + 1000$$

$$= 100(3x_2 - (2s_2 + 225)) = 0 \Rightarrow x_2 = \frac{2s_2 + 255}{3}$$

$240 \le \frac{2s_2 + 255}{3} \le 255 \Rightarrow 247.5 \le s_2 \le 270$. So, for

$247.5 \le s_2 \le 255$, $x_2^* = \frac{2s_2 + 225}{3}$ and

$$f_2^*(s_2) = 100\left(\frac{2s_2 + 225}{3} - s_2\right)^2 + 2000\left(\frac{2s_2 + 225}{3} - 240\right) + f_3^*\left(\frac{2s_2 + 225}{3}\right)$$

$$= \frac{100}{9}\left[(225 - s_2)^2 + (255 - s_2)^2 + (285 - s_2)^2 + 60(3s_2 - 615)\right].$$

Now for $220 \le s_2 \le 247.5$, $\frac{2s_2 + 225}{3} \le 240 \le x_2 \Rightarrow \frac{\partial f_2(s_2, x_2)}{\partial x_2} \ge 0 \Rightarrow x_2^* = 240$.

and $f_2^*(s_2) = 100(240 - s_2)^2 + 2000(240 - 240) + f_3^*(240) = 100(240 - s_2)^2 + 101,250$ or

s	$f_2^*(s_2)$	x_2^*
$220 \le s_2 \le 247.5$	$100(240 - s_2)^2 + 101,250$	240
$247.5 \le s_2 \le 255$	$\frac{100}{9}[(225 - s_2)^2 + (255 - s_2)^2 + (285 - s_2)^2 + 60(3s_2 - 165)]$	$\frac{2s_2 + 225}{3}$

$\underline{n=1}$: $f_1(255, x_1) = 100(x_1 - 255)^2 + 2000(x_1 - 220) + f_2^*(x_1)$

for $220 \le x_1 \le 247.5$ $\quad \frac{\partial f_1(255, x_1)}{\partial x_1} = 200(2x_1 - 485) = 0 \Rightarrow x_1^* = 242.5$

for $247.5 \le x_1 \le 255$ $\quad \frac{\partial f_1(255, x_1)}{\partial x_1} = \frac{800}{3}(x_1 - 240) > 0 \Rightarrow x_1^* = 247.5$

Hence, $x_1^* = 242.5$ and $f_1^*(255) = 100(242.5 - 255)^2 + 2000(242.5 - 220) + 100(240 - 242.5)^2 + 101,250 = 162,500$

or

s_1	$f_1^*(s_1)$	x_1^*
255	162,500	242.5

And so the optimal solution is:
with cost = 162,500

Summer	242.5
Autumn	240
Winter	242.5
Spring	255

10.3-12 let s_n be the amount of the RHS resource remaining (4 in stage 1)

$\underline{x_3}$ $\max_{0 \le x_3 \le s_3} (4x_3 - x_3^2)$ $\quad \frac{\partial}{\partial x_3} = 4 - 2x_3 = 0 \Rightarrow x_3 \stackrel{*}{=} 2$

$\frac{\partial^2}{\partial x_3^2} = -2 < 0$ so this is a maximum

$$\boxed{\begin{array}{l} 0 \le s_3 \le 2: \; x_3^* = s_3, \; f_3^* = 4s_3 - s_3^2 \\ 2 \le s_3 \le 4: \; x_3^* = 2, \; f_3^* = 4 \end{array}}$$

$\underline{x_2}$ $\max_{0 \le x_2 \le s_2} \{ 2x_2 + f_3^*(s_2 - x_2) \}$

$0 \le s_2 - x_2 \le 2$: $\max. (2x_2 + 4(s_2 - x_2) - (s_2 - x_2)^2)$

$\frac{\partial}{\partial x_2} = -2 + 2s_2 - 2x_2 = 0 \Rightarrow x_2^* = s_2 - 1$

$\frac{\partial^2}{\partial x_2^2} = -2 < 0$, so maximum

Thus if we choose x_2 between $s_2 - 2$ and s_2, we should choose $x_2^* = s_2 - 1$, leaving 1 for stage 3

$f_2^* = 2(s_2 - 1) + 4(1) - (1)^2 = 2s_2 + 1$

$2 \le s_2 - x_2 \le 4$: $\max (2x_2 + 4)$ so let x_2 be as large as possible, i.e. $x_2^* = s_2 - 2$

$f_2^* = 2(s_2 - 2) + 4 = 2s_2 < 2s_2 + 1$,

so this is inferior to $\boxed{\begin{array}{l} x_2^* = s_2 - 1 \\ f_2^* = 2s_2 + 1 \end{array}}$ $1 \le s_2 \le 4$

$\boxed{\text{If } 0 \le s_2 \le 1, \; x_2^* = 0, \; f_2^* = 4s_2 - s_2^2}$

$\underline{x_1}$ $\max_{0 \le x_1 \le 2} (2x_1^2 + f_2^*(4 - 2x_1))$

$0 \le 4 - 2x_1 \le 1$: $\max. \; 2x_1^2 + 4(4 - 2x_1) - (4 - 2x_1)^2$

$= -2x_1^2 + 8x_1$

$\frac{\partial}{\partial x_1} = -4x_1 + 8 = 0 \Rightarrow x_1 = -2$

but this is infeasible

Smallest feasible x_1 is $x_1^* = 3/2$ for $f_1^* = 7\frac{1}{2}$

(since we want to be near $x_1 = -2$)

(cont.)

