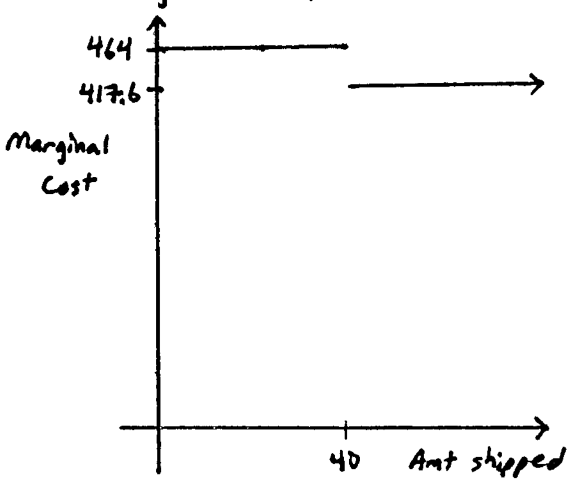
# Chapter 12: Nonlinear Programming

12.1-1. Maximize $f(x) = 100x_1^{2/3} + 10x_1 + 40x_2^{3/4} + 5x_2 + 50x_3^{1/2} + 5x_3$

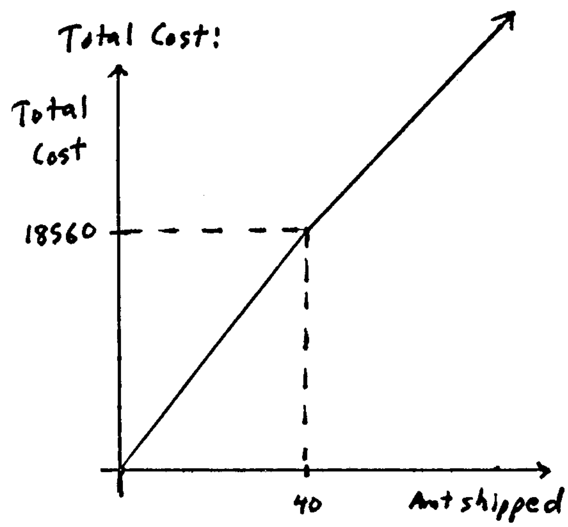
subject to:

$$9x_1 + 3x_2 + 5x_3 \le 500$$
$$5x_1 + 4x_2 \le 350$$
$$3x_1 + 2x_3 \le 150$$
$$x_3 \le 20$$
$$x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0$$

12.1-2. **Marginal Cost:**




**Total Cost:**




Each term in the objective function changes (as above) from $a_{ij}x_{ij}$ to

$$a_{ij}x_{ij} - 0.1\,a_{ij}(x_{ij}-40)\,\delta(x_{ij}-40)$$

where $a_{ij}$ is the shipping cost from cannery $i$ to warehouse $j$

and $\delta(x) = \begin{cases} 0 & x < 0 \\ 1 & x \ge 0 \end{cases}$

otherwise, the formulation is the same.

12.1-3

Let $S_1$ = number of blocks of stock 1 to purchase

$S_2$ = number of blocks of stock 2 to purchase.

Minimize Risk $= 4S_1^2 + 100S_2^2 + 5S_1S_2$,

subject to $20S_1 + 30S_2 \leq 50$

$5S_1 + 10S_2 \geq$ minimum acceptable expected return

and $S_1 \geq 0,\ S_2 \geq 0$.

12.2-1

$f(x) = f_1(x_1) + f_2(x_2) + f_3(x_3)$

with $f_1(x_1) = 100x_1^{2/3} + 10x_1$

$f_2(x_2) = 40x_2^{3/4} + 5x_2$

$f_3(x_3) = 50x_3^{1/2} + 5x_3$

$$\frac{d^2f_1}{dx_1^2} = -\frac{200}{9}x_1^{-4/3} \leq 0 \quad \text{for} \quad x_1 \geq 0$$

$$\frac{d^2f_2}{dx_2^2} = -\frac{120}{16}x_2^{-5/4} \leq 0 \quad \text{for} \quad x_2 \geq 0$$

$$\frac{d^2f_3}{dx_3^2} = -\frac{50}{4}x_3^{-3/2} \leq 0 \quad \text{for} \quad x_3 \geq 0$$

$f_1$, $f_2$ and $f_3$ are concave over the non-negative orthant so $f$ must be concave in the same region. The constraints are linear and, therefore, convex so the problem is a convex programming problem.

12.2-2

$$\left.\begin{array}{l}\dfrac{\partial^2 f}{\partial S_1^2}=8\geq 0\\[2ex] \dfrac{\partial^2 f}{\partial S_2^2}=200\geq 0\\[2ex] \dfrac{\partial^2 f}{\partial S_1 \partial S_2}=5\end{array}\right\} \frac{\partial^2 f}{\partial S_1^2}\frac{\partial^2 f}{\partial S_2^2}-\left[\frac{\partial^2 f}{\partial S_1 \partial S_2}\right]^2 = 1595$$

$\Rightarrow f$ is convex everywhere

12.2-3. Objective function: $z = 3x_1 + 5x_2$

$\Rightarrow x_2 = -3/5\, x_1 + 1/5\, z$

$\Rightarrow$ slope is $-3/5$

Constraint boundary: $9x_1^2 + 5x_2^2 = 216$

$\Rightarrow x_2 = \sqrt{1/5(216 - 9x_1^2)}$

$\Rightarrow \dfrac{\partial x}{\partial x_1} = -\dfrac{1}{5}\dfrac{9x_1}{\sqrt{1/5(216-9x_1^2)}}$

$= -3/5$ for $x_1 = 2$

So the objective function is tangent to this constraint at $(x_1, x_2) = (2, 6)$

12.2-4. Constraint boundary: $3x_1 + 2x_2 = 18$

$\Rightarrow g(x_1) = x_2 = -3/2\, x_1 + 9$

$\Rightarrow \dfrac{dg(x_1)}{dx_1} = -3/2$

Objective function at $(8/3, 5)$:

$(9x_1^2 - 126x_1 + 857) - 182x_2 + 13x^2 = 0$

$\Rightarrow f(x_1) = x_2 = \dfrac{182 - 2\sqrt{-2860 + 1638x_1 - 117x^2}}{26}$

$\Rightarrow \dfrac{df(x_1)}{dx_1} = -\dfrac{1}{26}\dfrac{1638 - 234x_1}{\sqrt{-2860 + 1638x_1 - 117x_1^2}}$

$\Rightarrow \dfrac{df(8/3)}{dx_1} = -3/2$

and $f(8/3) = 5$, $g(8/3) = 5$

so the objective function is tangent to this constraint at $(x_1, x_2) = (8/3, 5)$

12.2-5. (a) $\dfrac{df}{dx} = 48 - 120x + 3x^2 = 0$

$\Rightarrow x^* = \dfrac{120 \pm \sqrt{120^2 - 4\cdot 3\cdot 48}}{6}$

$= .4041$ or $39.596$

$\dfrac{d^2 f(.4041)}{dx^2} = -117.6 \Rightarrow f(.4041) = 2.475$ is a local maximum

$\dfrac{d^2 f(39.596)}{dx^2} = 117.6 \Rightarrow f(39.596) = .0253$ is a local minimum

12.2-5. (b) for $x > 39.596$ $\frac{df}{dx} > 0$ and $\frac{d^2f}{dx^2} = 6x - 120 > 0$

$\Rightarrow f$ is unbounded above.

for $x < .4041$ $\frac{df}{dx} < 0$ and $\frac{d^2f}{dx^2} < 0$

$\Rightarrow f$ is unbounded below.

12.2-6. (a) $\frac{d^2f}{dx^2} = -2 \quad \forall x \Rightarrow f$ is concave

(b) $\frac{d^2f}{dx^2} = 12x^2 + 12 > 0 \quad \forall x \Rightarrow f$ is convex

(c) $\frac{d^2f}{dx^2} = 12x - 6 \begin{cases} > 0 & \text{for } x > \frac{1}{2} \\ < 0 & \text{for } x < \frac{1}{2} \end{cases} \Rightarrow f$ is neither convex nor concave

(d) $\frac{d^2f}{dx^2} = 12x^2 + 2 > 0 \quad \forall x \Rightarrow f$ is convex

(e) $\frac{d^2f}{dx^2} = 6x + 12x^2 \begin{cases} > 0 & \text{for } x < -\frac{1}{2} \text{ or } x > 0 \\ < 0 & \text{for } -\frac{1}{2} < x < 0 \end{cases}$

$\Rightarrow f$ is neither convex nor concave.

12.2-7. (a) $\frac{\partial^2 f}{\partial x_1^2} = -2 < 0 \quad \forall (x_1, x_2)$

$\frac{\partial^2 f}{\partial x_2^2} = -2 < 0 \quad \forall (x_1, x_2)$

$\frac{\partial^2 f}{\partial x_2^2}\frac{\partial^2 f}{\partial x_1^2} - \left[\frac{\partial^2 f}{\partial x_1 \partial x_2}\right]^2 = 4 - 1^2 = 3 > 0 \quad \forall (x_1, x_2)$

$\Rightarrow f$ is concave

(b) $\frac{\partial^2 f}{\partial x_1^2} = 4 > 0 \quad \forall (x_1, x_2)$

$\frac{\partial^2 f}{\partial x_2^2} = 2 > 0 \quad \forall (x_1, x_2)$

$\frac{\partial^2 f}{\partial x_1^2}\frac{\partial^2 f}{\partial x_2^2} - \left[\frac{\partial^2 f}{\partial x_1 \partial x_2}\right]^2 = 8 - (2)^2 = 4 > 0 \quad \forall (x_1, x_2)$

$\Rightarrow f$ is convex

(c) $\frac{\partial^2 f}{\partial x_1^2} = 2 > 0 \quad \forall (x_1, x_2)$ $\quad \frac{\partial^2 f}{\partial x_2^2} = 4 > 0 \quad \forall (x_1, x_2)$

$\frac{\partial^2 f}{\partial x_1^2}\frac{\partial^2 f}{\partial x_2^2} - \left[\frac{\partial^2 f}{\partial x_1 \partial x_2}\right]^2 = -1 < 0$

$\Rightarrow f$ is neither convex nor concave

(d) $\frac{\partial^2 f}{\partial x_1^2} = \frac{\partial^2 f}{\partial x_2^2} = \frac{\partial^2 f}{\partial x_1 \partial x_2} = 0$

$\Rightarrow \frac{\partial^2 f}{\partial x_1^2}\frac{\partial^2 f}{\partial x_2^2} - \left[\frac{\partial^2 f}{\partial x_1 \partial x_2}\right]^2 = 0$

$\Rightarrow f$ is both convex and concave

12.2-7. (e) $\frac{\partial^2 f}{\partial x_1^2} = \frac{\partial^2 f}{\partial x_2^2} = 0 \quad \forall (x_1, x_2)$

$$\frac{\partial^2 f}{\partial x_1^2}\frac{\partial^2 f}{\partial x_2^2} - \left[\frac{\partial^2 f}{\partial x_1 \partial x_2}\right]^2 = -1 < 0$$

$\Rightarrow f$ is neither convex nor concave

12.2-8. Let $f = f_1 + f_2 + f_{34} + f_{56} + f_{67}$

where
$f_1 = 5x_1$
$f_2 = 2x_2^2$
$f_{34} = x_3^2 - 3x_3x_4 + 4x_4^2$
$f_{56} = x_5^2 + 3x_5x_6 + 3x_6^2$
$f_{67} = 3x_6^2 + 3x_6x_7 + x_7^2$

$\frac{d^2 f_1}{dx_1^2} = 0 \quad \forall x_1 \Rightarrow f_1$ is convex (and concave).

$\frac{d^2 f_2}{dx_2^2} = 4 \quad \forall x_2 \Rightarrow f_2$ is convex

$\frac{\partial^2 f_{34}}{\partial x_3^2} = 2 > 0 \quad \forall (x_3, x_4) \qquad \frac{\partial^2 f_{34}}{\partial x_4^2} = 8 > 0 \quad \forall (x_3, x_4)$

$$\frac{\partial^2 f_{34}}{\partial x_3^2}\frac{\partial^2 f_{34}}{\partial x_4^2} - \left[\frac{\partial^2 f_{34}}{\partial x_3 \partial x_4}\right]^2 = 16 - 3^2 = 7 > 0 \quad \forall (x_3, x_4)$$

$\Rightarrow f_{34}$ is convex

$\frac{\partial^2 f_{56}}{\partial x_5^2} = 2 > 0 \quad \forall (x_5, x_6) \qquad \frac{\partial^2 f_{56}}{\partial x_6^2} = 6 > 0 \quad \forall (x_5, x_6)$

$$\frac{\partial^2 f_{56}}{\partial x_5^2}\frac{\partial^2 f_{56}}{\partial x_6^2} - \left[\frac{\partial^2 f_{56}}{\partial x_5 \partial x_6}\right]^2 = 12 - 3^2 = 3 > 0 \quad \forall (x_5, x_6)$$

$\Rightarrow f_{56}$ is convex

$f_{67}(x_6, x_7) = f_{56}(x_7, x_6) \Rightarrow f_{67}$ is convex.

$\Rightarrow f$ is convex

12.2-9. (a) maximize $f(\underline{x}) = x_1 + x_2$
subject to $g(\underline{x}) = x_1^2 + x_2^2 \le 1$
$\underline{x} \ge 0$

$\frac{\partial^2 f}{\partial x_1^2} = \frac{\partial^2 f}{\partial x_2^2} = \frac{\partial^2 f}{\partial x_1 \partial x_2} = 0$

$$\Rightarrow \frac{\partial^2 f}{\partial x_1^2}\frac{\partial^2 f}{\partial x_2^2} - \left[\frac{\partial^2 f}{\partial x_1 \partial x_2}\right]^2 = 0$$

$\Rightarrow f$ is concave (and convex)

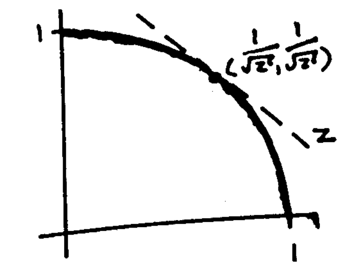
$\frac{\partial^2 g}{\partial x_1^2} = 2 > 0 \quad \forall (x_1, x_2) \qquad \frac{\partial^2 g}{\partial x_2^2} = 2 > 0 \quad \forall (x_1, x_2)$

$$\frac{\partial^2 g}{\partial x_1^2}\frac{\partial^2 g}{\partial x_2^2} - \left[\frac{\partial^2 g}{\partial x_1 \partial x_2}\right]^2 = 4 - 0^2 = 4 > 0 \quad \forall (x_1, x_2)$$

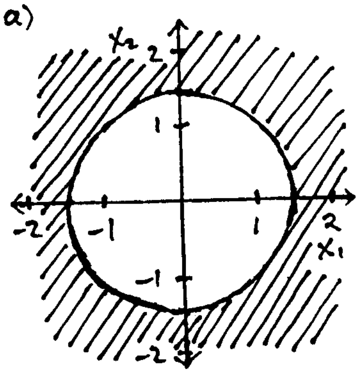
$\Rightarrow g_1$ is convex

$\Rightarrow$ the problem is a convex programming problem.

12.2-9. (b)




12.2-10. a)




$x_1^2 + x_2^2 \geq 2$

Clearly, this is not a convex feasible region (Take $(0, \sqrt{2})$ and $(0, -\sqrt{2})$, $(0,0) = \frac{1}{2}(0, \sqrt{2}) + \frac{1}{2}(0, -\sqrt{2})$ is not feasible)

b) Feasible region: $-x_1^2 - x_2^2 \leq -2$

$g_1(x_1) = -x_1^2$ and $g_2(x_2) = -x_2^2$ are both concave functions, not convex, so the feasible region does not need to be convex.

$\left(\frac{\partial^2}{\partial x_i^2} g_i(x_i) = -1 < 0\right)$

To prove that the feasible region is not convex, we need to show there are $y$ and $z$ both feasible, but $\alpha y + (1-\alpha) z$ not feasible $(0<\alpha<1)$, as shown in (a).

12.3-1. Since we are minimizing a concave function (see 13.1-2.) this is a nonconvex programming problem.

12.3-2. $\frac{df}{dx} = -6 + 6x - 6x^2 = 0$

$\Rightarrow \quad x = \frac{-6 \pm \sqrt{36 - 4 \cdot 36}}{12}$ has no real solution

$\frac{d^2 f}{dx^2} = 6 - 12x \begin{cases} >0, & x < \frac{1}{2} \\ <0, & x > \frac{1}{2} \end{cases}$

So the slope of $f$ increases from $-6$ at $x = 0$ to $-9/2$ at $x = \frac{1}{2}$ and decreases for all $x$ thereafter.

Thus, $x^* = 0$ is optimal.

12.3-3. a) Linearly constrained convex program:

$\frac{\partial^2 f}{\partial x_1^2} = -12x_1^2 - 4 < 0 \quad \forall(x_1,x_2) \quad \frac{\partial^2 f}{\partial x_2^2} = -8 < 0 \quad \forall(x_1,x_2)$

$\frac{\partial^2 f}{\partial x_1^2}\frac{\partial^2 f}{\partial x_2^2} - \left[\frac{\partial^2 f}{\partial x_1 \partial x_2}\right] = 96x_1^2 + 32 - (-2)^2 > 0 \quad \forall(x_1,x_2)$

$\Rightarrow f$ is concave

$g_1$ and $g_2$ are linear and, hence, convex.

Geometric Program:

$f(\underline{x}) = c_1 x_1^{a_{11}} x_2^{a_{12}} + c_2 x_1^{a_{21}} x_2^{a_{22}} + c_3 x_1^{a_{31}} x_2^{a_{32}} + c_4 x_1^{a_{41}} x_2^{a_{42}}$

with $c_1 = 1 \quad a_{11} = 4 \quad a_{12} = 0$
$c_2 = 2 \quad a_{21} = 2 \quad a_{22} = 0$
$c_3 = 2 \quad a_{31} = 1 \quad a_{32} = 1$
$c_4 = 4 \quad a_{41} = 0 \quad a_{42} = 2$

$g_1(\underline{x}) = c_1 x_1^{a_{11}} x_2^{a_{12}} + c_2 x_1^{a_{21}} x_2^{a_{22}}$
with $c_1 = -2 \quad a_{11} = 1 \quad a_{12} = 0$
$c_2 = -1 \quad a_{21} = 0 \quad a_{22} = 1$

$g_2(\underline{x}) = c_1 x_1^{a_{11}} x_2^{a_{12}} + c_2 x_1^{a_{21}} x_2^{a_{22}}$
with $c_1 = -1 \quad a_{11} = 1 \quad a_{12} = 0$
$c_2 = -2 \quad a_{21} = 0 \quad a_{22} = 1$

Fractional Program:
$f' = f_1/f_2$ where $f_1 = f$ & $f_2 = 1$

b) Let $y_1 = x_1 - 1$ and $y_2 = x_2 - 1$

$\Rightarrow$ Minimize $y_1^4 + 4y_1^3 + 8y_1^2 + 10y_1 + 2y_1y_2 + 4y_2^2 + 10y_2$
subject to
$2y_1 + y_2 \geq 7$
$y_1 + 2y_2 \geq 7$
$y_1 \geq 0,\ y_2 \geq 0$

12.3-4. a) Let $x_1 = e^{y_1}$ and $x_2 = e^{y_2}$

$\Rightarrow$ Minimize $2e^{-2y_1 - y_2} + e^{-y_1 - 2y_2}$
subject to $4e^{y_1 + y_2} + e^{2y_1 + 2y_2} \leq 12$
$e^{y_1} \geq 0,\ e^{y_2} \geq 0$

b) $\frac{\partial^2 f}{\partial y_1^2} = -8e^{-2y_1 - y_2} - e^{-y_1 - 2y_2} \leq 0 \quad \forall(y_1,y_2)$

$\frac{\partial^2 f}{\partial y_2^2} = -2e^{-2y_1 - y_2} - 4e^{-y_1 - 2y_2} \leq 0 \quad \forall(y_1,y_2)$

$\frac{\partial^2 f}{\partial y_1^2}\frac{\partial^2 f}{\partial y_2^2} - \left[\frac{\partial^2 f}{\partial y_1 \partial y_1}\right]^2 = 18e^{-3y_1 - 3y_2} \geq 0 \quad \forall(y_1,y_2)$

so $f$ is concave

12.3-4. b) (cont') for $g(y_1,y_2) = 4e^{y_1+y_2} + e^{2y_1+2y_2} - 12$

$$\frac{\partial^2 g}{\partial y_1^2} = 4e^{y_1+y_2} + 4e^{2y_1+2y_2} \geq 0 \quad \forall (y_1, y_2)$$

$$\frac{\partial^2 g}{\partial y_2^2} = 4e^{y_1+y_2} + 4e^{2y_1+2y_2} \geq 0 \quad \forall (y_1, y_2)$$

$$\frac{\partial^2 g}{\partial y_1^2}\frac{\partial^2 g}{\partial y_2^2} - \left[\frac{\partial^2 g}{\partial y_1 \partial y_2}\right]^2 = 0 \quad \forall (y_1, y_2)$$

so $g$ is convex

$e^{y_1} \geq 0$, $e^{y_2} \geq 0 \quad \forall (y_1, y_2)$

Now there is, in fact, no non-negativity constraints on the y's, and so, strictly speaking, according to the definition in the book this isn't quite a convex program. However, in general, the entire theory of convex programming applies as long as the feasible region is convex which is the case here.

12.3-5. (a) Maximize $10y_1 + 20y_2 + 10t$

subject to

$$y_1 + 3y_2 - 50t \leq 0$$
$$3y_1 + 2y_2 - 80t \leq 0$$
$$3y_1 + 4y_2 + 20t = 1$$
$$y_1 \geq 0,\ y_2 \geq 0,\ t \geq 0$$

(b)

| Bas Var | Eq No | Z | X1 | X2 | X3 | X4 | X5 | X6 | Right side |
|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | 1M | |
| Z | 0 | 1 | 3.269 | 0 | 0 | 1.385 | 0 | 3.962 | 3.962 |
| X2 | 1 | 0 | 0.654 | 1 | 0 | 0.077 | 0 | 0.192 | 0.192 |
| X5 | 2 | 0 | 3.231 | 0 | 0 | -1.38 | 1 | 0.538 | 0.538 |
| X3 | 3 | 0 | 0.019 | 0 | 1 | -0.02 | 0 | 0.012 | 0.012 |

Variables $(x_1, x_2, x_3)$ from the courseware solution correspond to $(y_1, y_2, t)$ for part (a) variables, respectively. Therefore, the optimal solution has $(y_1, y_2, t) = (0, 0.192, 0.012)$ with $z = 3.962$. The optimal solution to the original problem is $(x_1, x_2) = (0, 16.67)$ with $f(x) = 3.962$.

12.3-6. The KKT conditions may be rewritten:

$$Qx + A^T u - c = y$$
$$-Ax + b = v$$

$$x \geq 0,\ u \geq 0,\ y \geq 0,\ v \geq 0$$

$$x^T(Qx + A^T u - c) + u^T(-Ax + b) = 0$$

and this is the Linear Complementarity Problem with $z = \begin{pmatrix} x \\ u \end{pmatrix}$ $M = \begin{bmatrix} Q & A^T \\ -A & 0 \end{bmatrix}$ $q = \begin{pmatrix} -c \\ b \end{pmatrix}$

and $w = \begin{bmatrix} Qx + A^T u - c \\ -Ax + b \end{bmatrix}$

Problem 12-4.1 (a)

Interactive One-Dimensional Search Procedure:

Max f(X) = 1 X^3 + 2 X - 2 X^2 -0.25 X^4

df(X)/dX = 3 X^2 + 2 - 4 X - 1 X^3

Lower Bound: 0 Upper Bound: 2.4

| Iteration | df(X)/dX | X(L) | X(U) | New X' | f(X') |
|---|---|---|---|---|---|
| 0 | | 0 | 2.4 | 1.2 | 0.7296 |
| 1 | -0.208 | 0 | 1.2 | 0.6 | 0.6636 |
| 2 | +0.464 | 0.6 | 1.2 | 0.9 | 0.745 |
| 3 | +0.101 | 0.9 | 1.2 | 1.05 | 0.7487 |
| 4 | - 0.05 | 0.9 | 1.05 | 0.975 | 0.7497 |
| 5 | +0.025 | 0.975 | 1.05 | 1.0125 | 0.7499 |
| Stop | | | | | |

Solution: X = 1.0125

**Problem 12-4.1 (b)**

Newton's method

**Max f(x) = x^3 + 2x - 2 x^2 - 0.25 x^4**

**f'(x) = 3x^2 + 2 - 4 x - x^3**

**f''(x) = 6x - 4 - 3 x^2**

**error** 0.001

| Iteration i | $x_i$ | $f(x_i)$ | $f'(x_i)$ | $f''(x_i)$ | $x_{i+1}$ | $\|x_i - x_{i+1}\|$ |
|---|---|---|---|---|---|---|
| 1 | 1.2 | 0.7296 | -0.208 | -1.12 | 1.014286 | 0.185714 |
| 2 | 1.01428571 | 0.74989795 | -0.014289 | -1.000612 | 1.000006 | 0.01428 |
| 3 | 1.00000583 | 0.75 | -5.83E-06 | -1 | 1 | 5.83E-06 |

12.4-2. a)

| Iteration | df(X)/dX | X(L) | X(U) | New X' | f(X') |
|---|---|---|---|---|---|
| 0 | | 0 | 4.8 | 2.4 | 8.64 |
| 1 | + 1.2 | 2.4 | 4.8 | 3.6 | 8.64 |
| 2 | - 1.2 | 2.4 | 3.6 | 3 | 9 |
| 3 | + 0 | 3 | 3.6 | 3.3 | 8.91 |
| 4 | - 0.6 | 3 | 3.3 | 3.15 | 8.9775 |
| 5 | - 0.3 | 3 | 3.15 | 3.075 | 8.9944 |
| 6 | - 0.15 | 3 | 3.075 | 3.0375 | 8.9986 |
| Stop | | | | | |

b)

| Iteration | df(X)/dX | X(L) | X(U) | New X' | f(X') |
|---|---|---|---|---|---|
| 0 | | -4 | 1 | -1.5 | -1.688 |
| 1 | - 1.5 | -1.5 | 1 | -0.25 | -1.121 |
| 2 | +3.188 | -1.5 | -0.25 | -0.875 | -1.984 |
| 3 | +0.258 | -1.5 | -0.875 | -1.188 | -1.964 |
| 4 | -0.401 | -1.188 | -0.875 | -1.031 | -1.999 |
| 5 | -0.063 | -1.031 | -0.875 | -0.953 | -1.998 |
| 6 | +0.094 | -1.031 | -0.953 | -0.992 | -2 |
| Stop | | | | | |

12.4-3.(a)

| Iteration | df(X)/dX | X(L) | X(U) | New X' | f(X') |
|---|---|---|---|---|---|
| 0 | | -1 | 4 | 1.5 | -16.69 |
| 1 | - 100 | -1 | 1.5 | 0.25 | 0.3047 |
| 2 | +0.156 | 0.25 | 1.5 | 0.875 | 0.2482 |
| 3 | -0.923 | 0.25 | 0.875 | 0.5625 | 0.3125 |
| 4 | -0.001 | 0.25 | 0.5625 | 0.4063 | 0.3124 |
| 5 | +0.004 | 0.4063 | 0.5625 | 0.4844 | 0.3125 |
| Stop | | | | | |

**Problem 12-4.3 (b)**

Newton's method

**Max f(x) = 48 x^5 + 42x^3 + 3.5x -16 x^6 - 61 x^4 -16.5x^2**

**f'(x) = 240x^4 + 126 x^2 + 3.5 - 96 x^5 - 264 x^3 - 33 x**

**f''(x) = 960 x^3 + 252 x - 480 x^4 - 792 x^2 - 33**

**error** 0.001

| Iteration i | $x_i$ | $f(x_i)$ | $f'(x_i)$ | $f''(x_i)$ | $x_{i+1}$ | $\lvert x_i - x_{i+1}\rvert$ |
|---|---|---|---|---|---|---|
| 1 | 1 | 0 | -23.5 | -93 | 0.747312 | 0.252688 |
| 2 | 0.74731183 | 0.30509816 | -8.496421 | -36.0381 | 0.51155 | 0.235762 |
| 3 | 0.51154965 | 0.31249998 | -2.677284 | -15.70259 | 0.34105 | 0.170499 |
| 4 | 0.34105018 | 0.31160364 | -0.767583 | -7.588489 | 0.239899 | 0.101151 |
| 5 | 0.23989924 | 0.302969 | -0.091464 | -6.461815 | 0.225745 | 0.014154 |
| 6 | 0.22574474 | 0.30003409 | 0.001383 | -6.675803 | 0.225952 | 0.000207 |

12.4-4.(a)

| Iteration | df(X)/dX | X(L) | X(U) | New X' | f(X') |
|---|---|---|---|---|---|
| 0 | | 0 | 2 | 1 | 25 |
| 1 | + 13 | 1 | 2 | 1.5 | 20.109 |
| 2 | -44.81 | 1 | 1.5 | 1.25 | 26.068 |
| 3 | -6.748 | 1 | 1.25 | 1.125 | 26.146 |
| 4 | +4.844 | 1.125 | 1.25 | 1.1875 | 26.288 |
| Stop | | | | | |

**Problem 12-4.4 (b)**

Newton's method

**Max f(x) = x^3 + 30x - x^6 - 2 x^4 - 3 x^2**

**f'(x) = 3x^2 + 30 - 6 x^5 - 8 x^3 - 6 x**

**f''(x) = 6x - 30 x^4 - 24 x^2 - 6**

**error** 0.001

| Iteration i | $x_i$ | $f(x_i)$ | $f'(x_i)$ | $f''(x_i)$ | $x_{i+1}$ | $\lvert x_i - x_{i+1}\rvert$ |
|---|---|---|---|---|---|---|
| 1 | 1 | 25 | 13 | -54 | 1.240741 | 0.240741 |
| 2 | 1.24074074 | 26.1259522 | -5.748826 | -106.5981 | 1.186811 | 0.05393 |
| 3 | 1.18681083 | 26.2881521 | -0.395765 | -92.20147 | 1.182518 | 0.004292 |
| 4 | 1.18251843 | 26.2890048 | -0.00231 | -91.12675 | 1.182493 | 2.54E-05 |

12.4-5

(a) $f'(x) = 4x^3 + 2x - 4$

$f'(0) = -4$

$f'(1) = 2$

$f'(2) = 32$

since $f'(x)$ is continuous there must be a point in $0 \le x \le 1$ such that $f'(x) = 0$, and so since $f$ is a concave function (this is a convex program) the optimal solution must be in the interval $0 \le x \le 1$.

