Optimization in Spreadsheets
with
What's*Best!*

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Introduction to Optimization in Spreadsheets

1.1 Introduction

This document illustrates the use of optimization in spreadsheets for solving a variety of problems in business, industry, and government. We assume the reader is familiar with the basics of using the What'sBest! optimizer as described in the What'sBest! users manual.

Some of the material used herein is based on the text, *Optimization Modeling with LINGO*. That text is concerned with the use of a general purpose modeling language for formulating and solving optimization problems.

2

Portfolio Optimization

2.1 Introduction

Financial portfolio models are concerned with investments where there are typically two criteria: *expected return* and *risk*. The investor wants the former to be high and the latter to be low. There are a variety of measures of risk. The most popular measure of risk has been variance in return. Even though there are some problems with it, we will first look at it very closely.

2.2 The Markowitz Mean/Variance Portfolio Model

The portfolio model introduced by Markowitz (1959) (see also Roy (1952)), assumes an investor has two considerations when constructing an investment portfolio: *expected return* and *variance in return* (i.e., risk). Variance measures the variability in realized return around the expected return, giving equal weight to realizations below the expected and above the expected return. The Markowitz model might be mildly criticized in this regard because the typical investor is probably concerned only with variability below the expected return, so-called *downside risk*.

The Markowitz model requires two major kinds of information: (1) the estimated expected return for each candidate investment and (2) the covariance matrix of returns. The covariance matrix characterizes not only the individual variability of the return on each investment, but also how each investment's return tends to move with other investments. We assume the reader is somewhat familiar with the concepts of *variance* and *covariance* as described in most intermediate statistics texts.

2.2.1 Example

We will use some publicly available data from Markowitz (1959). The following table shows the increase in price, including dividends, for three stocks over a twelve-year period:

		Growth in		
Year	**S&P500**	**ATT**	**GMC**	**USX**
43	1.259	1.300	1.225	1.149
44	1.198	1.103	1.290	1.260
45	1.364	1.216	1.216	1.419
46	0.919	0.954	0.728	0.922
47	1.057	0.929	1.144	1.169
48	1.055	1.056	1.107	0.965
49	1.188	1.038	1.321	1.133
50	1.317	1.089	1.305	1.732
51	1.240	1.090	1.195	1.021
52	1.184	1.083	1.390	1.131
53	0.990	1.035	0.928	1.006
54	1.526	1.176	1.715	1.908

For reference later, we have also included the change each year in the Standard and Poor's/S&P 500 stock index. To illustrate, in the first year, *ATT* appreciated in value by 30%. In the second year, *GMC* appreciated in value by 29%.

Computing Covariances, Variances and Standard Deviations in Excel

Excel has the function COVAR() for computing covariances, VAR() for computing variances, *STDEV()* for computing standard deviations, and CORREL() for computing correlations. For reasons of numerical accuracy, we suggest that VAR() and STDEV() not be used. Below we discuss the usage of these functions. Given n observations on two random variables $\{X_i\}$ and $\{Y_i\}$, the population means are defined as the expectations:

$$\mu_X = \text{E}[X_i], \quad \text{and} \quad \mu_Y = \text{E}[Y_i].$$

The sample means are defined as the averages:

$$xbar = \sum_i X_i / n, \quad \text{and} \quad ybar = \sum_i Y_i / n.$$

The population covariance between X and Y is defined as the expectation:

$$\sigma^2_{XY} = \text{E}[(X_i - \mu_X)(Y_i - \mu_Y)].$$

With some effort it can be shown that this is algebraically equivalent to:

$$\sigma^2_{XY} = \text{E}[X_i \, Y_i\,] - \mu_X \, \mu_Y.$$

When X and Y are the same random variable, σ^2_{XX} is called the population variance.

The population standard deviation of X is defined as the square root of σ^2_{XX}, i.e.:

$$\sigma_X = (\sigma^2_{XX})^{0.5}.$$

The population correlation between X and Y is defined as

$$\rho_{XY} = (\sigma^2_{XY})/(\sigma_X \sigma_Y).$$

The sample covariance between X and Y is defined as the average:

$$s^2_{XY} = \sum_i [(X_i - xbar)(Y_i - ybar)]/n. \qquad (1)$$

With some effort it can be shown that this is algebraically equivalent to:

$$s^2_{XY} = \sum_i [X_i \, Y_i]/n - xbar * ybar. \qquad (2)$$

If in fact X and Y are the same random variable, s^2_{XX} is called the sample variance.

The sample standard deviation of X is defined as the square root of s^2_{XX}, i.e.:

$$s_X = (s^2_{XX})^{0.5}.$$

The sample correlation between X and Y is defined as

$$r_{XY} = (s^2_{XY})/(s_X s_Y).$$

Given weights w_X and w_Y, and the definition that $Z = w_X X + w_Y Y$, it can be shown that the variance of Z is:

$$\begin{aligned} \sigma^2_{ZZ} &= w_X^2 \, \sigma^2_{XX} + w_X \, w_Y \, \sigma^2_{XY} + w_Y \, w_X \, \sigma^2_{YX} + w_Y^2 \, \sigma^2_{YY} \\ &= w_X^2 \, \sigma^2_{XX} + 2 w_X \, w_Y \sigma^2_{XY} + w_Y^2 \, \sigma^2_{YY}. \end{aligned}$$

Although formulae (1) and (2) are algebraically equivalent, they are not numerically equivalent on a computer because of round-off error. Formula (1) is more accurate. In Excel, the functions VAR() and STDEV() are based on formula (2), whereas COVAR() and CORREL() are based on (1). In Excel, you can expect COVAR() and CORREL() to be accurate to at least six decimal places, whereas VAR() and STDEV() may have essentially no accuracy if *xbar* is large relative to s_X. To illustrate, suppose $n = 2$ with {X1, X2} = {123456789, 123456787}. If you use VAR()to compute the sample variance, or STDEV() to compute the sample standard deviation, they will both give an answer of 0.0, whereas it is easy to see that the sample variance should be $[(1)^2+(-1)^2]/2 = 1$.

One should also be interested in whether s^2_{XY} is a good estimator of the unknown parameter σ^2_{XY}. With a bit more algebra it can be shown that the expected value, $E(s^2_{XY}) = \sigma^2_{XY}(n-1)/n$. That is, s^2_{XY} underestimates σ^2_{XY}, especially when n is small. Thus, one typically applies a $n/(n-1)$ adjustment factor to s^2_{XY}. VAR() and STDEV() include the adjustment factor, but COVAR() does not. Because CORREL() is ratio of two estimators, the adjustment does not matter.

So, based on the twelve years of data, we use the COVAR() function in Excel to calculate the sample covariances for three stocks: ATT, GMC, and USX. Multiplying the results by 12/11 gives the following covariance matrix.

	ATT	GMC	USX
ATT	0.01080754	0.01240721	0.01307513
GMC	0.01240721	0.05839170	0.05542639
USX	0.01307513	0.05542639	0.09422681

From the same data, we estimate the expected return per year, including dividends, for *ATT*, *GMC*, and *USX* as 0.0890833, 0.213667, and 0.234583, respectively.

The correlation matrix makes it more obvious how two random variables move together. The correlation between two random variables equals the covariance between the two variables, divided by the product of the standard deviations of the two random variables. For our three investments, the sample correlation matrix is:

	ATT	GMC	USX
ATT	1.0		
GMC	0.493895589	1.0	
USX	0.409727718	0.747229121	1.0

The correlation can be between −1 and +1 with +1 being a high correlation between the two. Notice GMC and USX are highly correlated. ATT tends to move with GMC and USX, but not nearly so much as GMC moves with *USX*.

Let the symbols ATT, GMC, and USX represent the fraction of the portfolio devoted to each of the three stocks. Suppose, we desire a 15% yearly return. For the objective, we want to minimize the variance in the portfolio value after one year. In algebraic notation, what we want to do is:

```
Minimize
   0.01080754*ATT*ATT + 0.01240721*ATT*GMC + 0.01307513*ATT*USX +
   0.01240721*GMC*ATT + 0.05839170*GMC*GMC + 0.05542639*GMC*USX +
   0.01307513*USX*ATT + 0.05542639*USX*GMC + 0.09422681*USX*USX;
```

Use exactly 100% of the starting budget:

```
ATT + GMC + USX = 1;
```

Required wealth at end of period:

```
1.089083 * ATT + 1.213667 * GMC + 1.234583 * USX ≥ 1.15;
```

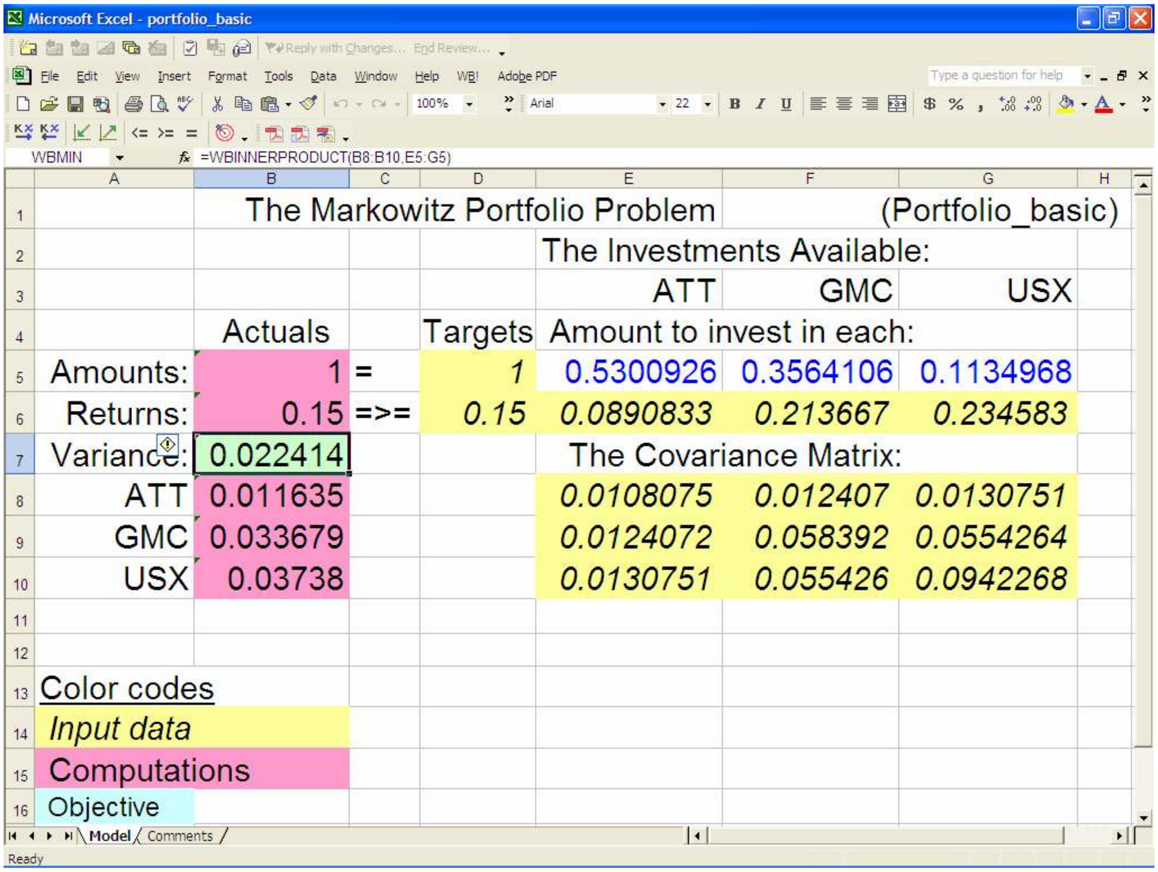
Note the two constraints are effectively in the same units. The first constraint is effectively a “beginning inventory” constraint, while the second constraint is an “ending inventory” constraint. Alternatively, we could have stated the expected return constraint just as easily as:

```
.0890833 * ATT + .213667 * GMC + .234583 * USX ≥ .15
```

Although perfectly correct, this latter style does not measure end-of-period state in quite the same way as start-of-period state. Fans of consistency may prefer the former style.

In preparation for writing the model in a spreadsheet, note that we can also write the objective as:

```
    ATT*(.01080754 * ATT +.01240721 * GMC +.01307513 * USX)
  + GMC*(.01240721 * ATT +.05839170 * GMC +.05542639 * USX)
  + USX*(.01307513 * ATT +.05542639 * GMC +.09422681 * USX);
```



In the spreadsheet, portfolio_basic, we calculate the expressions in parentheses in column B using the SUMPRODUCT() function, e.g., B8=SUMPRODUCT(E$5:G$5,E8:G8) in. We calculate the variance with cell B7=WBINNERPRODUCT(B8:B11,E5:G5). The WBINNERPRODUCT() function is similar to SUMPRODUCT(), except that it allows you to multiply a row vector by column vector. WBINNERPRODUCT expects one range to be a row range and the other a column range.