$1 \le 4-2x_1 \le 4$: max. $2x_1^2 + 1 + 2(4-2x_1) = 2x_1^2 - 4x_1 + 9$

$$\frac{\partial}{\partial x_1} = 4x_1 - 4 = 0 \Rightarrow x_1 = 1$$

$$\frac{\partial^2}{\partial x_1^2} = 4 > 0 \rightarrow \text{this is a } \underline{\text{minimum}}$$

So we look at the endpoints of the range $1 \le 4-2x_1 \le 4$
$0 \le x_1 \le 3/2$

$x_1 = 3/2 \rightarrow f_1^* = 7\frac{1}{2}$ we saw above

$x_1 = 0 \rightarrow f_1^* = 0 + 1 + 8 = 9$, $\underline{\text{best}}$

So optimal solution: $x_1^* = 0$, $x_2^* = 3$, $x_3^* = 1$, $z = 9$

10.3-13 $\underline{\underline{x_2}}$ min. $2x_2^2$
$x_2^2 \ge s_2$

where s_2 represents the amount of the original 2 that still must be used up by x_2^2

$\Rightarrow x_2^* = \sqrt{s_2}$, $f_2^* = 2s_2$

$\underline{\underline{x_1}}$ $\min_{x_1} \{x_1^4 + f_2^*(2-x_1^2)^+\}$ where $(2-x_1^2)^+$ denotes the positive part of $(2-x_1^2)$ i.e. $\max\{0, 2-x_1^2\}$

$x_1^4 + 2(2-x_1^2)$

$x_1^4 + 4 - 2x_1^2$

$$\frac{\partial}{\partial x_1} = 4x_1^3 - 4x_1 = 0$$
$$4x_1(x_1^2 - 1) = 0$$
$$x_1^* = 0, 1, -1$$

$$\frac{\partial^2}{\partial x_1^2} = 12x_1^2 - 4$$

$0 \rightarrow -$ not OK, max.
$1 \rightarrow +$ OK, min.
$-1 \rightarrow +$ OK, min.

So $(x_1, x_2)^* = \begin{cases} (1,1) \\ (1,-1) \\ (-1,1) \\ (-1,-1) \end{cases}$ all with $\underline{z=3}$

10.3-14
(a) Since the only integer factors of 4 are 1, 2 and 4, let s_n be the remaining factor of 4 entering stage n.

$\underline{n=3}$:

s_3	$f_3^*(s_3)$	x_3^*
1	16	1
2	32	2
4	64	3

(cont.)

10.3-14(a) (continued)

$n = 2$:

$s_2 \backslash x_2$	$f_2(s_2, x_2)$: 1	2	4	$f^*(s_2)$	x_2^*
1	20	—	—	20	1
2	36	32	—	36	1
4	68	48	80	80	4

$n = 1$:

$s_1 \backslash x_1$	$f_1(s_1, x_1)$: 1	2	4	$f_1^*(s_1)$	x_1^*
4	81	44	84	84	4

And so the optimal solution is $x_1^* = 4$, $x_2^* = 1$ and $x_3^* = 1$ with a value of 84.

(b) As above let s_n = the (not necessarily integer) factor remaining at stage n.

$f_3^*(s_3) = 16 s_3$ and $x_3^* = s_3$

$$f_2^*(s_2) = \max_{1 \le x_2 \le s_2} \{4x_2^2 + f_3^*(s_2/x_2)\} = \max_{1 \le x_2 \le s_2} \{4x_2^2 + 16 s_2/x_2\}$$

$$\frac{\partial f_2(x_2, s_2)}{\partial x_2} = 4x_2 - \frac{16 s_2}{x_2^2} \quad \text{and}$$

$$\frac{\partial^2 f_2(x_2, s_2)}{\partial x_2^2} = 4 + \frac{32 s_2}{x_2^3} > 0 \quad \text{when } x_2, s_2 \ge 0$$

Thus, $f_2(x_2, s_2)$ is a convex function of x_2 when $s_2, x_2 \ge 0$

And so the maximum of f_2 must occur when $x_2 = 1$ or $x_2 = s_2$ (the endpoints of the interval over which we are maximizing) but $f_2(1, s_2) = 4 + 16 s_2$ and $f_2(s_2, s_2) = 4s_2^2 + 16$

so $f_2(1, s_2) \ge f_2(s_2, s_2) \iff 4 + 16 s_2 \ge 4s_2^2 + 16$

$\iff 4 s_2^2 - 16 s_2 + 12 \le 0 \iff 4(s_2 - 1)(s_2 - 3) \le 0 \iff 1 \le s_2 \le 3$