(b)

| Iteration | df(X)/dX | X(L) | X(U) | New X' | f(X') |
|---|---|---|---|---|---|
| 0 | | 0 | 2 | 1 | -2 |
| 1 | + 2 | 0 | 1 | 0.5 | -1.688 |
| 2 | - 2.5 | 0.5 | 1 | 0.75 | -2.121 |
| 3 | -0.813 | 0.75 | 1 | 0.875 | -2.148 |
| 4 | + 0.43 | 0.75 | 0.875 | 0.8125 | -2.154 |
| 5 | -0.229 | 0.8125 | 0.875 | 0.8438 | -2.156 |
| 6 | + 0.09 | 0.8125 | 0.8438 | 0.8281 | -2.156 |
| Stop | | | | | |

**Problem 12-4.5 (C)**

Newton's method

**Max f(x) = x^4 + x^2 - 4 x      s.t. x>= 0, x<=2**

**f'(x) = 4x^3 + 2x - 4**

**f"(x) = 12x^2 - 2**

**error** 0.0001

| Iteration i | $x_i$ | $f(x_i)$ | $f'(x_i)$ | $f''(x_i)$ | $x_{i+1}$ | $\lvert x_i - x_{i+1}\rvert$ |
|---|---|---|---|---|---|---|
| 1 | 1 | -2 | 2 | 10 | 0.8 | 0.2 |
| 2 | 0.8 | -2.1504 | -0.352 | 5.68 | 0.861972 | 0.061972 |
| 3 | 0.86197183 | -2.1528497 | 0.285708 | 6.915945 | 0.82066 | 0.041312 |
| 4 | 0.82066031 | -2.1555781 | -0.147875 | 6.0818 | 0.844975 | 0.024314 |
| 5 | 0.84497469 | -2.1561459 | 0.103137 | 6.567787 | 0.829271 | 0.015703 |
| 6 | 0.82927123 | -2.1564755 | -0.060329 | 6.252289 | 0.83892 | 0.009649 |
| 7 | 0.83892031 | -2.1565774 | 0.039526 | 6.445448 | 0.832788 | 0.006132 |
| 8 | 0.83278785 | -2.1566242 | -0.024152 | 6.322427 | 0.836608 | 0.00382 |
| 9 | 0.83660793 | -2.1566409 | 0.015426 | 6.398954 | 0.834197 | 0.002411 |
| 10 | 0.83419717 | -2.1566479 | -0.009585 | 6.350619 | 0.835706 | 0.001509 |
| 11 | 0.83570643 | -2.1566506 | 0.00606 | 6.380863 | 0.834757 | 0.00095 |
| 12 | 0.83475674 | -2.1566517 | -0.00379 | 6.361826 | 0.835352 | 0.000596 |
| 13 | 0.83535244 | -2.1566521 | 0.002386 | 6.373764 | 0.834978 | 0.000374 |
| 14 | 0.83497803 | -2.1566523 | -0.001496 | 6.36626 | 0.835213 | 0.000235 |
| 15 | 0.83521306 | -2.1566523 | 0.000941 | 6.37097 | 0.835065 | 0.000148 |
| 16 | 0.83506541 | -2.1566524 | -0.00059 | 6.368011 | 0.835158 | 9.27E-05 |

12.4-6

a) Claim: $(\bar{x}_{n+1} - \underline{x}_{n+1}) = \frac{1}{2}(\bar{x}_n - \underline{x}_n)$. To see this consider the two cases:

Case 1: $\bar{x}_{n+1} = \bar{x}_n$, $\underline{x}_{n+1} = x_n'$, then $\bar{x}_{n+1} - \underline{x}_{n+1} = \bar{x}_n - x_n' = \bar{x}_n - \frac{1}{2}(\bar{x}_n + \underline{x}_n)$
$= \frac{1}{2}(\bar{x}_n - \underline{x}_n)$

Case 2: $\bar{x}_{n+1} = x_n'$, $\underline{x}_{n+1} = \underline{x}_n$, then $\bar{x}_{n+1} - \underline{x}_{n+1} = x_n' - \underline{x}_n = \frac{1}{2}(\bar{x}_n + \underline{x}_n) - \underline{x}_n$
$= \frac{1}{2}(\bar{x}_n - \underline{x}_n)$

Therefore, we have $(\bar{x}_{n+1} - \underline{x}_{n+1}) = \frac{1}{2}(\bar{x}_n - \underline{x}_n) = \dots = \frac{1}{2^{n+1}}(\bar{x}_0 - \underline{x}_0)$

so $\lim_{n\to\infty}(\bar{x}_{n+1} - \underline{x}_{n+1}) = \lim_{n\to\infty} 2^{-n}(\bar{x}_0 - \underline{x}_0) = 0$

Suppose the points generated by this procedure do not converge, then we know there is $\varepsilon > 0$ such that, no matter what $N$ we choose, there will be $n \geq N$ and $m \geq N$ so that $|x_n' - x_m'| > \varepsilon$. But if we choose $N$ so that $2^{-N}(\bar{x}_0 - \underline{x}_0) < \varepsilon$ then for all $n \geq N$ $x_n' \in [\bar{x}_N, \underline{x}_N]$ and so for $n \geq N$ and $m \geq N$ and $|x_n' - x_m'| > \varepsilon$ we have $\varepsilon < |x_n' - x_m'| \leq |\bar{x}_N - \underline{x}_N| = 2^{-N}(\bar{x}_0 - \underline{x}_0) < \varepsilon$, a contradiction. Thus, the sequence of trial solutions must converge.

(b) Let $\bar{x}$ be the limiting solution. Then we know $f'(x) \geq 0$ for $x < \bar{x}$ and $f'(x) \leq 0$ for $x > \bar{x}$. Suppose $\bar{x}$ is not the global maximum, but that $\hat{x}$ is. Then $f(\hat{x}) > f(\bar{x})$.

Case 1: $\hat{x} > \bar{x}$. By the Mean Value Theorm there is a $z$ with $\hat{x} > z > x$ such that $f(\hat{x}) - f(\bar{x}) = (\hat{x} - \bar{x})f'(z)$. The right hand side is non-positive, since $z > \bar{x}$, $f'(z) \leq 0$ and $\hat{x} > \bar{x} \Rightarrow (\hat{x} - \bar{x}) > 0$, so $f(\hat{x}) - f(\bar{x}) \leq 0$, a contradiction.

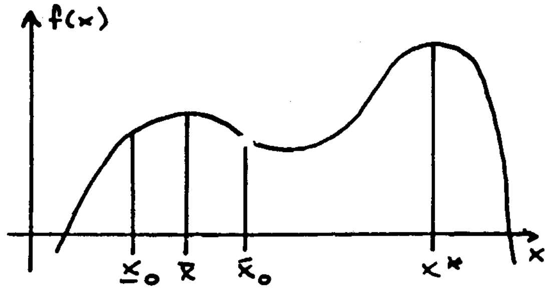
Case 2: $\hat{x} < \bar{x}$. Using the Mean Value Theorm, we have $z$ with $\hat{x} < z < \bar{x}$, $f(\bar{x}) - f(\hat{x}) = (\bar{x} - \hat{x})f'(z) \geq 0$ which implies $f(\bar{x}) \geq f(\hat{x})$, a contradiction.

So $\bar{x}$ must be the global maximum.

12.4-6

(c) The argument follows as in part (b) by observing that the z of the Mean Value Theorem (which is between $\bar{x}$ and $\hat{x}$) is in the part of the domain where f is concave.

(d)




The procedure would converge to $\hat{x}$, not to $x^*$ which is the global maximum.

(e) Suppose $f'(x) < 0$, but that a global maximum occurs at $\hat{x}$. Let $x < \hat{x}$. By the Mean Value Theorem, there exists a z with $\hat{x} > z > x$ and $f(\hat{x}) - f(x) = (\hat{x} - x) f'(z)$, or $f(x) = f(\hat{x}) + (x - \hat{x}) f'(z) > f(x')$, a contradiction. So a global maximumum cannot occur.

Suppose $f'(x) > 0$, but that a global maximum occurs at $\hat{x}$. Let $x > \hat{x}$. By the Mean Value Theorem, there exists a z with $x > z > \hat{x}$ and $f(\hat{x}) - f(x) = (\hat{x} - x) f'(z)$, or $f(x) = f(\hat{x}) + (x - \hat{x}) f'(z) > f(x')$, a contradiction. So a global maximum cannot occur.

(f) Suppose $f(x)$ is concave and $\lim_{x \to -\infty} f'(x) < 0$, but that an $\underline{x}_0$ exists. So $f'(\underline{x}_0) \geq 0$. But $f'(x)$ is monotone decreasing, so for $x < \underline{x}_0$, $f'(x) \geq 0$ and $\lim_{x \to -\infty} f'(x) \geq 0$, a contradiction. So no $\underline{x}_0$ exists.

Suppose $f(x)$ is concave and $\lim_{x \to \infty} f'(x) > 0$, but that an $\bar{x}_0$ exists. So $f'(\underline{x}_0) \leq 0$. But $f'(x)$ is monotone decreasing, so for $x > \bar{x}_0$, $f'(x) \leq 0$, and $\lim_{x \to \infty} f'(x) \leq 0$, a contradiction. So no $\underline{x}_0$ exists.

In either case, we know there is no global maximum from part (e).

12.4-7. $f(x) = f_1(x_1) + f_2(x_2)$

$f_1(x_1) = 32x_1 - x_1^4$

$f_2(x_2) = 50x_2 - 10x_2^2 + x_2^3 - x_2^4$

$\frac{df_1}{dx_1} = 32 - 4x_1^3 \equiv 0 \Rightarrow x_1 = 2$

Using the automatic one-dimensional search procedure ($\varepsilon = 0.001$) with initial bounds 0 and 4, and

$f(x) = 48 + 50x_2 - 10x_2^2 + x_2^3 - x_2^4$

we get : $x_2 = 1.8076$ , $f(x) = 100.936$

$3x_1 + x_2 = 7.8076 < 11$

$2x_1 + 5x_2 = 13.038 < 16$

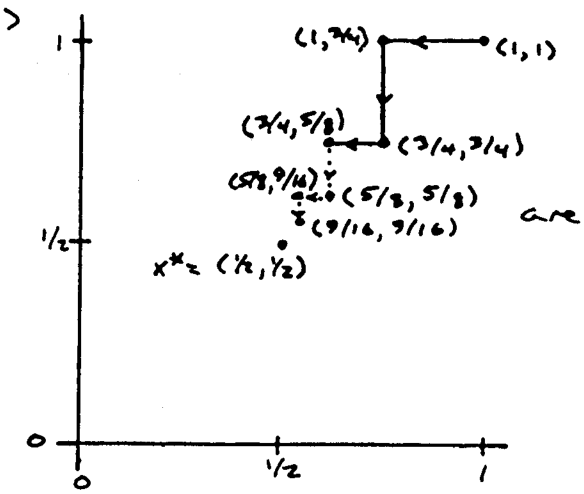
Since the optimum for the unconstrained problem is in the interior of the feasible region for the constrained problem, it is also optimal for the original constrained problem.

12.5-1. (a)

| It. | X' | grad f(X') | x' + t[grad f(X')] | t* | X'+t[grad f] |
|---|---|---|---|---|---|
| 1 | ( 1, 1) | ( 0, -1) | ( 1+ 0t, 1- 1t) | 0.25 | ( 1, 0.75) |
| 2 | ( 1, 0.75) | ( -0.5, 0) | ( 1- 0.5t, 0.75+ 0t) | 0.5 | ( 0.75, 0.75) |
| 3 | ( 0.75, 0.75) | ( 0, -0.5) | ( 0.75+ 0t, 0.75- 0.5t) | 0.25 | ( 0.75,0.625) |
| 4 | ( 0.75,0.625) | (-0.25, 0) | | | |

(b) $-2x_1 + 2x_2 = 0$
$-2x_1 + 4x_2 = 1$ $\Rightarrow$ $x_2 = 1/2$, $x_1 = 1/2$ so $(1/2, 1/2)$ is optimal.

(c)




are the next iterations

(d) The software gives: Solution:

(X1,X2) = (0.508,0.504)

grad f(X1,X2) = (-8e-3, 6e-8)

12.5-2

| It. | X' | grad f(X') | X' + t[grad f(X')] | t* | X'+t[grad f] |
|---|---|---|---|---|---|
| 1 | ( 1, 1) | ( 0, -2) | ( 1+ 0t, 1- 2t) | 0.167 | ( 1,0.667) |
| 2 | ( 1,0.667) | (-1.33, 0) | ( 1-1.33t,0.667+ 0t) | 0.25 | (0.667,0.667) |
| 3 | (0.667,0.667) | ( 0,-1.33) | | | |

The automatic routine ($\varepsilon = 0.01$) gives:

Solution:

$(X_1, X_2) = (0.005, 0.003)$

grad $f(X_1, X_2) = (-7e-3,\ 3e-8)$

$\nabla f = (4x_2 - 4x_1\ ,\ 4x_1 - 6x_2)$

$\nabla f = 0 \Rightarrow (x_1, x_2) = (0, 0)$ is the optimal solution.

## 12.5-3

| It. | X' | grad f(X') | X' + t[grad f(X')] | t* | X'+t[grad f] |
|---|---|---|---|---|---|
| 1 | ( 0, 0) | ( 8, -12) | ( 0+ 8t, 0- 12t) | 0.191 | (1.529, -2.29) |
| 2 | (1.529, -2.29) | (0.361, 0.219) | (1.529+0.36t, -2.29+0.22t) | 1.31 | (2.002, -2) |
| 3 | (2.002, -2) | ( -0, 0.003) | | | |

*Automatic routine ($\varepsilon = 0.01$):*

Solution:

$(X_1, X_2) = (1.997, -2)$

grad $f(X_1, X_2) = (0.002, 0.001)$

$\nabla f = (-2x_1 + 2x_2 + 8,\ 2x_1 - 4x_2 - 12)$

$\nabla f = 0 \Rightarrow (x_1, x_2) = (2, -2)$ is the optimal solution

## 12.5-4

| It. | X' | grad f(X') | X' + t[grad f(X')] | t* | X'+t[grad f] |
|---|---|---|---|---|---|
| 1 | ( 0, 0) | ( 6, -2) | ( 0+ 6t, 0- 2t) | 0.2 | ( 1.2, -0.4) |
| 2 | ( 1.2, -0.4) | ( 0.4, 1.2) | ( 1.2+ 0.4t, -0.4+ 1.2t) | 1 | ( 1.6, 0.8) |
| 3 | ( 1.6, 0.8) | ( 1.2, -0.4) | | | |

*Automatic routine ($\varepsilon = 0.01$)*

Solution:

$(X_1, X_2) = (1.994, 0.989)$

grad $f(X_1, X_2) = (0.003, 0.01)$

$\nabla f = (-4x_1 + 2x_2 + 6,\ 2x_1 - 2x_2 - 2)$

$\nabla f = 0 \Rightarrow (x_1, x_2) = (2, 1)$

## 12.5-5

$\nabla f(x_1, x_2) = (4 - 2x_1 - 4x_1^3 - 2x_2,\ 2 - 2x_1 - 2x_2)$

| Iter. | $x_n$ | $\nabla f(x)$ | $f(x_n + t\nabla f(x_n))$ | Iter. | $t'$ | $f(t)$ |
|---|---|---|---|---|---|---|
| 1 | (0,0) | (4,2) | $20t - 36t^2 - 256t^4$ | 1<br>2 | .5<br>.25<br>$t^* = .125$ | -144<br>-14 |
| (2) | (1/2, 1/4) | | | | | |

$\Rightarrow x + t^* \nabla f(x) = (0.500, 0.250)$ is our estimate.

12.5-6

a) We can rewrite f as $f = f_1(x_1, x_2) + f_2(x_2, x_3)$

with $f_1 = 3x_1x_2 - x_1^2 - 3x_2^2$

$f_2 = 3x_2x_3 - x_3^2 - 3x_2^2$

For any given $x_2$ (including the optimal one) we realize that by symmetry $x_1 = x_3$ at the maximizing point for f with $x_2$ at the given value. Therefore, to maximize f we can maximize $f_1$ (or $f_2$) and obtain $(x_1, x_2)$. From the solution to maximize $f_1$ we can set $x_3 = x_1$ and $f = 2f_1$ to find the solution to our original problem. We can solve maximize $f_1$ with the courseware.

b) Using $f(x) = 3x_1x_2 - x_1^2 - 3x_2^2$ :

| It. | **X**' | grad f(**X**') | **X**' + t[grad f(**X**')] | t* | **X**'+t*[grad f] |
|---|---|---|---|---|---|
| 1 | ( 1, 1) | ( 1, -3) | ( 1+ 1t, 1- 3t) | 0.135 | (1.135,0.595) |
| 2 | (1.135,0.595) | (-0.49,-0.16) | (1.14-0.49t,0.59-0.16t) | 1.616 | (0.343,0.336) |
| 3 | (0.343,0.336) | (0.323,-0.99) | (0.34+0.32t,0.34-0.99t) | 0.135 | (0.387,0.202) |
| 4 | (0.387,0.202) | (-0.17,-0.05) | (0.39-0.17t, 0.2-0.05t) | 1.427 | (0.144,0.131) |
| 5 | (0.144,0.131) | (0.103,-0.35) | (0.14+ 0.1t,0.13-0.35t) | 0.139 | (0.158,0.083) |
| 6 | (0.158,0.083) | (-0.07,-0.02) | (0.16-0.07t,0.08-0.02t) | 1.361 | (0.063,0.056) |
| 7 | (0.063,0.056) | (0.042,-0.15) | (0.06+0.04t,0.06-0.15t) | 0.135 | (0.069,0.036) |

Final solution: $(X_1, X_2)$ = (0.069,0.036)

The estimated solution to the original problem will be $(x_1, x_2, x_3) = (0.069, 0.036, 0.069)$

c) Automatic routine ($\varepsilon = 0.005$) gives:

Solution:

$(X_1, X_2)$ = (0.004,0.002)

grad $f(X_1, X_2)$ = (-2e-3,-6e-4)

12.5-7

| It. | X' | grad f(X') | X' + t[grad f(X')] | t* | X'+t[grad f] |
|---|---|---|---|---|---|
| 1 | ( 0, 0) | ( 0, 3) | ( 0+ 0t, 0+ 3t) | 0.5 | ( 0, 1.5) |
| 2 | ( 0, 1.5) | ( 1.5, 0) | ( 0+ 1.5t, 1.5+ 0t) | 0.5 | ( 0.75, 1.5) |
| 3 | ( 0.75, 1.5) | ( 0, 0.75) | | | |

Automatic routine ($\varepsilon = 0.01$):

Solution:

$(X_1, X_2)$ = (0.996,1.998)

grad $f(X_1, X_2)$ = (0.006,-2e-8)

12.6-1. Converting min to max, the KKT conditions are:

1) $-4x^3-2x+4-u \le 0$
2) $x(-4x^3-2x+4-u)=0$
3) $x-2 \le 0$
4) $u(x-2)=0$
5) $x \ge 0$
6) $u \ge 0$ (Also can get this from (1) and (6))

From 13.4-5. (a), $0 \le x \le 1$, so $x \ne 2$ and (4) $\Rightarrow u=0$.
From (1), we see that $x \ne 0$ ($4 \nleq 0$), so from (2), we have $-4x^3-2x+4=0$ (or $2x^3+x-2=0$)
Solving for the root (real) of this cubic, we get

$$x=\sqrt[3]{\tfrac{1}{2}+\sqrt{\tfrac{55}{216}}}+\sqrt[3]{\tfrac{1}{2}-\sqrt{\tfrac{55}{216}}} = .83512$$

12.6-2. The KKT conditions are.

1(j=1). $1-2ux_1 \le 0$
2(j=1). $x_1(1-2ux_1)=0$
1(j=2). $1-2ux_2 \le 0$
2(j=2). $x_2(1-2ux_2)=0$
3. $x_1^2+x_2^2-1 \le 0$
4. $u(x_1^2+x_2^2-1)=0$
5. $x_1 \ge 0,\ x_2 \ge 0$
6. $u \ge 0$

For $x=\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$, 2(j=1) $\Rightarrow 1-2u\frac{1}{\sqrt{2}}=0 \Rightarrow u=\frac{1}{\sqrt{2}}$
This satisfies all the KKT conditions, so $x=\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$, $u=\frac{1}{\sqrt{2}}$ is optimal.

12.6-3. KKT conditions:

1a) $-4x_1^3-4x_1-2x_2+2u_1+u_2 \le 0$
2a) $x_1(-4x_1^3-4x_1-2x_2+2u_1+u_2)=0$
1b) $-2x_1-8x_2+u_1+2u_2 \le 0$
2b) $x_2(-2x_1-8x_2+u_1+2u_2)=0$
3a) $2x_1+x_2 \ge 10$
3b) $x_1+2x_2 \ge 10$
4a) $u_1(-2x_1-x_2+10)=0$
4b) $u_2(-x_1-2x_2+10)=0$
5) $x_1 \ge 0,\ x_2 \ge 0$
6) $u_1 \ge 0,\ u_2 \ge 0$

for $(x_1,x_2)=(0,10)$ 2b) $\Rightarrow u_1+2u_2=80$
and 4b) $\Rightarrow u_2=0$ so $u_1=80$
but then 1a) does not hold
so $(x_1,x_2)=(0,10)$ is not optimal

12.6-4. (a) the KKT conditions:

1a) $24 - 2x_1 - u_1 \le 0$

2a) $x_1(24 - 2x_1 - u_1) = 0$

1b) $10 - 2x_2 - u_2 \le 0$

2b) $x_2(10 - 2x_2 - u_2) = 0$

3) $x_1 \le 8$, $x_2 \le 7$

4a) $u_1(x_1 - 8) = 0$

4b) $u_2(x_2 - 7) = 0$

5) $x_1 \ge 0$, $x_2 \ge 0$

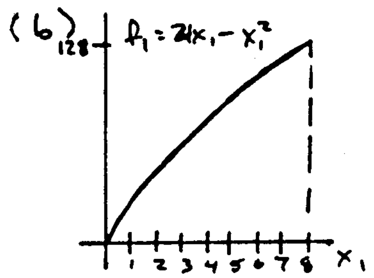
6) $u_1 \ge 0$, $u_2 \ge 0$

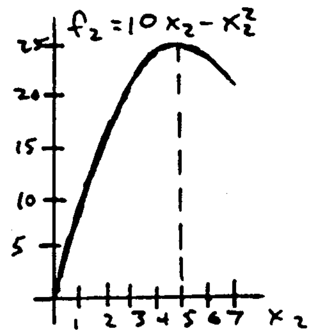
Take $x_1 = 8$ 2a) $\Rightarrow u_1 = 8$
then 1a), 3), 4a), 5) and 6) are satisfied

Take $u_2 = 0$ 2b) $\Rightarrow x_2 = 5$
then 1b), 3), 4b), 5) and 6) are satisfied

So $(x_1, x_2) = (8, 5)$ is optimal since this is a convex program.

(b)




$\frac{\partial f}{\partial x_1} = 24 - 2x_1 > 0$ $\forall\, 0 \le x_1 \le 8 \Rightarrow x_1 = 8$ is the maximum over the feasible region.

$\frac{\partial f}{\partial x_2} = 10 - 2x_2 = 0$ at $x_2 = 5$ and $\frac{\partial^2 f}{\partial x_2} = -2 \le 0$
so $x_2 = 5$ is a global maximum.

12.6-5. (a) $\frac{\partial^2 f}{\partial x_1^2} = -\frac{1}{(x_1+1)^2} \le 0 \quad \forall (x_1,x_2): x_1 \ne -1$

$\frac{\partial^2 f}{\partial x_2^2} = -2 \le 0 \quad \forall (x_1,x_2)$

$\frac{\partial^2 f}{\partial x_1^2}\frac{\partial^2 f}{\partial x_2^2} - \left[\frac{\partial f}{\partial x_1 \partial x_2}\right]^2 = \frac{2}{(x_1+1)^2} \ge 0 \quad \forall (x_1,x_2): x_1 \ne -1$

$\Rightarrow f$ is concave

$g = x_1 + 2x_2 - 3$ is linear, and, hence, convex.

so this is a convex program.

12.6-5 (b) The KKT conditions:

1 a) $\frac{1}{x_1+1} - u \le 0$

2 a) $x_1\left[\frac{1}{x_1+1} - u\right] = 0$

1 b) $-2x_2 - 2u \le 0$

2 b) $x_2[-2x_2 - 2u] \le 0$

3) $x_1 + 2x_2 \le 3$

4) $u(x_1 + 2x_2 - 3) = 0$

5) $x_1 \ge 0;\ x_2 \ge 0$

6) $u \ge 0$

Trying $u \ne 0$ 4) $\Rightarrow$ $x_1 + 2x_2 = 3$

Trying $x_2 = 0 \Rightarrow x_1 = 3$ 2 a) $\Rightarrow u = \frac{1}{4}$

and $(3, 0, \frac{1}{4})$ satisfies 1a), 1b), 2b), 3), 5) & 6)

so $(x_1, x_2) = (3, 0)$ is optimal.

(c) Since $-x_2^2$ is monotonically strictly decreasing for $x_2 \ge 0$ and $\ln(x_1+1)$ is monotonically strictly increasing, it is intuitively clear that one would like to increase $x_1$ as much as possible and decrease $x_2$ toward 0 as much as possible in an optimal solution. The feasible region is "nice" in this respect since if we take F to be the feasible region:

$$\max_{x_1}\left[\min_{x_2} F\right] = \min_{x_2}\left[\max_{x_1} F\right] = \{(3,0)\}.$$

12.6-6 The KKT conditions:

1a) $36 + 18x_1 - 18x_1^2 - u \le 0$

2a) $x_1(36 + 18x_1 - 18x_1^2 - u) = 0$

1b) $36 - 9x_2^2 - u \le 0$

2b) $x_2(36 - 9x_2^2 - u) = 0$

3) $x_1 + x_2 \le 3$

4) $u(x_1 + x_2 - 3) = 0$

5) $x_1 \ge 0,\ x_2 \ge 0$

6) $u \ge 0$

So for $(x_1, x_2) = (1, 2)$

2a) $\Rightarrow$ $u = 36$

2b) $\Rightarrow$ $u = 0$

Thus, the KKT conditions cannot be met at $(x_1, x_2) = (1, 2)$, and so $(1, 2)$ is not optimal.

12.6-7 (a) The KKT conditions:

1a) $\frac{1}{x_2+1} - u \le 0$

2a) $x_1\left[\frac{1}{x_2+1} - u\right] = 0$

1b) $-\frac{x_1}{(x_2+1)^2} + u \le 0$

2b) $x_2\left[-\frac{x_1}{(x_2+1)^2} + u\right] = 0$

3) $x_1 - x_2 \le 2$

4) $u[x_1 - x_2 - 2] = 0$

5) $x_1 \ge 0,\ x_2 \ge 0$

6) $u \ge 0$

for $(x_1, x_2) = (4, 2)$

2a) $\Rightarrow$ $u = 1/3$

2b) $\Rightarrow$ $u = 4/9$

So $(x_1, x_2) = (4, 2)$ does not satisfy the KKT condition, and, hence, is not optimal.