The "ABC's of Optimization" for this spreadsheet are:

A) Adjustable Cells or Decision Variables, specifying how much to invest in each asset appear in row 5, cells E5:G5;

B) The Best or objective cell, the portfolio variance to be minimized is cell B7. The most complicated computation for this model is the computation of the variance of the portfolio. If x_i is the amount invested in asset i, and σ^2_{ij} is the covariance between one unit of i and one unit of j, then the portfolio variance = $\Sigma_i\Sigma_j\, x_i * x_j * \sigma^2_{ij}$. This can be rewritten:

$$variance = \Sigma_i\, x_i\, \Sigma_j\, x_j * \sigma^2_{ij}.$$

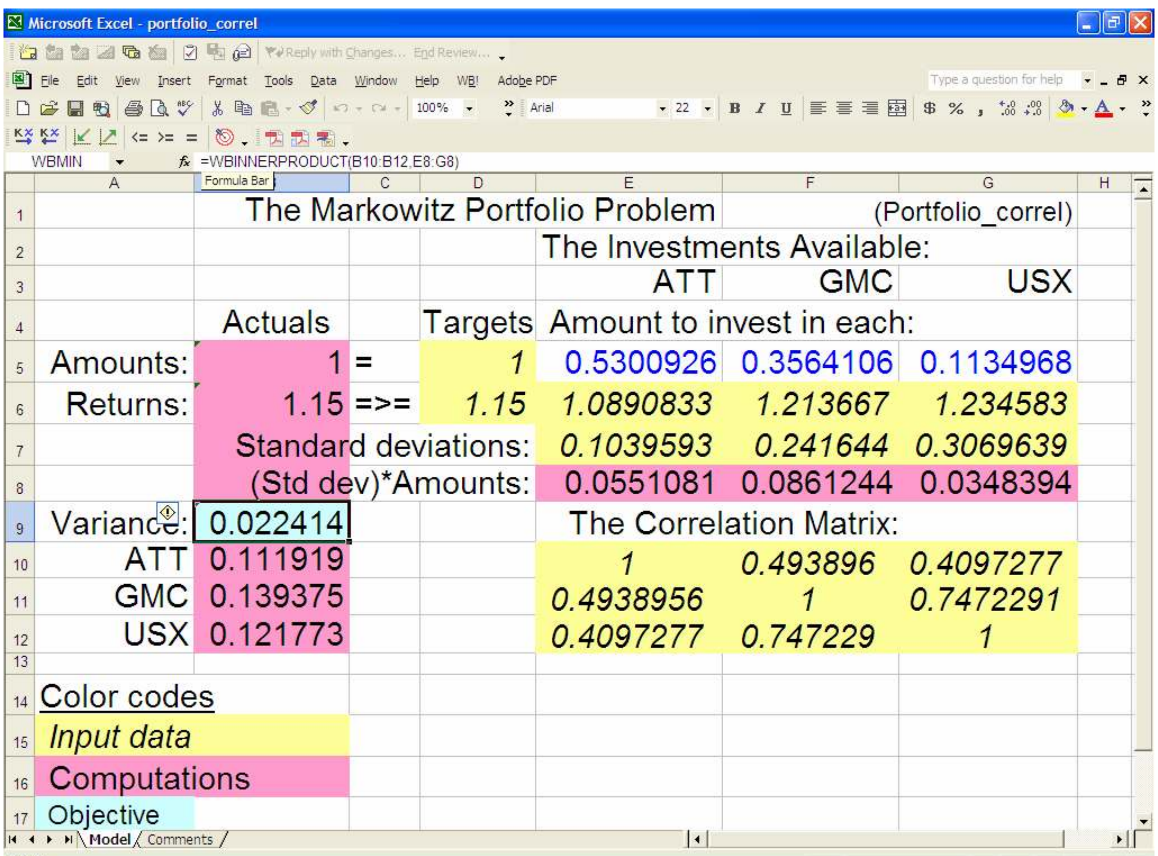
In the spreadsheet, Column B computes the inner summation, $\Sigma_j\, x_j * \sigma^2_{ij}$. For example, cell B8 contains the formula =SUMPRODUCT(E8:G8,E$5:G$5). The "$5" holds row 5 constant when the formula is copied down to cells B9:B10. The final summation, $\Sigma_i\, x_i\, \Sigma_j\!: x_j * \sigma^2_{ij}$, is done in cell B7.

C) Constraints: There are two constraints in this model. Cell C5, which contains =WB(B5,"=",D5), says the amount invested(computed in B5) must equal the target amount to invest given as input in D5. Cell C6, which contains =WB(B6,">=",D6), says the expected return(computed in B6) must be greater than or equal to the target return specified in D6.

The solution recommends about 53% of the portfolio be put in *ATT*, about 36% in *GMC* and just over 11% in *USX*. The expected return is 15%, with a variance of 0.02241381 or, equivalently, a standard deviation of about 0.1497123.

Using a Correlation Matrix

We based the previous model simply on straightforward statistical data based on yearly returns. In practice, it may be more typical to use monthly rather than yearly data as a basis for calculating covariances. Also, rather than use historical data for estimating the expected return of an asset, a decision maker might base the expected return estimate on more current, proprietary information about expected future performance of the asset. One may also wish to use considerable care in estimating the covariances and the expected returns. For example, one could use quite recent data to estimate the standard deviations. A larger set of data extending further back in time might be used to estimate the correlation matrix. Then, using the relationship between the correlation matrix and the covariance matrix, one could derive a covariance matrix. The version portfolio_correl, illustrates two alternative approaches to this problem: a) using the correlation matrix instead of the covariance matrix to describe how investments tend to move together, and b) and stating the desired return as a growth factor, 1.15, rather than a fraction return, 0.15.



Microsoft Excel - portfolio_correl

WBMIN =WBINNERPRODUCT(B10:B12,E8:G8)

	A	B	C	D	E	F	G
1		The Markowitz Portfolio Problem					(Portfolio_correl)
2					The Investments Available:		
3					ATT	GMC	USX
4		Actuals		Targets	Amount to invest in each:		
5	Amounts:	1	=	1	0.5300926	0.3564106	0.1134968
6	Returns:	1.15	=>=	1.15	1.0890833	1.213667	1.234583
7		Standard deviations:			0.1039593	0.241644	0.3069639
8		(Std dev)*Amounts:			0.0551081	0.0861244	0.0348394
9	Variance:	0.022414			The Correlation Matrix:		
10	ATT	0.111919			1	0.493896	0.4097277
11	GMC	0.139375			0.4938956	1	0.7472291
12	USX	0.121773			0.4097277	0.747229	1
13							
14	Color codes						
15	Input data						
16	Computations						
17	Objective						

Model / Comments

The most significant difference between this formulation and the previous one is in the computation of the portfolio variance. Here we exploit the fact that the variance can be written in terms of the correlations and the standard deviations as:

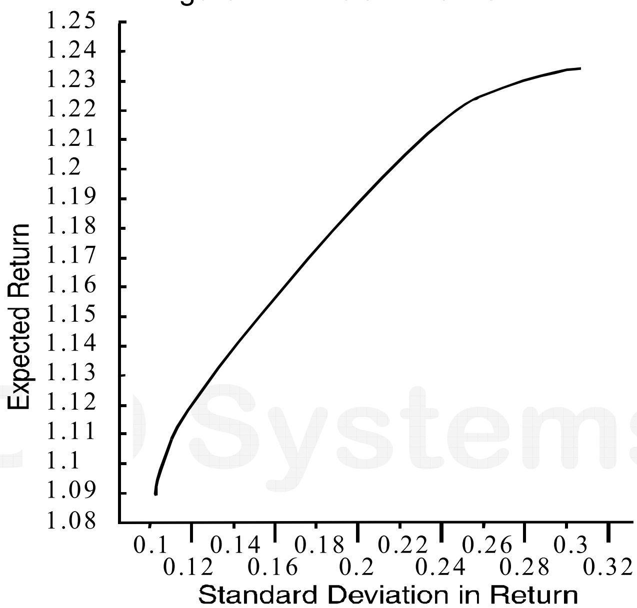
$$\text{variance} = \Sigma_i \Sigma_j\, x_i * x_j * \sigma_i * \sigma_j * \rho_{ij}. = \Sigma_i\, x_i * \sigma_i\, \Sigma_j\, x_j * \sigma_j * \rho_{ij}.$$

In row 8 we compute the term, $x_j * \sigma_j$, e.g., with formulae such as: E8=E5*E7. In column B we compute the inner sum, $\Sigma_j\, x_j * \sigma_j * \rho_{ij}$, with formulae such as B10=SUMPRODUCT(E10:G10,E$8:G$8). The outer summation is computed in cell B9 with the formula: B9=WBINNERPRODUCT(B10:B13,E8:H18). Observe that the same solution is obtained.

2.3 Dualing Objectives: Efficient Frontier and Parametric Analysis

There is no obvious way for an investor to determine the “correct” tradeoff between risk and return. Thus, one is frequently interested in looking at the tradeoff between the two. If an investor wants a higher expected return, she generally has to “pay for it” with higher risk. In finance terminology, we would like to trace out the efficient frontier of return and risk. If we solve for the minimum variance portfolio over a range of values for the expected return, ranging from 0.0890833 to 0.234583, we get the following plot or tradeoff curve for our little three-asset example:

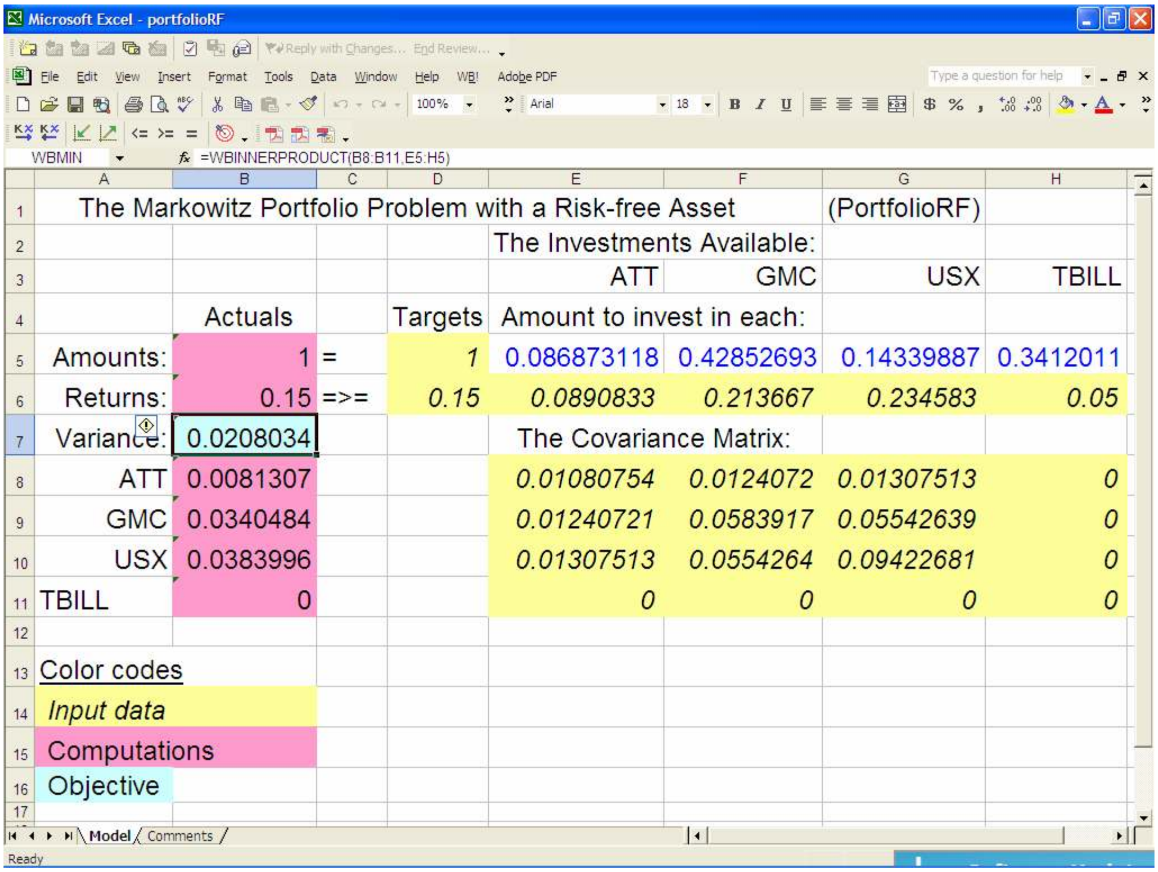
Figure 2.1 Efficient Frontier



Notice the “knee” in the curve as the required expected return increases past 1.21894. This is the point where ATT drops out of the portfolio.

2.3.1 Portfolios with a Risk-Free Asset

When one of the investments available is risk free, then the optimal portfolio composition has a particularly simple form. Suppose the opportunity to invest money risk free (e.g., in government treasury bills) at 5% per year has just become available. Working with our previous example, we now have a fourth investment instrument that has zero variance and zero covariance. There is no limit on how much can be invested at 5%. We ask the question: How does the portfolio composition change as the desired rate of return changes from 15% to 5%?