Thus, $x_2^* = \begin{cases} 1 & 1 \le s_2 \le 3 \\ s_2 & 3 \le s_2 \le 4 \end{cases}$

$$f_2^*(s_2) = \begin{cases} 4 + 16 s_2 & 1 \le s_2 \le 3 \\ 4s_2^2 + 16 & 3 \le s_2 \le 4 \end{cases}$$

$$f_1^*(s_1) = \max_{1 \le x_1 \le 4} \{x_1^3 + f^*(\tfrac{4}{x_1})\}$$

$$= \max\left\{\max_{1 \le x_1 \le \frac{4}{3}} \{x_1^3 + 4(\tfrac{16}{x_1^2}) + 16\}, \max_{\frac{4}{3} \le x_1 \le 4} \{x_1^3 + 4 + 16(\tfrac{4}{x_1})\}\right\}$$

$$\frac{\partial^2 [x_1^3 + \frac{64}{x_1^2} + 16]}{\partial x_1^2} = 6x_1 + \frac{204}{x_1^4} > 0 \quad \text{when } x_1 \ge 0 \quad \text{and}$$

$$\frac{\partial^2 [x_1^3 + 4 + \frac{64}{x_1}]}{\partial x_1^2} = 6x_1 + \frac{128}{x_1^2} > 0 \quad \text{when } x_1 \ge 0$$

and so we need only check the end points of each interval for the maximum.

10.3-14 (b) (continued)

$f_1^*(s_1) = \max\{80, 54\tfrac{10}{27}, 54\tfrac{10}{27}, 84\} = 84$ which occurs when $x_1^* = 4$, $x_2^* = 1$ and $x_3^* = 1$ as in the integer case.

10.3-15 Let s_n be the slack remaining in the constraint $x_1 - x_2 + x_3 \le 1$ entering the n^{th} stage.

$$f_3^*(s_3) = \max_{0 \le x_3 \le s_3} \{x_3\} = s_3 \quad \text{and} \quad x_3^* = s_3$$

$$f_2^*(s_2) = \max_{x_2 \ge \max\{-s_2, 0\}} \{(1-x_2) f_3^*(s_2 + x_2)\}$$

$$= \max_{x_2 \ge \max\{-s_2, 0\}} \{(1-x_2)(s_2 + x_2)\} = \max_{x_2 \ge \max\{-s_2, 0\}} \{-x_2^2 + (1-s_2)x_2 + s_2\}$$

$\dfrac{\partial^2 f_2(x_2, s_2)}{\partial s^2} = -2$ and so f_2 is concave with respect to x_2 for fixed s_2, and thus, the maximum of f_2 will occur at the endpoint or when $\dfrac{\partial f_2}{\partial x_1} = 0$.

$$\frac{\partial f_2(x_2, s_2)}{\partial x_2} = -2x_2 + (s_2 - 1) = 0 \Rightarrow x_2 = \frac{1 - s_2}{2}$$

but x_2 must be greater than $-s_2$ for this to be the maximum

$\Rightarrow \dfrac{1 - s_2}{2} \ge -s_2 \Rightarrow s_2 \ge -1$ and so

$$f_2^*(s_2) = \begin{cases} \dfrac{(1-s_2)^2}{4} + s_2 & \text{if } s_2 \ge -1 \\ 0 & \text{if } s_2 \le -1 \end{cases}$$

and $$x_2^* = \begin{cases} \dfrac{1 - s_2}{2} & \text{if } s_2 \ge -1 \\ -s_2 & \text{if } s_2 \le -1 \end{cases}$$

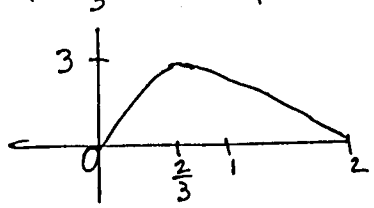
$$f_1^*(s_1) = \max_{x_1 \ge 0} \{x_1 f_1^*(1 - x_1)\} = \max\left\{\max_{0 \le x_1 \le 2} \left\{x_1\left(\frac{x_1^2}{4} + (1 - x_1)\right)\right\}, 0\right\}$$

$$= \max_{0 \le x_1 \le 2} \left\{\frac{x_1^3}{4} - x_1^2 + x_1\right\}$$

$$\frac{\partial\left[\frac{x^3}{4} - x_1^2 + x_1\right]}{\partial x_1} = \frac{3x_1^2}{4} - 2x_1 + 1 = 0$$

$$\Rightarrow x_1 = \frac{2 \pm \sqrt{4-3}}{\frac{3}{2}} = \frac{4}{3} \pm \frac{2}{3} \quad \text{and so}$$

this function has a relative maximum and minimum at $x_1 = \frac{2}{3}$ and $x_1 = 2$. Plotting the function we find