(b) Try $x_2 = 0$, $u \ne 0$ 2a) $\Rightarrow$ $u = 1$
4) $\Rightarrow$ $x_1 = 2$

and $(2, 0, 1)$ satisfies 1a), 1b), 2a), 3), 5) & 6)
so $(x_1, x_2) = (2, 0)$ satisfies the KKT conditions

(c) $\frac{\partial^2 f}{\partial x_1^2} = 0 \quad \forall (x_1, x_2)$

$\frac{\partial^2 f}{\partial x_2^2} = \frac{2x_1}{(x_2+1)^3} \ge 0 \quad \forall x_1 \ge 0, x_2 \ge 0$

Thus, f cannot be concave, and so this is not a convex program.

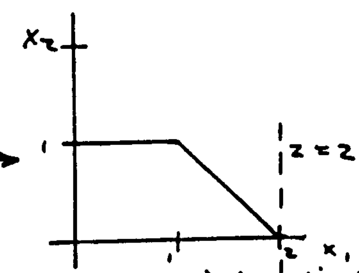
12.6-7. (d) $f(x)$ is a monotone strictly increasing function of $x_2$ and is monotone strictly decreasing in $x_2$ for $x_2 > -1$, and, thus, any optimal solution in a bounded feasible region with $x_2 > -1$ will have $x_1$ increased as high as possible and $x_2$ decreased toward $-1$ as much as possible. The feasible region here allows $x_1$ to be increased without bound. However, then $x_2$ can only be decreased to the line $x_1 - x_2 = 2$ or $x_1 = 2 + x_2$:

$$f(x_2+2, x_2) = \frac{x_2+2}{x_2+1} \longrightarrow 1 \text{ as } x_2 \to \infty$$
$$= 2 \text{ at } x_2 = 0$$

Conversely, if we then decrease $x_2$ to 0 we can increase $x_1$ to $x_1 = 2$. Thus, the feasible region is "nice" and the optimal solution is at $(x_1, x_2) = (2, 0)$.

(e) Maximize $x_1$
subject to $x_1 - x_2 - 2t \le 0$
$x_2 + t = 1$
$x_1 \ge 0,\ x_2 \ge 0,\ t \ge 0$

$\Leftrightarrow$ Maximize $x_1$
subject to $x_1 + x_2 \le 2$
$x_2 \le 1$
$x_1 \ge 0,\ x_2 \ge 0$




$(x_1, x_2) = (2, 0)$ is optimal

12.6-8. (a) The KKT conditions:

1a) $1 - u \le 0$
2a) $x_1(1 - u) = 0$
1b) $2 - 3x_2^2 - u \le 0$
2b) $x_2(2 - 3x_2^2 - u) = 0$
3) $x_1 + x_2 \le 1$
4) $u(x_1 + x_2 - 1) = 0$
5) $x_1 \ge 0,\ x_2 \ge 0$
6) $u \ge 0$

It is easy to check that $(x_1, x_2, u) = \left(1 - \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, 1\right)$ satisfies the KKT conditions. Thus, $(x_1, x_2) = \left(1 - \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right)$ is optimal since this is a convex program.

12.6-8. (b) The KKT conditions:

1a) $20 - u_1 2x_1 - u_2 \le 0$

2a) $x_1(20 - u_1 2x_1 - u_2) = 0$

1b) $10 - u_1 2x_2 - 2u_2 \le 0$

2b) $x_2(10 - u_1 2x_2 - 2u_2) = 0$

3a) $x_1^2 + x_2^2 \le 1$

b) $x_1 + 2x_2 \le 2$

4a) $u_1(x_1^2 + x_2^2 - 1) = 0$

b) $u_2(x_1 + 2x_2 - 2) = 0$

5) $x_1 \ge 0,\ x_2 \ge 0$

6) $u_1 \ge 0,\ u_2 \ge 0$

It is easy to check that $(x_1, x_2, u_1, u_2) = \left(\frac{2\sqrt{5}}{5}, \frac{\sqrt{5}}{5}, 5\sqrt{5}, 0\right)$ satisfies the KKT conditions, and so since we have a convex program again $(x_1, x_2) = \left(\frac{2\sqrt{5}}{5}, \frac{\sqrt{5}}{5}\right)$ is optimal.

12.6-9.

Minimize $f(\underline{x})$ subject to $g_i(\underline{x}) \ge b_i$, $\underline{x} \ge 0$ $\iff$ Maximize $-f(\underline{x})$ subject to $-g_i(\underline{x}) \le -b_i$, $\underline{x} \ge 0$

has KKT conditions:

1) $\sum_{i=1}^{m} u_i \frac{\partial g_i}{\partial x_j} - \frac{\partial f}{\partial x_j} \le 0$

2) $x_j^* \left( \sum_{i=1}^{m} u_i \frac{\partial g_i}{\partial x_j} - \frac{\partial f}{\partial x_j} \right) = 0$

(1, 2) at $x_j = x_j^*$ for $j = 1, \ldots, n$

3) $g_i(x^*) \ge b_i$

4) $u_i(b_i - g_i(x^*)) = 0$

(3, 4) for $i = 1, \ldots, m$

5) $x_j^* \ge 0 \quad j = 1, \ldots, n$

6) $u_i \ge 0 \quad i = 1, \ldots, m$

12.6-10

(a) An equivalent nonlinear programming problem is:

$$\text{Maximize } Z = -2x_1^2 - x_2^2$$

subject to:

$$x_1 + x_2 \le 10$$

$$-x_1 - x_2 \le -10 \qquad x_1 \ge 0 \quad x_2 \ge 0.$$

This nonlinear program fits the following types:

- Linearly Constrained Optimization Problem, because all constraints are linear.
- Quadratic Programming Problem, because all constraints are linear and the objective only involves the squares of variables.
- Convex Programming Problem, because the objective is concave as shown below and the constraints are linear and, therefore, convex

  Using the test in the appendix $\frac{\partial^2 f}{\partial x_1^2}\frac{\partial^2 f}{\partial x_2^2} - \left[\frac{\partial^2 f}{\partial x_1 \partial x_2}\right]^2 = (-4)(-2) - 0 = 8 \ge 0$

  so $f$ is concave
- Geometric Programming Problem, because the first constraint can be written as $g_1(x_1, x_2) = c_1 P_1(x_1, x_2) + c_2 P_2(x_1, x_2)$ with $c_1 = 1 = c_2$ $P_1 = x_1$ and $P_2 = x_2$. Similarly for the second constraint and the objective function.
- Fractional Programming Problem, because $f(x_1, x_2) = -2x_1^2 - x_2^2 = \frac{f_1(x)}{f_2(x)}$ with $f_1(x) = -2x_1^2 - x_2^2$ and $f_2(x) = 1$.

b)

The KKT conditions are:

1(a) $-4x_1 - u_1 + u_2 \le 0$

(b) $-2x_2 - u_1 + u_2 \le 0$

2(a) $x_1(-4x_1 - u_1 + u_2) = 0$

(b) $x_2(-2x_2 - u_1 + u_2) = 0$

3(a) $x_1 + x_2 - 10 \le 0$

(b) $-x_1 - x_2 + 10 \le 0$

4(a) $u_1(x_1 + x_2 - 10) = 0$

(b) $u_2(-x_1 - x_2 + 10) = 0$

5 $x_1 \ge 0$, $x_2 \ge 0$

6 $u_1 \ge 0$, $u_2 \ge 0$

c) From 3(a) + (b), $x_1 + x_2 = 10$, so 4(a) + (b) are automatically satisfied. Let us try $x_1, x_2 \ne 0$. Then 2(a) + (b) give

$$-4x_1 - u_1 + u_2 = 0 = -2x_2 - u_1 + u_2$$

$$\Rightarrow x_2 = 2x_1$$

$$\Rightarrow x_1 + 2x_1 = 10 \Rightarrow x_1 = \frac{10}{3},\ x_2 = \frac{20}{3}$$

Now 2(a) gives $-u_1 + u_2 = \frac{40}{3}$

$u_1 = 0$, $u_2 = \frac{40}{3}$ satisfies this

(actually $u_1 = c \ge 0$, $u_2 = \frac{40}{3} + c$ works)

$(x_1, x_2) = (\frac{10}{3}, \frac{20}{3})$, $(u_1, u_2) = (0, \frac{40}{3})$ satisfies all KKT conditions, so $(x_1, x_2) = (\frac{10}{3}, \frac{20}{3})$ is optimal.

12.6-11. (a) An equivalent nonlinear programming problem is:

Maximize $f(y) = -(y_1+1)^3 - 4(y_2+1)^2 - 16(y_3+1)$

subject to: $y_1 + y_2 + y_3 \le 2$

$-y_1 - y_2 - y_3 \le -2$

$y_1 \ge 0 \quad y_2 \ge 0 \quad y_3 \ge 0$

(b) The KKT Conditions are:

1 (a) $-3(y_1+1)^2 - u_1 + u_2 \le 0$

(b) $-8(y_2+1) - u_1 + u_2 \le 0$

(c) $-16 - u_1 + u_2 \le 0$

2 (a) $y_1(-3(y_1+1)^2 - u_1 + u_2) = 0$

(b) $y_2(-8(y_2+1)^2 - u_1 + u_2) = 0$

(c) $y_3(-16 - u_1 + u_2) = 0$

3 (a) $y_1 + y_2 + y_3 - 2 \le 0$

(b) $-y_1 - y_2 - y_3 + 2 \le 0$

4 (a) $u_1(y_1 + y_2 + y_3 - 2) = 0$

(b) $u_2(-y_1 - y_2 - y_3 + 2) = 0$

5 $y_1 \ge 0 \quad y_2 \ge 0 \quad y_3 \ge 0$

6 $u_1 \ge 0 \quad u_2 \ge 0$.

(c) For $x = (2,1,2)$ we have $y = (1,0,1)$

2 (a) implies $-3(2^2) - u_1 + u_2 = 0$ or $-u_1 + u_2 = 12$

2 (c) implies $-16 - u_1 + u_2 = 0$ or $-u_1 + u_2 = 16$

The KKT Conditions are not satisfied by $x = (2,1,2)$

12.6-12 (a) The KKT conditions:

1 a) $6 - 2x_1 - u \le 0$

2 a) $x_1(6 - 2x_1 - u) = 0$

1 b) $3 - 3x_2^2 - u \le 0$

2 b) $x_2(3 - 3x_2^2 - u) = 0$

3) $x_1 + x_2 \le 1$

4) $u(x_1 + x_2 - 1) = 0$

5) $x_1 \ge 0,\ x_2 \ge 0$

6) $u \ge 0$.

(b) for $(x_1, x_2) = (1/2, 1/2)$

2 a) $\Rightarrow u = 5$

2 b) $\Rightarrow u = 9/4$

so $(x_1, x_2) = (1/2, 1/2)$ is not optimal.

(c) It is easy to check that $(x_1, x_2, u) = (1, 0, 4)$ satisfies the KKT condition, and once again we have a convex program so $(x_1, x_2) = (1, 0)$ is optimal.

12.6-13 (a) The KKT conditions:

1a) $8 - 2x_1 - u \le 0$
2a) $x_1(8 - 2x_1 - u) = 0$
1b) $2 - 3u \le 0$
2b) $x_2(2 - 3u) = 0$
1c) $1 - 2u \le 0$
2c) $x_3(1 - 2u) = 0$
3) $x_1 + 3x_2 + 2x_3 \le 12$
4) $u(x_1 + 3x_2 + 2x_3 - 12) = 0$
5) $x_1 \ge 0, x_2 \ge 0, x_3 \ge 0$
6) $u \ge 0$

for $(x_1, x_2, x_3) = (2, 2, 2)$ 2a) $\Rightarrow$ $u = 4$
2b) $\Rightarrow$ $u = 2/3$
2c) $\Rightarrow$ $u = 1/2$

so $(2, 2, 2)$ cannot be an optimal solution.

(b) It is easy to check that $(x_1, x_2, x_3, u) = (11/3, 25/9, 0, 2/3)$ satisfies the KKT conditions, and so $(x_1, x_2, x_3) = (11/3, 25/9, 0)$ is optimal since the program is convex.

12.6-14 The KKT conditions:

1a) $-2 - u(-2x_1) \le 0$
2a) $x_1(-2 + u2x_1) = 0$
1b) $-3x_2^2 - u(-4x_2) \le 0$
2b) $x_2(-3x_2^2 + u4x_2) = 0$
1c) $-2x_3 - u(-2x_3) \le 0$
2c) $x_3(-2x_3 + u2x_3) = 0$
3) $x_1^2 + 2x_2^2 + x_3^2 \ge 4$
4) $u(4 - x_1^2 - 2x_2^2 - x_3^2) = 0$
5) $x_1 \ge 0, x_2 \ge 0, x_3 \ge 0$
6) $u \ge 0$

For $(x_1, x_2, x_3) = (1, 1, 1)$ 2a) $\Rightarrow$ $u = 1$
2b) $\Rightarrow$ $u = 3/4$
2c) $\Rightarrow$ $u = 1$

so $(x_1, x_2, x_3) = (1, 1, 1)$ cannot be optimal

12.6-15. KKT conditions:

1a) $-4x_1^3 + 2x_1u \le 0$
2a) $x_1(-4x_1^3 + 2x_1u) = 0$
1b) $-4x_2 + 2x_2u \le 0$
2b) $x_2(-4x_2 + 2x_2u) = 0$
3) $-x_1^2 - x_2^2 + 2 \le 0$
4) $u(-x_1^2 - x_2^2 + 2) = 0$
5) $x_1 \ge 0, x_2 \ge 0$
6) $u \ge 0$

If $x_1 = x_2 = 1$, 2a) $\Rightarrow$ $-4 + 2u = 0$ or $u = 2$

$(x_1, x_2) = (1, 1)$, $u = 2$ satisfies all KKT conditions, so $(1, 1)$ is optimal.

12.6-16 The KKT conditions:

1a) $32 - 4x_1^3 - 3u_1 - 2u_2 \le 0$

2a) $x_1(32 - 4x_1^3 - 3u_1 - 2u_2) = 0$

1b) $50 - 20x_2 + 3x_2^2 - 4x_2^3 - u_1 - 5u_2 \le 0$

2b) $x_2(50 - 20x_2 + 3x_2^2 - 4x_2^3 - u_1 - 5u_2) = 0$

3a) $3x_1 + x_2 \le 11$

4a) $u_1(3x_1 + x_2 - 11) = 0$

3b) $2x_1 + 5x_2 \le 16$

4b) $u_2(2x_1 + 5x_2 - 16) = 0$

5) $x_1 \ge 0,\ x_2 \ge 0$

6) $u_1 \ge 0,\ u_2 \ge 0$

for $(x_1, x_2) = (2, 2)$ 2a) & 6) $\Rightarrow u_1 = u_2 = 0$
2b) $\Rightarrow u_1 + 5u_2 = -10$

$\Rightarrow (x_1, x_2) = (2, 2)$ is not optimal.

12.7-1. (a) $\frac{\partial^2 f}{\partial x_1^2} = -4 < 0 \quad \forall (x_1, x_2)$

$\frac{\partial^2 f}{\partial x_2^2} = -8 < 0 \quad \forall (x_1, x_2)$

$\frac{\partial^2 f}{\partial x_1^2}\frac{\partial^2 f}{\partial x_2^2} - \left[\frac{\partial^2 f}{\partial x_1 \partial x_2}\right]^2 = 16 > 0 \quad \forall (x_1, x_2)$

$\Rightarrow f$ is strictly concave.

(b) $x^T Q x = 4x_1^2 - 8x_1x_2 + 8x_2^2 = 4(x_1 - x_2)^2 + 4x_2^2 > 0$
$\forall (x_1, x_2) \ne (0, 0)$
$\Rightarrow Q$ is positive definite.

(c) The KKT conditions:

1a) $15 + 4x_2 - 4x_1 - u \le 0$

2a) $x_1(15 + 4x_2 - 4x_1 - u) = 0$

1b) $30 + 4x_1 - 8x_2 - 2u \le 0$

2b) $x_2(30 + 4x_1 - 8x_2 - 2u) = 0$

3) $x_1 + 2x_2 \le 30$

4) $u(x_1 + 2x_2 - 30) = 0$

5) $x_1 \ge 0,\ x_2 \ge 0$

6) $u \ge 0$

It is easy to verify that $(x_1, x_2, u) = (12, 9, 3)$ satisfies these conditions.

12.7-2. (a) The KKT conditions:

1 a) $8 - 2x_1 - u \le 0$
2 a) $x_1(8 - 2x_1 - u) = 0$
1 b) $4 - 2x_2 - u \le 0$
2 b) $x_2(4 - 2x_2 - u) = 0$
3) $x_1 + x_2 \le 2$
4) $u(x_1 + x_2 - 2) = 0$
5) $x_1 \ge 0, \; x_2 \ge 0$
6) $u \ge 0$

It is easy to verify that $(x_1, x_2, u) = (2, 0, 4)$ satisfies the above conditions; hence, since this is a convex program $(x_1, x_2) = (2, 0)$ is optimal.

(b) The objective function in vector notation is:

Maximize $(8, 4)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \frac{1}{2}(x_1, x_2)\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

Hence, the equivalent problem is:

Minimize $z_1 + z_2$

subject to

$$\begin{aligned} 2x_1 - y_1 + y_3 + z_1 &= 8 \\ 2x_2 - y_2 + y_3 + z_2 &= 4 \\ x_1 + x_2 + x_3 &= 2 \end{aligned}$$

$x_1 \ge 0, \; x_2 \ge 0, \; x_3 \ge 0$
$y_1 \ge 0, \; y_2 \ge 0, \; y_3 \ge 0$
$z_1 \ge 0, \; z_2 \ge 0$

The complementarity constraint is: $x_1 y_1 + x_2 y_2 + x_3 y_3 = 0$

(c)

(0)

| Bas Var | Eq No | Z | X1 | X2 | X3 | X4 | X5 | X6 | X7 | X8 | Right side |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Z | 0 | 1 | -2 | -2 | -2 | 1 | 1 | 0 | 0 | 0 | -12 |
| X6 | 1 | 0 | 2 | 0 | 1 | -1 | 0 | 1 | 0 | 0 | 8 |
| X7 | 2 | 0 | 0 | 2 | 1 | 0 | -1 | 0 | 1 | 0 | 4 |
| X8 | 3 | 0 | 1* | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 2 |

(1)

| Bas Var | Eq No | Z | X1 | X2 | X3 | X4 | X5 | X6 | X7 | X8 | Right side |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Z | 0 | 1 | 0 | 0 | -2 | 1 | 1 | 0 | 0 | 2 | -8 |
| X6 | 1 | 0 | 0 | -2 | 1* | -1 | 0 | 1 | 0 | -2 | 4 |
| X7 | 2 | 0 | 0 | 2 | 1 | 0 | -1 | 0 | 1 | 0 | 4 |
| X1 | 3 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 2 |

12.7-2. (c) (cont.)

(2)

| Bas Var | Eq No | Z | X1 | X2 | X3 | X4 | X5 | X6 | X7 | X8 | Right side |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Z | 0 | 1 | 0 | -4 | 0 | -1 | 1 | 2 | 0 | -2 | 0 |
| X3 | 1 | 0 | 0 | -2 | 1 | -1 | 0 | 1 | 0 | -2 | 4 |
| X7 | 2 | 0 | 0 | 4* | 0 | 1 | -1 | -1 | 1 | 2 | 0 |
| X1 | 3 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 2 |

(3)

| Bas Var | Eq No | Z | X1 | X2 | X3 | X4 | X5 | X6 | X7 | X8 | Right side |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Z | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 |
| X3 | 1 | 0 | 0 | 0 | 1 | -0.5 | -0.5 | 0.5 | 0.5 | -1 | 4 |
| X2 | 2 | 0 | 0 | 1 | 0 | 0.25 | -0.25 | -0.25 | 0.25 | 0.5 | 0 |
| X1 | 3 | 0 | 1 | 0 | 0 | -0.25 | 0.25 | 0.25 | -0.25 | 0.5 | 2 |

This provides the optimal solution $x_1 = 2$, $x_2 = 0$, $u = 4$.

d) Using the Excel Solver we get $X_1 = 2$, $X_2 = 0$

12.7-3. (a) The objective function in vector notation is:

Maximize $(20, 50)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \frac{1}{2}(x_1, x_2)\begin{pmatrix} 40 & -20 \\ -20 & 10 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

Hence, the equivalent problem is:

Minimize $z_1 + z_2$

subject to

$$40x_1 - 20x_2 - y_1 + y_3 + y_4 + z_1 = 20$$
$$-20x_1 + 10x_2 - y_2 + y_3 + 4y_4 + z_2 = 50$$
$$x_1 + x_2 + x_3 = 6$$
$$x_1 + 4x_2 + x_4 = 18$$
$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0, y_1 \geq 0, y_2 \geq 0, y_3 \geq 0, y_4 \geq 0, z_1 \geq 0, z_2 \geq 0$$

The enforced complementarity constraint is:

$$x_1 y_1 + x_2 y_2 + x_3 y_3 + x_4 y_4 = 0$$

12.7-3 (b)

(0)

| Bas Var | Eq No | Z | X1 | X2 | X3 | X4 | X5 | X6 | X7 | X8 | X9 | X10 | Right side |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Z | 0 | 1 | -20 | 10 | -2 | -5 | 1 | 1 | 0 | 0 | 0 | 0 | -70 |
| X7 | 1 | 0 | 40* | -20 | 1 | 1 | -1 | 0 | 1 | 0 | 0 | 0 | 20 |
| X8 | 2 | 0 | -20 | 10 | 1 | 4 | 0 | -1 | 0 | 1 | 0 | 0 | 50 |
| X9 | 3 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 6 |
| Xe | 4 | 0 | 1 | 4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 18 |

(1)

| Bas Var | Eq No | Z | X1 | X2 | X3 | X4 | X5 | X6 | X7 | X8 | X9 | X10 | Right side |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Z | 0 | 1 | 0 | 0 | -1.5 | -4.5 | 0.5 | 1 | 0.5 | 0 | 0 | 0 | -60 |
| X1 | 1 | 0 | 1 | -0.5 | 0.025 | 0.025 | -0.03 | 0 | 0.025 | 0 | 0 | 0 | 0.5 |
| X8 | 2 | 0 | 0 | 0 | 1.5 | 4.5 | -0.5 | -1 | 0.5 | 1 | 0 | 0 | 60 |
| X9 | 3 | 0 | 0 | 1.5* | -0.03 | -0.03 | 0.025 | 0 | -0.03 | 0 | 1 | 0 | 5.5 |
| Xe | 4 | 0 | 0 | 4.5 | -0.03 | -0.03 | 0.025 | 0 | -0.03 | 0 | 0 | 1 | 17.5 |

12.7-3. (b) (cont.)

(2)

| Bas Var | Eq No | Z | X1 | X2 | X3 | X4 | X5 | X6 | X7 | X8 | X9 | X10 | Right side |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Z | 0 | 1 | 0 | 0 | -1.5 | -4.5 | 0.5 | 1 | 0.5 | 0 | 0 | 0 | -60 |
| X1 | 1 | 0 | 1 | 0 | 0.017 | 0.017 | -0.02 | 0 | 0.017 | 0 | 0.333 | 0 | 2.333 |
| X8 | 2 | 0 | 0 | 0 | 1.5 | 4.5 | -0.5 | -1 | 0.5 | 1 | 0 | 0 | 60 |
| X2 | 3 | 0 | 0 | 1 | -0.02 | -0.02 | 0.017 | 0 | -0.02 | 0 | 0.667 | 0 | 3.667 |
| Xe | 4 | 0 | 0 | 0 | 0.05* | 0.05 | -0.05 | 0 | 0.05 | 0 | -3 | 1 | 1 |

(3)

| Bas Var | Eq No | Z | X1 | X2 | X3 | X4 | X5 | X6 | X7 | X8 | X9 | X10 | Right side |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Z | 0 | 1 | 0 | 0 | 0 | -3 | -1 | 1 | 2 | 0 | -90 | 30 | -30 |
| X1 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1.353 | -0.34 | 1.993 |
| X8 | 2 | 0 | 0 | 0 | 0 | 3* | 1 | -1 | -1 | 1 | 90 | -30 | 30 |
| X2 | 3 | 0 | 0 | 1 | 0.003 | 0.003 | -0 | 0 | 0.003 | 0 | -0.53 | 0.4 | 4.067 |
| X3 | 4 | 0 | 0 | 0 | 1 | 1 | -1 | 0 | 1 | 0 | -60 | 20 | 20 |

(4)

| Bas Var | Eq No | Z | X1 | X2 | X3 | X4 | X5 | X6 | X7 | X8 | X9 | X10 | Right side |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Z | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 |
| X1 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1.353 | -0.34 | 1.993 |
| X4 | 2 | 0 | 0 | 0 | 0 | 1 | 0.333 | -0.33 | -0.33 | 0.333 | 30 | -10 | 10 |
| X2 | 3 | 0 | 0 | 1 | 0.003 | 0 | -0 | 0.001 | 0.004 | -0 | -0.63 | 0.433 | 4.033 |
| X3 | 4 | 0 | 0 | 0 | 1 | 0 | -1.33 | 0.333 | 1.333 | -0.33 | -90 | 30 | 10 |

This provides the optimal solution $(x_1, x_2) = (1.993, 4.033)$
$(u_1, u_2) = (10, 10)$

12.7-4. a) The KKT conditions:

1a) $2 - 2x_1 - u \le 0$

2a) $x_1(2 - 2x_1 - u) = 0$

1b) $4 - 3x_2 - u \le 0$

2b) $x_2(4 - 3x_2 - u) = 0$

3) $x_1 + x_2 \le 2$

4) $u(x_1 + x_2 - 2) = 0$

5) $x_1 \ge 0, x_2 \ge 0$

6) $u \ge 0$

If you plot the points obtained in (a) it is pretty clear that you are converging to a boundary point solution. So $x_1, x_2, u \neq 0$ $\Rightarrow (x_1, x_2, u) = (.8, 1.2, .4)$ satisfies the above conditions. So $(x_1, x_2) = (.8, 1.2)$ is optimal.