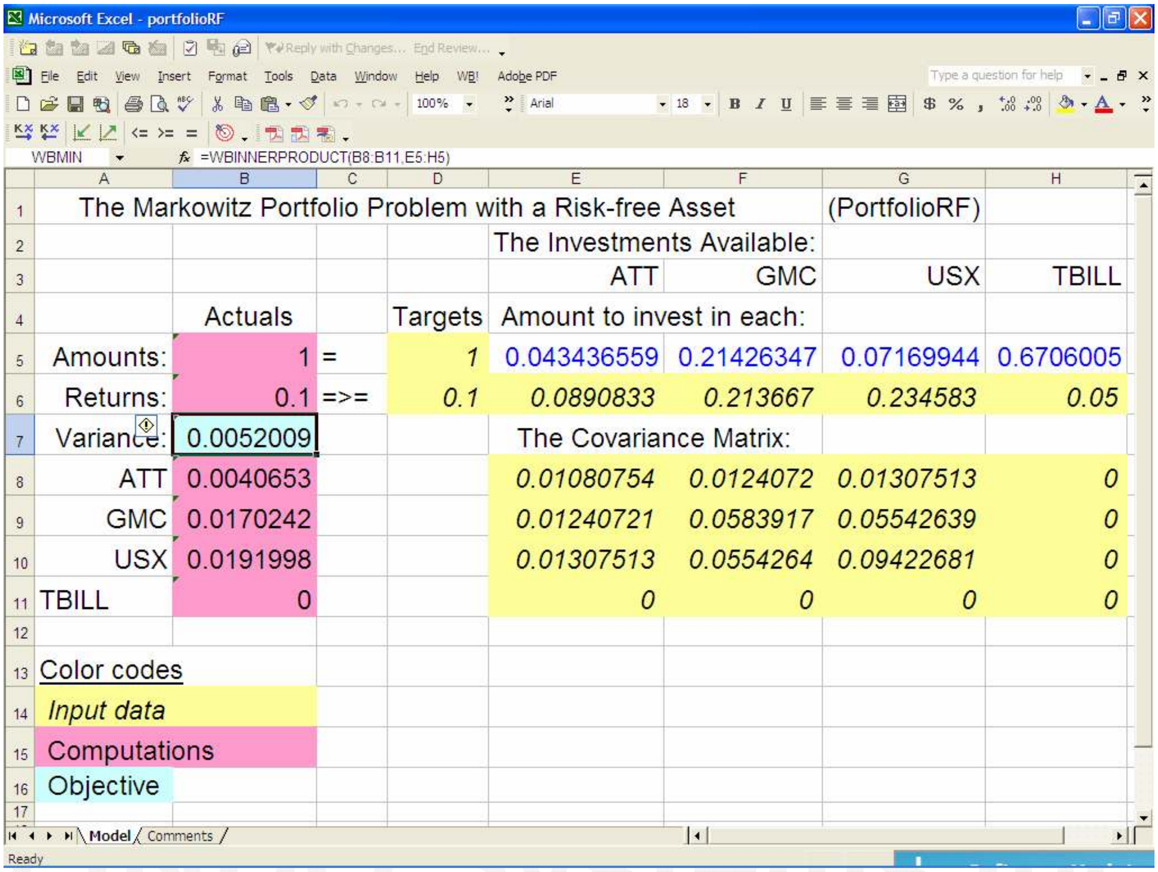
Microsoft Excel - portfolioRF

WBMIN =WBINNERPRODUCT(B8:B11,E5:H5)

	A	B	C	D	E	F	G	H
1	The Markowitz Portfolio Problem with a Risk-free Asset						(PortfolioRF)	
2					The Investments Available:			
3					ATT	GMC	USX	TBILL
4		Actuals		Targets	Amount to invest in each:			
5	Amounts:	1	=	1	0.086873118	0.42852693	0.14339887	0.3412011
6	Returns:	0.15	=>=	0.15	0.0890833	0.213667	0.234583	0.05
7	Variance:	0.0208034			The Covariance Matrix:			
8	ATT	0.0081307			0.01080754	0.0124072	0.01307513	0
9	GMC	0.0340484			0.01240721	0.0583917	0.05542639	0
10	USX	0.0383996			0.01307513	0.0554264	0.09422681	0
11	TBILL	0			0	0	0	0
12								
13	Color codes							
14	Input data							
15	Computations							
16	Objective							

Notice that more than 34% of the portfolio was invested in the risk-free investment, the T-bill, even though its return rate, 5%, is less than the target of 15%. Further, the variance has dropped to about 0.0208 from about 0.0224.

What happens as we decrease the target return towards 5%? Clearly, at 5%, we would put zero in *ATT*, *GMC*, and *USX*. A simple form of solution would be to keep the same proportions in *ATT*, *GMC*, and *USX*, but just change the allocation between the risk-free asset and the risky ones. Let us check an intermediate point. When we decrease the required return to 10%, we get the following solution:



	A	B	C	D	E	F	G	H
1	The Markowitz Portfolio Problem with a Risk-free Asset						(PortfolioRF)	
2					The Investments Available:			
3					ATT	GMC	USX	TBILL
4		Actuals		Targets	Amount to invest in each:			
5	Amounts:	1	=	1	0.043436559	0.21426347	0.07169944	0.6706005
6	Returns:	0.1	=>=	0.1	0.0890833	0.213667	0.234583	0.05
7	Variance:	0.0052009			The Covariance Matrix:			
8	ATT	0.0040653			0.01080754	0.0124072	0.01307513	0
9	GMC	0.0170242			0.01240721	0.0583917	0.05542639	0
10	USX	0.0191998			0.01307513	0.0554264	0.09422681	0
11	TBILL	0			0	0	0	0
12								
13	Color codes							
14	Input data							
15	Computations							
16	Objective							

This solution supports our conjecture:

> *As we change our required return, the relative proportions devoted to risky investments do not change. Only the allocation between the risk-free asset and the risky asset change.*

From the above solution, we observe that, except for round-off error, the amount invested in *ATT*, *GMC*, and *USX* is allocated in the same way for both solutions. Thus, two investors with different risk preferences would nevertheless both carry the same mix of risky stocks in their portfolio. Their portfolios would differ only in the proportion devoted to the risk-free asset. Our observation from the above example in fact holds in general. Thus, the decision of how to allocate funds among stocks, given the amount to be invested, can be separated from the questions of risk preference. Tobin received the Nobel Prize in 1981, largely for noticing the above feature, the so-called Separation Theorem. So, if you noticed it, you must be Nobel Prize caliber.

2.3.2 The Sharpe Ratio

For some portfolio p, of risky assets, excluding the risk-free asset, let:

R_p = its expected return,
s_p = its standard deviation in return, and
r_0 = the return of the risk-free asset.

A plausible single measure (as opposed to the two measures, risk and return) of attractiveness of portfolio p is the Sharpe ratio:

$(R_p - r_0) / s_p$.

In words, it measures how much additional return we achieved for the additional risk we took on, relative to putting all our money in the risk-free asset.

It happens the portfolio that maximizes this ratio has a certain well-defined appeal. Suppose:

t = our desired target return,
w_p = fraction of our wealth we place in portfolio p
(the rest placed in the risk-free asset).

To meet our return target, we must have:

$(1 - w_p) * r_0 + w_p * R_p = t.$

The standard deviation of our total investment is:

$w_p * s_p.$

Solving for wp in the return constraint, we get:

$w_p = (t - r_0) /(R_p - r_0).$

Thus, the standard deviation of the portfolio is:

$w_p * s_p = [(t - r_0) /(R_p - r_0)] * s_p.$

Minimizing the portfolio standard deviation means:

$\text{Min } [(t - r_0) /(R_p - r_0)] * s_p$
or
$\text{Min } [(t - r_0) * s_p /(R_p - r_0)].$

This is equivalent to:

$\text{Max } (R_p - r_0) /s_p.$

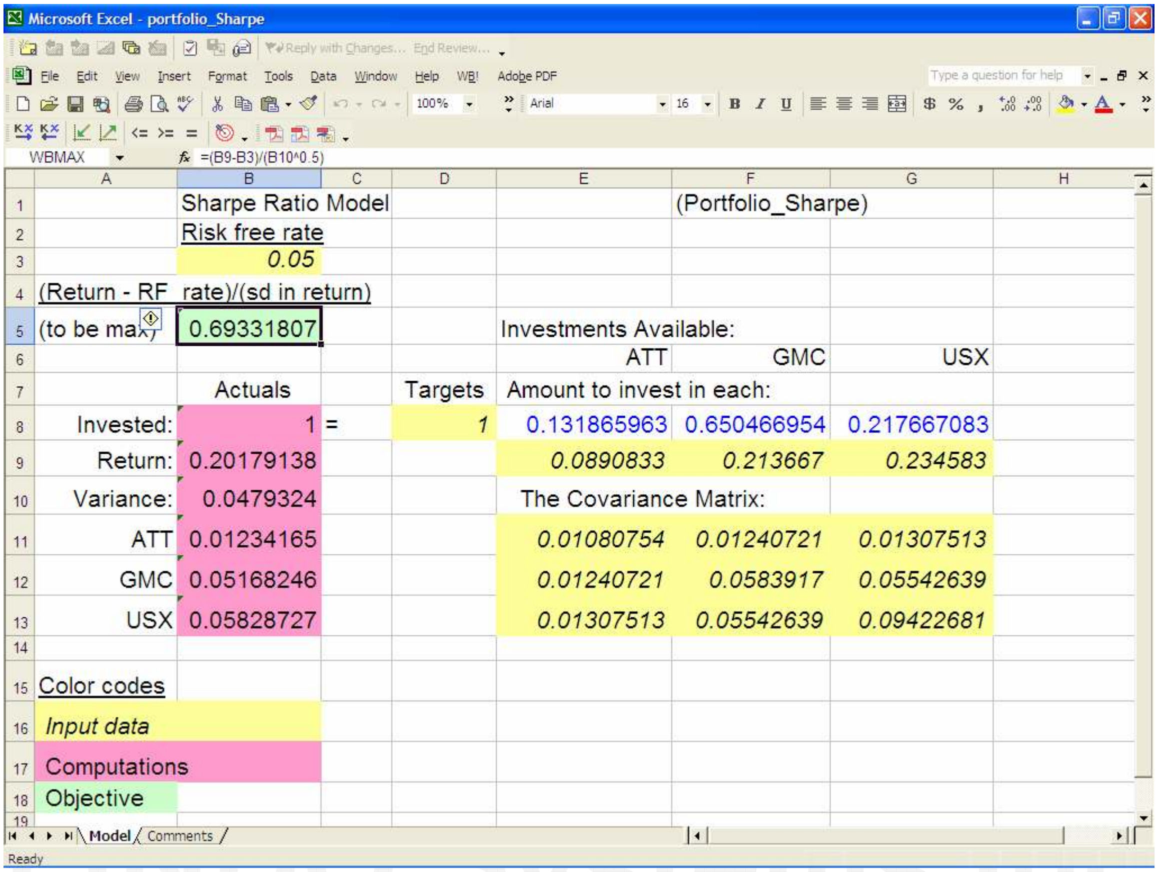
So, regardless of our risk/return preference, the money we invest in risky assets should be invested in the risky portfolio that maximizes the Sharpe ratio.

Algebraically, if the risk free rate is 5%, then what we would like to do is:

```
! Maximize the Sharpe ratio;
 MAX =
(1.089083*ATT + 1.213667*GMC + 1.234583*USX - 1.05)/
 ((.01080754*ATT*ATT + .01240721*ATT*GMC + .01307513*ATT*USX
 + .01240721*GMC*ATT + .05839170*GMC*GMC + .05542639*GMC*USX
 + .01307513*USX*ATT + .05542639*USX*GMC + .09422681*USX*USX)^.5);

! Use exactly 100% of the starting budget;
  ATT + GMC + USX = 1;
```

The spreadsheet *portfolio_sharpe* illustrates. The crucial differences from the previous models are: a) There is no target return constraint, and b) the Sharpe ratio is computed with: B5=(B9-B3)/(B10^0.5).



Notice the relative proportions of ATT, GMC, and USX are the same as in the previous model where we explicitly included a risk free asset with a return of 5%. E.g., except for round-off error:

```
0.131865963/0.6504669543 = 0.086873118/0.42852693.
```

The formulae in the spreadsheet Portfolio_Sharpe are essentially the same as in the previous except for the objective function in cell B5. It is B5=(B9-B3)/(B10^0.5), that is,

(expected_return – risk_free_rate)/(square_root_of_portfolio_variance).

2.4 Important Variations of the Portfolio Model

There are several issues that may concern you when you think about applying the Markowitz model in its simple form:

a) As we increase the number of assets to consider, the size of the covariance matrix becomes overwhelming. For example, 1000 assets implies 1,000,000 covariance terms, or at least 500,000 if symmetry is exploited.

b) If the model were applied every time new data become available (e.g., weekly), we would “rebalance” the portfolio frequently, making small, possibly unimportant adjustments in the portfolio.

c) There are no upper bounds on how much can be held of each asset. In practice, there might be legal or regulatory reasons for restricting the amount of any one asset to no more than, say, 5% of the total portfolio. Some portfolio managers may set the upper limit on a stock to one day’s trading volume for the stock. The reasoning being, if the manager wants to “unload” the stock quickly, the market price would be affected significantly by selling so much.

Two approaches for simplifying the covariance structure have been proposed: the scenario approach and the factor approach. For the issue of portfolio “nervousness”, the incorporation of transaction costs is useful.

2.4.1 Portfolios with Transaction Costs

The models above do not tell us much about how frequently to adjust our portfolio as new information becomes available, e.g., new estimates of expected return and new estimates of variance. If we applied the above models every time new information became available, we would be constantly adjusting our portfolio. This might make our broker happy because of all the commission fees, but that should be a secondary objective at best. The important observation is that there are costs associated with buying and selling. There are the obvious commission costs, and the not so obvious bid-ask spread. The bid-ask spread is effectively a transaction cost for buying and selling.

The method we will describe assumes transaction costs are paid at the beginning of the period. It is a straightforward exercise to modify the model to handle the case of transaction costs paid at the end of the period. The major modifications to the basic portfolio model are:

a) We must introduce two additional variables for each asset, an "amount bought" variable and an "amount sold" variable.

b) The budget constraint must be modified to include money spent on commissions.

c) An additional constraint must be included for each asset to enforce the requirement:
amount invested in asset i = (initial holding of i) + (amount bought of i) – (amount sold of i).