$\Rightarrow x_1^* = \frac{2}{3}$, $x_2^* = \frac{1}{3}$, $x_3^* = \frac{2}{3}$ and

$$Z^* = \frac{8}{27}$$

10.3-16

Let $S = (R_1, R_2)$ where R_i is the slack in the i^{th} constraint

$f_2(R_1, R_2, x_2) = 2x_2 \quad 0 \le x_2 \le \min\{\frac{R_1}{2}, R_2\}$

n=2:

S	$f_2^*(S)$	x_2^*
$\begin{bmatrix} R_1 \\ R_2 \end{bmatrix}$	$10\min\{\frac{R_1}{2}, R_2\}$	$\min\{\frac{R_1}{2}, R_2\}$

n=1: $f_1(6,8,x_1) = 15x_1 + f_2^*(6-x_1, 8-3x_1)$

$$= 15x_1 + 10\min\left\{\frac{6-x_1}{2}, 8-3x_1\right\} \quad 0 \le x_1 \le \frac{8}{3}$$

$0 \le x_1 \le 2 \Rightarrow \min\{\frac{6-x_1}{2}, 8-3x_1\} = \frac{6-x_1}{2}$

$\Rightarrow f_1(6,8,x_1) = 15x_1 + 30 - 5x_1 = 10x_1 + 30$

$\Rightarrow \max\limits_{0 \le x_1 \le 2} [f_1(6,8,x_1)] = 50$ at $x_1 = 2$

$2 \le x_1 \le \frac{8}{3} \Rightarrow \min\left\{\frac{6-x_1}{2}, 8-3x_1\right\} = 8-3x_1$

$\Rightarrow f_1(6,8,x_1) = 15x_1 + 80 - 30x_2 = 80 - 15x_1$

$\Rightarrow \max\limits_{2 \le x_1 \le \frac{8}{3}} [f_1(6,8,x_1)] = 50$ at $x_1 = 2$

$\Rightarrow f_1^*(6,8) = \max\limits_{0 \le x_1 \le \frac{8}{3}} [f_1(6,8,x_1)] = 50$ at $x_1^* = 2$

$\Rightarrow x_1^* = 2$ and $x_2^* = 2$ is the optimal solution with objective function value $Z^* = 50$.

Let $s = (R_1, R_2)$ where R_i is the slack in the i^{th} constraint

10.3-17

$$f_3(R_1,R_2,x_3) = \begin{cases} 0 & \text{if } x_3 = 0 \\ -1+x_3 & \text{if } x_3 > 0 \end{cases}$$

$$\Rightarrow f_3^*(R_1,R_2) = \max\left\{0, \max_{0 \le x_3 \le \frac{R_i}{2}} [-1 + x_3]\right\}$$

$$= \max\{0, (R_1/2) - 1\}$$

$$= \begin{cases} 0 & \text{if } R_1 \ge 2 \\ \frac{R_1}{2} - 1 & \text{if } 0 \le R_1 \le 2 \end{cases}$$

$$x_3^* = \begin{cases} 0 & \text{if } R_1 \ge 2 \\ \frac{R_1}{2} & \text{if } 0 \le R_1 \le 2 \end{cases}$$

$$f_2^*(R_1,R_2) = \max_{0 \le x_2 \le \min\{\frac{R_1}{3}, R_2\}} [7x_2 + f_3^*(R_1 - 3x_2, R_2)]$$

$$\Rightarrow f_2(R_1,R_2,x_2) = \begin{cases} 7x_2 & \text{if } R_1 - 3x_2 \ge 2 \\ 7x_2 + \frac{R_1 - 3x_2}{2} - 1 & \text{if } 0 \le R_1 - 3x_2 \le 2 \end{cases}$$

$$\Rightarrow f_2^*(R_1,R_2) = \begin{cases} \frac{17R_2}{2} + \frac{R_1}{2} - 1 & \text{if } R_2 \le \frac{R_1 - 2}{3} \\ 7R_2 & \text{if } \frac{R_1-2}{3} \le R_2 \le \frac{R_1}{3} \\ \frac{7R_1}{3} & \text{if } \frac{R_1}{3} \le R_2 \end{cases}$$

$$x_2^* = \begin{cases} R_2 & \text{if } R_2 \le \frac{R_1 - 2}{3} \\ R_2 & \text{if } \frac{R_1-2}{3} \le R_2 \le \frac{R_1}{3} \\ \frac{R_1}{3} & \text{if } \frac{R_1}{3} \le R_2 \end{cases}$$

$$f_3^*(R_1,R_2) = \max_{0 \le x_1 \le 5} [3x_1 + f_2^*(6 - x_1, 5 - x_1)]$$

$$= \max\left\{ \max_{0 \le x_1 \le \frac{9}{2}} \left[3x_1 + \frac{7(6-x_1)}{3}\right], \max_{\frac{9}{2} \le x_1 \le 5} [3x_1 + 7(5-x_1)]\right\}$$

$$= \max\left\{ \max_{0 \le x_1 \le \frac{9}{2}} \left[\frac{2x_1}{3} + 14\right], \max_{\frac{9}{2} \le x_1 \le 5} [35 - 2x_1]\right\} = 17$$

at $x_1^* = \frac{9}{2}$

$\Rightarrow x_1^* = \frac{9}{2}$, $x_2^* = \frac{1}{2}$ and $x_1^* = 0$ with $Z^* = 17$

is optimal.