12.7-4. b) Minimize $z_1 + z_2$

subject to

$$2x_1 - y_1 + y_3 + z_1 = 2$$
$$3x_2 - y_2 + y_3 + z_2 = 4$$
$$x_1 + x_2 + x_3 = 2$$
$$x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0,\ y_1 \ge 0,\ y_2 \ge 0,\ y_3 \ge 0,\ z_1 \ge 0,\ z_2 \ge 0$$

The enforced complementarity constraint is:

$$x_1 y_1 + x_2 y_2 + x_3 y_3 = 0$$

c) for $x_1 = .8$, $x_2 = 1.2$, $U = y_3 = .4$

the first constraint becomes $-y_1 + z_1 = 0$

the second constraint becomes $-y_2 + z_2 = 0$

the third constraint becomes $x_3 = 0$

and, hence, the complementarity constraint becomes $.8y_1 + 1.2y_2 = 0$, but $y_1 \ge 0$ & $y_2 \ge 0$

$\Rightarrow y_1 = 0,\ y_2 = 0$

So $z_1 = z_2 = 0 \Rightarrow Z = 0$

Thus, $(.8, 1.2) = (x_1, x_2)$ is optimal

d)

(0)

| Bas Var | Eq No | Z | X1 | X2 | X3 | X4 | X5 | X6 | X7 | X8 | Right side |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Z | 0 | 1 | -2 | -3 | -2 | 1 | 1 | 0 | 0 | 0 | -6 |
| X6 | 1 | 0 | 2 | 0 | 1 | -1 | 0 | 1 | 0 | 0 | 2 |
| X7 | 2 | 0 | 0 | 3* | 1 | 0 | -1 | 0 | 1 | 0 | 4 |
| X8 | 3 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 2 |

(1)

| Bas Var | Eq No | Z | X1 | X2 | X3 | X4 | X5 | X6 | X7 | X8 | Right side |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Z | 0 | 1 | -2 | 0 | -1 | 1 | 0 | 0 | 1 | 0 | -2 |
| X6 | 1 | 0 | 2 | 0 | 1 | -1 | 0 | 1 | 0 | 0 | 2 |
| X2 | 2 | 0 | 0 | 1 | 0.333 | 0 | -0.33 | 0 | 0.333 | 0 | 1.333 |
| X8 | 3 | 0 | 1* | 0 | -0.33 | 0 | 0.333 | 0 | -0.33 | 1 | 0.667 |

(2)

| Bas Var | Eq No | Z | X1 | X2 | X3 | X4 | X5 | X6 | X7 | X8 | Right side |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Z | 0 | 1 | 0 | 0 | -1.67 | 1 | 0.667 | 0 | 0.333 | 2 | -0.67 |
| X6 | 1 | 0 | 0 | 0 | 1.667* | -1 | -0.67 | 1 | 0.667 | -2 | 0.667 |
| X2 | 2 | 0 | 0 | 1 | 0.333 | 0 | -0.33 | 0 | 0.333 | 0 | 1.333 |
| X1 | 3 | 0 | 1 | 0 | -0.33 | 0 | 0.333 | 0 | -0.33 | 1 | 0.667 |

12.7-4. d) (cont')

| | Bas Var | Eq No | Z | X1 | X2 | X3 | X4 | X5 | X6 | X7 | X8 | Right side |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | Z | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 |
| (3) | X3 | 1 | 0 | 0 | 0 | 1 | -0.6 | -0.4 | 0.6 | 0.4 | -1.2 | 0.4 |
| | X2 | 2 | 0 | 0 | 1 | 0 | 0.2 | -0.2 | -0.2 | 0.2 | 0.4 | 1.2 |
| | X1 | 3 | 0 | 1 | 0 | 0 | -0.2 | 0.2 | 0.2 | -0.2 | 0.6 | 0.8 |

This provides the optimal solution $(x_1, x_2) - (0.8, 1.2)$

$u = 0.4$

e) Using the Excel Solver we get $x_1 = 0.8$, $x_2 = 1.2$

12.7-5. a) The KKT conditions:

1a) $126 - 18x_1 - u_1 - 3u_3 \le 0$

2a) $x_1(126 - 18x_1 - u_1 - 3u_3) = 0$

1b) $182 - 26x_2 - 2u_2 - 2u_3 \le 0$

2b) $x_2(182 - 26x_2 - 2u_2 - 2u_3) = 0$

3a) $x_1 \le 4$

4a) $u_1(x_1 - 4) = 0$

3b) $2x_2 \le 12$

4b) $u_2(2x_2 - 12) = 0$

3c) $3x_1 + 2x_2 \le 18$

4c) $u_3(3x_1 + 2x_2 - 18) = 0$

It is easy to verify that $(x_1, x_2, u_1, u_2, u_3) = (8/5, 5, 0, 0, 26)$ satisfies the above conditions, and, hence, $(x_1, x_2) = (8/3, 5)$ is optimal.

b) Minimize $z_1 + z_2$

subject to

$$18x_1 - y_1 + y_3 + 3y_5 + z_1 = 126$$
$$26x_2 - y_2 + 2y_4 + 2y_5 + z_2 = 182$$
$$x_1 + x_3 = 4$$
$$2x_2 + x_4 = 12$$
$$3x_1 + 2x_2 + x_5 = 18$$

$x_i \ge 0 \quad i = 1, \ldots, 5 \quad y_i \ge 0 \quad i = 1, \ldots, 5 \quad z_1 \ge 0, z_2 \ge 0$

12.7-5 c)

for $(8/3, 5, 26) = (x_1, x_2, u_3) = (x_1, x_2, y_5)$
the third constraint $\Rightarrow x_3 = 1/3$
the fourth constraint $\Rightarrow x_4 = 2$
the fifth constraint $\Rightarrow x_5 = 0$
the first constraint $\Rightarrow y_3 - y_1 + z_1 = 0$
the second constraint $\Rightarrow 2y_4 - y_2 + z_2 = 0$
the complementary constraint $\Rightarrow 8/3 y_1 + 5y_2 + 1/3 y_3 + 2y_4 = 0$
so non-negativity $\Rightarrow y_1 = y_2 = y_3 = y_4 = 0$
$\Rightarrow z_1 = z_2 = 0$
$\Rightarrow (8/3, 5) = (x_1, x_2)$ is optimal

12.7-6 a) & b)

Minimum acceptable expected return = 13

| Factor | Amount Per Block Stock 1 | Stock 2 | Totals | Right-Hand Side |
|---|---|---|---|---|
| Budget | 20 | 30 | 50 | 50 |
| Expected Return | 5 | 10 | 13.00 | 13 |
| Risk | 4 | 100 | 25.56 | |
| Solution | 2.2 | 0.2 | | |

| Joint Risk | Stock 1 | Stock 2 |
|---|---|---|
| Stock 1 | | 5 |
| Stock 2 | | |

Minimum acceptable expected return = 14

| Factor | Amount Per Block Stock 1 | Stock 2 | Totals | Right-Hand Side |
|---|---|---|---|---|
| Budget | 20 | 30 | 50 | 50 |
| Expected Return | 5 | 10 | 14 00 | 14 |
| Risk | 4 | 100 | 51.04 | |
| Solution | 1.6 | 0.6 | | |

| Joint Risk | Stock 1 | Stock 2 |
|---|---|---|
| Stock 1 | | 5 |
| Stock 2 | | |

Minimum acceptable expected return = 15

| Factor | Amount Per Block Stock 1 | Stock 2 | Totals | Right-Hand Side |
|---|---|---|---|---|
| Budget | 20 | 30 | 50 | 50 |
| Expected Return | 5 | 10 | 15.00 | 15 |
| Risk | 4 | 100 | 109.00 | |
| Solution | 1.0 | 1.0 | | |

| Joint Risk | Stock 1 | Stock 2 |
|---|---|---|
| Stock 1 | | 5 |
| Stock 2 | | |

Minimum acceptable expected return = 16

| Factor | Amount Per Block Stock 1 | Stock 2 | Totals | Right-Hand Side |
|---|---|---|---|---|
| Budget | 20 | 30 | 50 | 50 |
| Expected Return | 5 | 10 | 16.00 | 16 |
| Risk | 4 | 100 | 199.44 | |
| Solution | 0.4 | 1.4 | | |

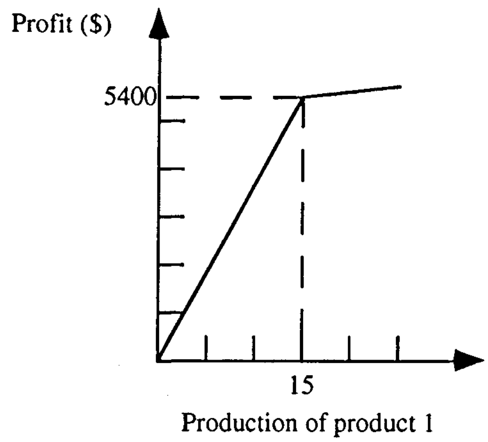
| Joint Risk | Stock 1 | Stock 2 |
|---|---|---|
| Stock 1 | | 5 |
| Stock 2 | | |

c)

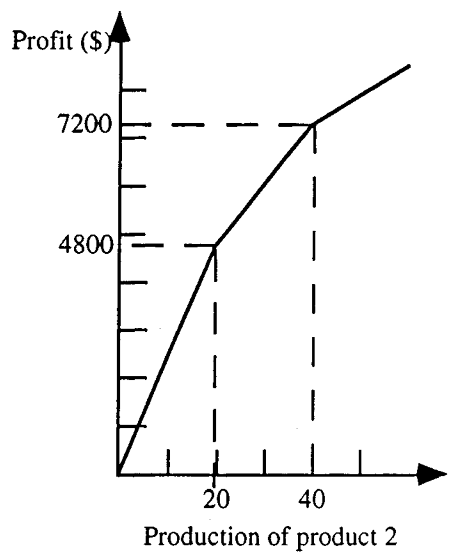
| $\mu$ | $\sigma$ | $\mu-\sigma$ | $\mu-3\sigma$ |
|---|---|---|---|
| 13 | 5.06 | 7.94 | -2.18 |
| 14 | 7.14 | 6.86 | -7.42 |
| 15 | 10.44 | 4.56 | -16.32 |
| 16 | 14.12 | 1.88 | -26.36 |

12.8-1

a) The profit graph for product 1 is shown below:




The profit graph for product 2 is shown below:

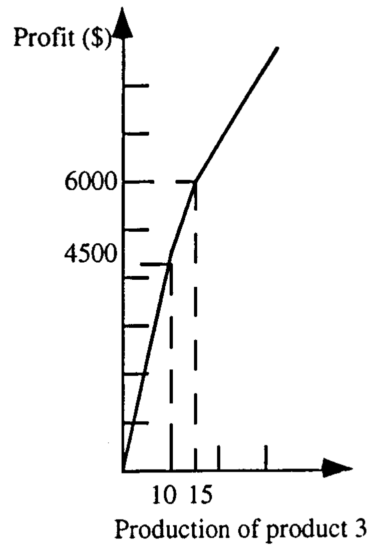



(CONT'D)

12.8-1 (CONT'D)

a)

The profit graph for product 3 is shown below:




b) Maximize $360x_{11}+30x_{12}+240x_{21}+120x_{22}+90x_{23}+450x_{31}+300x_{32}+180x_{33}$

subject to

$$x_{11}+x_{12}+x_{21}+x_{22}+x_{23}+x_{31}+x_{32}+x_{33} \le 60$$
$$3x_{11}+3x_{12}+2x_{21}+2x_{22}+2x_{23} \le 200$$
$$x_{11}+x_{12}+x_{31}+x_{32}+x_{33} \le 70$$
$$0 \le x_{11} \le 15,\ 0 \le x_{12}$$
$$0 \le x_{21} \le 20,\ 0 \le x_{22} \le 20,\ 0 \le x_{23}$$
$$0 \le x_{31} \le 10,\ 0 \le x_{32} \le 5,\ 0 \le x_{33}$$

where $x_1 = x_{11}+x_{12}$, $x_2 = x_{21}+x_{22}+x_{23}$ & $x_3 = x_{31}+x_{32}+x_{33}$

c) Solving automatically by the Simplex Method:

**Optimal Solution**

Value of the
Objective Function: Z = 18000

| Variable | | Value |
|---|---|---|
| $X_1$ | ($x_{11}$) | 15 |
| $X_2$ | ($x_{12}$) | 0 |
| $X_3$ | ($x_{21}$) | 20 |
| $X_4$ | ($x_{22}$) | 0 |
| $X_5$ | ($x_{23}$) | 0 |
| $X_6$ | ($x_{31}$) | 10 |
| $X_7$ | ($x_{32}$) | 5 |
| $X_8$ | ($x_{33}$) | 10 |

(CONT'D)

12.8-1 c) CONT'D)

In terms of the original variables, we have

$X_1 = X_{11} + X_{12} = 15$, $X_2 = X_{21} + X_{22} + X_{23} = 20$, $X_3 = X_{31} + X_{32} + X_{33} = 25$.

d) The restriction on profit from products 1 and 2 can be modeled by adding the following constraint to the linear program in part (b):

$$360x_{11} + 30x_{12} + 240x_{21} + 120x_{22} + 90x_{23} \geq 9000$$

e) The Simplex Method gives the optimal solution:

Value of the
Objective Function: Z = 18000

| Variable | Value |
|---|---|
| $X_1$ | 15 |
| $X_2$ | 0 |
| $X_3$ | 20 |
| $X_4$ | 0 |
| $X_5$ | 0 |
| $X_6$ | 10 |
| $X_7$ | 5 |
| $X_8$ | 10 |

Note that the optimal hasn't changed. This is because $\underline{X}$ still is feasible with the new constraint. (i.e. $360x_{11} + 30x_{12} + 240x_{21} + 120x_{22} + 90x_{23} = 10200 > 9000$)

In terms of the original variables, we (still) have

$X_1 = 15$, $X_2 = 20$, $X_3 = 25$.

12.8-2 a) The KKT conditions:

1a) $4 - 3x_1^2 - u_1 - 5u_2 \leq 0$

2a) $x_1(4 - 3x_1^2 - u_1 - 5u_2) = 0$

1b) $6 - 4x_2 - 3u_1 - 2u_2 \leq 0$

2b) $x_2(6 - 4x_2 - 3u_1 - 2u_2) = 0$

3a) $x_1 + 3x_2 \leq 8$

4a) $u_1(x_1 + 3x_2 - 8) = 0$

3b) $5x_1 + 2x_2 \leq 14$

4b) $u_2(5x_1 + 2x_2 - 14) = 0$

5) $x_1 \geq 0$, $x_2 \geq 0$

6) $u_1 \geq 0$, $u_2 \geq 0$

It is easy to verify that $(x_1, x_2, u_1, u_2) = (\frac{2}{\sqrt{3}}, \frac{3}{2}, 0, 0)$ satisfies the above conditions, and so $(x_1, x_2) = (\frac{2}{\sqrt{3}}, \frac{3}{2})$ is optimal with $z = 7.58$

12.8-2

b) 3,000 power saws and 7,000 power drills should be produced in November.

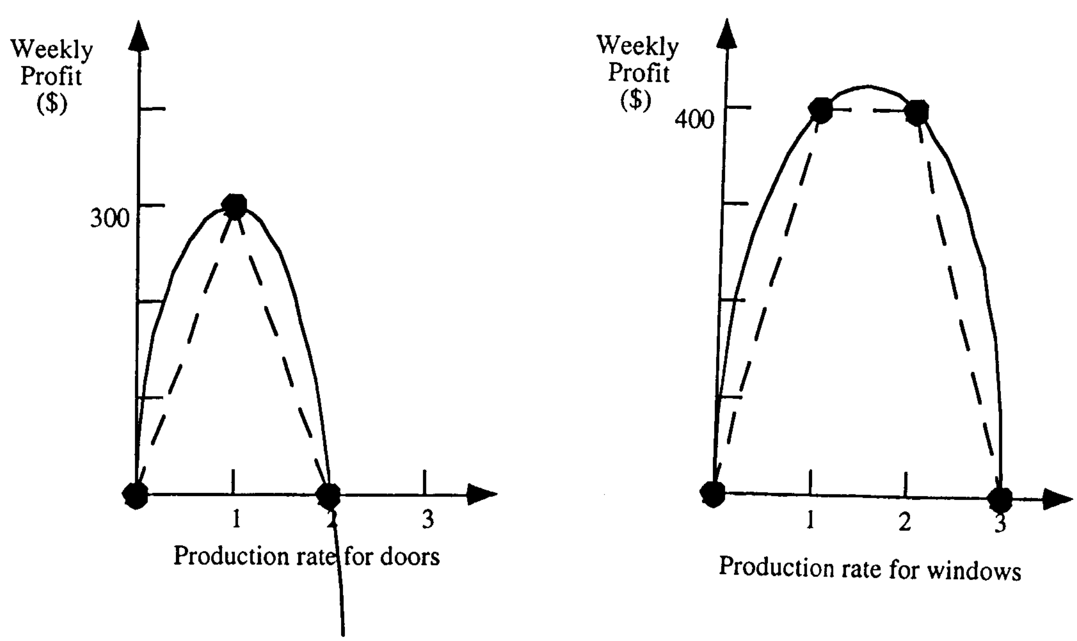
Profit data for doors when marketing costs are considered:

| Production Rate | Gross Profit | Marketing Cost | Net Profit | Incremental Net Profit |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | — |
| 1 | \$400 | \$100 | \$300 | \$300 |
| 2 | \$800 | \$800 | \$0 | -\$300 |
| 3 | \$1200 | \$2700 | -\$1900 | -\$1900 |
| $D$ | $4D$ | $D^3$ | $4D - D^3$ | |

Profit data for windows when marketing costs are considered:

| Production Rate | Gross Profit | Marketing Cost | Net Profit | Incremental Net Profit |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | — |
| 1 | \$600 | \$200 | \$400 | \$400 |
| 2 | \$1200 | \$800 | \$400 | \$0 |
| 3 | \$1800 | \$1800 | 0 | -\$400 |
| $W$ | $6W$ | $2W^2$ | $6W - 2W^2$ | |

c) The profit graphs for doors and windows are shown below:



12.8-2

d) Let $x_1 = x_{11}+x_{12}+x_{13}$ , $x_2 = x_{21}+x_{22}+x_{23}$, $f_1(x_1) = 4x_1 - x_1^3$, $f_2(x_2) = 6x_2 - 2x_2^2$

$f_1(0) = 0$, $f_1(1) = 3$, $f_1(2) = 0$, $f_1(3) = -15$

$f_2(0) = 0$, $f_2(1) = 4$, $f_2(2) = 4$, $f_2(3) = 0$

$s_{11} = \frac{3-0}{1-0} = 3$, $s_{12} = \frac{0-3}{2-1} = -3$, $s_{13} = -15$

$s_{21} = 4$, $s_{22} = 0$, $s_{23} = -4$

The approximate linear programming model is:

Maximize $3x_{11} - 3x_{12} - 15x_{13} + 4x_{21} - 4x_{23}$

subject to: $x_{11} + x_{12} + x_{13} + 3x_{21} + 3x_{22} + 3x_{23} \leq 8$

$5x_{11} + 5x_{12} + 5x_{13} + 2x_{21} + 2x_{22} + 2x_{23} \leq 14$

$0 \leq x_{ij} \leq 1$ for $i = 1, 2$ and $j = 1, 2, 3$.

e) The Simplex method gives the solution:

Value of the
Objective Function: Z = 7

| Variable | | Value |
|---|---|---|
| X1 | ($x_{11}$) | 1 |
| X2 | ($x_{12}$) | 0 |
| X3 | ($x_{13}$) | 0 |
| X4 | ($x_{21}$) | 1 |
| X5 | ($x_{22}$) | 0 |
| X6 | ($x_{23}$) | 0 |

or $x_1 = x_{11}+x_{12}+x_{13} = 1$

$x_2 = x_{21}+x_{22}+x_{23} = 1$

(also $x_2 = 2$ gives optimal objective value $Z = 7$)

$x_{12} = 0 \Rightarrow x_{13} = 0$ (also $x_{11} = 0 \Rightarrow x_{12} = 0$ since $x_{11} \neq 0$

$x_{22} = 0 \Rightarrow x_{23} = 0$ $x_{21} = 0 \Rightarrow x_{22} = 0$ since $x_{21} \neq 0$ )

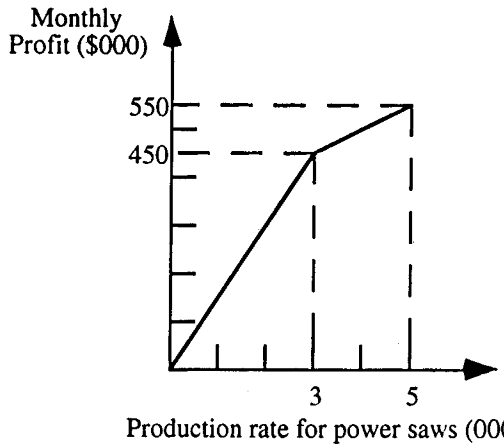
So special restriction for the model is satisfied.

The approximate solution is fairly close: (1,1) or (1,2) ≈ (1.155, 1.5) optimal

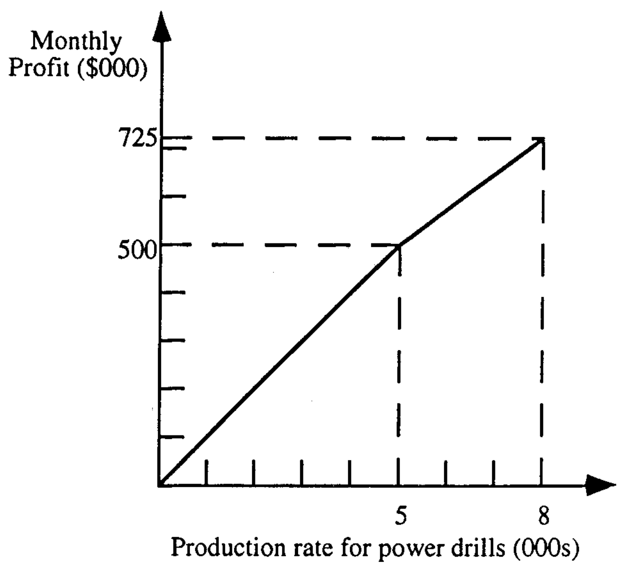
+ Z is close.

12.8-3

a) The profit graph for power saws is shown below:




The profit graph for power drills is shown below:




b) Maximize $150x_{11} + 50x_{12} + 100x_{21} + 75x_{22}$

subject to:

$0 \le x_{11} \le 3000$ ; $0 \le x_{12} \le 2000$

$0 \le x_{21} \le 5000$ ; $0 \le x_{22} \le 3000$

$x_{11} + x_{12} + x_{21} + x_{22} \le 10{,}000$

$2x_{11} + 2x_{12} + x_{21} + x_{22} \le 15{,}000$

## 12.8-3

c)

| Resource | Power Saws Regular | Power Saws Overtime | Power Drills Regular | Power Drills Overtime | Totals | | Resource Available |
|---|---|---|---|---|---|---|---|
| | *Resource Usage Per Unit of Each Activity* | | | | | | |
| power supplies | 1 | 1 | 1 | 1 | 10000 | ≤ | 10000 |
| gear assemblies | 2 | 2 | 1 | 1 | 13000 | ≤ | 15000 |
| Unit Profit | 150 | 50 | 100 | 75 | $ 1,100,000 | | |
| Solution | 3000 | 0 | 5000 | 2000 | | | |
| Maximum | 3000 | 2000 | 5000 | 3000 | | | |

## 12.8-4

(a) Let $x_1 = x_{11} + x_{12} + x_{13}$, $x_2 = x_{21} + x_{22} + x_{23}$, $f_1(x_1) = 32x_1 - x_1^4$ and

$f_2(x_2) = 50x_2 - 10x_2^2 + x_2^3 - x_2^4$.

We will have $s_{11} = 31$, $s_{12} = 17$, $s_{13} = -33$, $s_{21} = 40$, $s_{22} = 12$ and $s_{23} = -46$.

The approximate linear programming model is:

Maximize $31x_{11} + 17x_{12} - 33x_{13} + 40x_{21} + 12x_{22} - 46x_{23}$

subject to: $3x_{11} + 3x_{12} + 3x_{13} + x_{21} + x_{22} + x_{23} \le 11$

$2x_{11} + 2x_{12} + 2x_{13} + 5x_{21} + 5x_{22} + 5x_{23} \le 16$

$0 \le x_{ij} \le 1$ for $i = 1,2$ and $j = 1,2,3$.

b) The automatic routine (Simplex method) gives the solution:

Value of the
Objective Function: Z = 100

| Variable | | Value |
|---|---|---|
| X1 | $(x_{11})$ | 1 |
| X2 | $(x_{12})$ | 1 |
| X3 | $(x_{13})$ | 0 |
| X4 | $(x_{21})$ | 1 |
| X5 | $(x_{22})$ | 1 |
| X6 | $(x_{23})$ | 0 |

In terms of the original variables, we have:

$x_1 = x_{11} + x_{12} + x_{13} = 2$, $x_2 = x_{21} + x_{22} + x_{23} = 2$.

12.8-5

Let $f_1(x_1) = \begin{cases} 5x_1 & 0 \le x_1 \le 2 \\ 2+4x_1 & 2 \le x_1 \le 5 \\ 12+2x_1 & 5 \le x_1 \end{cases}$

$f_2(x_2) = \begin{cases} 4x_2 & 0 \le x_2 \le 3 \\ 9+x_2 & 3 \le x_2 \le 4 \end{cases}$

So Maximize $f_1(x_1)+f_2(x_2)$

subject to

$$3x_1 + 2x_2 \le 25$$
$$2x_1 - x_2 \le 10$$
$$x_2 \le 4$$
$$x_1 \ge 0,\ x_2 \ge 0$$

Possibly, the $f_i(x_i)$ are piecewise-linear approximations of the true objective function.

12.8-6

a) Assume that in the optimal solution of the linear program there exists an $x_{ij}$ such that $x_{ij} < u_{ij}$ and $x_{i(j+1)} > 0$. Create a new solution with $x'_{ij} = \min\{u_{ij}, x_{ij}+x_{i(j+1)}\}$, $x'_{i(j+1)} = \max\{0, x_{ij}+x_{i(j+1)}-u_{ij}\}$. This solution is feasible since the $g_i$'s are all linear and $x_{ij}+x_{i(j+1)} = x'_{ij}+x'_{i(j+1)}$.

But $s_{ij}x'_{ij} + s_{i(j+1)}x'_{i(j+1)} = \begin{cases} s_{ij}(x_{ij}+x_{i(j+1)}) & x_{ij}+x_{i(j+1)} \le u_{ij} \\ s_{ij}u_{ij}+s_{i(j+1)}(x_{ij}+x_{i(j+1)}-u_{ij}) & \text{otherwise.} \end{cases}$

Clearly, $s_{ij}(x_{ij}+x_{i(j+1)}) > s_{ij}x_{ij}+s_{i(j+1)}x_{i(j+1)}$ since $s_{ij} \ge s_{i(j+1)}$

Furthermore, $(s_{ij}-s_{i(j+1)})u_{ij} > (s_{ij}-s_{i(j+1)})x_{ij}$ from $x_{ij}<u_{ij}$

$$\Rightarrow s_{ij}u_{ij} + s_{i(j+1)}(x_{ij}-u_{ij}) > s_{ij}x_{ij}$$
$$\Rightarrow s_{ij}u_{ij} + s_{i(j+1)}(x_{ij}+x_{i(j+1)}-u_{ij}) > s_{ij}x_{ij}+s_{i(j+1)}x_{i(j+1)}$$

So $s_{ij}x'_{ij} + s_{i(j+1)}x'_{i(j+1)} > s_{ij}x_{ij} + s_{i(j+1)}x_{i(j+1)}$

Thus, the original solution was not optimal.

(b) Make the same assumptions as above and construct $x'$ from $x$ as above. For the linear approximate of $g_i$ we will have $\ldots a_{ij}x_{ij}+a_{i(j+1)}x_{i(j+1)}\ldots \le b_i$ with $a_{ij} \le a_{i(j+1)}$ since $g_i$ was convex. Thus, it can be shown by the analysis above if we reverse the inequalities in the appropriate places:

$$a_{ij}x'_{ij} + a_{i(j+1)}x'_{i(j+1)} < a_{ij}x_{ij} + a_{i(j+1)}x_{i(j+1)}$$

So $x'$ is feasible. Further, by above

$$s_{ij}x'_{ij} + s_{i(j+1)}x'_{i(j+1)} > s_{ij}x_{ij} + s_{i(j+1)}x_{i(j+1)}$$

So $x$ was not optimal.

12.8-7 Let $f_1(x_1) = \begin{cases} 15x_1 & 0 \le x_1 \le 2000 \\ 25x_1 - 20{,}000 & 2000 \le x_1 \end{cases}$

$f_2(x) = \begin{cases} 16x_2 & 0 \le x_2 \le 1000 \\ 24x_2 - 8000 & 1000 \le x_1 \end{cases}$

The (non-linear) programming problem is:

Maximize $Z = x_1 + x_2$

subject to $f_1(x_1) + f_2(x_2) \le 60{,}000$

$x_1 \le 3000$

$x_2 \le 1500$

$x_1 \ge 0,\ x_2 \ge 0$

(a) Let $x_i^R$, $x_i^O$ denote regular and overtime production at plant $i$. The LP is:

Maximize $Z = x_1^R + x_1^O + x_2^R + x_2^O$

subject to $15x_1^R + 25x_1^O + 16x_2^R + 24x_2^O \le 60{,}000$

$x_1^R \le 2000$

$x_1^O \le 1000$

$x_2^R \le 1000$

$x_2^O \le 500$

$x_1^R \ge 0, x_1^O \ge 0, x_2^R \ge 0, x_2^O \ge 0$

(b) Since overtime production is more costly than regular time, the objective of maximizing total production time will force the regular time to be used first.

12.8-8

(a) The objective function is linear and, therefore, concave.

$\frac{\partial^2 g_1}{\partial x_1^2} \cdot \frac{\partial^2 g_1}{\partial x_2^2} - \left(\frac{\partial^2 g_1}{\partial x_1 \partial x_2}\right)^2 = 4 \cdot 0 - 0^2 = 0 \ge 0$ so from the test in the appendix $g_1(x_1, x_2)$ is convex. Similarly

$\frac{\partial^2 g_2}{\partial x_1^2} \cdot \frac{\partial^2 g_2}{\partial x_2^2} - \left(\frac{\partial^2 g_2}{\partial x_1 \partial x_2}\right)^2 = 2 \cdot 0 - 0^2 = 0 \ge 0$ so $g_2(x_1, x_2)$ is also convex.

12.8-8

(b) Let $x_1 = x_{11} + x_{12} + x_{13}$. From constraint 1 we know $2x_1^2 \le 13$ or $x_1 \le \sqrt{\frac{13}{2}} \approx 2.55$, therefore, using integer breakpoints will require 3 linear pieces. Let $g_{11}(x_1) = 2x_1^2$, $g_{12}(x_2) = x_2$, $g_{21}(x_1) = x_1^2$ and $g_{22}(x_2) = x_2$

Constraint 1 = $g_{11}(x_1) + g_{12}(x_2)$ and Constraint 2 = $g_{21}(x_1) + g_{22}(x_2)$

$g_{11}(0) = 0$, $g_{11}(1) = 2$, $g_{11}(2) = 8$, $g_{11}(3) = 18$, $g_{21}(0) = 0$, $g_{21}(1) = 1$, $g_{21}(2) = 4$ and $g_{21}(3) = 9$.