2.4.2 Example

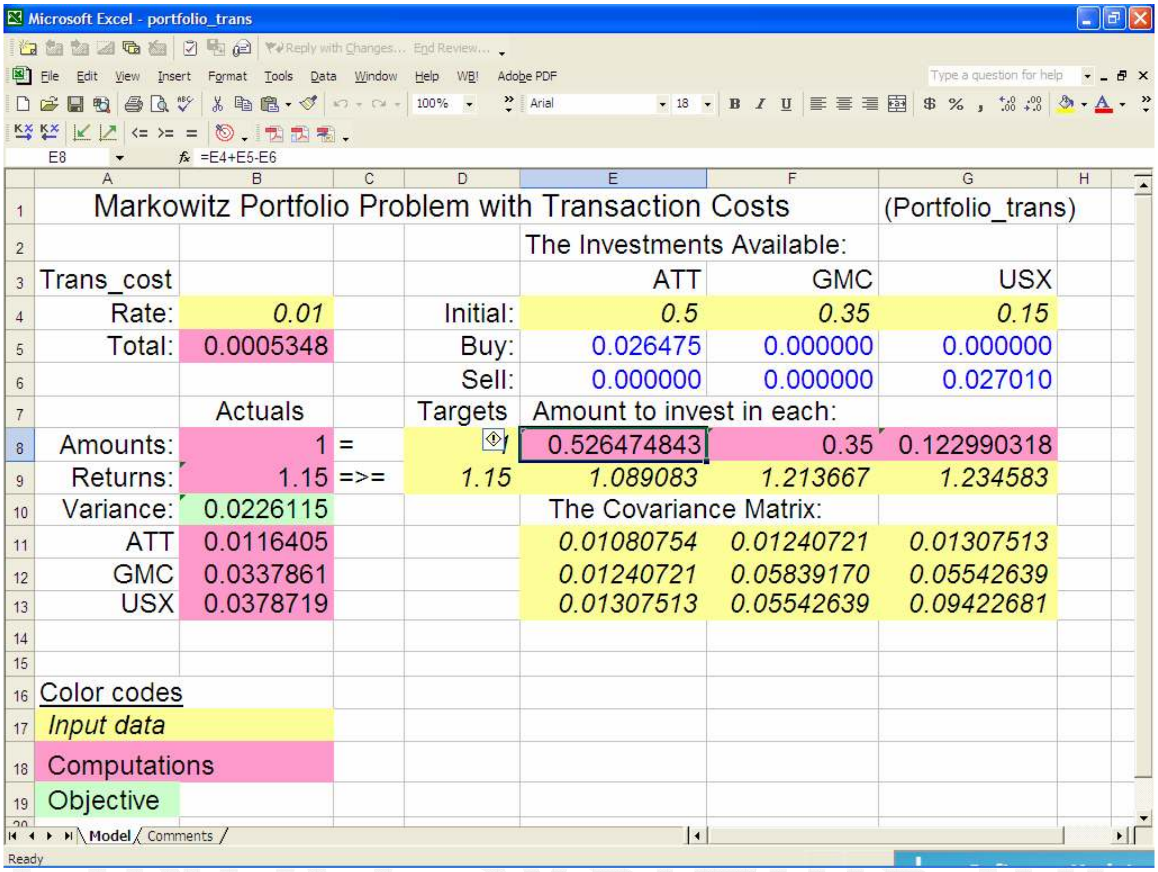
Suppose we have to pay a 1% transaction fee on the amount bought or sold of any stock and our current portfolio is 50% *ATT*, 35% *GMC*, and 15% *USX*. This is pretty close to the optimal mix. Should we incur the cost of adjusting? The following is the relevant model:


```
MIN = .01080754 * ATT * ATT +.01240721 * ATT * GMC +.01307513 * ATT *
 USX +.01240721 * GMC * ATT +.05839170 * GMC * GMC +.05542639 * GMC *
 USX +.01307513 * USX * ATT +.05542639 * USX * GMC +.09422681 * USX *
 USX;
ATT + GMC + USX + .01 * ( BA + BG + BU + SA + SG + SU) = 1;
1.089083 * ATT + 1.213667 * GMC + 1.234583 * USX >= 1.15;
ATT = .50 + BA - SA;
GMC = .35 + BG - SG;
USX = .15 + BU – SU;
```

The first constraint says the total uses of funds must equal 1. Another way of interpreting this constraint is to subtract each of the next three constraints from it. We then get:

```
.01 * (BA + BG + BU + SA + SG + SU) + BA + BG + BU=SA + SG + SU;
```

It says any purchases plus transaction fees must be funded by selling. The spreadsheet model is:



	A	B	C	D	E	F	G
1	Markowitz Portfolio Problem with Transaction Costs						(Portfolio_trans)
2					The Investments Available:		
3	Trans_cost				ATT	GMC	USX
4	Rate:	0.01		Initial:	0.5	0.35	0.15
5	Total:	0.0005348		Buy:	0.026475	0.000000	0.000000
6				Sell:	0.000000	0.000000	0.027010
7		Actuals		Targets	Amount to invest in each:		
8	Amounts:	1	=		0.526474843	0.35	0.122990318
9	Returns:	1.15	=>=	1.15	1.089083	1.213667	1.234583
10	Variance:	0.0226115			The Covariance Matrix:		
11	ATT	0.0116405			0.01080754	0.01240721	0.01307513
12	GMC	0.0337861			0.01240721	0.05839170	0.05542639
13	USX	0.0378719			0.01307513	0.05542639	0.09422681
14							
15							
16	Color codes						
17	Input data						
18	Computations						
19	Objective						

The solution recommends buying a little bit more *ATT*, neither buy nor sell any *GMC*, and sell a little *USX*.

The ABC's of this spreadsheet are:

A) The Adjustable cells are the Buy variables in row 5, and the Sell variables in row 6.

B) The "Best" or objective cell is cell B10=WBINNERPRODUCT(B11:B13,E8:G8),

i.e., the variance in the end of period portfolio value.

C) There are two constraints:

C8 contains =WB(B8,"=",D9), and C9 contains =WB(B8,">=",D9).

The crucial formulae are:

Row 8 computes the amount held of each asset after transactions, e.g.,

E8=E4+E5-E6.

Column B computes the first half of the variance calculation, e.g.,

B11=SUMPRODUCT(E11:G11,E$8:G$8).

Cell B10 completes the variance calculation with

B10=WBINNERPRODUCT(B11:B13,E8:G8),

Cell B5 computes total transaction expenses from both buying and selling:

B5=B4*SUM(E5:G6);

Cell B8 computes the total uses of funds, i.e., transactions expense + amount in assets after transactions:

B8=B5+SUM(E8:G8);

Cell B9 computes the expected portfolio value at the end of the period:

B9=SUMPRODUCT(E9:G9,E$8:G$8);

2.4.3 Portfolios with Taxes

Taxes are an unpleasant complication of investment analysis that should be considered. The effect of taxes on a portfolio is illustrated by the following results during one year for two similar "growth-and-income" portfolios from the Vanguard company. Portfolio S was managed without (Sans) regard to taxes. Portfolio T was managed with after-tax performance in mind:

	Distributions		Initial	
Portfolio	**Income**	**Gain-from-sales**	**Share-price**	**Return**
S	$0.41	$2.31	$19.85	33.65%
T	$0.28	$0.00	$13.44	34.68%

The tax managed portfolio, probably just by chance, in fact had a higher before tax return. It looks even more attractive after taxes. If the tax rate for both dividend income and capital gains is 30%, then the tax paid at year end per dollar invested in portfolio *S* is $.3 \times (.41 + 2.31) / 19.85 = 4.1$ cents; whereas, the tax per dollar invested in portfolio *S* is $.3 \times .28/13.44 = 0.6$ of a cent.

Below is a generalization of the Markowitz model to take into account taxes. As input, it requires in particular:

a) number of shares held of each kind of asset,

b) price per share paid for each asset held, and

c) estimated dividends per share for each kind of asset.

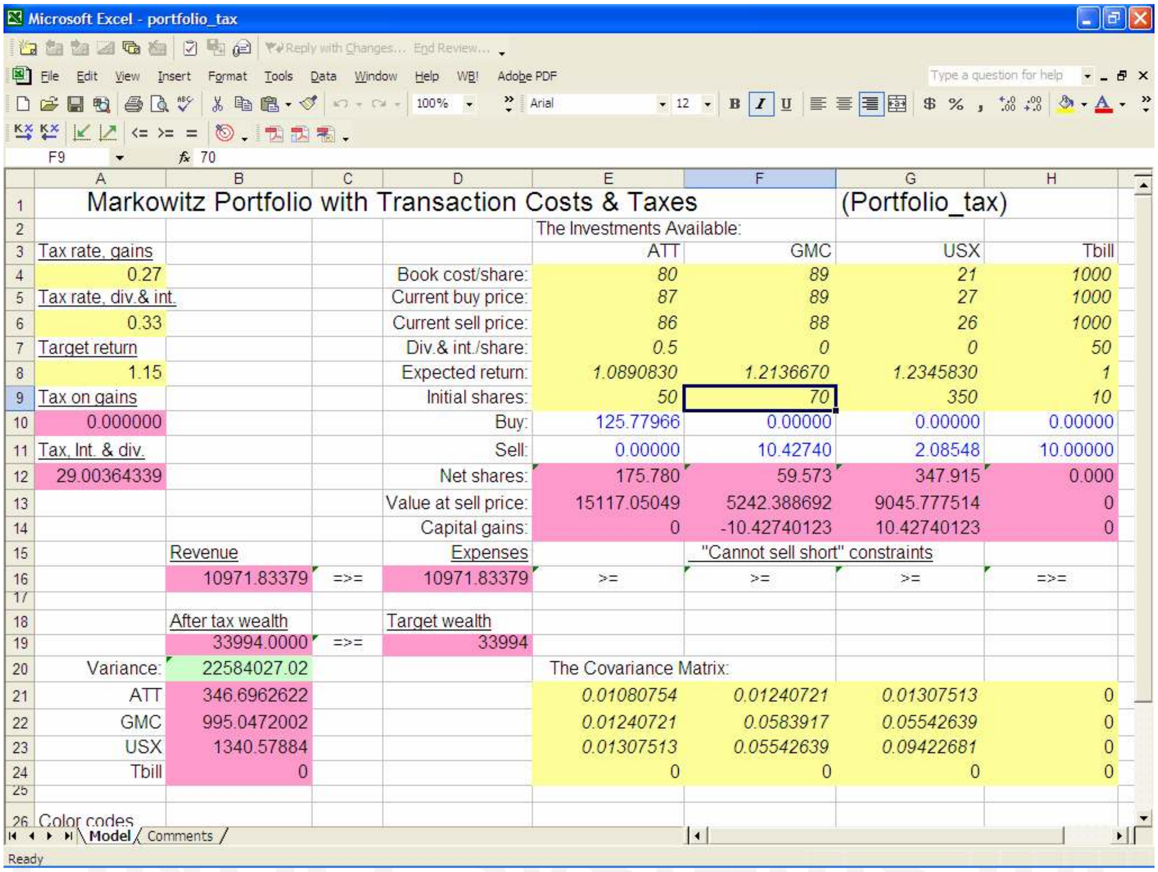
The results from this model will differ from a model that does not consider taxes in that this model, when considering equally attractive assets, will tend to:

i. purchase the asset that does not pay dividends, so as to avoid the immediate tax on dividends,

ii. sell the asset that pays dividends, and

iii. sell the asset whose purchase cost was higher, so as to avoid more tax on capital gains.

This is all given that two assets are otherwise identical (presuming rates of return are computed including dividends). For completeness, this model also includes transaction costs.



	A	B	C	D	E	F	G	H
1	Markowitz Portfolio with Transaction Costs & Taxes						(Portfolio_tax)	
2					The Investments Available:			
3	Tax rate, gains				ATT	GMC	USX	Tbill
4	0.27			Book cost/share:	80	89	21	1000
5	Tax rate, div.& int.			Current buy price:	87	89	27	1000
6	0.33			Current sell price:	86	88	26	1000
7	Target return			Div.& int./share:	0.5	0	0	50
8	1.15			Expected return:	1.0890830	1.2136670	1.2345830	1
9	Tax on gains			Initial shares:	50	70	350	10
10	0.000000			Buy:	125.77966	0.00000	0.00000	0.00000
11	Tax, Int. & div.			Sell:	0.00000	10.42740	2.08548	10.00000
12	29.00364339			Net shares:	175.780	59.573	347.915	0.000
13				Value at sell price:	15117.05049	5242.388692	9045.777514	0
14				Capital gains:	0	-10.42740123	10.42740123	0
15		Revenue		Expenses		"Cannot sell short" constraints		
16		10971.83379	=>=	10971.83379	>=	>=	>=	=>=
17								
18		After tax wealth		Target wealth				
19		33994.0000	=>=	33994				
20	Variance:	22584027.02			The Covariance Matrix:			
21	ATT	346.6962622			0.01080754	0.01240721	0.01307513	0
22	GMC	995.0472002			0.01240721	0.0583917	0.05542639	0
23	USX	1340.57884			0.01307513	0.05542639	0.09422681	0
24	Tbill	0			0	0	0	0
25								
26	Color codes							

Notice that the solution recommends selling 2.08548 shares of USX at $26/share. Because these shares were bought at 21, this generates a capital gain of 10.4274. This gain, however, is exactly cancelled out by selling 10.4274 shares of GMC at $88/share. These shares were bought at $87, so this generates a capital loss of 10.4274, so the portfolio does not have to pay any capital gains tax.

There are no constraints in the model to prevent both selling and buying a given stock or instrument. In fact, in some instances the model may recommend doing this so as to recognize or claim a capital loss. This is called a "wash sale" and U.S. tax rules prevent you from claiming the capital loss. The general rule is that if you sell a security and also buy the same security within the 30 days before, the same day, or the 30 days after the sale, then you cannot claim a capital loss from the sale. To the extent that wash sales are recommended by the model, it does not accurately model U.S. tax rules.