10.4-1 Let S_n be the current fortune of the player
Let A be the event "have \$100 at the end."
Let X_n be the amount bet at the n^{th} match

$$f_3^*(S_3) = \max_{0 \le X_3 \le S_3} \{ P\{A \mid S_3\}\}$$

if $0 \le S_3 < 50$ $\quad f_3(S_3) = 0$

if $50 \le S_3 < 100$ $\quad f_3(S_3) = \begin{cases} 0 & \text{if } X_3 \ne 100 - S_3 \\ 1/2 & \text{if } X_3 = 100 - S_3 \end{cases}$

if $S_3 = 100$ $\quad f_3(S_3) = \begin{cases} 0 & \text{if } X_3 > 0 \\ 1 & \text{if } X_3 = 0 \end{cases}$

if $S_3 > 100$ $\quad f_3(S_3) = \begin{cases} 0 & \text{if } X_3 \ne S_3 - 100 \\ 1/2 & \text{if } X_3 = S_3 - 100 \end{cases}$

S_3	$f_3^*(S_3)$	X_3^*
$0 \le S_3 \le 50$	0	$0 \le X_3^* \le 50$
$50 \le S_3 < 100$	1/2	$100 - S_3$
100	1	0
$100 < S_3$	1/2	$S_3 - 100$

$$f_2^*(S_2) = \max_{0 \le X_2 \le S_2} \left[f^*(S_2 - X_2)\tfrac{1}{2} + f^*(S_2 + X_2)\tfrac{1}{2} \right]$$

S_2	$f_2(S_2, X_2)$	X_2
$0 \le S_2 < 25$	0	$0 \le X_2 \le S_2$
$25 \le S_2 < 50$	0	$0 \le X_2 < 50 - S_2$
	$\frac{1}{4}$	$50 - S_2 \le X_2 \le S_2$
$S_2 = 50$	$\frac{1}{4}$	$0 \le X_2 < 50$
	1/2	$X_2 = 50$
$50 \le S_2 < 75$	1/2	$0 \le X_2 \le S_2 - 50$
	1/4	$S_2 - 50 < X_2 < 100 - S_2$
	1/2	$X_2 = 100 - S_2$
	1/4	$100 - S_2 < X_2 \le S_2$
$S_2 = 75$	1/2	$0 \le X_2 < 25$
	3/4	$X_2 = 25$
	1/4	$25 \le X_2 \le 75$
$75 < S_2 < 100$	1/2	$0 \le X_2 < 100 - S_2$
	3/4	$X_2 = 100 - S_2$
	1/2	$100 - S_2 < X_2 \le S_2 - 50$
	1/4	$S_2 - 50 < X_2 \le S_2$
$S_2 = 100$	1	$X_2 = 0$
	1/2	$0 < X_2 \le 50$
	1/4	$50 \le X_2 \le 100$
$S_2 > 100$	1/2	$0 \le X_2 < S_2 - 100$
	3/4	$X_2 = S_2 - 100$
	1/2	$S_2 - 100 < X_2 \le S_2 - 50$
	1/4	$S_2 - 50 < X_2 \le S_2$

10.4-1 (CONT) ⇒

s_2	$f_2^*(s_2)$	x_2^*
$0 \le s_2 < 25$	0	$0 \le x_2 < s_2$
$25 \le s_2 < 50$	$1/4$	$20 - s_2 \le x_2^* \le s_2$
50	$1/2$	50
$50 < s_2 < 75$	$1/2$	$0 \le x_2^* \le s_2 - 50$ or $100 - s_2$
75	$3/4$	25
$75 < s_2 < 100$	$3/4$	$100 - s_2$
100	1	0
$40 < s_2$	$3/4$	$s_2 - 100$

$$f_1^*(75) = \max_{0 \le x_1 \le 75} \left[f_2^*(s - x_1)\tfrac{1}{2} + f_2^*(s_1 + x_1)\tfrac{1}{2} \right]$$

$$f_1(75, x_1) = \begin{cases} 3/4 & \text{if } x_1 = 0 \\ 5/8 & \text{if } 0 < x_1 < 25 \\ 3/4 & \text{if } x_1 = 25 \\ 1/2 & \text{if } 25 < x_1 \le 50 \\ 3/8 & \text{if } 50 < x_1 \le 75 \end{cases}$$

s_1	$f_1^*(s_1)$	x_1^*
75	3/4	0 or 25

policy #	x_1	won 1st bet	lost 1st bet	won 2nd bet	lost 2nd bet
1	0	25	25	0	50
2	25	0	50	0	0

10.4-2
(a) Let x_n be the investment made in year n; that is, $x_n = 0, A, B$.
Let s_n be the amount of money on hand at the begining of the year.
Let $f_n(s_n, x_n)$ be the maximum expected amount of money at the end of the third year given s_n and x_n in year n.