Let $s_{11} = \frac{2-0}{1-0} = 2$ $\quad s_{12} = \frac{8-2}{2-1} = 6$ $\quad s_{13} = \frac{18-8}{3-2} = 10$

$s_{21} = \frac{1-0}{1-0} = 1$ $\quad s_{22} = \frac{4-1}{2-1} = 3$ $\quad s_{23} = \frac{9-4}{3-2} = 5$

The approximate linear program is:

Maximize $5x_{11} + 5x_{12} + 5x_{13} + x_2$

subject to: $2x_{11} + 6x_{12} + 10x_{13} + x_2 \le 13$

$x_{11} + 3x_{12} + 5x_{13} + x_2 \le 9$

$0 \le x_{11} \le 1 \quad 0 \le x_{12} \le 1 \quad x_{13} \ge 0 \quad x_2 \ge 0.$

We could have $0 \le x_{13} \le 1$ but the constraints will enforce the upper bound.

(c)

| Bas Var | Eq No | Z | X1 | X2 | X3 | X4 | X5 | X6 | X7 | X8 | Right side |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Z | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 4 | 2 | 15 |
| X3 | 1 | 0 | 0 | 0 | 1 | 0 | 0.2 | -0.2 | -0.2 | -0.6 | 0 |
| X4 | 2 | 0 | 0 | 0 | 0 | 1 | -1 | 2 | 0 | 0 | 5 |
| X1 | 3 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 |
| X2 | 4 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |

In terms of the original variables, the optimal solution to the approximate problem is: $X_1 = X_{11} + X_{12} + X_{13} = 1 + 1 + 0 = 2$

$X_2 = 5$

12.8-9 a) Let $X_1 = X_{11} + X_{12} + X_{13}$ , $X_2 = X_{21} + X_{22} + X_{23}$

Obj: $f_1(x_1) = 32x_1 - x_1^4 \quad \left(\frac{\partial^2 f_1}{\partial x_1^2} = -12x_1^2 \le 0\right)$ } concave

$f_2(x_2) = 4x_2 - x_2^2 \quad \left(\frac{\partial^2 f_2}{\partial x_2^2} = -2 < 0\right)$

$f_1(0) = 0,\ f_1(1) = 31,\ f_1(2) = 48,\ f_1(3) = 15$

$f_2(0) = 0,\ f_2(1) = 3,\ f_2(2) = 4,\ f_2(3) = 3$

So, $s_{11} = \frac{31-0}{1-0} = 31,\ s_{12} = 17,\ s_{13} = -33$

$s_{21} = 3,\ s_{22} = 1,\ s_{23} = -1$

Constraints: $g_{11}(x_1) = x_1^2,\ g_{12}(x_2) = x_2^2 \quad \left(\frac{\partial^2 g_{11}}{\partial x_1^2} = \frac{\partial^2 g_{12}}{\partial x_2^2} = 2 > 0\right.$, convex$\left.\right)$

$g_{11}(0) = 0,\ g_{11}(1) = 1,\ g_{11}(2) = 4,\ g_{11}(3) = 9$

$g_{12}(0) = 0,\ g_{12}(1) = 1,\ g_{12}(2) = 4,\ g_{12}(3) = 9$

$t_{11} = \frac{1-0}{1-0} = 1,\ t_{12} = \frac{4-1}{2-1} = 3,\ t_{13} = 5$

$t_{21} = 1,\ t_{22} = 3,\ t_{23} = 5$

The approximate linear program is:

Maximize $31x_{11} + 17x_{12} - 33x_{13} + 3x_{21} + x_{22} - x_{23}$

subject to $x_{11} + 3x_{12} + 5x_{13} + x_{21} + 3x_{22} + 5x_{23} \le 9$

$x_{11} \ge 0,\ x_{12} \ge 0,\ x_{13} \ge 0,\ x_{21} \ge 0,\ x_{22} \ge 0,\ x_{23} \ge 0$

$x_{11} \le 1,\ x_{12} \le 1,\ (x_{13} \le 1),\ x_{21} \le 1,\ x_{22} \le 1,\ (x_{23} \le 1)$

b) The Simplex Method gives the solution:

Value of the
Objective Function: Z = 52

| Variable | | Value |
|---|---|---|
| X1 | ($x_{11}$) | 1 |
| X2 | ($x_{12}$) | 1 |
| X3 | ($x_{13}$) | 0 |
| X4 | ($x_{21}$) | 1 |
| X5 | ($x_{22}$) | 1 |
| X6 | ($x_{23}$) | 0 |

In terms of the original variables, we have

$x_1 = x_{11} + x_{12} + x_{13} = 2$ , $x_2 = x_{21} + x_{22} + x_{23} = 2$ .

c) The KKT conditions:

1 a) $32 - 4x_1^3 - 2x_1 u \le 0$

2 a) $x_1(32 - 4x_1^3 - 2x_1 u) = 0$

1 b) $4 - 2x_2 - 2x_2 u \le 0$

2 b) $x_2(4 - 2x_2 - 2x_2 u) = 0$

3) $x_1^2 + x_2^2 - 9 \le 0$

4) $u(x_1^2 + x_2^2 - 9) = 0$

5) $x_1 \ge 0,\ x_2 \ge 0$

6) $u \ge 0$

From (4), $x_1 = 2 = x_2 \Rightarrow u = 0$

These values satisfy all of the KKT conditions

So the solution to the linear approximation is, in fact, the optimal to the original problem.

12.8-10 a) $f(x) = f_1(x_1) + f_2(x_2)$

$f_1(x_1) = 3x_1^2 - x_1^3$

$f_2(x_2) = 5x_2^2 - x_2^3$

$\frac{\partial^2 f_1}{\partial x_1^2} = 6 - 6x_1 \quad \leftarrow \quad > 0$ if $0 \le x_1 < 1$

$\frac{\partial^2 f_2}{\partial x_2^2} = 10 - 6x_2 \quad \leftarrow \quad > 0$ if $0 \le x_2 < \frac{5}{3}$

Neither $f_1$ nor $f_2$ are concave (we only need to show one of these are not concave), so $f(x)$ is not concave.

b) Let $x_1 = x_{11} + x_{12} + x_{13} + x_{14}$

$x_2 = x_{21} + x_{22}$

$f_1(0) = 0$, $f_1(1) = 2$, $f_1(2) = 4$, $f_1(3) = 0$, $f_1(4) = -16$

$f_2(0) = 0$, $f_2(1) = 4$, $f_2(2) = 12$

So, $s_{11} = \frac{2-0}{1-0} = 2$, $s_{12} = \frac{4-2}{2-1} = 2$, $s_{13} = -4$, $s_{14} = -16$

$s_{21} = 4$, $s_{22} = 8$

We need special restrictions

i) $x_{12} = 0$ if $x_{11} < 1$

ii) $x_{13} = 0$ if $x_{12} < 1$

iii) $x_{14} = 0$ if $x_{13} < 1$

iv) $x_{22} = 0$ if $x_{21} < 1$

Since $s_{12} > s_{13} > s_{14}$, (ii) and (iii) are automatically satisfied upon optimizing.

The approximate BIP formulation is:

Maximize $2x_{11} + 2x_{12} - 4x_{13} - 16x_{14} + 4x_{21} + 8x_{22}$

subject to: $x_{11} + x_{12} + x_{13} + x_{14} + 2x_{21} + 2x_{22} \le 4$

$-x_{11} + x_{12} \le 0$

$-x_{21} + x_{22} \le 0$

$x_{ij}$ binary $(i,j) = (1,1), \ldots, (2,2)$

c) The solution (by BIP automatic routine) is:

$x_{11} = 0, x_{12} = 0, x_{13} = 0, x_{14} = 0, x_{21} = 1, x_{22} = 1$ and $Z = 12$

In terms of the original variables, we have

$x_1 = x_{11} + x_{12} + x_{13} + x_{14} = 0$, $x_2 = x_{21} + x_{22} = 2$

(An alternate solution is $x_1 = 2$, $x_2 = 1$ which also gives the optimal objective value $Z = 12$)

12.9-1. $\nabla f(x_1,x_2) = \left(\frac{1}{x_1+1}, -2x_2\right)$

$\nabla f(0,0) = (1,0)$

Solving: $\max\ x_1$

s.t. $x_1 + 2x_2 \le 3$ $\Rightarrow$ $x_1 = 3, x_2 = 0$

$x_1 \ge 0, x_2 \ge 0$

$x^{(1)} = (0,0) + t(3,0)$

$t^* = 1$ ( $f(x)$ increases with $t$ )

$x^{(1)} = (3,0)$ (the solution found in 13.6-6 (b) )

$2^{nd}$ iter: $\nabla f(3,0) = (\frac{1}{4}, 0)$

Solving LP: $\max\ \frac{1}{4}x_1$

s.t. $x_1 + 2x_2 \le 3$ $\Rightarrow$ $x_1 = 3,\ x_2 = 0$

$x_1 \ge 0, x_2 \ge 0$

$x^{(2)} = (3,0) + t(0,0)$

So $x = (3,0)$ is optimal.

## 12.9-2.

| k | $\mathbf{x}^{(k-1)}$ | c1 | c2 | $\mathbf{x}_{LP}^{(k)}$ | t* | $\mathbf{x}^{(k)}$ |
|---|---|---|---|---|---|---|
| 1 | ( 0, 0) | -6 | -3 | ( 1, 0) | 1 | ( 1, 0) |
| 2 | ( 1, 0) | -4 | -3 | ( 1, 0) | 1e-8 | ( 1, 0) |

Final solution: ( 1, 0).

$\nabla f(x_1, x_2) = (2x_1 - 6,\ 3x_2^2 - 3)$

but since $x_1 \le 1, x_2 \le 1$ (since $x_1 + x_2 \le 1,\ x_1 \ge 0, x_2 \ge 0$)

$2x_1 - 6 < 3x_2^2 - 3$ $\quad (2x_1 - 6 \le -4 < -3 \le 3x_2^2 - 3)$

So the resulting LP will be of the form

$\min\ c_1 x_1 + c_2 x_2$ $\quad c_1 < c_2$

s.t. $x_1 + x_2 \le 1$ $\quad \Rightarrow$ (1,0) will always be the

$x_1 \ge 0, x_2 \ge 0$ $\quad$ optimal solution.

$x^{(1)} = (x_1^{(0)}, x_2^{(0)}) + t(1 - x_1^{(0)}, -x_2^{(0)})$

So at $t^* = 1$, $x^{(1)} = (1,0)$, the optimal

## 12.9-3.

| k | $\mathbf{x}^{(k-1)}$ | c1 | c2 | c3 | $\mathbf{x}_{LP}^{(k)}$ | t* | $\mathbf{x}^{(k)}$ |
|---|---|---|---|---|---|---|---|
| 1 | ( 0, 0, 0) | 8 | 2 | 1 | ( 12, 0, 0) | 0.33 | ( 4, 0, 0) |
| 2 | ( 4, 0, 0) | 0 | 2 | 1 | ( 0, 4, 0) | 0.25 | ( 3, 1, 0) |

Final solution: ( 3, 1, 0).

## 12.9-4

| k | $\mathbf{x}^{(k-1)}$ | c1 | c2 | $\mathbf{x}_{LP}^{(k)}$ | t* | $\mathbf{x}^{(k)}$ |
|---|---|---|---|---|---|---|
| 1 | ( 5, 5) | 15 | 10 | ( 30, 0) | 0.088 | (7.196,4.561) |
| 2 | (7.196,4.561) | 4.459 | 22.3 | ( 0, 15) | 0.119 | (6.337,5.807) |
| 3 | (6.337,5.807) | 12.88 | 8.89 | ( 30, 0) | 0.07 | (7.996, 5.4) |
| 4 | (7.996, 5.4) | 4.615 | 18.79 | ( 0, 15) | 0.089 | (7.283,6.256) |
| 5 | (7.283,6.256) | 10.89 | 9.082 | ( 30, 0) | 0.054 | (8.514,5.917) |
| 6 | (8.514,5.917) | 4.611 | 16.72 | ( 0, 15) | 0.072 | (7.903,6.569) |
| 7 | (7.903,6.569) | 9.666 | 9.056 | ( 30, 0) | 0.045 | (8.887,6.277) |

Final solution: (8.8866, 6.277).

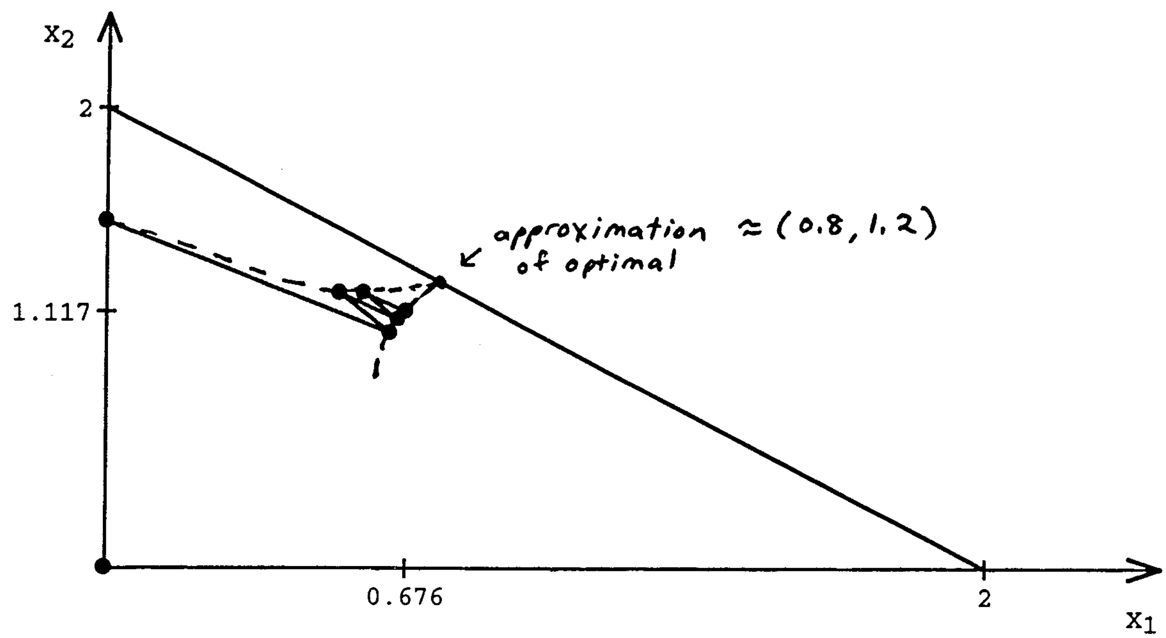
12.9-5

a)

| k | x(k-1) | $c_1$ | $c_2$ | $x_{LP}^{(k)}$ | t* | x(k) |
|---|---|---|---|---|---|---|
| 1 | ( 0, 0) | 2 | 3 | ( 0, 2) | 0.75 | ( 0, 1.5) |
| 2 | ( 0, 1.5) | 2 | 0 | ( 2, 0) | 0.32 | ( 0.64, 1.02) |
| 3 | ( 0.64, 1.02) | 0.72 | 0.96 | ( 0, 2) | 0.175 | (0.528,1.192) |
| 4 | (0.528,1.192) | 0.944 | 0.617 | ( 2, 0) | 0.092 | (0.663,1.082) |
| 5 | (0.663,1.082) | 0.674 | 0.835 | ( 0, 2) | 0.126 | (0.579,1.198) |
| 6 | (0.579,1.198) | 0.842 | 0.603 | ( 2, 0) | 0.068 | (0.676,1.117) |

Final solution: ( 0.676,1.1166).

b)



12.9-6.

| k | $x^{(k-1)}$ | $c_1$ | $c_2$ | $x_{LP}^{(k)}$ | t* | $x^{(k)}$ |
|---|---|---|---|---|---|---|
| 1 | ( 0, 0) | 32 | 50 | ( 3, 2) | 0.729 | (2.188,1.458) |
| 2 | (2.188,1.458) | -9.87 | 14.81 | ( 0, 3.2) | 0.131 | (1.902,1.686) |
| 3 | (1.902,1.686) | 4.499 | 5.634 | ( 3, 2) | 0.111 | (2.024,1.721) |
| 4 | (2.024,1.721) | -1.15 | 4.078 | ( 0, 3.2) | 0.028 | (1.966,1.763) |

Final solution: (1.9662,1.7629).

12.9-7

| k | X^(k-1) | c1 | c2 | X[LP]^k | t* | X^k |
|---|---|---|---|---|---|---|
| 1 | ( 0, 0) | 40 | 30 | ( 3, 0) | 0.616 | (1.847, 0) |
| 2 | (1.847, 0) | 0.001 | 35.54 | ( 0, 2) | 0.406 | (1.097,0.812) |

12.9-8. (a)

| k | X^(k-1) | c1 | c2 | X[LP]^k | t* | X^k |
|---|---|---|---|---|---|---|
| 1 | ( 0.25, 0.25) | 2.813 | 3.5 | ( 0, 1) | 1 | ( 0, 1) |
| 2 | ( 0, 1) | 3 | 2 | ( 1, 0) | 0.333 | (0.333,0.667) |
| 3 | (0.333,0.667) | 2.667 | 2.667 | ( 1, 0) | 0.001 | (0.334,0.666) |

(b) The KKT conditions:

1a) $3 - 3x_1^2 - u \le 0$

2a) $x_1(3 - 3x_1^2 - u) = 0$

1b) $4 - 2x_2 - u \le 0$

2b) $x_2(4 - 2x_2 - u) = 0$

3) $x_1 + x_2 \le 1$

4) $u(x_1 + x_2 - 1) = 0$

5) $x_1 \ge 0,\ x_2 \ge 0$

6) $u \ge 0$

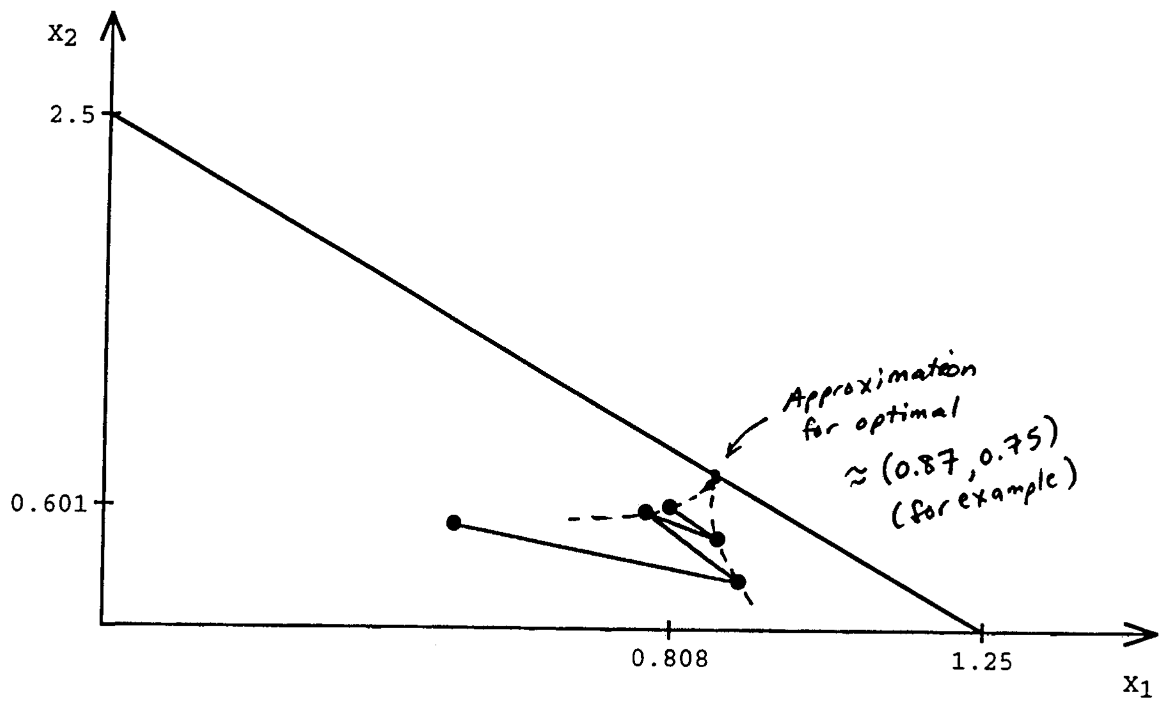
It is easy to verify that $(x_1, x_2, u) = (1/3, 2/3, 8/3)$ satisfies these conditions, so $(x_1, x_2) = (1/3, 2/3)$, the estimated solution from part (a), is optimal.

12.9-9. a)

| k | $\mathbf{x}^{(k-1)}$ | $c_1$ | $c_2$ | $\mathbf{x}_{LP}^{(k)}$ | $t^*$ | $\mathbf{x}^{(k)}$ |
|---|---|---|---|---|---|---|
| 1 | ( 0.5, 0.5) | 3.5 | 1 | ( 1.25, 0) | 0.541 | (0.906,0.229) |
| 2 | (0.906,0.229) | 1.027 | 1.541 | ( 0, 2.5) | 0.148 | (0.771,0.566) |
| 3 | (0.771,0.566) | 2.164 | 0.867 | ( 1.25, 0) | 0.216 | (0.875,0.444) |
| 4 | (0.875,0.444) | 1.323 | 1.112 | ( 0, 2.5) | 0.076 | (0.808,0.601) |

Final solution: (0.8079,0.6011).

b)




(c) The KKT conditions:

1a) $4 - 4x_1^3 - 4u \le 0$

2a) $x_1(4 - 4x_1^3 - 4u) = 0$

1b) $2 - 2x_2 - 2u \le 0$

2b) $x_2(2 - 2x_2 - 2u) = 0$

3) $4x_1 + 2x_2 \le 5$

4) $u(4x_1 + 2x_2 - 5) = 0$

5) $x_1 \ge 0,\ x_2 \ge 0$

6) $u \ge 0$

Are satisfied by $(x_1, x_2, u) = (.8934, .7131, .5737)$

So $(x_1, x_2) = (.8934, .7131)$ is optimal.

12.9-10 a) $P(\underline{x};r) = 3x_1 + 4x_2 - x_1^3 - x_2^2 - r\left[\frac{1}{1-x_1-x_2} + \frac{1}{x_1} + \frac{1}{x_2}\right]$

b) $\nabla P(\underline{x};r) = \begin{bmatrix} 3 - 3x_1^2 + r\left[\frac{-1}{(1-x_1-x_2)^2} + \frac{1}{x_1^2}\right] \\ 4 - 2x_2 + r\left[\frac{-1}{(1-x_1-x_2)^2} + \frac{1}{x_2^2}\right] \end{bmatrix}$

$\Rightarrow \nabla P((\tfrac{1}{4},\tfrac{1}{4});1) = \begin{bmatrix} 14\frac{13}{16} \\ 15\frac{1}{2} \end{bmatrix}$

$x' + t\nabla P(x';1) = (\tfrac{1}{4} + 14\tfrac{13}{16}t, \tfrac{1}{4} + 15\tfrac{1}{2}t)$

$t^* = 0.006606$

New $x' = (0.3479, 0.3524)$

c)

| k | r | $x_1$ | $x_2$ | f(**x**) |
|---|---|---|---|---|
| 0 | | 0.25 | 0.25 | 1.672 |
| 1 | 1 | 0.343 | 0.357 | 2.29 |
| 2 | 0.01 | 0.322 | 0.619 | 3.023 |
| 3 | 0.0001 | 0.331 | 0.663 | 3.169 |

d) The true solution is $(\frac{1}{3}, \frac{2}{3})$. The percentage error

$x_1$ is $\frac{|1/3 - .331|}{1/3} = 0.70\%$

$x_2$ is $\frac{|2/3 - .663|}{2/3} = 0.55\%$

$f(x)$ is $\frac{|3\frac{5}{27} - 3.169|}{3\frac{5}{27}} = 0.51\%$

12.9-11. a) $P(\underline{x};r) = 4x_1 - x_1^4 + 2x_2 - x_2^2 - r\left[\frac{1}{5-4x_1-2x_2} + \frac{1}{x_1} + \frac{1}{x_2}\right]$

b) $\nabla P(\underline{x};r) = \begin{bmatrix} 4 - 4x_1^3 + r\left[\frac{-4}{(5-4x_1-2x_2)^2} + \frac{1}{x_1^2}\right] \\ 2 - 2x_2 + r\left[\frac{-2}{(5-4x_1-2x_2)^2} + \frac{1}{x_2^2}\right] \end{bmatrix}$

$\Rightarrow \nabla P((\tfrac{1}{2},\tfrac{1}{2});1) = \begin{bmatrix} 6\frac{1}{2} \\ 4\frac{1}{2} \end{bmatrix}$

$x' + t\nabla P(x';1) = (\tfrac{1}{2} + 6\tfrac{1}{2}t, \tfrac{1}{2} + 4\tfrac{1}{2}t)$

$t^* = 0.03167$

New $x' = (0.7058, 0.6425)$

12.9-11 c)

| k | r | $x_1$ | $x_2$ | f(**x**) |
|---|---|---|---|---|
| 0 | | 0.5 | 0.5 | 2.688 |
| 1 | 1 | 0.669 | 0.716 | 3.395 |
| 2 | 0.01 | 0.871 | 0.671 | 3.801 |
| 3 | 0.0001 | 0.891 | 0.708 | 3.849 |
| 4 | 0.000001 | 0.894 | 0.712 | 3.854 |

12.9-12 a) $P(\underset{\sim}{x};r) = -x_1^4 - 2x_1^2 - 2x_1x_2 - 4x_2^2 - r\left[\frac{1}{2x_1+x_2-10} + \frac{1}{x_1+2x_2-10} + \frac{1}{x_1} + \frac{1}{x_2}\right]$

b) $$\nabla P(\underset{\sim}{x};100) = \begin{bmatrix} -4x_1^3 - 4x_1 - 2x_2 + 100\left[\frac{2}{(2x_1+x_2-10)^2} + \frac{1}{(x_1+2x_2-10)^2} + \frac{1}{x_1^2}\right] \\ -2x_1 - 8x_2 + 100\left[\frac{1}{(2x_1+x_2-10)^2} + \frac{2}{(x_1+2x_2-10)^2} + \frac{1}{x_2^2}\right] \end{bmatrix}$$

$$\Rightarrow \nabla P((5,5);100) = \begin{bmatrix} -514 \\ -34 \end{bmatrix}$$

$x' + t\,\nabla P(x';100) = (5 - 514t,\ 5 - 34t)$

$t^* = 0.003529$

New $x' = (3.1862,\ 4.8802)$

c)

| k | r | $x_1$ | $x_2$ | f(**x**) |
|---|---|---|---|---|
| 0 | | 5 | 5 | -825 |
| 1 | 100 | 2.725 | 6.072 | -251 |
| 2 | 1 | 2.587 | 4.976 | -183 |
| 3 | 0.01 | 2.562 | 4.891 | -177 |
| 4 | 0.0001 | 2.557 | 4.888 | -176 |

Note that because of minimization, we converted

Min $f(x)$ to Max $-f(x)$

Also, $g(x) \geq b$

becomes $-g(x) \leq -b$ .

12.9-13. (a) The KKT conditions:

1a) $x_2 - 2u_1x_1 \le 0$

2a) $x_1(x_2 - 2u_1x_1) = 0$

1b) $x_1 - u_1 \le 0$

2b) $x_2(x_1 - u_1) = 0$

3) $x_1^2 + x_2 \le 3$

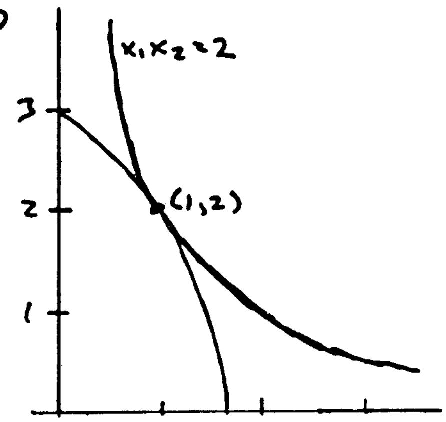
4) $u_1(x_1^2 + x_2 - 3) = 0$

5) $x_1 \ge 0,\ x_2 \ge 0$

6) $u_1 \ge 0$

$(x_1, x_2) = (1, 2)$ satisfies these with $u_1 = 1$.