The ABC's of this spreadsheet are:

A) The Adjustable cells are the Buy variables E11:H11, and the Sell variables in row E12:H12.

B) The "Best" or objective cell is B20=WBINNERPRODUCT(B21:B24,E$13:H$13),

i.e., the variance in the end of period portfolio value.

C) The constraints are:

C16=WB(B16,">=",D16)
C19=WB(B19,">=",D19),

Cannot sell short, i.e., hold negative quantities of an asset, cells E16:H16.

E16=WB(12,">=",0),

The crucial formulae are:

A10=A4*MAX(0,SUM(E14:H14),

A12=A6*SUMPRODUCT(E12:H14)
B16=SUMPRODUCT(E11:H11,E6:H6),

B19 computes the expected portfolio value at the end of the period:

B19=SUMPRODUCT(E8:H8,E13:H13),

Column B computes the first half of the variance calculation, e.g.,

B21=SUMPRODUCT(E21:H21,E$13:H$13),

Cell B20 completes the variance calculation with

B20=WBINNERPRODUCT(B21:B24,E13:G13),
D16=SUMPRODUCT(E10:H10,E5:H5)+A1,
D19=A8*SUMPRODUCT(E6:H6,E9:H9)

Row 12 computes the amount held of each asset after transactions, e.g.,

E12=E9+E10-E11,
E13=E12*E6,
E14=(E6-E4)*E11,

2.4.4 Factors Model for Simplifying the Covariance Structure

Sharpe (1963) introduced a substantial simplification to the modeling of the random behavior of stock market prices. He proposed that there is a “market factor” that has a significant effect on the movement of a stock. The market factor might be the Dow-Jones Industrial average, the S&P 500 average, or the Nikkei index. If we define:

M = the market factor,
m_0 = E(M),
s_0^2 = var(M),
e_i = random movement specific to stock i,
s_i^2 = var(e_i).

Sharpe’s approximation assumes (where E( ) denotes expected value):

$E(e_i) = 0$
$E(e_i e_j) = 0$ for $i \neq j$,
$E(e_i M) = 0$.

Then, according to the Sharpe single factor model, the return of one dollar invested in stock or asset i is:

$u_i + b_i M + e_i$.

The parameters u_i and b_i are obtained by regression (e.g., least squares, of the return of asset i on the market factor). The parameter b_i is known as the “beta” of the asset. Let:

X_i = amount invested in asset i and
define the variance in return of the portfolio as:

$$\begin{aligned}&\text{var}[\sum X_i(u_i + b_i M + e_i)]\\&= \text{var}(\sum X_i b_i M) + \text{var}(\sum X_i e_i)\\&= (\sum X_i b_i)^2 s_o^2 + \sum X_i^2 s_i^2.\end{aligned}$$

Thus, our problem can be written:

$$\begin{aligned}&\text{Minimize} \quad Z^2 s_o^2 + \sum X_i^2 s_i^2\\&\text{subject to}\\&\quad Z - \sum X_i b_i = 0\\&\quad \sum X_i = 1\\&\quad \sum X_i (u_i + b_i m_o) \geq r.\end{aligned}$$

So, at the expense of adding one constraint and one variable, we have reduced a dense covariance matrix to a diagonal covariance matrix.

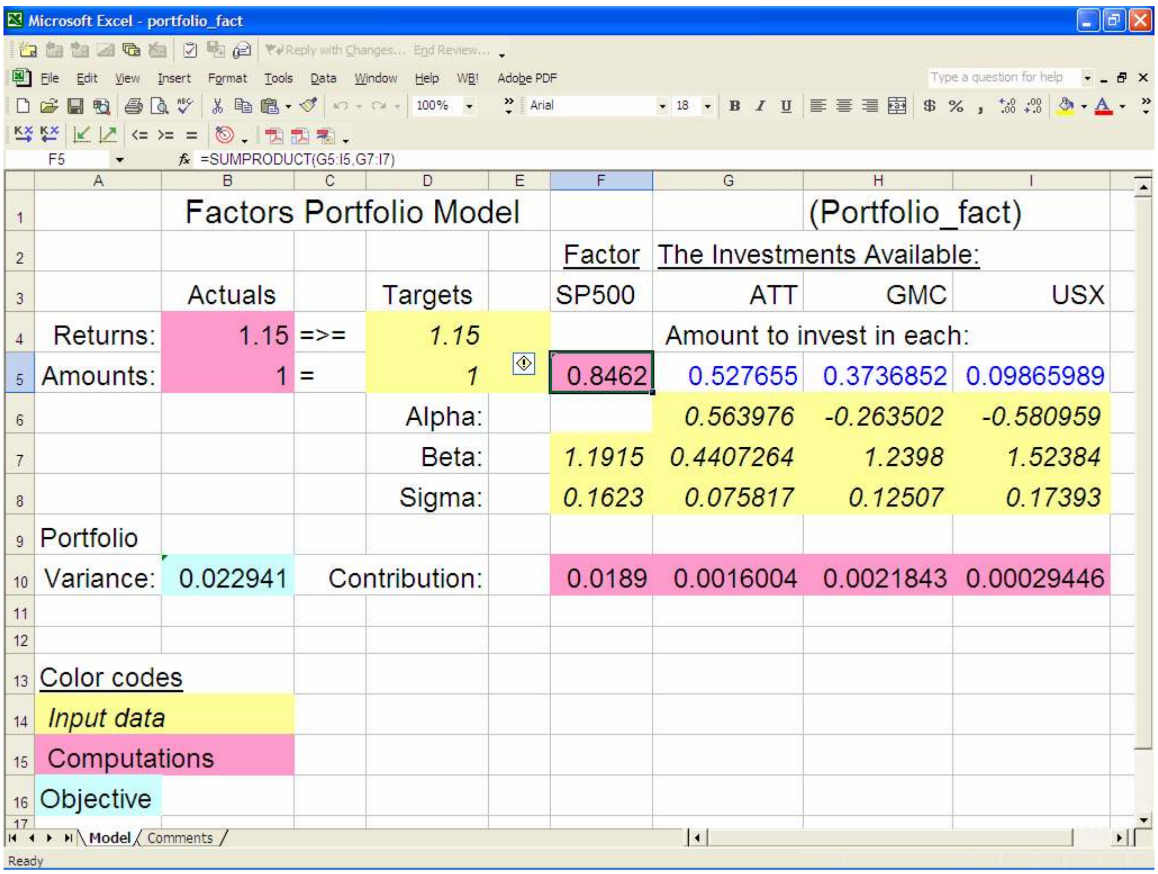
In practice, perhaps a half dozen factors might be used to represent the “systematic risk”. That is, the return of an asset is assumed to be correlated with a number of indices or factors. Typical factors might be a market index such as the S&P 500, interest rates, inflation, defense spending, energy prices, gross national product, correlation with the business cycle, various industry indices, etc. For example, bond prices are very affected by interest rate movements.

2.4.5 Example of the Factor Model

The Factor Model represents the variance in return of an asset as the sum of the variances due to the asset’s movement with one or more factors, plus a factor-independent variance.

To illustrate the factor model, we used multiple regression to regress the returns of *ATT*, *GMC*, and *USX* on the S&P 500 index for the same period. The stocks were regressed on the factor, SP500, based on the formula: Return(i) = Alpha(i) + Beta(i) * SP500 + error(i). The results were:

```
ASSET =     ATT         GMC       USX;
ALPHA = .563976   -.263502  -.580959;
BETA  = .4407264  1.23980   1.52384;
SIGMA = .075817    .125070   .173930;
```



	A	B	C	D	E	F	G	H	I
1		Factors Portfolio Model						(Portfolio_fact)	
2						Factor	The Investments Available:		
3		Actuals		Targets		SP500	ATT	GMC	USX
4	Returns:	1.15	=>=	1.15			Amount to invest in each:		
5	Amounts:	1	=	1		0.8462	0.527655	0.3736852	0.09865989
6				Alpha:			0.563976	-0.263502	-0.580959
7				Beta:		1.1915	0.4407264	1.2398	1.52384
8				Sigma:		0.1623	0.075817	0.12507	0.17393
9	Portfolio								
10	Variance:	0.022941	Contribution:			0.0189	0.0016004	0.0021843	0.00029446
11									
12									
13	Color codes								
14	Input data								
15	Computations								
16	Objective								

Notice the portfolio makeup is slightly different. However, the estimated variance of the portfolio is very close to our original portfolio.

The important formulae are:

B4=SUMPRODUCT(G5:I5,G6:I6)+F5*F7,
B5=SUM(G5:I5),
C4=WB(B4,”>=”,D4),
C5=(B5,”=”,D5),
F5=SUMPRODUCT(G5:I5,G7:I7),
F10=(F8*F5)^2,
B10=SUM(F10:I10).

2.4.6 Scenario Model for Representing Uncertainty

The scenario approach to modeling uncertainty assumes the possible future situations can be represented by a small number of “scenarios”. The smallest number used is typically three (e.g., “optimistic,” “most likely,” and “pessimistic”). Some of the original ideas underlying the scenario approach come from the approach known as stochastic programming; see Madansky (1962), for example. For a discussion of the scenario approach for large portfolios, see Markowitz and Perold (1981) and Perold (1984). For a thorough discussion of the general approach of stochastic programming, see Infanger (1992). Eppen, Martin, and Schrage (1988) use the scenario approach for capacity planning in the automobile industry.

Let:

P_s = Probability scenario s occurs,
u_{is} = return of asset i if the scenario is s,
X_i = investment in asset i,
Y_s = deviation of actual return from the mean if the scenario is s;
$= \sum_i X_i(u_{is} - \sum_q P_q u_{iq})$.

Our problem in algebraic form is:

Minimize $\sum_s P_s Y_s^2$
subject to
$Y_s - \sum_i X_i(u_{is} - \sum_q P_q u_{iq}) = 0$ (deviation from mean of each scenario, s)
$\sum_i X_i = 1$ (budget constraint)
$\sum_i X_i \sum_s P_s u_{is} \geq r$ (desired return).

If asset i has an inherent variability v_i^2, the objective generalizes to:

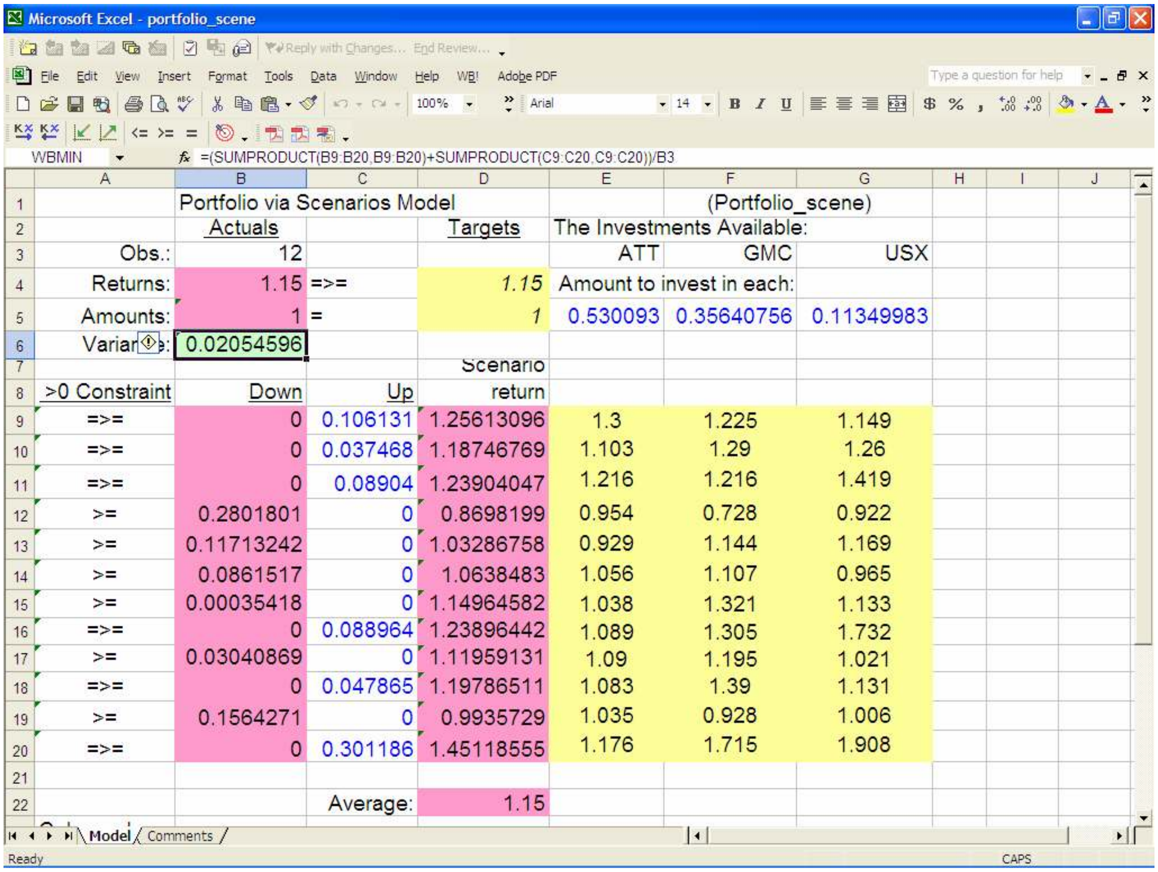
$$\text{Min } \sum_i X_i^2 v_i^2 + \sum_s P_s Y_s^2$$

The key feature is that, even though this formulation has a few more constraints, the covariance matrix is diagonal and, thus, very sparse.