For $s_n \ge 5000$
$$f_n(s_n, x_n) = \begin{cases} f_{n+1}^*(s_n) & x_n = 0 \\ .3 f_{n+1}^*(s_n - 5000) + .7 f_{n+1}^*(s_n + 5000) & x_n = A \\ .9 f_{n+1}^*(s_n) + .1 f_{n+1}^*(s_n + 5000) & x_n = B \end{cases}$$

For $0 \le s_n < 5000$ $\quad f_n(s_n, x_n) = f_{n+1}^*(s_n)$ and $x_n = 0$
(one cannot invest less than 5000)

$n = 3$:

s_3	$f_3^*(s_3)$	x_3^*
$0 \le s_3 < 5000$	s_3	0
$s_3 \ge 5000$	$s_3 + 2000$	A

$n = 2$:

s_2 \ x_2	$f_2(s_2, x_2)$: 0	A	B	$f_2^*(s_2)$	x_2^*
$0 \le s_2 < 5000$	s_2	—	—	s_2	0
$5000 \le s_2 < 10000$	$s_2 + 2000$	$s_2 + 3400$	$s_2 + 2500$	$s_2 + 3400$	A
$10000 \le s_2$	$s_2 + 2000$	$s_2 + 4000$	$s_2 + 2500$	$s_2 + 4000$	A

10.4-2 cont.

n = 1:

s_1 \ x_1	$f_1(s_1,x_1)$: 0	A	B	$f_1^*(s_1)$	x_1^*
5000	8400	9800	8150	9800	A

The optimal policy is to always invest in A with an expected fortune after three years of \$9800.

(b) Let x_n and s_n be as in part (a).
Let $f_n(s_n, x_n)$ be the maximum probability of having at least \$10,000 after 3 years given s_n and x_n.

n = 3:

s_3 \ x_3	$f_3(s_3,x_3)$: 0	A	B	$f_3^*(s_3)$	x_3^*
$0 \le s_3 < 5000$	0	—	—	0	0
$5000 \le s_3 < 10000$	0	.7	.1	.7	A
$10000 \le s_3 < 15000$	1	.7	1	1	0 or B
$15000 \le s_3$	1	1	1	1	0, A or B

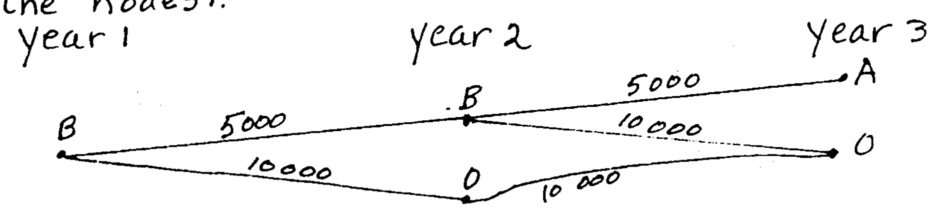
n = 2:

s_2 \ x_2	$f_2(s_2,x_2)$: 0	A	B	$f_2^*(s_2)$	x_2^*
$0 \le s_2 < 5000$	0	—	—	0	0
$5000 \le s_2 < 10000$	.7	.7	.73	.73	B
$10000 \le s_2$	1	.73	1	1	0 or B

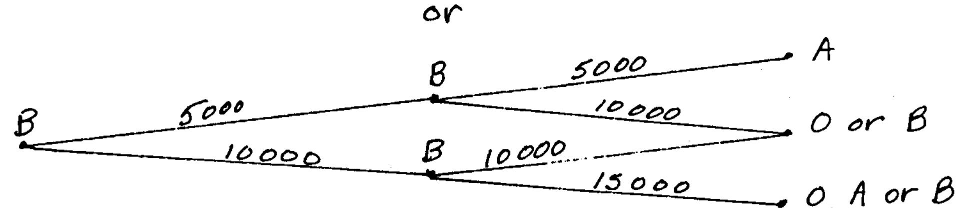
n = 1:

s_1 \ x_1	$f_1(s_1,x_1)$: 0	A	B	$f_1^*(s_1)$	x_1^*
5000	.73	.7	.757	.757	B

Hence, the optimal policies are (using the numbers on the arcs to represent the return on the investment indicated at the nodes).



or



and the maximum probability of having at least \$10,000 at the end of three years is .757.