(b)




12.9-14. a) $P(\underline{x}; r) = -2x_1 - (x_2-3)^2 - r\left(\frac{1}{x_1-3} + \frac{1}{x_2-3}\right)$

b) $\frac{\partial P(x;r)}{\partial x_1} = -2 + \frac{r}{(x_1-3)^2} = 0 \Rightarrow x_1 = \sqrt{\frac{r}{2}} + 3$

$\frac{\partial P(x;r)}{\partial x_2} = -2x_2 + 6 + \frac{r}{(x_2-3)^2} = 0 \Rightarrow x_2 = \sqrt[3]{\frac{r}{2}} + 3$

| $r$ | $x_1$ | $x_2$ |
|---|---|---|
| 1 | 3.7071 | 3.7937 |
| $10^{-2}$ | 3.0707 | 3.1710 |
| $10^{-4}$ | 3.0071 | 3.0368 |
| $10^{-6}$ | 3.0007 | 3.0079 |

Note $(x_1, x_2) \to (3, 3)$ as $r \to 0$, so $(x_1, x_2) = (3,3)$ is optimal

c)

| k | r | X1 | X2 | f(X) |
|---|---|---|---|---|
| 0 | | 4 | 4 | -9 |
| 1 | 1 | 3.707 | 3.794 | -8.044 |
| 2 | 0.01 | 3.07 | 3.179 | -6.172 |
| 3 | 0.0001 | 3.007 | 3.056 | -6.017 |
| 4 | 0.000001 | 3.001 | 3.011 | -6.002 |

12.9-15 $P(x;r) = -x_1^2 - x_2^2 - x_1 - x_2 + x_1 x_2 - r/x_2$

| k | r | $X_1$ | $X_2$ | f(**X**) |
|---|---|---|---|---|
| 0 | | 1 | 1 | -3 |
| 1 | 1 | -0.18 | 0.638 | -1.01 |
| 2 | 0.01 | -0.46 | 0.079 | 0.127 |
| 3 | 0.0001 | -0.5 | 0.008 | 0.238 |

12.9-16 $P(x;r) = 2x_1 + 3x_2 - x_1^2 - x_2^2 - r\left(\frac{1}{2 - x_1 - x_2} + \frac{1}{x_1} + \frac{1}{x_2}\right)$

| k | r | $X_1$ | $X_2$ | f(**X**) |
|---|---|---|---|---|
| 0 | | 0.5 | 0.5 | 2 |
| 1 | 1 | 0.649 | 0.781 | 2.61 |
| 2 | 0.01 | 0.691 | 1.184 | 3.055 |
| 3 | 0.0001 | 0.743 | 1.243 | 3.118 |
| 4 | 0.000001 | 0.749 | 1.249 | 3.124 |

12.9-17 $P(x;r) = 126x_1 - 9x_1^2 + 182x_2 - 13x_2^2 - r\left(\frac{1}{4 - x_1} + \frac{1}{12 - 2x_2} + \frac{1}{18 - 3x_1 - 2x_2} + \frac{1}{x_1} + \frac{1}{x_2}\right)$

| k | r | $X_1$ | $X_2$ | f(**X**) |
|---|---|---|---|---|
| 0 | | 2 | 3 | 645 |
| 1 | 100 | 2.292 | 4.523 | 798.8 |
| 2 | 1 | 2.62 | 4.972 | 851.9 |
| 3 | 0.01 | 2.661 | 4.999 | 856.5 |
| 4 | 0.0001 | 2.665 | 5.002 | 856.9 |

12.9-18(a) $P(x;r) = \sin 3x_1 + \cos 3x_2 + \sin(x_1+x_2) + r\left(\frac{1}{1+x_1^2-10x_2} + \frac{1}{100-10x_1-x_2^2} + \frac{1}{x_1} + \frac{1}{x_2}\right)$

(b) SUMT can be used to obtain the global minimum if "enough" different starting points are used and SUMT is run from each. If we choose a lattice of points over the feasible region so that the adjacent points don't change by more than $\frac{2}{3}\pi$ then this set of points will work for $f(x)$. Since sin and cos have period $2\pi$, choosing lattice points with grid size not greater than $\frac{2}{3}\pi$ will ensure that the arguments of the sin and cos terms in f will not change by more than $2\pi$ between adjacent lattice points. Since the second constraint ensures $x_1 \le 10$ and $x_2 \le 10$, we can see that at most $\left(\frac{10}{2\pi/3}\right)^2 \approx 23$ starting points are required if chosen well.

12.9-18 (c) Use LINGO Global Solver;

```
MIN =    @SIN(3*X1) + @COS(3*X2) + @SIN(X1+X2);
         X1^2 - 10 * X2        >= -1;
         10*X1  + X2^2         <= 100;
            X1                 >= 0;
                    X2         >= 0;
```

Global optimal solution found at iteration: 6

Objective value: -2.999999

| Variable | Value | Reduced Cost |
|---|---|---|
| X1 | 3.665418 | 0.000000 |
| X2 | 1.046684 | 0.000000 |

| Row | Slack or Surplus | Dual Price |
|---|---|---|
| 1 | -2.999999 | -1.000000 |
| 2 | 3.968452 | 0.000000 |
| 3 | 62.25027 | 0.000000 |
| 4 | 3.665418 | 0.000000 |
| 5 | 1.046684 | 0.000000 |

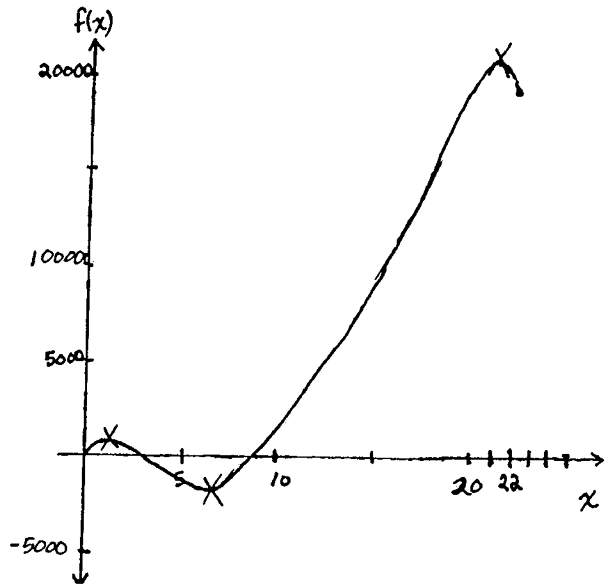
12.10-1 (a) Solving for the roots of $x^2+x-500=0$ we find $x$ is feasible in the range $[0, \frac{-1+\sqrt{2001}}{2}]$ or $[0, 21.866]$

$f'(x) = 1000 - 800x + 120x^2 - 4x^3$

$f''(x) = -800 + 240x - 12x^2$

$f'''(x) = 240 - 24x$

A rough sketch of f(x):




X corresponds to a local maximum or minimum

(b)

| Iteration | df(X)/dX | X(L) | X(U) | New X' | f(X') |
|---|---|---|---|---|---|
| 0 | | 0 | 5 | 2.5 | 585.94 |
| 1 | -312.5 | 0 | 2.5 | 1.25 | 700.68 |
| 2 | +179.7 | 1.25 | 2.5 | 1.875 | 720.06 |
| 3 | -104.5 | 1.25 | 1.875 | 1.5625 | 732.56 |
| 4 | +27.71 | 1.5625 | 1.875 | 1.7188 | 731.48 |
| 5 | -40.82 | 1.5625 | 1.7188 | 1.6406 | 733.36 |
| 6 | -7.166 | 1.5625 | 1.6406 | 1.6016 | 733.3 |
| Stop | | | | | |

| Iteration | df(X)/dX | X(L) | X(U) | New X' | f(X') |
|---|---|---|---|---|---|
| 0 | | 18 | 21.866 | 19.933 | 19931 |
| 1 | + 1053 | 19.933 | 21.866 | 20.899 | 20546 |
| 2 | +180.4 | 20.899 | 21.866 | 21.383 | 20509 |
| 3 | -346.2 | 20.899 | 21.383 | 21.141 | 20559 |
| 4 | - 75.1 | 20.899 | 21.141 | 21.02 | 20560 |
| 5 | +54.58 | 21.02 | 21.141 | 21.081 | 20562 |
| 6 | -9.778 | 21.02 | 21.081 | 21.051 | 20561 |
| Stop | | | | | |

There is a local maximum near 1.6016 and a global maximum near 21.051

**Problem 12-10.1 (c)**

Newton's method

**Max f(x) = 1000 x - 400 x^2 +40 x^3 - x^4 s.t. x^2 + x <= 500, x>= 0**

**f'(x) = 1000 - 800 x + 120 x^2 - 4 x^3**

**f''(x) = -800 + 240 x - 12 x^2**

**error** 0.001

**Starting with x = 3**

| Iteration i | $x_i$ | $f(x_i)$ | $f'(x_i)$ | $f''(x_i)$ | $x_{i+1}$ | $\lvert x_i - x_{i+1}\rvert$ |
|---|---|---|---|---|---|---|
| 1 | 3 | 399 | -428 | -188 | 0.723404 | 2.276596 |
| 2 | 0.72340426 | 528.947606 | 482.56 | -632.6627 | 1.486149 | 0.762744 |
| 3 | 1.48614867 | 729.11001 | 62.98815 | -469.828 | 1.620215 | 0.134066 |
| 4 | 1.62021508 | 733.414048 | 1.826677 | -442.6495 | 1.624342 | 0.004127 |
| 5 | 1.62434177 | 733.417819 | 0.001712 | -441.8198 | 1.624346 | 3.88E-06 |

local maximum $x^* = 1.6243$

**Starting with x = 15**

| Iteration i | $x_i$ | $f(x_i)$ | $f'(x_i)$ | $f''(x_i)$ | $x_{i+1}$ | $\lvert x_i - x_{i+1}\rvert$ |
|---|---|---|---|---|---|---|
| 1 | 20 | 20000 | 1000 | -800 | 21.25 | 1.25 |
| 2 | 21.25 | 20544.4336 | -195.3125 | -1118.75 | 21.07542 | 0.174581 |
| 3 | 21.075419 | 20561.721 | -4.093317 | -1071.979 | 21.0716 | 0.003818 |
| 4 | 21.0716005 | 20561.7289 | -0.001938 | -1070.964 | 21.0716 | 1.81E-06 |
| 5 | 21.0715987 | 20561.7289 | -4.51E-10 | -1070.964 | 21.0716 | 4.23E-13 |
| 6 | 21.0715987 | 20561.7289 | 0 | -1070.964 | 21.0716 | 0 |
| 7 | 21.0715987 | 20561.7289 | 0 | -1070.964 | 21.0716 | 0 |
| 8 | 21.0715987 | 20561.7289 | 0 | -1070.964 | 21.0716 | 0 |
| 9 | 21.0715987 | 20561.7289 | 0 | -1070.964 | 21.0716 | 0 |
| 10 | 21.0715987 | 20561.7289 | 0 | -1070.964 | 21.0716 | 0 |
| 11 | 21.0715987 | 20561.7289 | 0 | -1070.964 | 21.0716 | 0 |
| 12 | 21.0715987 | 20561.7289 | 0 | -1070.964 | 21.0716 | 0 |
| 13 | 21.0715987 | 20561.7289 | 0 | -1070.964 | 21.0716 | 0 |

local maximum $x^* = 21.0716$

12.10-1 d)

| k | r | $x_1$ | f(x) |
|---|---|---|---|
| 0 | | 3 | 399 |
| 1 | 1000 | 2.171 | 672.8 |
| 2 | 100 | 1.704 | 732 |
| 3 | 10 | 1.633 | 733.4 |
| 4 | 1 | 1.625 | 733.4 |
| 0 | | 15 | 9375 |
| 1 | 1000 | 21.04 | 20561 |
| 2 | 100 | 21.07 | 20562 |
| 3 | 10 | 21.07 | 20562 |
| 4 | 1 | 21.07 | 20562 |

Initial trial solution X = 3 gives X = 1.625 (f(x) = 733.4) as max

Initial trial solution X = 15 gives X = 21.07 (f(x) = 20562) as max

The global maximum is $X^* = 21.07$

**Problem 12-10.1 (f)**
**Evolutionary Solver**
**Max f(x) = 1000 x - 400 x^2 +40 x^3 - x^4 s.t. x^2 + x <= 500, x>= 0**

| | | | | |
|---|---|---|---|---|
| Max | f(x) | 20561.73 | | |
| s.t. | x | 21.0716 | >= | 0 |
| | g(x) | 465.0839 | <= | 500 |

$x^* = 21.0716$

**Problem 12-10.1 (e)**
**Solver Table**
**Max f(x) = 1000 x - 400 x^2 +40 x^3 - x^4 s.t. x^2 + x <= 500, x>= 0**

| | | | | |
|---|---|---|---|---|
| Max | f(x) | 733.4178 | | |
| s.t. | x | 1.624346 | >= | 0 |
| | g(x) | 4.262844 | <= | 500 |

| Starting poi | optimal x* | profit f(x*) |
|---|---|---|
| | 1.624 | 733.418 |
| 0 | 1.624 | 733.418 |
| 5 | 1.624 | 733.418 |
| 10 | 21.072 | 20561.729 |
| 15 | 21.072 | 20561.729 |
| 20 | 21.072 | 20561.729 |
| 25 | 21.072 | 20561.729 |

```
Problem 12-10.1 (g) Lingo with Global Solver
! Nonlinear constraint;
MAX = 1000 * X  -400*X^2 + 40*X^3 - X^4        ;
          X                               >=   0;
          X^2 + X                         <= 500;


  Global optimal solution found at iteration:          33
  Objective value:                                 20561.73


                    Variable           Value        Reduced Cost
                           X        21.07159            0.000000

                         Row    Slack or Surplus      Dual Price
                           1        20561.73            1.000000
                           2        21.07159            0.000000
                           3        34.91636            0.000000
```

12.10-2. (a) $P(x;r) = 3x_1x_2 - 2x_1^2 - x_1^2 - r\left(\frac{1}{4-x_1^2-2x_2^2} + \frac{1}{x_2-2x_1} + \frac{1}{x_1} + \frac{1}{x_2}\right) - \frac{(2-x_1x_2^2-x_1^2x_2)^2}{\sqrt{r}}$

(b)

| k | r | X1 | X2 | f(X) |
|---|---|---|---|---|
| 0 | | 1 | 1 | 0 |
| 1 | 1 | 0.915 | 1.007 | 0.0758 |
| 2 | 0.01 | 0.848 | 1.169 | 0.1692 |
| 3 | 0.0001 | 0.843 | 1.175 | 0.1697 |

**Problem 12-10.2 (c)**
**Evolutionary Solver**

| | | | | |
|---|---|---|---|---|
| Max | f(x) | 0.171564 | | |
| s.t. | g1(x) | 3.519555 | <= | 4 |
| | g2(x) | 0.504927 | <= | 3 |
| | g3(x) | 2.030303 | = | 2 |
| | x1 | 0.844707 | >= | 0 |
| | x2 | 1.184488 | >= | 0 |

12-10.2 (d) Use global optimizer feature of LINGO

```
! Nonlinear constraint;
MAX = 3 * X1 * X2  - 2 * X1^2 - X2^2;
          X1^2  + X2^2          <= 4;
          2*X1  - X2            <= 3;
         X1* X2^2  + X1^2* X2    = 2;
         X1                     >= 0;
                X2              >= 0;
```

```
Global optimal solution found at iteration:          13
Objective value:                              0.1698892

                    Variable           Value        Reduced Cost
                          X1       0.8382396            0.000000
                          X2        1.181385            0.000000

                         Row    Slack or Surplus      Dual Price
                           1       0.1698892            1.000000
                           2        1.901685            0.000000
                           3        2.504905            0.000000
                           4        0.000000            0.000000
                           5       0.8382396            0.000000
                           6        1.181385            0.000000
```

12.10-3(a) $P(x;r) = x_1^3 + 4x_2^2 + 16x_3 + r\left(\frac{1}{x_1-1} + \frac{1}{x_2-1} + \frac{1}{x_3-1}\right) + \frac{(5-x_1-x_2-x_3)^2}{\sqrt{r}}$

(b)

| k | r | X1 | X2 | X3 | f(X) |
|---|---|---|---|---|---|
| 0 | | 1.5 | 1.5 | 2 | -44.38 |
| 1 | 0.01 | 1.95 | 1.434 | 1.047 | -32.38 |
| 2 | 0.0001 | 2.179 | 1.743 | 1.007 | -38.62 |
| 3 | 0.000001 | 2.208 | 1.784 | 1.001 | -39.51 |
| 4 | 0.00000001 | 2.21 | 1.786 | 1.002 | -39.6 |

**Problem 12-10.3 (c)**
**Standard Excel Solver**

| | | | | |
|---|---|---|---|---|
| Min | f(x) | 39.608 | | |
| s.t. | g1(x) | 5.000 | = | 5 |
| | x1 | 2.194 | >= | 1 |
| | x2 | 1.806 | >= | 1 |
| | x3 | 1.000 | >= | 1 |

**Problem 12-10.3 (d)**
**Evolutionary Solver**

| | | | | |
|---|---|---|---|---|
| Min | f(x) | 38.796 | | |
| s.t. | g1(x) | 4.941 | = | 5 |
| | x1 | 2.146 | >= | 1 |
| | x2 | 1.783 | >= | 1 |
| | x3 | 1.012 | >= | 1 |

Problem 12-10.3(e) - use LINGO

```
MIN =     X1^3 + 4 * X2^2 + 16*X3;
          X1   + X2 + X3          = 5;
          X1                     >= 1;
                X2               >= 1;
                     X3          >= 1;
```

```
 Local optimal solution found at iteration:          34
 Objective value:                              39.60766

                       Variable           Value        Reduced Cost
                             X1        2.194335            0.000000
                             X2        1.805665            0.000000
                             X3        1.000000            0.000000

                            Row    Slack or Surplus      Dual Price
                              1        39.60766           -1.000000
                              2        0.000000           -14.44532
                              3        1.194335            0.000000
                              4       0.8056651            0.000000
                              5        0.000000           -1.554676
```

## Problem 12-10 4

a)

| | A | B | C | D | E | F |
|---|---|---|---|---|---|---|
| 1 | | 0 | | Starting | | |
| 2 | | <= | | Point | x* | Profit* |
| 3 | x = | 3.537 | | | 3.537 | 6.801 |
| 4 | | <= | | 0 | 0.405 | 10.735 |
| 5 | | 5 | | 1 | 0.405 | 10.735 |
| 6 | | | | 2 | 3.537 | 6.801 |
| 7 | Profit = | $x^5 - 13x^4 + 59x^3 - 107x^2 + 61x$ | | 3 | 3.537 | 6.801 |
| 8 | = | 6.801 | | 4 | 3.537 | 6.801 |
| 9 | | | | 5 | 5 | 5 |

b)

| | A | B |
|---|---|---|
| 1 | | 0 |
| 2 | | <= |
| 3 | x = | 0.405 |
| 4 | | <= |
| 5 | | 5 |
| 6 | | |
| 7 | Profit = | $x^5 - 13x^4 + 59x^3 - 107x^2 + 61x$ |
| 8 | = | 10.735 |

## Problem 12-10. 5

a)

| | A | B | C | D | E | F |
|---|---|---|---|---|---|---|
| 1 | | 0 | | Starting | | |
| 2 | | <= | | Point | x* | Profit* |
| 3 | x = | 1.187 | | | 1.187 | 753.451 |
| 4 | | <= | | 0 | 0 | 0 |
| 5 | | 5 | | 1 | 1.187 | 753.451 |
| 6 | | | | 2 | 1.187 | 753.451 |
| 7 | Profit = | $100x^6 - 1{,}359x^5 + 6{,}836x^4 - 15{,}670x^3 + 15{,}870x^2 - 5{,}095x$ | | 3 | 3.184 | 906.902 |
| 8 | = | 753.451 | | 4 | 3.184 | 906.902 |
| 9 | | | | 5 | 5 | 650 |

b)

| | A | B |
|---|---|---|
| 1 | | 0 |
| 2 | | <= |
| 3 | x = | 3.184 |
| 4 | | <= |
| 5 | | 5 |
| 6 | | |
| 7 | Profit = | $100x^6 - 1{,}359x^5 + 6{,}836x^4 - 15{,}670x^3 + 15{,}870x^2 - 5{,}095x$ |
| 8 | = | 906.902 |

Problem 12-10. 6

| | A | B | C | D | E | F | G | H | I | J | K | L | M | N | O | P |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | City | Democrat | Republican | Total | | | | District | | | | | | | | |
| 2 | 1 | 152 | 62 | 214 | | 1 | <= | 3 | <= | 10 | | | Min District Population | | 150 | |
| 3 | 2 | 81 | 59 | 140 | | 1 | <= | 4 | <= | 10 | | | Max District Population | | 350 | |
| 4 | 3 | 75 | 83 | 158 | | 1 | <= | 8 | <= | 10 | | | Number of Districts | | 10 | |
| 5 | 4 | 34 | 52 | 86 | | 1 | <= | 6 | <= | 10 | | | | | | |
| 6 | 5 | 62 | 87 | 149 | | 1 | <= | 5 | <= | 10 | | | | | | |
| 7 | 6 | 38 | 87 | 125 | | 1 | <= | 5 | <= | 10 | | | | | | |
| 8 | 7 | 48 | 69 | 117 | | 1 | <= | 7 | <= | 10 | | District | Democrat | Republican | Total | Winner |
| 9 | 8 | 74 | 49 | 123 | | 1 | <= | 1 | <= | 10 | | 1 | 119 | 131 | 250 | Republican |
| 10 | 9 | 98 | 62 | 160 | | 1 | <= | 7 | <= | 10 | | 2 | 140 | 151 | 291 | Republican |
| 11 | 10 | 66 | 72 | 138 | | 1 | <= | 9 | <= | 10 | | 3 | 152 | 62 | 214 | Democrat |
| 12 | 11 | 83 | 75 | 158 | | 1 | <= | 6 | <= | 10 | | 4 | 174 | 127 | 301 | Democrat |
| 13 | 12 | 86 | 82 | 168 | | 1 | <= | 9 | <= | 10 | | 5 | 100 | 174 | 274 | Republican |
| 14 | 13 | 72 | 83 | 155 | | 1 | <= | 10 | <= | 10 | | 6 | 117 | 127 | 244 | Republican |
| 15 | 14 | 28 | 53 | 81 | | 1 | <= | 2 | <= | 10 | | 7 | 146 | 131 | 277 | Democrat |
| 16 | 15 | 112 | 98 | 210 | | 1 | <= | 2 | <= | 10 | | 8 | 75 | 83 | 158 | Republican |
| 17 | 16 | 45 | 82 | 127 | | 1 | <= | 1 | <= | 10 | | 9 | 152 | 154 | 306 | Republican |
| 18 | 17 | 93 | 68 | 161 | | 1 | <= | 4 | <= | 10 | | 10 | 144 | 181 | 325 | Republican |
| 19 | 18 | 72 | 98 | 170 | | 1 | <= | 10 | <= | 10 | | | | Total Republican Districts | | 7 |
| 20 | Total | 1,319 | 1,321 | | | | | | | | | | | | | |

Problem 12-10. 7

a)

| | B | C | D | E | F | G |
|---|---|---|---|---|---|---|
| 3 | | Doors | Windows | | | |
| 4 | Unit Profit | $300 | $500 | | | |
| 5 | | | | Hours | | Hours |
| 6 | | Hours Used Per Unit Produced | | Used | | Available |
| 7 | Plant 1 | 1 | 0 | 2 | Š | 4 |
| 8 | Plant 2 | 0 | 2 | 12 | Š | 12 |
| 9 | Plant 3 | 3 | 2 | 18 | Š | 18 |
| 10 | | | | | | |
| 11 | | Doors | Windows | | | Total Profit |
| 12 | Units Produced | 2 | 6 | | | $3,600 |

b)

| | B | C | D | E | F | G |
|---|---|---|---|---|---|---|
| 3 | | Doors | Windows | | | |
| 4 | Unit Profit | $300 | $500 | | | |
| 5 | | | | Hours | | Hours |
| 6 | | Hours Used Per Unit Produced | | Used | | Available |
| 7 | Plant 1 | 1 | 0 | 1.96 | Š | 4 |
| 8 | Plant 2 | 0 | 2 | 11.80 | Š | 12 |
| 9 | Plant 3 | 3 | 2 | 17.68 | Š | 18 |
| 10 | | | | | | |
| 11 | | Doors | Windows | | | Total Profit |
| 12 | Units Produced | 1.959 | 5.902 | | | $3,538 |
| 13 | | <= | <= | | | |
| 14 | | 10 | 10 | | | |

c) The standard Solver gives a better solution and finds it *much* more quickly. The standard Solver is much better suited to linear programs than is the Evolutionary Solver.

12.11-1. a) Yes. $f(\underset{\sim}{x}) = f_1(x_1) + f_2(x_2)$

$$f_1(x_1) = 4x_1 - x_1^2 \qquad \frac{\partial^2 f_1}{\partial x_1^2} = -2 < 0$$
$$f_2(x_2) = 10x_2 - x_2^2 \qquad \frac{\partial^2 f_2}{\partial x_2^2} = -2 < 0$$

So $f(\underset{\sim}{x})$ is concave.

$g(x) = x_1^2 + 4x_2^2$ is convex (again $g = g_1(x_1) + g_2(x_2)$, $g_1(x_1) = x_1^2$, $g_2(x_2) = 4x_2^2$ both convex)

So this is a convex programming problem.

b) No. This is not a quadratic programming problem since the constraints are not linear.

c) No. The Frank-Wolfe algorithm in 13.9 requires linear constraints so cannot be used for this problem.

d) KKT conditions are:

1a) $4 - 2x_1 - 2x_1 u \le 0$
1b) $x_1(4 - 2x_1 - 2x_1 u) = 0$
2a) $10 - 2x_2 - 8x_2 u \le 0$
2b) $x_2(10 - 2x_2 - 8x_2 u) = 0$
3) $x_1^2 + 4x_2^2 - 16 \le 0$
4) $u(x_1^2 + 4x_2^2 - 16) = 0$
5) $x_1 \ge 0,\ x_2 \ge 0$
6) $u \ge 0$

If $x_1 = 1$, (1b) $\Rightarrow u = 1$ but this violates (4) so $(x_1, x_2) = (1, 1)$ cannot be optimal.

e) Let $x_1 = x_{11} + x_{12} + x_{13} + x_{14}$, $x_2 = x_{21} + x_{22}$

Obj: $f_1(x_1) = 4x_1 - x_1^2$, $f_2(x_2) = 10x_2 - x_2^2$

$f_1(0) = 0,\ f_1(1) = 3,\ f_1(2) = 4,\ f_1(3) = 3,\ f_1(4) = 0$

$f_2(0) = 0,\ f_2(1) = 9,\ f_2(2) = 16$

$s_{11} = \frac{3-0}{1-0} = 3,\ s_{12} = \frac{4-3}{2-1} = 1,\ s_{13} = -1,\ s_{14} = -3$

$s_{21} = 9,\ s_{22} = 7$

Constraints: $g_1(x_1) = x_1^2$, $g_2(x_2) = 4x_2^2$

$g_1(0) = 0,\ g_1(1) = 1,\ g_1(2) = 4,\ g_1(3) = 9,\ g_1(4) = 16$

$g_2(0) = 0,\ g_2(1) = 4,\ g_2(2) = 16$

$t_{11} = \frac{1-0}{1-0} = 1,\ t_{12} = \frac{4-1}{2-1} = 3,\ t_{13} = 5,\ t_{14} = 7$

$t_{21} = 4,\ t_{22} = 12$

12.11-1. e) (cont')

The approximate LP is:

Maximize $3X_{11} + X_{12} - X_{13} - 3X_{14} + 9X_{21} + 7X_{22}$

subject to: $X_{11} + 3X_{12} + 5X_{13} + 7X_{14} + 4X_{21} + 12X_{22} \le 16$

$0 \le X_{ij} \le 1$ (all $i,j$)

f) The Simplex Method gives the solution:

Value of the
Objective Function: Z = 18.4166667

| Variable | | Value |
|---|---|---|
| $x_1$ | $(X_{11})$ | 1 |
| $x_2$ | $(X_{12})$ | 0 |
| $x_3$ | $(X_{13})$ | 0 |
| $x_4$ | $(X_{14})$ | 0 |
| $x_5$ | $(X_{21})$ | 1 |
| $x_6$ | $(X_{22})$ | 0.91667 |

In terms of the original variables, we have

$X_1 = X_{11}+X_{12}+X_{13}+X_{14} = 1$ , $X_2 = X_{21}+X_{22} = 1.91667$

g) $P(\underline{x}; r) = 4x_1 - x_1^2 + 10x_2 - x_2^2 - r\left[\frac{1}{16 - x_1^2 - 4x_2^2} + \frac{1}{x_1} + \frac{1}{x_2}\right]$

h)

| k | r | $x_1$ | $x_2$ | f(**x**) |
|---|---|---|---|---|
| 0 | | 2 | 1 | 13 |
| 1 | 1 | 1.504 | 1.754 | 18.22 |
| 2 | 0.01 | 1.409 | 1.862 | 18.8 |
| 3 | 0.0001 | 1.41 | 1.871 | 18.86 |
| 4 | 0.000001 | 1.411 | 1.871 | 18.86 |

**Problem 12-11.1 (i)**
**Standard Solver**

| | | | | |
|---|---|---|---|---|
| Max | f(x) = | 4x1-x1^2+10x2-x2^2 | | |
| | = | 18.865 | | |
| s.t. | g1(x) = | x1^2 + 4*x2^2 | | |
| | = | 16.000 | <= | 16 |
| | x1 = | 1.411 | >= | 0 |
| | x2 = | 1.871 | >= | 0 |