You will generally also want to put upper limits on what fraction of the portfolio is invested in each asset. Otherwise, if there are no upper bounds or inherent variabilities specified, the optimization will tend to invest in only as many assets as there are scenarios.

2.4.7 Example: Scenario Model for Representing Uncertainty

We will use the original data from Markowitz once again. We simply treat each of the 12 years as being a separate scenario, independent of the other 11 years.



	A	B	C	D	E	F	G
1		Portfolio via Scenarios Model				(Portfolio_scene)	
2		Actuals		Targets	The Investments Available:		
3	Obs.:	12			ATT	GMC	USX
4	Returns:	1.15	=>=	1.15	Amount to invest in each:		
5	Amounts:	1	=	1	0.530093	0.35640756	0.11349983
6	Variance:	0.02054596					
7				Scenario			
8	>0 Constraint	Down	Up	return			
9	=>=	0	0.106131	1.25613096	1.3	1.225	1.149
10	=>=	0	0.037468	1.18746769	1.103	1.29	1.26
11	=>=	0	0.08904	1.23904047	1.216	1.216	1.419
12	>=	0.2801801	0	0.8698199	0.954	0.728	0.922
13	>=	0.11713242	0	1.03286758	0.929	1.144	1.169
14	>=	0.0861517	0	1.0638483	1.056	1.107	0.965
15	>=	0.00035418	0	1.14964582	1.038	1.321	1.133
16	=>=	0	0.088964	1.23896442	1.089	1.305	1.732
17	>=	0.03040869	0	1.11959131	1.09	1.195	1.021
18	=>=	0	0.047865	1.19786511	1.083	1.39	1.131
19	>=	0.1564271	0	0.9935729	1.035	0.928	1.006
20	=>=	0	0.301186	1.45118555	1.176	1.715	1.908
21							
22			Average:	1.15			

The solution should be familiar. The alert reader may have noticed the solution suggests the same portfolio (except for round-off error) as our original model based on the covariance matrix (based on the same 12 years of data as in the above scenario model). This, in fact, is a general result. In other words, if the covariance matrix and expected returns are calculated directly from the original data by the traditional statistical formulae, then the covariance model and the scenario model, based on the same data, will recommend exactly the same portfolio.

The careful reader will have noticed the objective function from the scenario model (0.02056) is slightly less than that of the covariance model (.02241). The exceptionally perceptive reader may have noticed 12 × 0.02054597/11 is, except for round-off error, equal to 0.002241. The difference in objective value is a result simply of the fact that standard statistics packages tend to divide by $N - 1$ rather than N when computing variances and covariances, where N is the number of observations. Thus, a slightly more general statement is, if the covariance matrix is computed using a divisor of N rather than $N - 1$, then the covariance model and the scenario model will give the same solution, including objective value.

The crucial formulae are:

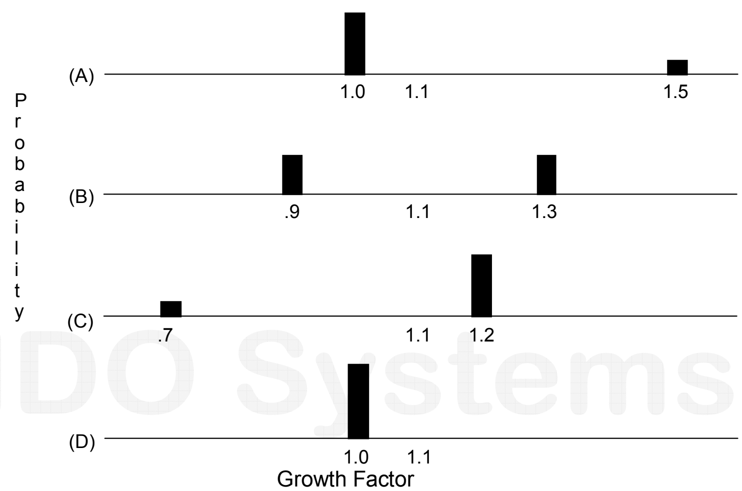
B4=D22,
B5=SUM(E5:G5),
B6=(SUMPRODUCT(B9:B20,B9:B20)+SUMPRODUCT(C9:C20,C9:C20))/B3,
B9=C9-D9+$D$22,
D9=SUMPRODUCT(E9:G9,E$5:G$5),
D22=AVERAGE(D9:D20).

2.5 Measures of Risk other than Variance

The most common measure of risk is variance (or its square root, the standard deviation). This is a reasonable measure of risk for assets that have a symmetric distribution and are traded in a so-called “efficient” market. If these two features do not hold, however, variance has some drawbacks. Consider the four possible growth distributions in Figure 2.2.

Investments *A*, *B*, and *C* are equivalent according to the variance measure because each has an expected growth of 1.10 (an expected return of 10%) and a variance of 0.04 (standard deviation around the mean of 0.20). Risk-averse investors would, however, probably not be indifferent among the three. Under distribution (*A*), you would never lose any of your original investment, and there is a 0.2 probability of the investment growing by a factor of 1.5 (i.e., a 50% return). Distribution (*C*), on the other hand, has a 0.2 probability of an investment decreasing to 0.7 of its original value (i.e., a negative 30% return). Risk-averse investors would tend to prefer (*A*) most and to prefer (*C*) least. This illustrates variance need not be a good measure of risk if the distribution of returns is not symmetric:

Figure 2.2 Possible Growth Factor Distributions



Investment (*D*) is an inefficient investment. It is dominated by (*A*). Suppose the only investments available are (*A*) and (*D*) and our goal is to have an expected return of at least 5% (i.e., a growth factor of 1.05) and the lowest possible variance. The solution is to put 50% of our investment in each of (*A*) and (*D*). The resulting variance is 0.01 (standard deviation = 0.1). If we invested 100% in (*A*), the standard deviation would be 0.20. Nevertheless, we would prefer to invest 100% in (*A*). It is true the return is more random. However, our profits are always at least as high under every outcome. (If the randomness in profits is an issue, we can always give profits to a worthy educational institution when our profits are high to reduce the variance.) Thus, the variance objective may cause us to choose inefficient investments.

In active and efficient markets such as major stock markets, you will tend not to find investments such as (*D*) because investors will realize (*A*) dominates (*D*). Thus, the market price of (*D*) will drop until its return approaches competing investments. In investment decisions regarding new physical facilities, however, there are no strong market forces making all investment candidates “efficient”, so the variance risk measure may be less appropriate in such situations.

2.5.1 Utility Functions

A variety of utility functions have been proposed for measuring expected risk. If *w* is our wealth at the end of the period then the utility function *U*(*w*) measures the utility of that wealth. Sensible utility functions have two features: a) they are increasing in *w*, or at least non-decreasing(more wealth cannot hurt), and b) they are concave(each additional $ of wealth is no more valuable than the previous one, maybe less). Some commonly proposed utility functions are:

1) Downside risk: *U*(*w*) = *w* – max(*w*-*t*, 0), where *t* is the threshold,
2) Log: *U*(*w*) = Log(*w*), sometimes called the Kelly criterion,

3) Quadratic: $U(w) = a*w - b*w^2$,
4) Exponential: $U(w)= -\exp(-a*w)$,
5) Power: $U(w) = w^{(1-r)}/(1-r)$,
6) Hyperbolic: $U(w) = [(1-\gamma)/\gamma]*[a*w/(1-\gamma)+b)]^{\gamma}$.
The Hyperbolic includes the quadratic, exponential, and power utilities as special cases.

In the next section we set what kind of anomalous situations can arise if we do not use a “sensible” utility function in the above sense.

2.5.2 Maximizing the Minimum Return

A very conservative investor might react to risk by maximizing the minimum return over scenarios. There are some curious implications from this. Suppose the only investments available are A and C above and the two scenarios are:

Scenario	Probability	Payoff from A	Payoff from C
1	0.8	1.0	1.2
2	0.2	1.5	0.7

If we wish to maximize the minimum possible wealth, the probability of a scenario does not matter, as long as the probability is positive. Thus, the following LP is appropriate:

```
  MAX = WMIN;
!  Initial budget constraint;
             A +       C = 1;
!  Wealth under scenario 1;
      WMIN   <=       A + 1.2 * C > 0;
!  Wealth under scenario 2;
      WMIN  <= 1.5 * A + 0.7 * C > 0;
```

It is not difficult to deduce that the solution is:

```
Variable           Value
    WMIN        1.100000
       A       0.5000000
       C       0.5000000
```


Given that both investments have an expected return of 10%, it is not surprising the expected growth factor is 1.10. That is, a return of 10%. The possibly surprising thing is there is no risk. Regardless of which scenario occurs, the $1 initial investment will grow to $1.10 if 50 cents is placed in each of *A* and *C*.

Now, suppose an extremely reliable friend provides us with the interesting news that, “if scenario 1 occurs, then investment *C* will payoff 1.3 rather than 1.2”. This is certainly good news. The expected return for *C* has just gone up, and its downside risk has certainly not gotten worse. How should we react to it? We make the obvious modification in our model:

```
  MAX = WMIN;
!  Initial budget constraint;
                     A       + C = 1;
!  Wealth under scenario 1;
      WMIN  <=       A + 1.3 * C ;
!  Wealth under scenario 2;
      WMIN  <= 1.5 * A + 0.7 * C ;

and re-solve it to find:

Variable           Value
    WMIN        1.136364
       A       0.5454545
       C       0.4545455
```

This is a bit curious. We have decreased our investment in *C*. This is as if our friend had continued on: "I have this very favorable news regarding stock *C*. Let's sell it before the market has a chance to react". Why the anomaly? The problem is we are basing our measure of goodness on a single point among the possible payoffs. In this case, it is the worst possible. For a further discussion of these issues, see Clyman (1995).

2.5.2 Value at Risk

In 1994, J.P. Morgan popularized the "Value at Risk" (VaR) concept with the introduction of their RiskMetrics™ system. To use VaR, you must specify two numbers: 1) an interval of time (e.g., one day) over which you are concerned about losing money, and 2) a probability threshold (e.g., 5%) beyond which you care about harmful outcomes. VaR is then defined as that amount of loss in one day that has at most a 5% probability of being exceeded. A comprehensive survey of VaR is Jorion (2001).

Example

Suppose that one day from now we think that our portfolio will have appreciated in value by $12,000. The actual value, however, has a Normal distribution with a standard deviation of $10,000. From a Normal table, we can determine that a left tail probability of 5% corresponds to an outcome that is 1.644853 standard deviations below the mean. Now:

$$12000 - 1.644853 * 10000 = -4448.50.$$

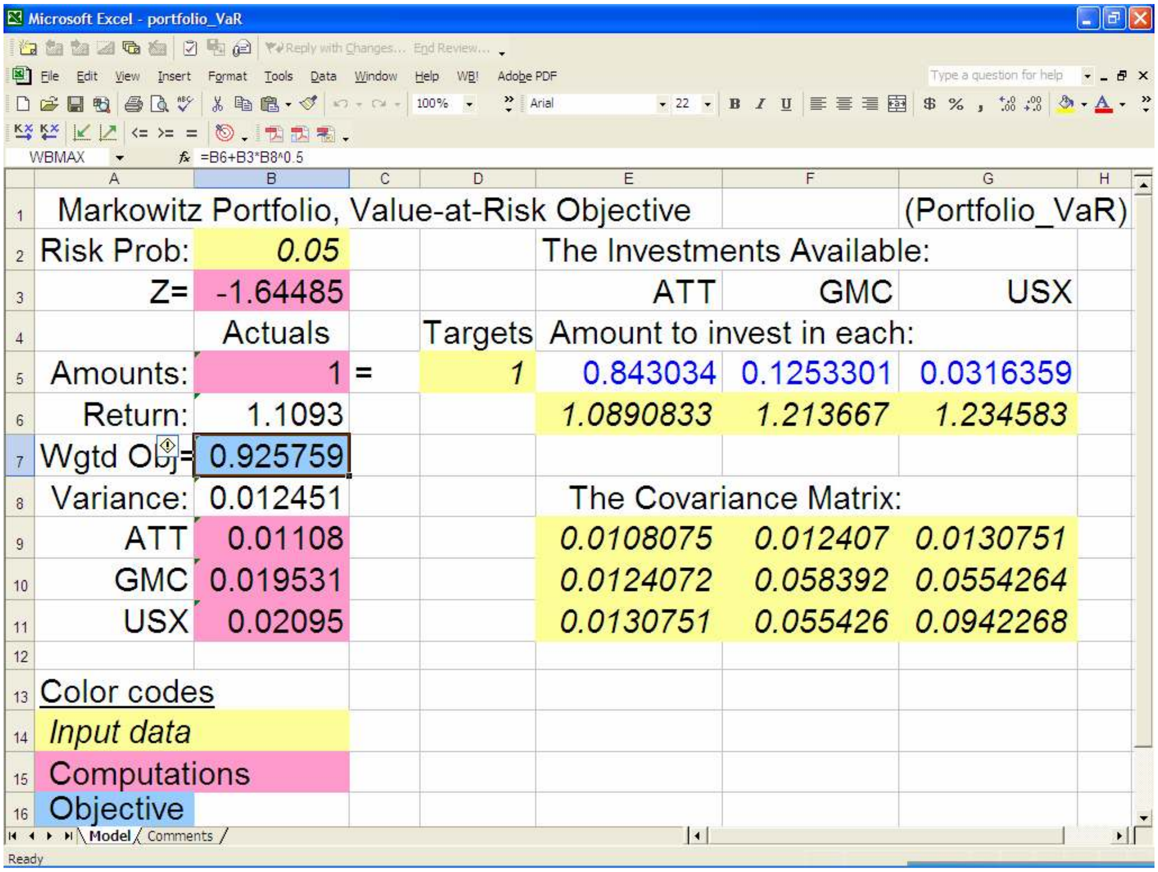
So, we would say that the value at risk is $4448.50.