10.4-3 $f_n(1, x_n) = K + x_n + (\tfrac{1}{3})^{x_n} \cdot f_{n+1}^*(1) + [1 - (\tfrac{1}{3})^{x_n}]\, f_{n+1}^*(0)$
$= K + x_n + (\tfrac{1}{3})^{x_n} f_{n+1}^*(1)$ since $f_{n+1}^*(0) = 0$
where $f_3^*(1) = 16$, $f_3^*(0) = 0$ and $K = \begin{cases} 0 & \text{if } x_n = 0 \\ 3 & \text{if } x_n > 0 \end{cases}$

10.4-3 (CONT)

$n=2$:

s_2 \ x_2	$f_2(s_2,x_2)$: 0	1	2	3	4	$f_2^*(s_2)$	x_2^*
						0	0

$n=1$:

s_1 \ x_1	$f_1(s_1,x_1)$: 0	1	2	3	4	$f_1^*(s_1)$	x_1^*
1	6.77	6.26	5.75	6.25	7.08	5.75	2

The optimal policy is to produce two on the first run and if none are acceptable, produce two on the second run. The minimum expected cost = \$5.75.

10.4-4 $$f_n^*(s_n) = \max_{0 \le x_n} \left\{ \tfrac{1}{3} f_{n+1}^*(s_n - x_n) + \tfrac{2}{3} f_{n+1}^*(s_n + x_n) \right\}$$

where $f_6^*(s_6) = 0$ for $s_6 < 5$ and $f_6^*(s_6) = 1$ for $s_6 \ge 5$

$n=5$:

s_5	$f_5^*(s_5)$	x_5^*
0	0	0
1	0	0
2	0	0
3	$\frac{2}{3}$	$x_5^* \ge 2$
4	$\frac{2}{3}$	$x_5^* \ge 1$
$s_5 \ge 5$	1	$x_5^* \le s_5 - 5$

$n=4$:

s_4 \ x_4	$f_4(s_4,x_4)$: 0	1	2	3	4	$f_4^*(s_4)$	x_4^*
0	0	—	—	—	—	0	0
1	0	0	—	—	—	0	0
2	0	4/9	4/9	—	—	4/9	1,2
3	2/3	4/9	2/3	2/3	—	2/3	0,2,3
4	2/3	8/9	2/3	2/3	2/3	8/9	1
$s_4 \ge 5$	1	—	—	—	—	1	$\le s_4 - 5$

$n=3$:

s_3 \ x_3	$f_3(s_3,x_3)$: 0	1	2	3	4	$f_3^*(s_3)$	x_3^*
0	0	—	—	—	—	0	0
1	0	8/27	—	—	—	8/27	1
2	4/9	4/9	16/27	—	—	16/27	2
3	2/3	20/27	2/3	2/3	—	20/27	1
4	8/9	8/9	22/27	2/3	2/3	22/27	0,1
$s_3 \ge 5$	1	—	—	—	—	1	$\le s_3 - 5$

$n=2$:

s_2 \ x_2	$f_2(s_2,x_2)$: 0	1	2	3	4	$f_2^*(s_2)$	x_2^*
0	0	—	—	—	—	0	0
1	8/27	32/81	—	—	—	32/81	1
2	16/27	48/81	48/81	—	—	48/81	0,1,2
3	20/27	64/81	62/81	2/3	—	64/81	1
4	24/27	74/81	70/81	62/81	2/3	74/81	1
$s_2 \ge 5$	1	—	—	—	—	1	$\le s_2 - 5$

$n=1$:

s_1 \ x_1	$f_1(s_1,x_1)$: 0	1	2	$f_1^*(s_1)$	x_1^*
2	48/81	160/243	124/243	160/243	1

So Probability{winning bet using above policy} = $\frac{160}{243}$ = .658

10.4-5
Let $x_n = a$ and $x_n = d$ denote the decision to advertise or discontinue the product, respectively, for quarter n ($n=1,2,3$).
Let s_n denote the level of sales above ($s_n \ge 0$) or below ($s_n \le 0$) the breakeven point (in millions) for quarter ($n-1$).
Let $f_n(s_n, x_n)$ denote the maximum expected discounted profit (in millions) from the beginning of period n onward given state s_n and decision x_n.

10.4-5 cont. Then $f_n(s_n, x_n) =$

$$-30 + 5\left[s_n + \frac{1}{b_n - a_n}\int_{a_n}^{b_n} t\,dt\right] + \frac{1}{b_n - a_n}\int_{a_n}^{b_n} f_{n+1}^*(s+t)\,dt$$

where a_n and b_n are given by

n	a_n	b_n
1	1	5
2	0	4
3	-1	3

Hence, for $1 \le n \le 3$, $f_n(s_n, a) = -30 + 5\left[s_n + \frac{a_n + b_n}{2}\right] + \frac{1}{b_n - a_n}\int_{a_n}^{b_n} f_{n+1}^*(s_n + t)\,dt$

$f_n(s_n, d) = -20$ and the process stops.