**Problem 12-11.1 (j)**
**Evolutionary Solver**

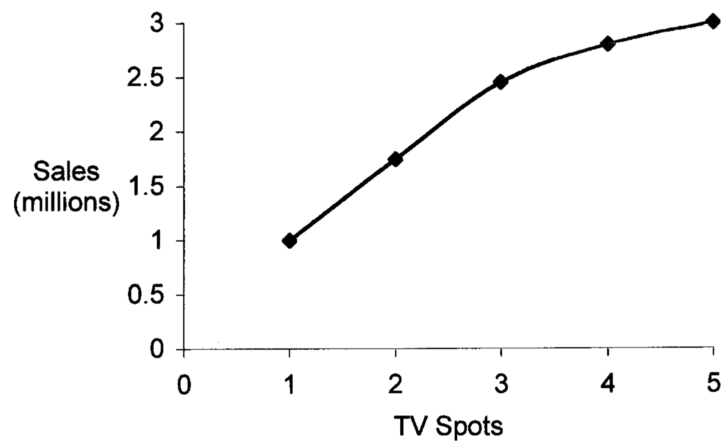
| | | | | |
|---|---|---|---|---|
| Max | f(x) = | 4x1-x1^2+10x2-x2^2 | | |
| | = | 18.865 | | |
| s.t. | g1(x) = | x1^2 + 4*x2^2 | | |
| | = | 16.000 | <= | 16 |
| | x1 = | 1.407 | >= | 0 |
| | x2 = | 1.872 | >= | 0 |

```
Problem 12-11.1(k) Use LINGO Solver;

MAX =    4*X1 - X1^2 + 10*X2 - X2^2;
         X1^2 + 4 * X2^2       <= 16;
         X1                    >= 0;
                    X2         >= 0;

 Local optimal solution found at iteration:           63
 Objective value:                                18.86516


                      Variable           Value        Reduced Cost
                            X1        1.410531            0.000000
                            X2        1.871524         0.1287136E-07

                           Row    Slack or Surplus      Dual Price
                             1        18.86516            1.000000
                             2        0.000000           0.4179049
                             3        1.410531            0.000000
                             4        1.871524            0.000000
```

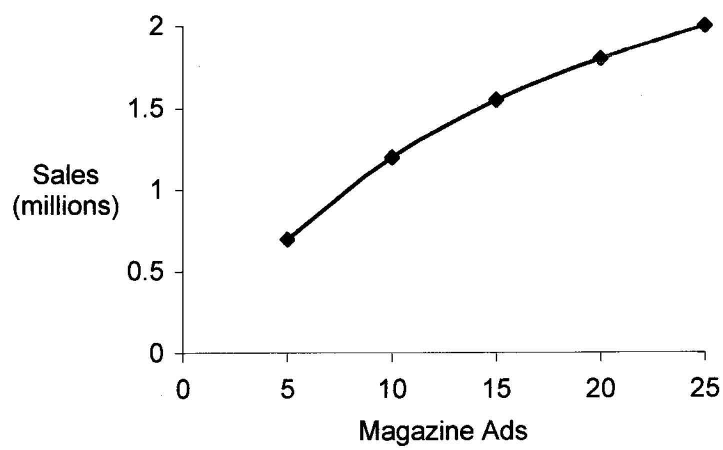
## Cases

12.1 a) TV Spots

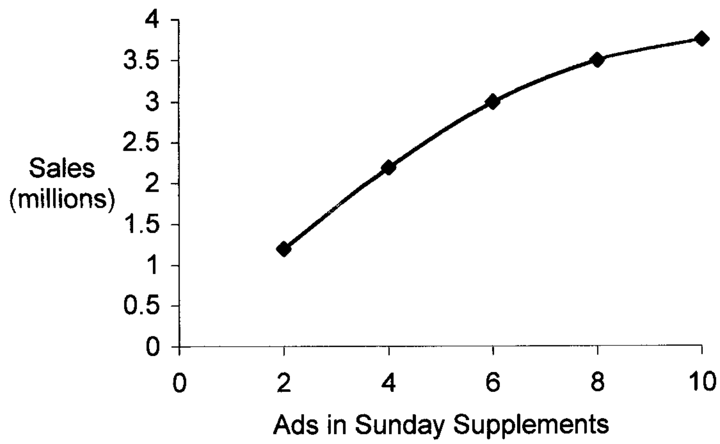
| TV Spots | Sales (millions) |
|---|---|
| 1 | 1 |
| 2 | 1.75 |
| 3 | 2.45 |
| 4 | 2.8 |
| 5 | 3 |




Magazine Ads

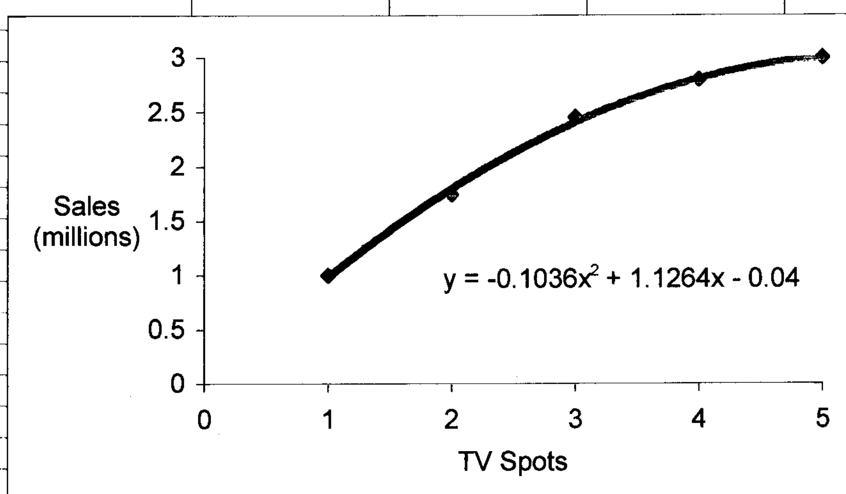


Ads in Sunday Supplements

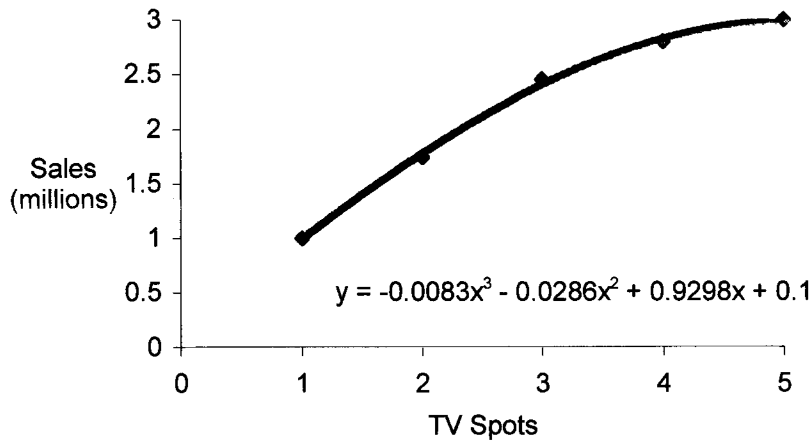



b) TV Spots (Polynomial of Order 2)

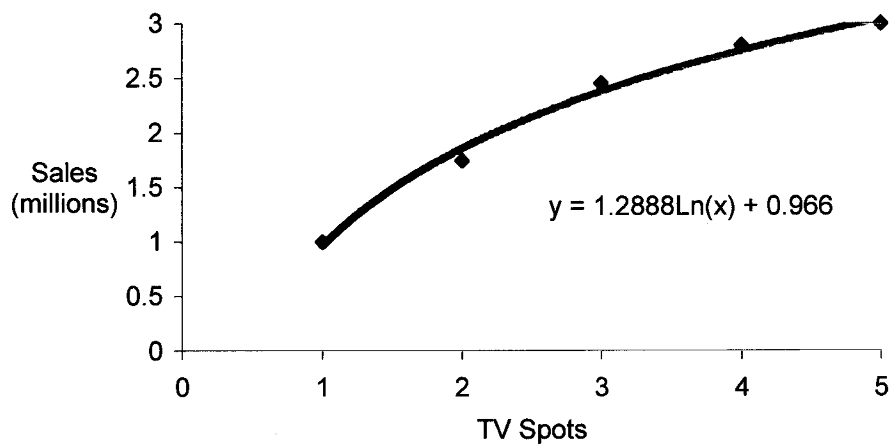
| | A | B | C | D | E |
|---|---|---|---|---|---|
| 1 | | Sales | | | |
| 2 | TV Spots | (millions) | | | |
| 3 | 1 | 1 | | | |
| 4 | 2 | 1.75 | | | |
| 5 | 3 | 2.45 | | | |
| 6 | 4 | 2.8 | | | |
| 7 | 5 | 3 | | | |
| 8 | | | | | |
| 9 | | | | | |
| 10 | | | | | |
| 11 | | | | | |
| 12 | | | | | |
| 13 | | | | | |
| 14 | | | | | |
| 15 | | | | | |
| 16 | | | | | |
| 17 | | | | | |
| 18 | | | | | |
| 19 | | | | | |
| 20 | | | | | |
| 21 | | | | | |
| 22 | | | | | |
| 23 | | | | | |
| 24 | | | | | |
| 25 | | | | | |



TV Spots (Polynomial of Order 3)

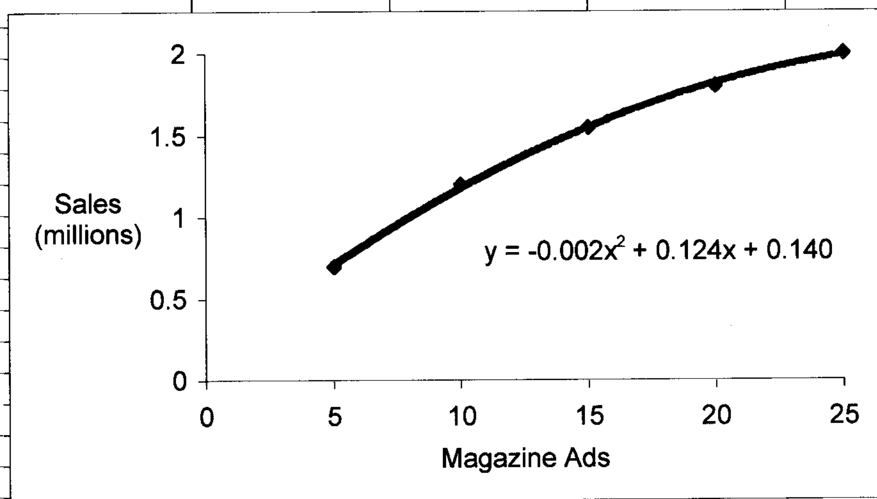



TV Spots (Logarithmic Form)

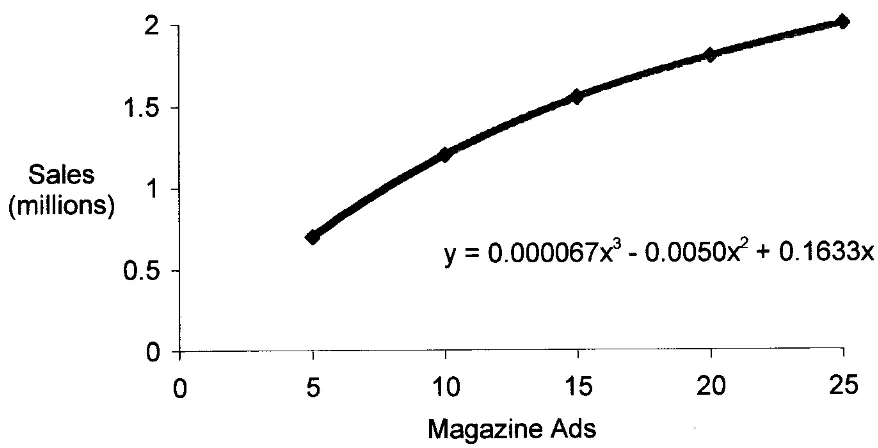


Magazine Ads (Polynomial of Order 2)

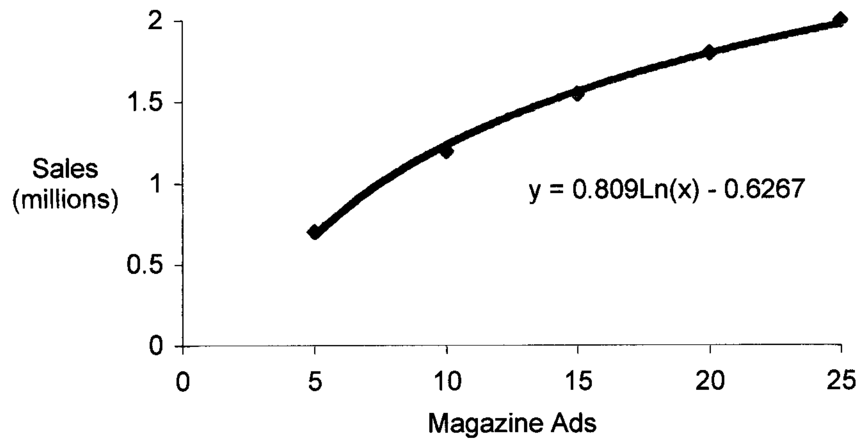
| | A | B | C | D | E |
|---|---|---|---|---|---|
| 1 | | Sales | | | |
| 2 | Magazine Ads | (millions) | | | |
| 3 | 5 | 0.7 | | | |
| 4 | 10 | 1.2 | | | |
| 5 | 15 | 1.55 | | | |
| 6 | 20 | 1.8 | | | |
| 7 | 25 | 2 | | | |
| 8 | | | | | |
| 9 | | | | | |
| 10 | | | | | |
| 11 | | | | | |
| 12 | | | | | |
| 13 | | | | | |
| 14 | | | | | |
| 15 | | | | | |
| 16 | | | | | |
| 17 | | | | | |
| 18 | | | | | |
| 19 | | | | | |
| 20 | | | | | |
| 21 | | | | | |
| 22 | | | | | |
| 23 | | | | | |
| 24 | | | | | |
| 25 | | | | | |




Magazine Ads (Polynomial of Order 3)

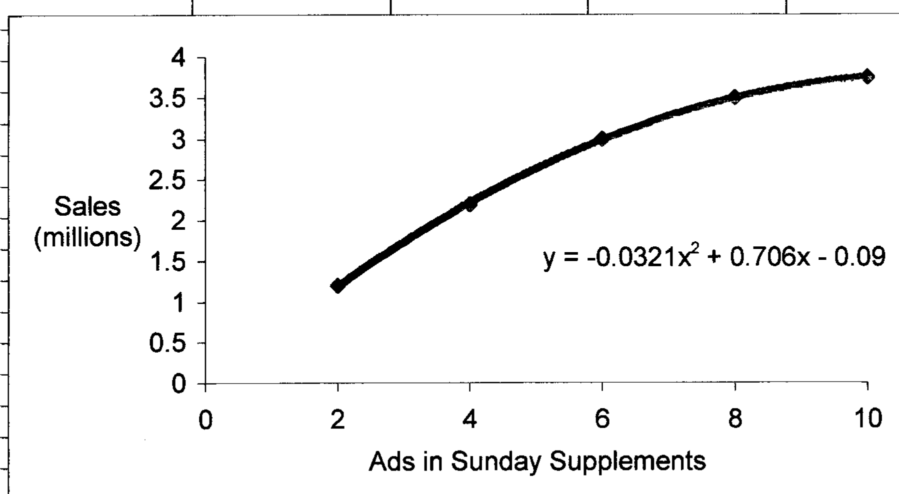


Magazine Ads (Logarithmic Form)

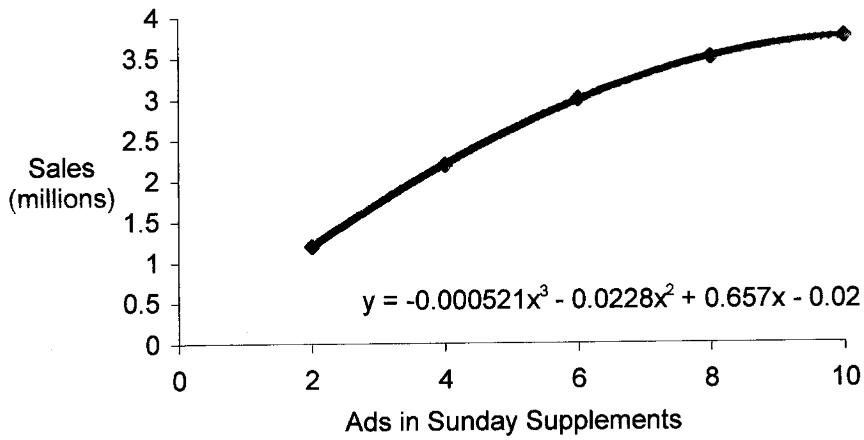



Ads in Sunday Supplements (Polynomial of Order 2)

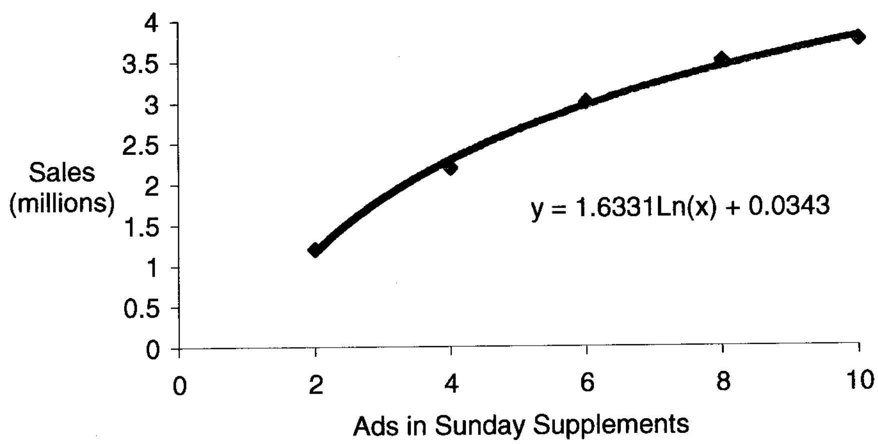
| | A | B | C | D | E |
|---|---|---|---|---|---|
| 1 | Ads in Sunday | Sales | | | |
| 2 | Supplements | (millions) | | | |
| 3 | 2 | 1.2 | | | |
| 4 | 4 | 2.2 | | | |
| 5 | 6 | 3 | | | |
| 6 | 8 | 3.5 | | | |
| 7 | 10 | 3.75 | | | |
| 8 | | | | | |
| 9 | | | | | |
| 10 | | | | | |
| 11 | | | | | |
| 12 | | | | | |
| 13 | | | | | |
| 14 | | | | | |
| 15 | | | | | |
| 16 | | | | | |
| 17 | | | | | |
| 18 | | | | | |
| 19 | | | | | |
| 20 | | | | | |
| 21 | | | | | |
| 22 | | | | | |
| 23 | | | | | |
| 24 | | | | | |
| 25 | | | | | |



Ads in Sunday Supplements (Polynomial of Order 3)




Ads in Sunday Supplements (Logarithmic Form)




In all three cases, the quadratic form is a close fit. The order-3 polynomial is also a good fit. The logarithmic form is not a bad fit, but not as close as the polynomial forms. We will use the quadratic form for the remainder of the case.

c) Let $TV$ = number of TV spots
$M$ = number of magazine ads
$SS$ = number of ads in sunday supplements

Based on the results in part *b*, using the quadratic form in each case,

Sales (in millions) = $-0.1036TV^2 + 1.1264TV - 0.04 - 0.002M^2 + 0.124M + 0.14 - 0.0321SS^2 + 0.706SS - 0.09$

Cost of Ads ($million) = $0.3TV + 0.15M + 0.1SS$

Planning Cost ($million) = $0.09TV + 0.03M + 0.04SS$

and then

Profit = ($0.75)(Sales) – Cost of Ads – Planning Cost

d) The total sales generated are calculated in row 7 using the nonlinear equations from part *b*. Then, the gross profit from sales are calculated in H20. The TotalProfit (H23) is the gross profit minus the cost of ads minus the planning cost. Maximizing the TotalProfit yields the following solution.

| | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|
| 3 | Sales per Ad = ax^2 + bx + k, where | TV Spots | Magazine Ads | SS Ads | | | |
| 4 | a= | -0.1036 | -0.002 | -0.0321 | | | |
| 5 | b= | 1.1264 | 0.124 | 0.706 | | | |
| 6 | k = | -0.0400 | 0.14 | -0.09 | Total | | Gross Profit per Sale |
| 7 | Sales Generated (millions) | 2.8296 | 0.5600 | 3.7903 | 7.1799 | | $0.75 |
| 8 | | | | | | | |
| 9 | | Cost per Ad ($thousands) | | | Budget Spent | | Budget Available |
| 10 | Ad Budget | 300 | 150 | 100 | 2,884 | <= | 4,000 |
| 11 | Planning Budget | 90 | 30 | 40 | 923 | <= | 1,000 |
| 12 | | | | | | | |
| 13 | | Number Reached per Ad (millions) | | | Total Reached | | Minimum Acceptable |
| 14 | Young Children | 1.2 | 0.1 | 0 | 5.25 | >= | 5 |
| 15 | Parents of Young Children | 0.5 | 0.2 | 0.2 | 5.00 | >= | 5 |
| 16 | | | | | | | |
| 17 | | TV Spots | Magazine Ads | SS Ads | Total Redeemed | | Required Amount |
| 18 | Coupon Redemption per Ad | 0 | 40 | 120 | 1,490 | = | 1,490 |
| 19 | ($thousands) | | | | | | |
| 20 | | | | | Gross Profit | | 5.385 |
| 21 | | TV Spots | Magazine Ads | SS Ads | Cost of Ads | | 2.884 |
| 22 | Number of Ads | 4.075 | 3.596 | 11.218 | Planning Cost | | 0.923 |
| 23 | | <= | | | Total Profit | | 1.578 |
| 24 | Maximum TV Spots | 5 | | | | | ($million) |

| | B | C | D |
|---|---|---|---|
| 7 | Sales Generated (millions) | =a*(NumberOfAds)^2+b*NumberOfAds+k | =a*(NumberOfAds)^2+b*NumberOfAds+k |

| Range Name | Cells |
|---|---|
| a | C4:E4 |
| b | C5:E5 |
| BudgetAvailable | H9:H10 |
| BudgetSpent | F9:F10 |
| CostPerAd | C9:E10 |
| CouponRedemptionPerAd | C17:E17 |
| GrossProfitPerSale | H7 |
| k | C6:E6 |
| MaxTVSpots | C23 |
| MinimumAcceptable | H13:H14 |
| NumberOfAds | C21:E21 |
| NumberReachedPerAd | C13:E14 |
| RequiredAmount | H17 |
| SalesGenerated | C7:E7 |
| Total Profit | H21 |
| TotalReached | F13:F14 |
| TotalRedeemed | F17 |
| TotalSales | F7 |
| TVSpots | C21 |

| | G | H |
|---|---|---|
| 20 | Gross Profit | =GrossProfitPerSale*TotalSales |
| 21 | Cost of Ads | =F10/1000 |
| 22 | Planning Cost | =F11/1000 |
| 23 | Total Profit | =H20-H21-H22 |
| 24 | | ($million) |

e) The separable programming formulation is as follows.

| | B | C | D | E | F | G | H | I |
|---|---|---|---|---|---|---|---|---|
| 3 | Sales per Ad | TV Spots | Magazine Ads | SS Ads | | | | |
| 4 | Group 1 | 1 | 0.14 | 0.6 | | | | |
| 5 | Group 2 | 0.75 | 0.1 | 0.5 | | | | |
| 6 | Group 3 | 0.7 | 0.07 | 0.4 | | | | |
| 7 | Group 4 | 0.35 | 0.05 | 0.25 | | | | |
| 8 | Group 5 | 0.2 | 0.04 | 0.125 | | | | |
| 9 | | | | | | | | |
| 10 | | | | | | | Budget | |
| 11 | | | Cost per Ad ($thousands) | | Budget Spent | | Available | |
| 12 | Ad Budget | 300 | 150 | 100 | 3,156 | <= | 4,000 | |
| 13 | Planning Budget | 90 | 30 | 40 | 938 | <= | 1,000 | |
| 14 | | | | | | | | |
| 15 | | | Number Reached per Ad (millions) | | Total Reached | | Min. Acceptable | |
| 16 | Young Children | 1.2 | 0.1 | 0 | 5.00 | >= | 5 | |
| 17 | Parents of Young Children | 0.5 | 0.2 | 0.2 | 5.23 | >= | 5 | |
| 18 | | | | | | | | |
| 19 | | TV Spots | Magazine Ads | SS Ads | Total Redeemed | | Req. Amount | |
| 20 | Coupon Redemption per Ad | 0 | 40 | 120 | 1,490 | = | 1,490 | |
| 21 | ($thousands) | | | | | | | |
| 22 | | | | | | | Maximum | |
| 23 | Number of Ads | TV Spots | Magazine Ads | SS Ads | | TV Spots | Magazine Ads | SS Ads |
| 24 | Group 1 | 1.000 | 5.000 | 2.000 | <= | 1 | 5 | 2 |
| 25 | Group 2 | 1.000 | 2.250 | 2.000 | <= | 1 | 5 | 2 |
| 26 | Group 3 | 1.000 | 0.000 | 2.000 | <= | 1 | 5 | 2 |
| 27 | Group 4 | 0.563 | 0.000 | 2.000 | <= | 1 | 5 | 2 |
| 28 | Group 5 | 0.000 | 0.000 | 2.000 | <= | 1 | 5 | 2 |
| 29 | Total | 3.563 | 7.250 | 10.000 | | | | |
| 30 | | <= | | | | Total Sales | 7.3219 | |
| 31 | Maximum TV Spots | 5 | | | | Gross Profit per Sale | $0.75 | |
| 32 | | | | | | | | |
| 33 | | | | | | Gross Profit | 5.491 | |
| 34 | | | | | | Cost of Ads | 3.156 | |
| 35 | | | | | | Planning Cost | 0.938 | |
| 36 | | | | | | Total Profit | 1.397 | |
| 37 | | | | | | | ($million) | |

| Range Name | Cells |
|---|---|
| BudgetAvailable | H12:H13 |
| BudgetSpent | F12:F13 |
| CostPerAd | C12:E13 |
| CouponRedemptionPerAd | C20:E20 |
| GrossProfitPerSale | H31 |
| Maximum | G24:I28 |
| MaxTVSpots | C31 |
| MinimumAcceptable | H16:H17 |
| NumberOfAds | C24:E28 |
| RequiredAmount | H20 |
| SalesPerAd | C4:E8 |
| TotalAds | C29:E29 |
| TotalProfit | H36 |
| TotalReached | F16:F17 |
| TotalRedeemed | F20 |
| TotalSales | H30 |
| TVSpots | C29 |

| | G | H |
|---|---|---|
| 30 | Total Sales | =SUMPRODUCT(SalesPerAd,NumberOfAds) |
| 31 | Gross Profit per Sale | 0.75 |
| 32 | | |
| 33 | Gross Profit | =GrossProfitPerSale*TotalSales |
| 34 | Cost of Ads | =F12/1000 |
| 35 | Planning Cost | =F13/1000 |
| 36 | Total Profit | =H33-H34-H35 |
| 37 | | ($million) |

f) In part *d*, 4.075 TV ads, 3.596 magazine ads, and 11.218 ads in Sunday supplements are placed. In part *e*, 3.563 TV ads, 7.25 magazine ads, and 10 ads in Sunday supplements are placed. In Figure 4.6, 3 TV ads, 14 magazine ads, and 7.75 ads in Sunday supplements are placed. Unlike the linear programming solution, the nonlinear and separable programs take into account the diminishing returns from repeated advertisements. Since the solution is fairly different, it certainly appears that it was worthwhile to refine the linear programming model used in Figure 4.6.

Case 12-2 a) If Lydia wants to ignore the risk of her investment she should invest all her money into the stock that promises the highest expected return. According to the predictions of the investment advisors the expected returns equal 20% for Bigbell, 42% for Lotsofplace, 100% for Internetlife, 50% for Healthtomorrow, 46% for Quicky, and 30% for Automobile Alliance. Therefore, she should invest 100% of her money into Internetlife. The risk (variance) of this portfolio equals 0.333.

b) Lydia should put 40% of her money into the stock with the highest expected returns, 40% into the stock with the second highest expected returns, and 20% into the stock with the third highest expected returns.