2.5.3 Example of VaR

Let us apply the VAR approach to our standard example, the ATT/GMC/USC model. Suppose that our time interval of interest is one year and our risk tolerance is 5% and we want to minimize the value at risk of our portfolio. This is equivalent to maximizing that threshold, so the probability our wealth is below this threshold is at most .05.

Analysis:

A left tail probability of 5% corresponds to the probability threshold. We want to consider the point that is 1.64485 standard deviations below the mean. Minimizing the value at risk corresponds to choosing the mean and standard deviation of the portfolio, so the (mean – 1.64485 * (standard deviation)) is maximized. The following model will do this:



Microsoft Excel - portfolio_VaR

WBMAX =B6+B3*B8^0.5

	A	B	C	D	E	F	G
1	Markowitz Portfolio, Value-at-Risk Objective						(Portfolio_VaR)
2	Risk Prob:	0.05			The Investments Available:		
3	Z=	-1.64485			ATT	GMC	USX
4		Actuals		Targets	Amount to invest in each:		
5	Amounts:	1	=	1	0.843034	0.1253301	0.0316359
6	Return:	1.1093			1.0890833	1.213667	1.234583
7	Wgtd Obj=	0.925759					
8	Variance:	0.012451			The Covariance Matrix:		
9	ATT	0.01108			0.0108075	0.012407	0.0130751
10	GMC	0.019531			0.0124072	0.058392	0.0554264
11	USX	0.02095			0.0130751	0.055426	0.0942268
12							
13	Color codes						
14	Input data						
15	Computations						
16	Objective						

Model / Comments

Ready

Note that, if we invested solely in ATT, the portfolio variance would be .01080754. So, the standard deviation would be .103959, and the VAR would be 1 - (1.089083 - 1.644853 * .103959) = .0818.

The portfolio is efficient because it is maximizing a weighted combination of the expected return and (a negatively weighted) standard deviation. Thus, if there is a portfolio that has both higher expected return and lower standard deviation, then the above solution would not maximize the objective function above.

Note, if you use: *PROB* = .1988, you get essentially the original portfolio considered for the ATT/GMC/USX problem.

The crucial formulae are:

```
B3==NORMSINV(B2),
B5=SUM(E5:G5),
B7=B6+B3*B8^0.5,
B9=SUMPRODUCT(E9:G9,E$5:G$5)
C5=WB(B5,"=",D5).
```

2.6 Scenario Model and Minimizing Downside Risk

Minimizing the variance in return is appropriate if either:

1) the actual return is Normal-distributed or
2) the portfolio owner has a quadratic utility function.

In practice, it is difficult to show either condition holds. Thus, it may be of interest to use a more intuitive measure of risk. One such measure is the downside risk, which intuitively is the expected amount by which the return is less than a specified target return. The approach can be described if we define:

T = user specified target threshold. When risk is disregarded, this is typically less than the maximum expected return and greater than the return under the worst scenario.

Y_s = amount by which the return under scenario s falls short of target.

$= \max\{0, T - \sum X_i u_{is}\}$

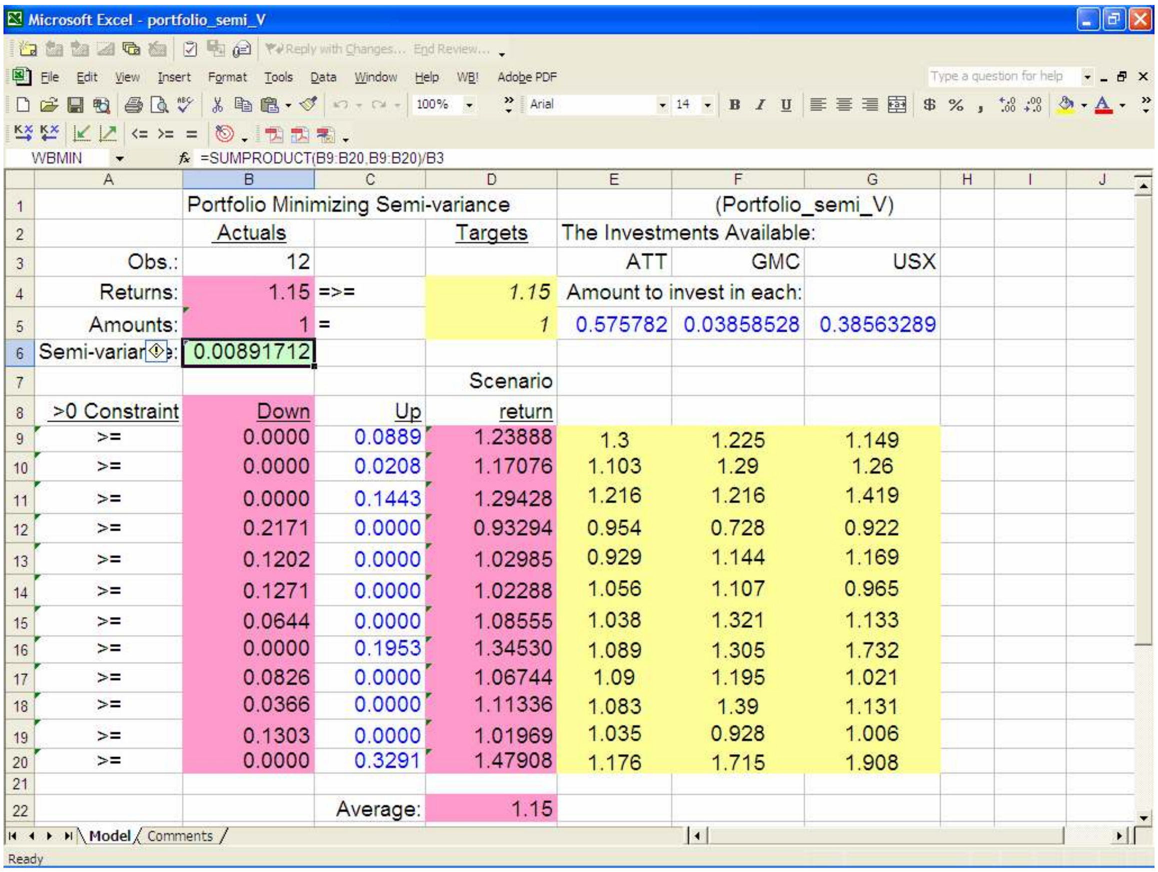
The model in algebraic form is then:

Min $\sum P_s Y_s$! Minimize expected downside risk
subject to
(compute deviation below target of each scenario, s):
$Y_s - T + \sum X_i u_{is} \geq 0$
$\sum X_i = 1$! (budget constraint)
$\sum X_i \sum P_s u_{is} \geq r$! (desired return).

Notice this is just a linear program.

2.6.1 Semi-variance and Downside Risk

The most common alternative suggested to variance as a measure of risk is some form of downside risk. One such measure is semi-variance. It is essentially variance, except only deviations below the mean are counted as risk. The scenario model is well suited to such measures. The previous scenario model needs only a slight modification to convert it to a semi-variance model.



Microsoft Excel - portfolio_semi_V

WBMIN =SUMPRODUCT(B9:B20,B9:B20)/B3

	A	B	C	D	E	F	G
1		Portfolio Minimizing Semi-variance				(Portfolio_semi_V)	
2		Actuals		Targets	The Investments Available:		
3	Obs.:	12			ATT	GMC	USX
4	Returns:	1.15	=>=	1.15	Amount to invest in each:		
5	Amounts:	1	=	1	0.575782	0.03858528	0.38563289
6	Semi-variance:	0.00891712					
7				Scenario			
8	>0 Constraint	Down	Up	return			
9	>=	0.0000	0.0889	1.23888	1.3	1.225	1.149
10	>=	0.0000	0.0208	1.17076	1.103	1.29	1.26
11	>=	0.0000	0.1443	1.29428	1.216	1.216	1.419
12	>=	0.2171	0.0000	0.93294	0.954	0.728	0.922
13	>=	0.1202	0.0000	1.02985	0.929	1.144	1.169
14	>=	0.1271	0.0000	1.02288	1.056	1.107	0.965
15	>=	0.0644	0.0000	1.08555	1.038	1.321	1.133
16	>=	0.0000	0.1953	1.34530	1.089	1.305	1.732
17	>=	0.0826	0.0000	1.06744	1.09	1.195	1.021
18	>=	0.0366	0.0000	1.11336	1.083	1.39	1.131
19	>=	0.1303	0.0000	1.01969	1.035	0.928	1.006
20	>=	0.0000	0.3291	1.47908	1.176	1.715	1.908
21							
22			Average:	1.15			

Notice the objective value is less than half that of the variance model. We would expect it to be at most half, because it considers only the down (not the up) deviations. The most noticeable change in the portfolio is substantial funds have been moved to *USX* from *GMC*. This is not surprising if you look at the original data. In the years in which *ATT* performs poorly, *USX* tends to perform better than *GMC*.

The formulae and constraints are essentially as with the model Portfolio_scene, except for the objective cell.

The crucial formulae are:

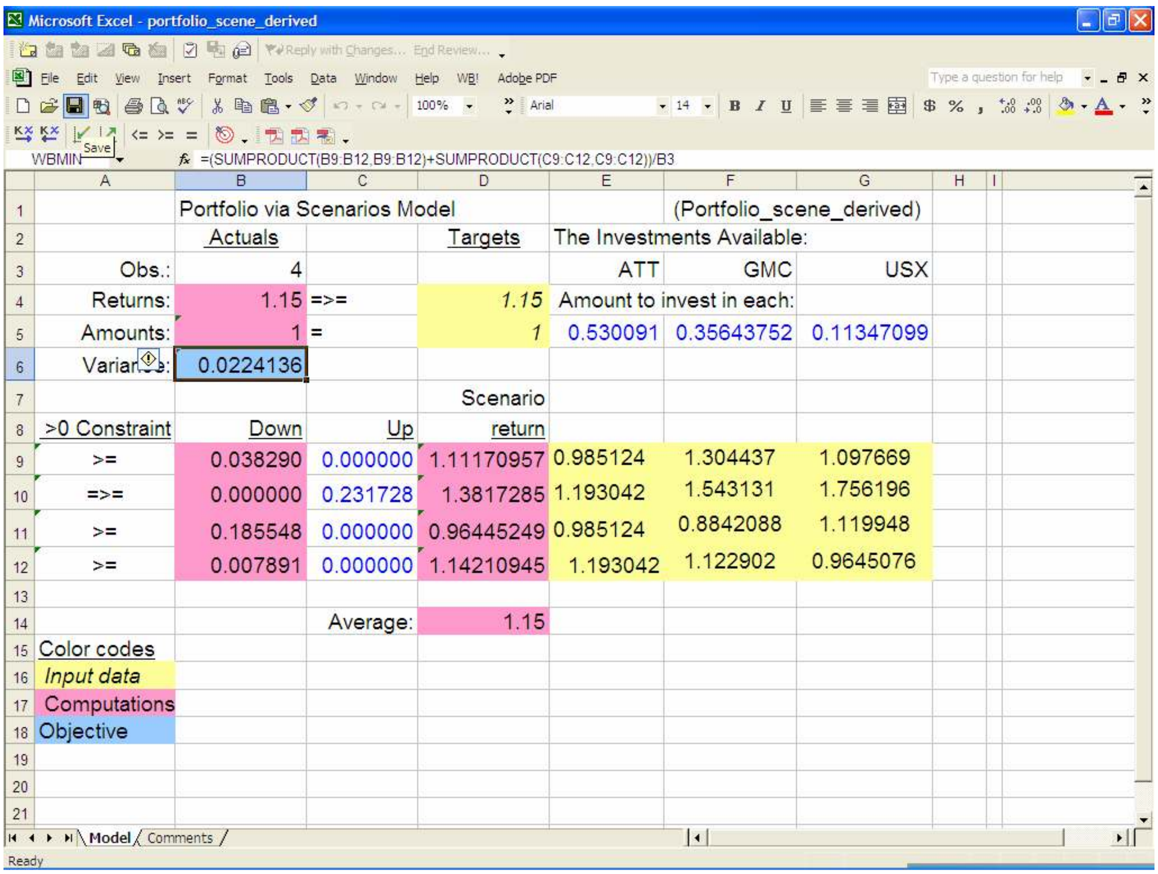
B4=D22,
B5=SUM(E5:G5),
B6=SUMPRODUCT(B9:B20,B9:B20)/B3,
B9=C9-D9+$D$22,
D9=SUMPRODUCT(E9:G9,E$5:G$5),
D22=AVERAGE(D9:D20).

2.6.2 Downside Risk and MAD

If the threshold for determining downside risk is the mean return, then minimizing the downside risk is equivalent to minimizing the mean absolute deviation (MAD) about the mean. This follows easily because the sum of deviations (not absolute) about the mean must be zero. Thus, the sum of deviations above the mean equals the sum of deviations below the mean. Therefore, the sum of absolute deviations is always twice the sum of the deviations below the mean. Thus, minimizing the downside risk below the mean gives exactly the same recommendation as minimizing the sum of absolute deviations below the mean. Konno and Yamazaki (1991) use the MAD measure to construct portfolios from stocks on the Tokyo stock exchange.