Then $f_n^*(s_n) = \max\{f_n(s_n, a), f_n(s_n, d)\}$ and $f_4^*(s_4) = \begin{cases} -20 & \text{if } 0 > s_4 \\ -40 s_4 & \text{if } s_4 \ge 0 \end{cases}$

$\underline{n=3}$:

$f_3(s_3, d) = -20$

$f_3(s_3, a) = -30 + 5(s_3 + 1) + \frac{1}{4}\int_{-1}^{3} f_4^*(s_3 + t)\,dt$

For $-3 \le s_3 \le 1$, $f_3(s_3, a) = -30 + 5(s_3 + 1) + \frac{1}{4}\left[\int_{-1}^{-s_3} -20\,dt + \frac{1}{4}\int_{-5}^{3} 40(s_3 + t)\,dt\right]$

$= 5(s_3 + 4)^2 - 65$

For $1 \le s_3 \le 5$, $f_3(s_3, a) = -30 + 5(s_3 + 1) + \frac{1}{4}\int_{-1}^{3} 40(s_3 + t)\,dt = 15 + 45 s_3$.

Hence, for $-3 \le s_3 \le 1$, $f_3^*(s_3) = \max\{-20, 5(s_3 + 4)^2 - 65\}$

$= \begin{cases} -20 & \text{if } -3 \le s_3 \le -1, \ x_3^* = d \\ 5(s_3 + 4)^2 - 65 & \text{if } -1 \le s_3 \le 1, \ x_3^* = a \end{cases}$

summarizing:

s_3	$f_3^*(s_3)$	x_3^*
$-3 \le s_3 \le -1$	-20	d
$-1 < s_3 \le 1$	$5(s_3 + 4)^2 - 65$	a
$1 < s_3 \le 5$	$15 + 45 s_3$	a

$\underline{n=2}$: for $-3 \le s_2 \le -1$,

$f_2(s_2, a) = -30 + 5(s_2 + 2) + \frac{1}{4}\left[\int_0^{-s_2 - 1} -20\,dt + \int_{-s_2 - 1}^{1 - s_2} (5(s_2 + t + 4)^2 - 65)\,dt\right.$

$\left. + \int_{1 - s_2}^{4} (45(s_2 + t) + 15)\,dt\right]$

$= \frac{5}{4}\left(\frac{9}{2} s_2^2 + 47 s_2 + \frac{427}{6}\right)$

Since $f_2(-3, a) < f_2(s_2, d) = -20 < f_2(-1, a)$, we must find where

$f_2(s_2, a) = f_2(s_2, d)$, that is, $\frac{5}{4}\left(\frac{9}{2} s_2^2 + 47 s_2 + \frac{427}{6}\right) = -20$.

$s_2^* = \frac{-47 + 8\sqrt{10}}{9} \doteq -2.411$

10.4-5 (CONT)

For $-1 \le S_2 \le 1$, $f_2(S_2, a) = -30 + 5(S_2+2) + \frac{1}{4}\left[\int_0^{1-S_2}(5(S_2+t+4)^2 - 65)\,dt + \int_{1-S_2}^{4}(45(S_2+t)+15)\,dt\right.$

$= \frac{5}{4}\left(-(S_2+4)^3/3 + 9(S_2+4)^2/2 + 20S_2 + 103/6\right)$

Since $f_2(-1,a) = 35\ 5/6$, and $f_2(S_2,a)$ is an increasing function of S_2 over $-1 \le S_2 \le 1$, $x_2^* = a$ is the optimal decision in this interval.

S_2	$f_2^*(S_2)$	x_2^*
$-3 \le S_2 \le S_2^*$	-20	d
$S_2^* < S_2 \le -1$	$5/4\left(\frac{9}{2}S_2^2 + 47S_2 + \frac{427}{6}\right)$	a
$-1 < S_2 \le 1$	$5/4\left[-(S_2+4)^3/3 + \frac{9}{2}(S_2+4)^2 + 20S_2 + \frac{103}{6}\right]$	a

n=1:

$f_1(-20, d) = -20$

$f_1(-20, a) = -30 + 5(-4+3) + \frac{1}{4}\int_1^5 f_2^*(-4+t)\,dt$

$= -35 + \frac{1}{4}\left[\int_1^{S_2^*+4} -20\,dt + \frac{5}{4}\int_{S_2^*+4}^{3}\left(\frac{9}{2}(-4+t)^2 + 47(-4+t) + \frac{427}{6}\right)dt\right]$

$+ \frac{1}{4}\left[\frac{5}{4}\int_3^5\left(\frac{t^3}{3} + \frac{9t^2}{2} + 20(-4+t) + \frac{103}{6}\right)dt\right] = 4.77$

Summarizing:

S_1	$f_1^*(S_1)$	x_1^*
-4	4.77	a

So the optimal policy is:

First quarter - advertise

Second quarter - if $S_2 \le S_2^*$, discontinue. If $S_2 > S_2^*$, advertise

Third quarter - if $S_3 \le -1$, discontinue. If $S > -1$, advertise.

where $S_2^* \doteq -2.411$.