We can also find this intuitive solution by solving a simple linear programming problem, in which we maximize

Max ExpectedReturn = SUMPRODUCT(Portfolio, StockExpectedReturn)

subject to

Total = OneHundredPercent

Portfolio ≤ MaxInSingleStock

The spreadsheet formulation and solution follows. on the next page.

c) The risk of Lydia's portfolio is a quadratic function of her decision variables. We, therefore, apply quadratic programming to her decision problem.

| | A | B | C | D | E | F | G | H | I | J |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | | BB | LOP | ILI | HEAL | QUI | AUA | | | |
| 2 | **Expected Return** | 20% | 42% | 100% | 50% | 46% | 30% | | | |
| 3 | | | | | | | | | | |
| 4 | **Covariance Matrix** | | | | | | | | | |
| 5 | **(Variance on Diagonal)** | BB | LOP | ILI | HEAL | QUI | AUA | | | |
| 6 | BB | 0.032 | 0.005 | 0.030 | -0.031 | -0.027 | 0.010 | | | |
| 7 | LOP | 0.005 | 0.1 | 0.085 | -0.07 | -0.05 | 0.020 | | | |
| 8 | ILI | 0.030 | 0.085 | 0.333 | -0.11 | -0.02 | 0.042 | | | |
| 9 | HEAL | -0.031 | -0.07 | -0.11 | 0.125 | 0.05 | -0.060 | | | |
| 10 | QUI | -0.027 | -0.05 | -0.02 | 0.05 | 0.065 | -0.020 | | | |
| 11 | AUA | 0.010 | 0.020 | 0.042 | -0.060 | -0.020 | 0.08 | | | |
| 12 | | | | | | | | | | |
| 13 | | BB | LOP | ILI | HEAL | QUI | AUA | **Total** | | |
| 14 | **Portfolio** | 0% | 0% | 40% | 40% | 20% | 0% | 100% | = | 100% |
| 15 | | ≤ | ≤ | ≤ | ≤ | ≤ | ≤ | | | |
| 16 | Max in Single Stock | 40% | 40% | 40% | 40% | 40% | 40% | | | |
| 17 | | | | | | | | | | |
| 18 | | **Portfolio** | | | | | | | | |
| 19 | **Expected Return =** | 69.2% | | | | | | | | |
| 20 | | | | | | | | | | |
| 21 | **Risk (Variance) =** | 0.04548 | | | | | | | | |

| Range Name | Cells |
|---|---|
| CovarianceMatrix | B6:G11 |
| MaxInSingleStock | B16:G16 |
| OneHundredPercent | J14 |
| Portfolio | B14:G14 |
| PortfolioExpectedReturn | B19 |
| StockExpectedReturn | B2:G2 |
| Total | H14 |
| Variance | B21 |

| | H |
|---|---|
| 13 | **Total** |
| 14 | =SUM(Portfolio) |

| | A | B |
|---|---|---|
| 18 | | **Portfolio** |
| 19 | **Expected Return =** | =SUMPRODUCT(StockExpectedReturn,Portfolio) |

| | A | B |
|---|---|---|
| 21 | Risk (Variance) = | =SUMPRODUCT(MMULT(Portfolio,CovarianceMatrix),Portfolio) |

The total expected return of her portfolio is 69.2% with a total variance of 0.04548.

d) The return of Lydia's portfolio is no longer a part of the objective but now becomes part of a new constraint: PortfolioExpectedReturn (C21) ≥ 35% (MinimumExpectedReturn). The objective is now to minimize the risk.

| | A | B | C | D | E | F | G | H | I | J |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | | BB | LOP | ILI | HEAL | QUI | AUA | | | |
| 2 | **Expected Return** | 20% | 42% | 100% | 50% | 46% | 30% | | | |
| 3 | | | | | | | | | | |
| 4 | **Covariance Matrix** | | | | | | | | | |
| 5 | **(Variance on Diagonal)** | BB | LOP | ILI | HEAL | QUI | AUA | | | |
| 6 | BB | 0.032 | 0.005 | 0.030 | -0.031 | -0.027 | 0.010 | | | |
| 7 | LOP | 0.005 | 0.1 | 0.085 | -0.07 | -0.05 | 0.020 | | | |
| 8 | ILI | 0.030 | 0.085 | 0.333 | -0.11 | -0.02 | 0.042 | | | |
| 9 | HEAL | -0.031 | -0.07 | -0.11 | 0.125 | 0.05 | -0.060 | | | |
| 10 | QUI | -0.027 | -0.05 | -0.02 | 0.05 | 0.065 | -0.020 | | | |
| 11 | AUA | 0.010 | 0.020 | 0.042 | -0.060 | -0.020 | 0.08 | | | |
| 12 | | | | | | | | | | |
| 13 | | BB | LOP | ILI | HEAL | QUI | AUA | **Total** | | |
| 14 | **Portfolio** | 31.8% | 19.9% | 0.0% | 16.8% | 20.9% | 10.6% | 100% | = | 100% |
| 15 | | 2 | 2 | 2 | 2 | 2 | 2 | | | |
| 16 | Max in Single Stock | 40% | 40% | 40% | 40% | 40% | 40% | | | |
| 17 | | | | | | | | | | |
| 18 | | | | Minimum | | | | | | |
| 19 | | | | Expected | | | | | | |
| 20 | | Portfolio | | Return | | | | | | |
| 21 | **Expected Return =** | 35.9% | 3 | 35% | | | | | | |
| 22 | | | | | | | | | | |
| 23 | **Risk (Variance) =** | 0.00136 | | | | | | | | |

Lydia's optimal portfolio consists of 31.8% BB, 19.9% LOP, 16.8% HEAL, 20.9% QUI, and 10.6% AUA. Her expected return equals 35.9% with a risk of 0.00136.

e) Since the return constraint in not binding in part *d*, decreasing the right-hand-side will not change the optimal solution. The minimum risk for a minimum expected return of 25% is the same as for a minimum expected return of 35% (0.00136).

For a minimum expected return of 40% we obtain a new portfolio:

| | A | B | C | D | E | F | G | H | I | J |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | | BB | LOP | ILI | HEAL | QUI | AUA | | | |
| 2 | **Expected Return** | 20% | 42% | 100% | 50% | 46% | 30% | | | |
| 3 | | | | | | | | | | |
| 4 | **Covariance Matrix** | | | | | | | | | |
| 5 | **(Variance on Diagonal)** | BB | LOP | ILI | HEAL | QUI | AUA | | | |
| 6 | BB | 0.032 | 0.005 | 0.030 | -0.031 | -0.027 | 0.010 | | | |
| 7 | LOP | 0.005 | 0.1 | 0.085 | -0.07 | -0.05 | 0.020 | | | |
| 8 | ILI | 0.030 | 0.085 | 0.333 | -0.11 | -0.02 | 0.042 | | | |
| 9 | HEAL | -0.031 | -0.07 | -0.11 | 0.125 | 0.05 | -0.060 | | | |
| 10 | QUI | -0.027 | -0.05 | -0.02 | 0.05 | 0.065 | -0.020 | | | |
| 11 | AUA | 0.010 | 0.020 | 0.042 | -0.060 | -0.020 | 0.08 | | | |
| 12 | | | | | | | | | | |
| 13 | | BB | LOP | ILI | HEAL | QUI | AUA | **Total** | | |
| 14 | **Portfolio** | 22.9% | 21.0% | 3.4% | 22.0% | 18.8% | 11.9% | 100% | = | 100% |
| 15 | | 2 | 2 | 2 | 2 | 2 | 2 | | | |
| 16 | Max in Single Stock | 40% | 40% | 40% | 40% | 40% | 40% | | | |
| 17 | | | | | | | | | | |
| 18 | | | | Minimum | | | | | | |
| 19 | | | | Expected | | | | | | |
| 20 | | Portfolio | | Return | | | | | | |
| 21 | **Expected Return =** | 40.0% | 3 | 40% | | | | | | |
| 22 | | | | | | | | | | |
| 23 | **Risk (Variance) =** | 0.00233 | | | | | | | | |

Lydia's optimal portfolio consists of 22.9% BB, 21.0% LOP, 3.4% ILI, 22.0% HEAL, 18.8% QUI, and 11.9% AUA. Her expected return equals 40% with a risk of 0.00233.

f) Lydia's approach is very risky. She puts a lot of confidence in the advice of the two investment experts. Of course, Lydia cannot expect to find an optimal investment strategy with her model if the estimates she uses for the input parameters are wrong.

Case 12.3 a) When Charles sells a portion of his B-Bonds in a given year, the first DM 6100 of interest are tax-free, but the interest earnings exceeding DM 6100 are levied a 30 percent tax. Therefore, Charles encounters decreasing marginal returns, and we can use separable programming to solve this problem.

Define the following variables:

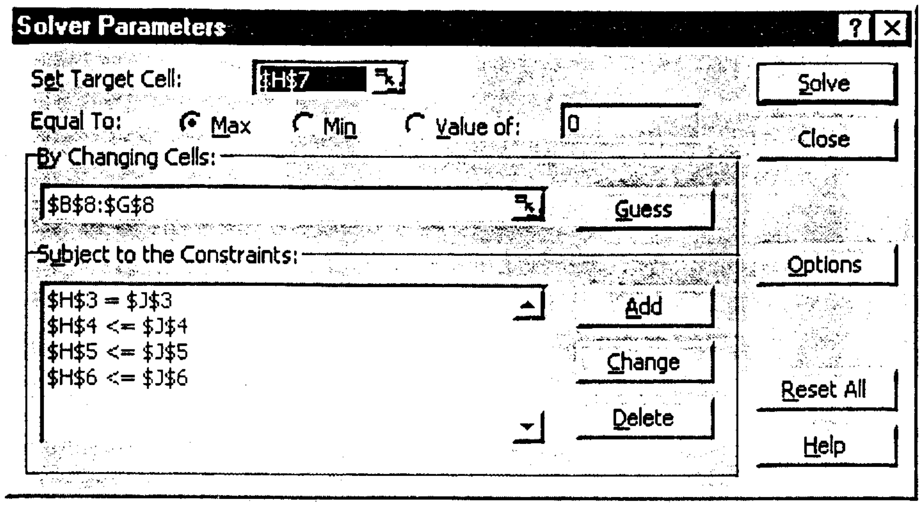
NoTax5 = The base amount of B-Bonds Charles sells in the fifth year that yield untaxed interest

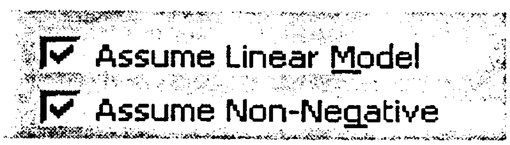
Tax5 = The base amount of B-Bonds Charles sells in the fifth year that yield taxed interest

Similarly, NoTax6, Tax6, NoTax7, and Tax7 are defined.

The sum of these six variables must equal the total of DM 30,000 that Charles invested at the beginning of year 1. When Charles sells B-Bonds with the base amount NoTax5, he earns 0.5001*NoTax5 in interest. In order for him not to pay any taxes on this amount, the interest must be less than or equal DM 6100. We have to include this constraint. Any additional base amount of B-bonds sold in year 5 yields Charles only 0.7*0.5001=0.35007. A similar reasoning applies to the variables for the other years. The objective is to maximize Charles' interest income.

| | A | B | C | D | E | F | G | H | I | J |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | | | | | | | | | | |
| 2 | Server | NoTax5 | Tax5 | NoTax6 | Tax6 | NoTax7 | Tax7 | Totals | | |
| 3 | Selling | 1 | 1 | 1 | 1 | 1 | 1 | 30000 | = | 30000 |
| 4 | Untaxed5 | 0.5001 | 0 | 0 | 0 | 0 | 0 | 0 | <= | 6100 |
| 5 | Untaxed6 | 0 | 0 | 0.6351 | 0 | 0 | 0 | 6100 | <= | 6100 |
| 6 | Untaxed7 | 0 | 0 | 0 | 0 | 0.7823 | 0 | 6100 | <= | 6100 |
| 7 | Interest | 0.5001 | 0.35007 | 0.6351 | 0.44457 | 0.7823 | 0.54761 | 19098.62 | | |
| 8 | Solution | 0 | 0 | 9604.79 | 0 | 7797.52 | 12597.69 | | | |
| 9 | | | | | | | | | | |
| 10 | | Formula in cell H3: | | | "=SUMPRODUCT(B3:G3,B8:G8)" | | | | | |
| 11 | | Formula in cell H4: | | | "=SUMPRODUCT(B4:G4,B8:G8)" | | | | | |
| 12 | | Formula in cell H5: | | | "=SUMPRODUCT(B5:G5,B8:G8)" | | | | | |
| 13 | | Formula in cell H6: | | | "=SUMPRODUCT(B6:G6,B8:G8)" | | | | | |
| 14 | | Formula in cell H7: | | | "=SUMPRODUCT(B7:G7,B8:G8)" | | | | | |




b) The optimal investment strategy for Charles is to sell a base amount of DM 9604.79 at the end of year 6 and the remaining DM 20395.21 at the end of year 7. His total after-tax interest income equals DM 19098.62.

c) When Charles sells all B-Bonds in the seventh year, then he must pay 30% of taxes on the amount of interest income exceeding DM 6100. This amount is earned interest not only from the last year, but it includes interest from all the previous years. So Charles does not pay 30% tax on the 9 percent interest he earned in the last year, but he effectively pays tax on the total interest of all the years. This tax payment decreases his after-tax interest so much that it pays for him to sell some of his bonds in the sixth year in order to take advantage of the yearly tax-free income of DM 6100. Compare the total amount of interest Charles earns if he sells tax-free after year 6 and taxed after year 7: In the former case his total interest equals 63.51% while in the latter case it is only 54.761%. Therefore, it is better to sell some bonds at the end of the sixth year than to keep them until the end of the last year.

d) The following observation greatly simplifies the analysis of this problem: The interest rate on the CD is much lower than the yearly interest rates on the B-Bonds. Therefore, it can never be optimal for Charles to sell B-Bonds in year 5 in order to buy a CD for year 6 if he does not take advantage of the maximal tax-free amount of selling B-Bonds in year 6. Put differently. Charles will only buy a CD for year 6 if he already plans to sell B-Bonds in year 6 to obtain at least the maximal tax-free amount of interest. The same argument applies to year 7. This observation implies that Charles will never earn untaxed interest on a CD. Therefore, his yearly interest on the CD will always be 0.7*0.04 = 0.028 = 2.8%.

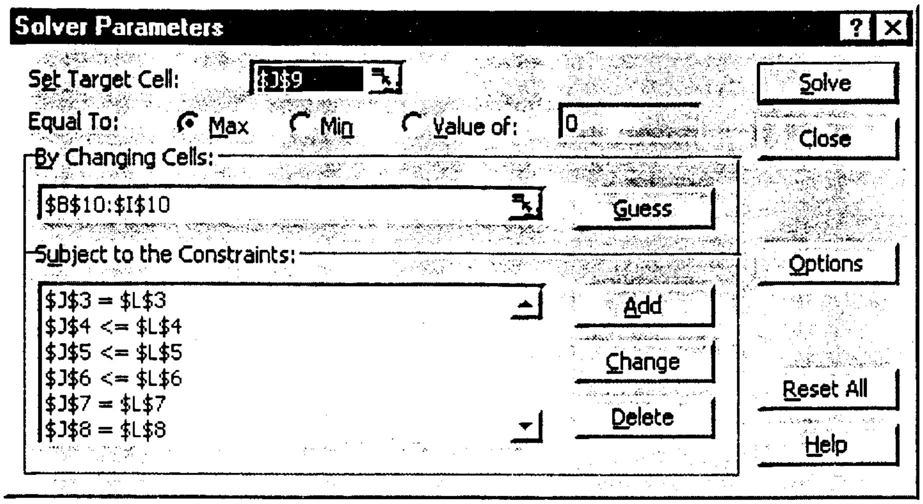
Define

CD6 = Amount invested in a CD in year 6

CD7 = Amount invested in a CD in year 7

The amount of money Charles can invest in a CD in year 6 equals the base amount of B-Bonds sold in year 5 plus the total after-tax interest earned on the base amount. This condition results in the constraint CD6 = 1.5001*NoTax5 + 1.35007*Tax5. The corresponding constraint for CD7 is CD7 = 1.6351*NoTax6 + 1.44457*Tax6 + 1.028*CD6.

| | A | B | C | D | E | F | G | H | I | J | K | L |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | | | | | | | | | | | | |
| 2 | Server | NoTax5 | Tax5 | CD6 | NoTax6 | Tax6 | CD7 | NoTax7 | Tax7 | Totals | | |
| 3 | Selling | 1 | 1 | 0 | 1 | 1 | 0 | 1 | 1 | 30000 | = | 30000 |
| 4 | Untaxed5 | 0.5001 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 6100 | <= | 6100 |
| 5 | Untaxed6 | 0 | 0 | 0 | 0.6351 | 0 | 0 | 0 | 0 | 6100 | <= | 6100 |
| 6 | Untaxed7 | 0 | 0 | 0 | 0 | 0 | 0 | 0.7823 | 0 | 6100 | <= | 6100 |
| 7 | CDInvest5 | 1.5001 | 1.35007 | -1 | 0 | 0 | 0 | 0 | 0 | 0 | = | 0 |
| 8 | CDInvest6 | 0 | 0 | 1.028 | 1.6351 | 1.44457 | -1 | 0 | 0 | 0 | = | 0 |
| 9 | Interest | 0.5001 | 0.35007 | 0.028 | 0.6351 | 0.44457 | 0.028 | 0.7823 | 0.5476 | 19997.86 | | |
| 10 | Solution | 12197.56 | 0 | 18297.56 | 9604.79 | 0 | 34514.68 | 7797.5 | 2400.133 | | | |
| 11 | | | | | | | | | | | | |
| 12 | | Formula in cell J3: | | "=SUMPRODUCT(B3:I3,B10:I10)" | | | | | | | | |
| 13 | | Formula in cell J4: | | "=SUMPRODUCT(B4:I4,B10:I10)" | | | | | | | | |
| 14 | | Formula in cell J5: | | "=SUMPRODUCT(B5:I5,B10:I10)" | | | | | | | | |
| 15 | | Formula in cell J6: | | "=SUMPRODUCT(B6:I6,B10:I10)" | | | | | | | | |
| 16 | | Formula in cell J7: | | "=SUMPRODUCT(B7:I7,B10:I10)" | | | | | | | | |
| 17 | | Formula in cell J8: | | "=SUMPRODUCT(B8:I8,B10:I10)" | | | | | | | | |
| 18 | | Formula in cell J9: | | "=SUMPRODUCT(B9:I9,B10:I10)" | | | | | | | | |




The options for the solver are the same as before.

Charles should sell the maximal base amount of B-bonds in year 5 that yields tax-free interest and then invest this money (base amount + interest) into a one-year CD for year 6. In year 6 he should sell again the maximal base amount of B-bonds that yields tax-free interest and then invest this money (base amount + interest) and the money from his CD into a one-year CD for year 7. In year 7 he should sell the remainder of the base amount of B-bonds. He again takes advantage of the maximum tax-free amount, but he also sells a base amount of DM 400.13 for which he must pay taxes on the interest earnings.

e) The right-hand side of the selling constraint needs to be changed:

| | A | B | C | D | E | F | G | H | I | J | K | L |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | | | | | | | | | | | | |
| 2 | Server | NoTax5 | Tax5 | CD6 | NoTax6 | Tax6 | CD7 | NoTax7 | Tax7 | Totals | | |
| 3 | Selling | 1 | 1 | 0 | 1 | 1 | 0 | 1 | 1 | 50000 | = | 50000 |
| 4 | Untaxed5 | 0.5001 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 6100 | <= | 6100 |
| 5 | Untaxed6 | 0 | 0 | 0 | 0.6351 | 0 | 0 | 0 | 0 | 6100 | <= | 6100 |
| 6 | Untaxed7 | 0 | 0 | 0 | 0 | 0 | 0 | 0.7823 | 0 | 6100 | <= | 6100 |
| 7 | CDInvest5 | 1.5001 | 1.35007 | -1 | 0 | 0 | 0 | 0 | 0 | 0 | = | 0 |
| 8 | CDInvest6 | 0 | 0 | 1.028 | 1.6351 | 1.44457 | -1 | 0 | 0 | 0 | = | 0 |
| 9 | Interest | 0.5001 | 0.35007 | 0.028 | 0.6351 | 0.44457 | 0.028 | 0.7823 | 0.54761 | 30950.06 | | |
| 10 | Solution | 12197.56 | 0 | 18297.56 | 9604.79 | 0 | 34514.68 | 7797.52 | 20400.1 | | | |
| 11 | | | | | | | | | | | | |
| 12 | | Formula in cell J3: | | | "=SUMPRODUCT(B3:I3,B10:I10)" | | | | | | | |
| 13 | | Formula in cell J4: | | | "=SUMPRODUCT(B4:I4,B10:I10)" | | | | | | | |
| 14 | | Formula in cell J5: | | | "=SUMPRODUCT(B5:I5,B10:I10)" | | | | | | | |
| 15 | | Formula in cell J6: | | | "=SUMPRODUCT(B6:I6,B10:I10)" | | | | | | | |
| 16 | | Formula in cell J7: | | | "=SUMPRODUCT(B7:I7,B10:I10)" | | | | | | | |
| 17 | | Formula in cell J8: | | | "=SUMPRODUCT(B8:I8,B10:I10)" | | | | | | | |
| 18 | | Formula in cell J9: | | | "=SUMPRODUCT(B9:I9,B10:I10)" | | | | | | | |

The optimal investment strategy is similar to the previous one, except that Charles must now pay taxes on the interest earned from selling a base amount of DM 20400.13 in year 7.

f) The right-hand sides of the Untaxed5, Untaxed6, and Untaxed7 constraints need to be changed.

| | A | B | C | D | E | F | G | H | I | J | K | L |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | | | | | | | | | | | | |
| 2 | Server | NoTax5 | Tax5 | CD6 | NoTax6 | Tax6 | CD7 | NoTax7 | Tax7 | Totals | | |
| 3 | Selling | 1 | 1 | 0 | 1 | 1 | 0 | 1 | 1 | 30000 | = | 30000 |
| 4 | Untaxed5 | 0.5001 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | <= | 12200 |
| 5 | Untaxed6 | 0 | 0 | 0 | 0.6351 | 0 | 0 | 0 | 0 | 9148.59 | <= | 12200 |
| 6 | Untaxed7 | 0 | 0 | 0 | 0 | 0 | 0 | 0.7823 | 0 | 12200 | <= | 12200 |
| 7 | CDInvest 5 | 1.5001 | 1.35007 | -1 | 0 | 0 | 0 | 0 | 0 | 0 | = | 0 |
| 8 | CDInvest 6 | 0 | 0 | 1.028 | 1.6351 | 1.44457 | -1 | 0 | 0 | 0 | = | 0 |
| 9 | Interest | 0.5001 | 0.35007 | 0.028 | 0.6351 | 0.44457 | 0.028 | 0.7823 | 0.54761 | 22008.09 | | |
| 10 | Solution | 0 | 0 | 0 | 14404.96 | 0 | 23553.55 | 15595.04 | 0 | | | |
| 11 | | | | | | | | | | | | |
| 12 | | Formula in cell J3: | | | "=SUMPRODUCT(B3:I3,B10:I10)" | | | | | | | |
| 13 | | Formula in cell J4: | | | "=SUMPRODUCT(B4:I4,B10:I10)" | | | | | | | |
| 14 | | Formula in cell J5: | | | "=SUMPRODUCT(B5:I5,B10:I10)" | | | | | | | |
| 15 | | Formula in cell J6: | | | "=SUMPRODUCT(B6:I6,B10:I10)" | | | | | | | |
| 16 | | Formula in cell J7: | | | "=SUMPRODUCT(B7:I7,B10:I10)" | | | | | | | |
| 17 | | Formula in cell J8: | | | "=SUMPRODUCT(B8:I8,B10:I10)" | | | | | | | |
| 18 | | Formula in cell J9: | | | "=SUMPRODUCT(B9:I9,B10:I10)" | | | | | | | |

By getting married in the fifth year, Charles can increase his interest income by (22008.09 - 19997.86) = 2010.23 German marks. He should sell the maximal base amount of B-Bonds earning tax-free interest in year 7 (DM 15595.04). The remainder of DM 14404.96 he should sell at the end of year 6. His entire interest income on this base amount will be tax-free. He then should invest the total amount (base amount + interest) in a CD for the seventh year.

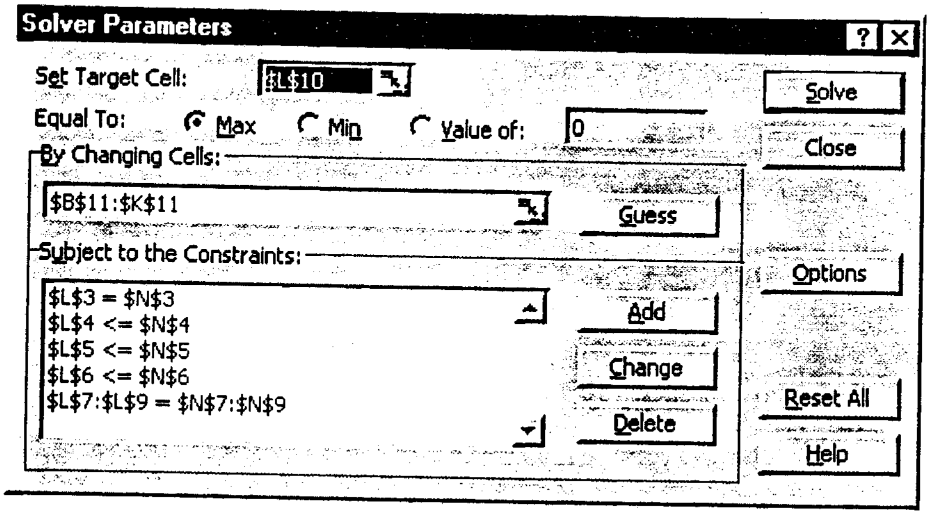
g) Instead of maximizing his interest income Charles now wants to maximize the expected dollar amount he will have at the end of the seventh year. He considers exchanging marks for dollars either at the end of year 5 or at the end of year 7. We define

CD-US = The amount of money (in dollars) Charles invests in a two-year CD at the end of year 5

US = The amount of dollars Charles converts at the end of year 7

The total amount of money in dollars Charles has at the end of year 7 equals (1.036)^2*CD-US + US; this function is the new objective. At the end of year 5, 1 US$ is assumed to equal DM 1.50, so Charles can exchange marks for dollars at this rate in year 5; this condition must be included as a constraint. Similarly, we include a constraint for the currency conversion at the end of the last year.

| | A | B | C | D | E | F | G | H | I | J | K | L | M | N |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | | | | | | | | | | | | | | |
| 2 | Server | NoTax5 | Tax5 | CD6 | CD-US | NoTax6 | Tax6 | CD7 | NoTax7 | Tax7 | US | Totals | | |
| 3 | Selling | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 1 | 1 | 0 | 30000 | = | 30000 |
| 4 | Untaxed5 | 0.5001 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 6100 | <= | 6100 |
| 5 | Untaxed6 | 0 | 0 | 0 | 0 | 0.6351 | 0 | 0 | 0 | 0 | 0 | 0 | <= | 6100 |
| 6 | Untaxed7 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0.7823 | 0 | 0 | 6100 | <= | 6100 |
| 7 | CDInvest5 | 1.5001 | 1.35007 | -1 | -1.5 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | = | 0 |
| 8 | CDInvest6 | 0 | 0 | 1.028 | 0 | 1.6351 | 1.44457 | -1 | 0 | 0 | 0 | 0 | = | 0 |
| 9 | Conversion | 0 | 0 | 0 | 0 | 0 | 0 | 1.028 | 1.7823 | 1.5461 | -1.8 | 0 | = | 0 |
| 10 | Dollars | 0 | 0 | 0 | 1.073296 | 0 | 0 | 0 | 0 | 0 | 1 | 30478.23 | | |
| 11 | Solution | 12197.56 | 10004.92 | 0 | 21203.27 | 0 | 0 | 0 | 7797.52 | 0 | 7720.84 | | | |
| 12 | | | | | | | | | | | | | | |
| 13 | | | | | | | | | | | | | | |
| 14 | | | Formula in cell L3: | | | "=SUMPRODUCT(B3:K3,B11:K11)" | | | | | | | | |
| 15 | | | Formula in cell L4: | | | "=SUMPRODUCT(B4:K4,B11:K11)" | | | | | | | | |
| 16 | | | Formula in cell L5: | | | "=SUMPRODUCT(B5:K5,B11:K11)" | | | | | | | | |
| 17 | | | Formula in cell L6: | | | "=SUMPRODUCT(B6:K6,B11:K11)" | | | | | | | | |
| 18 | | | Formula in cell L7: | | | "=SUMPRODUCT(B7:K7,B11:K11)" | | | | | | | | |
| 19 | | | Formula in cell L8: | | | "=SUMPRODUCT(B8:K8,B11:K11)" | | | | | | | | |
| 20 | | | Formula in cell L9: | | | "=SUMPRODUCT(B9:K9,B11:K11)" | | | | | | | | |
| 21 | | | Formula in cell L10: | | | "=SUMPRODUCT(B10:K10,B11:K11)" | | | | | | | | |




Charles converts DM (1.5001*12197.56 + 1.35007*10004.92) to dollars at the end of year 5. With the exchange rate of DM 1.50 for $1, he is able to invest $21203.27 in the American CD. At the end of the seventh year he converts the remaining DM (1.7823*7797.52) to dollars. The total amount of his investments at the end of year 7 will be $30478.23.