2.6.3 Scenarios Based Directly Upon a Covariance Matrix

If only a covariance matrix is available, rather than original data, then, not surprisingly, it is nevertheless possible to construct scenarios that match the covariance matrix. The following example uses just four scenarios to represent the possible returns from the three assets: ATT, GMC, and USX. These scenarios have been constructed, using the methods of section 2.8.2, so they mimic behavior consistent with the original covariance matrix:



	A	B	C	D	E	F	G
1		Portfolio via Scenarios Model				(Portfolio_scene_derived)	
2		Actuals		Targets	The Investments Available:		
3	Obs.:	4			ATT	GMC	USX
4	Returns:	1.15	=>=	1.15	Amount to invest in each:		
5	Amounts:	1	=	1	0.530091	0.35643752	0.11347099
6	Variance:	0.0224136					
7				Scenario			
8	>0 Constraint	Down	Up	return			
9	>=	0.038290	0.000000	1.11170957	0.985124	1.304437	1.097669
10	=>=	0.000000	0.231728	1.3817285	1.193042	1.543131	1.756196
11	>=	0.185548	0.000000	0.96445249	0.985124	0.8842088	1.119948
12	>=	0.007891	0.000000	1.14210945	1.193042	1.122902	0.9645076
13							
14			Average:	1.15			
15	Color codes						
16	Input data						
17	Computations						
18	Objective						

Notice the objective function value and the allocation of funds over *ATT*, *GMC*, and *USX* are essentially identical to our original portfolio example.

The crucial formulae are:

B4=D14
C4=WB(B4,">=",D4)
B5=SUM(E5:G5)
C5=WB(B5,"=",D5)
B6=(SUMPRODUCT(B9:B12,B9:B12)+SUMPRODUCT(C9:C12,C9:C12))/B3

```
B9=C9-D9+$D$14
D9=SUMPRODUCT(E9:G9,E$5:G$5)
D14=AVERAGE(D9:D12)
```

2.7 Hedging, Matching and Program Trading

2.7.1 Portfolio Hedging

Given a "benchmark" portfolio *B*, we say we hedge *B* if we construct another portfolio *C* such that, taken together, *B* and *C* have essentially the same return as *B*, but lower risk than *B*. Typically, our portfolio *B* contains certain components that cannot be removed. Thus, we want to buy some components negatively correlated with the existing ones. Examples are:

a) An airline knows it will have to purchase a lot of fuel in the next three months. It would like to be insulated from unexpected fuel price increases.
b) A farmer is confident his fields will yield $200,000 worth of corn in the next two months. He is happy with the current price for corn. Thus, would like to "lock in" the current price.

2.7.2 Portfolio Matching, Tracking, and Program Trading

Given a benchmark portfolio *B*, we say we construct a matching or tracking portfolio if we construct a new portfolio *C* that has stochastic behavior very similar to ***B***, but excludes certain instruments in *B*. Example situations are:

a) A portfolio manager does not wish to look bad relative to some well-known index of performance such as the S&P 500, but for various reasons cannot purchase certain instruments in the index.
b) An arbitrageur with the ability to make fast, low-cost trades wants to exploit market inefficiencies (i.e., instruments mispriced by the market). If he can construct a portfolio that perfectly matches the future behavior of the well-defined portfolio, but costs less today, then he has an arbitrage profit opportunity (if he can act before this "mispricing" disappears).
c) A retired person is concerned mainly about inflation risk. In this case, a portfolio that tracks inflation is desired.

As an example of (a), a certain so-called "green" mutual fund will not include in its portfolio companies that derive more than 2% of their gross revenues from the sale of military weapons, own directly or operate nuclear power plants, or participate in business related to the nuclear fuel cycle.

The following table, for example, compares the performance of six Vanguard portfolios with the indices the portfolios were designed to track; see Vanguard (1995):

Total Return Six Months Ended June 30, 1995

Vanguard Portfolio Name	Portfolio Growth	Comparative Growth Index	Index Name
500 Portfolio	+20.1%	+20.2%	S&P500
Growth Portfolio	+21.1	+21.2	S&P500/BARRA Growth
Value Portfolio	+19.1	+19.2	S&P500/BARRA Value
Extended Market Portfolio	+17.1%	+16.8%	Wilshire 4500 Index
SmallCap Portfolio	+14.5	+14.4	Russell 2000 Index
Total Stock Market Portfolio	+19.2%	+19.2%	Wilshire 5000 Index

Notice, even though there is substantial difference in the performance of the portfolios, each matches its benchmark index quite well.

2.8 Methods for Constructing Benchmark Portfolios

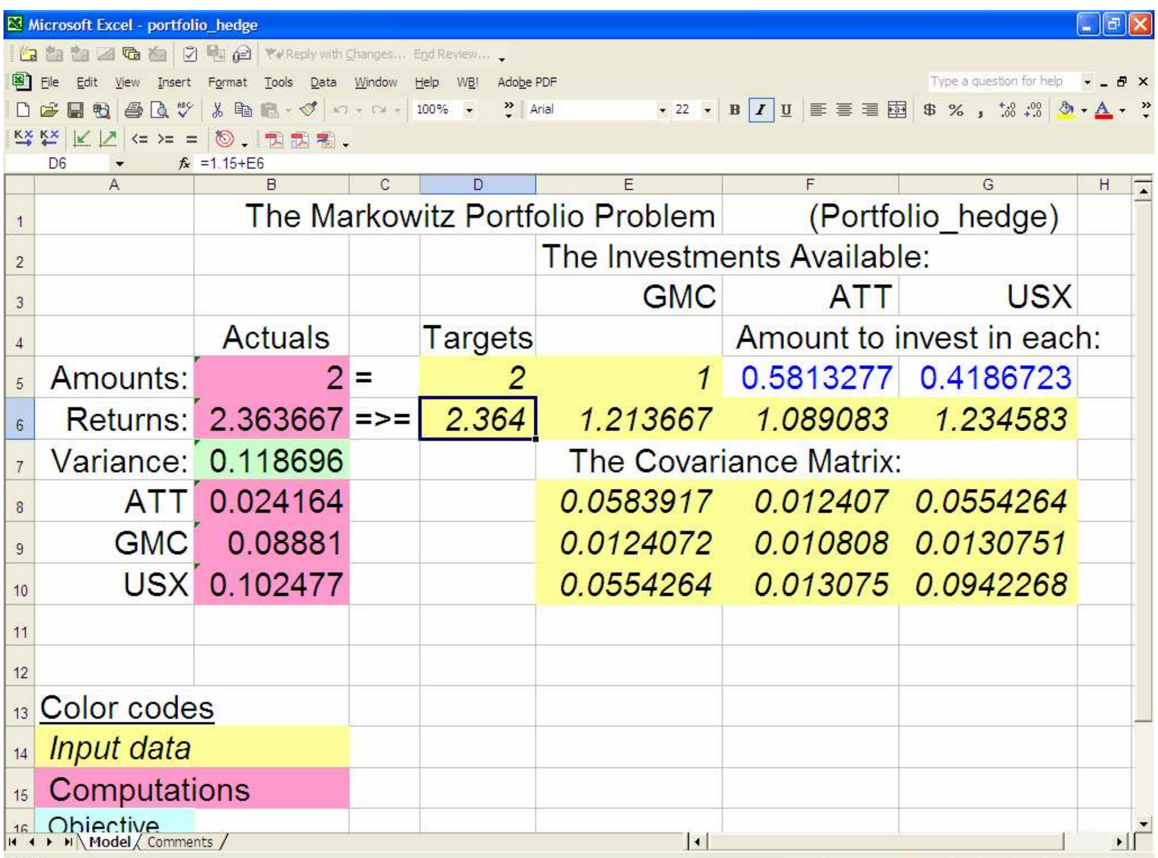
A variety of approaches has been used for constructing hedging and matching portfolios. For matching portfolios, an intuitive approach has been to generalize the Markowitz model, so the objective is to minimize the variance in the difference in return between the target portfolio and the tracking portfolio.

A useful way to think about hedging or matching of a benchmark is to think of it as our being forced to include the benchmark or its negative in our portfolio. Suppose the benchmark is a simple index such as the S&P500. If our measure of risk is variance, then proceed as follows:

1. Include the benchmark in the covariance matrix just like any other instrument, except do not include it in the budget constraint. We presume we have a budget of $1 to invest in the controllable, non-benchmark portion of our portfolio.
2. To get a “matching” portfolio (e.g., one that mimics the S&P 500), set the value of the benchmark factor to -1. The essential effect is the off diagonal covariance terms are negated in the row/column of the benchmark factor. Effectively, we have shorted the factor. If we can get the total variance to zero, we have perfectly matched the randomness of the benchmark.
3. To get a “hedging” portfolio (e.g., one as negatively correlated with the S&P 500 as possible), set the value of the benchmark factor to +1. Thus, we will compose the rest of the portfolio to counteract the effect of the factor we are stuck with having in the portfolio.

One might even want to drop the budget constraint. The solution will then tell you how much to invest in the controllable portfolio to get the best possible hedge or match per $ of the benchmark.

The following model illustrates the extension of the Markowitz approach to the hedging case where we want to “cancel out” some benchmark. In the case of *GMC*, it could be that our decision maker works for *GMC* and thus has his fortunes unavoidably tied to those of *GMC*. He might wish to find a portfolio negatively correlated with *GMC*:



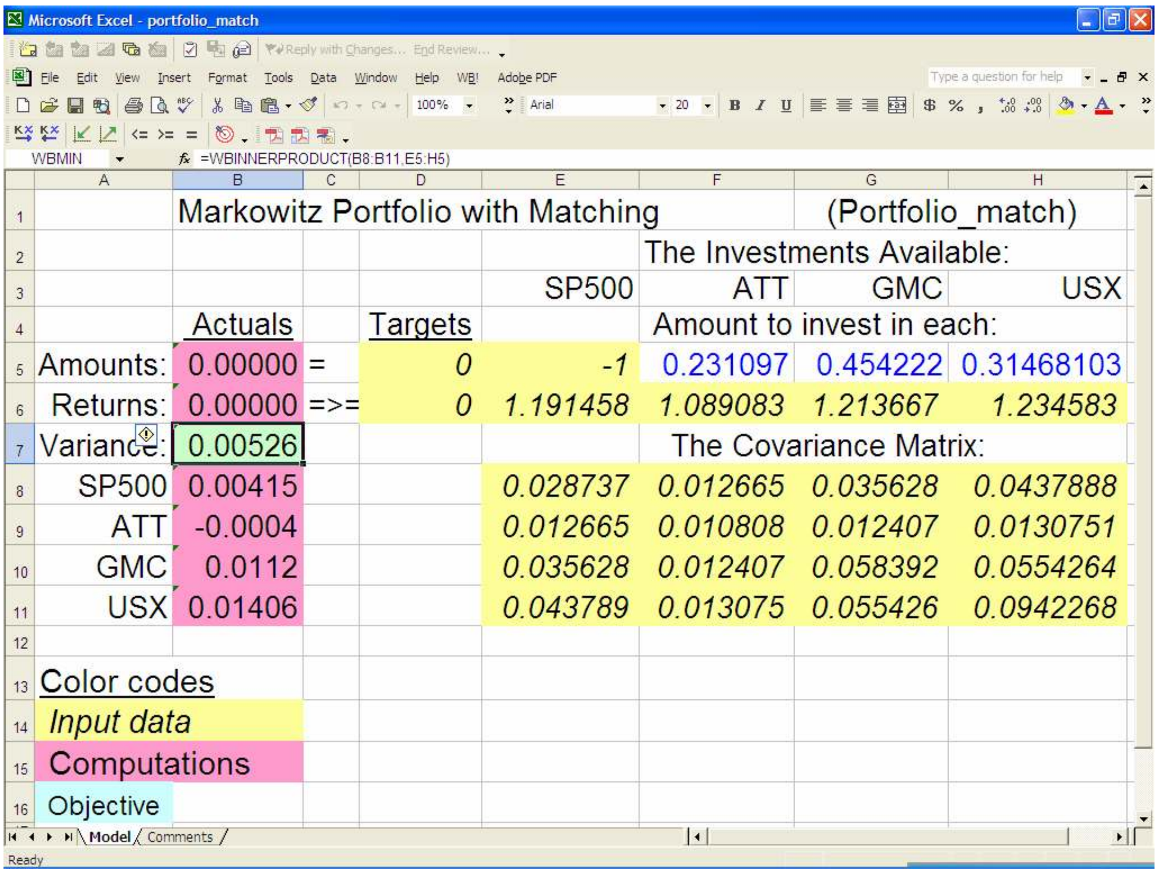
	A	B	C	D	E	F	G
1		The Markowitz Portfolio Problem				(Portfolio_hedge)	
2					The Investments Available:		
3					GMC	ATT	USX
4		Actuals		Targets		Amount to invest in each:	
5	Amounts:	2	=	2	1	0.5813277	0.4186723
6	Returns:	2.363667	=>=	2.364	1.213667	1.089083	1.234583
7	Variance:	0.118696			The Covariance Matrix:		
8	ATT	0.024164			0.0583917	0.012407	0.0554264
9	GMC	0.08881			0.0124072	0.010808	0.0130751
10	USX	0.102477			0.0554264	0.013075	0.0942268
11							
12							
13	Color codes						
14	Input data						
15	Computations						
16	Objective						

Thus, our investor puts more of the portfolio in *ATT* than in *USX* (whose fortunes are more closely tied to those of *GMC*).

The crucial formulae are:

B5=SUM(E5:G5),
C5=WB(B5,"=",D5),
B6=SUMPRODUCT(E6:G6,E5:G5),
B7=WBINNERPRODUCT(B8:B10,E5:G5),
B8=SUMPRODUCT(E9:G9,E$5:G$5)

The following model illustrates the extension of the Markowitz approach to the matching case where we want to construct a portfolio that *mimics or matches* a benchmark portfolio. In this case, we want to match the S&P500, but limit ourselves to investing in only *ATT*, *GMC*, and *USX*.



Microsoft Excel - portfolio_match

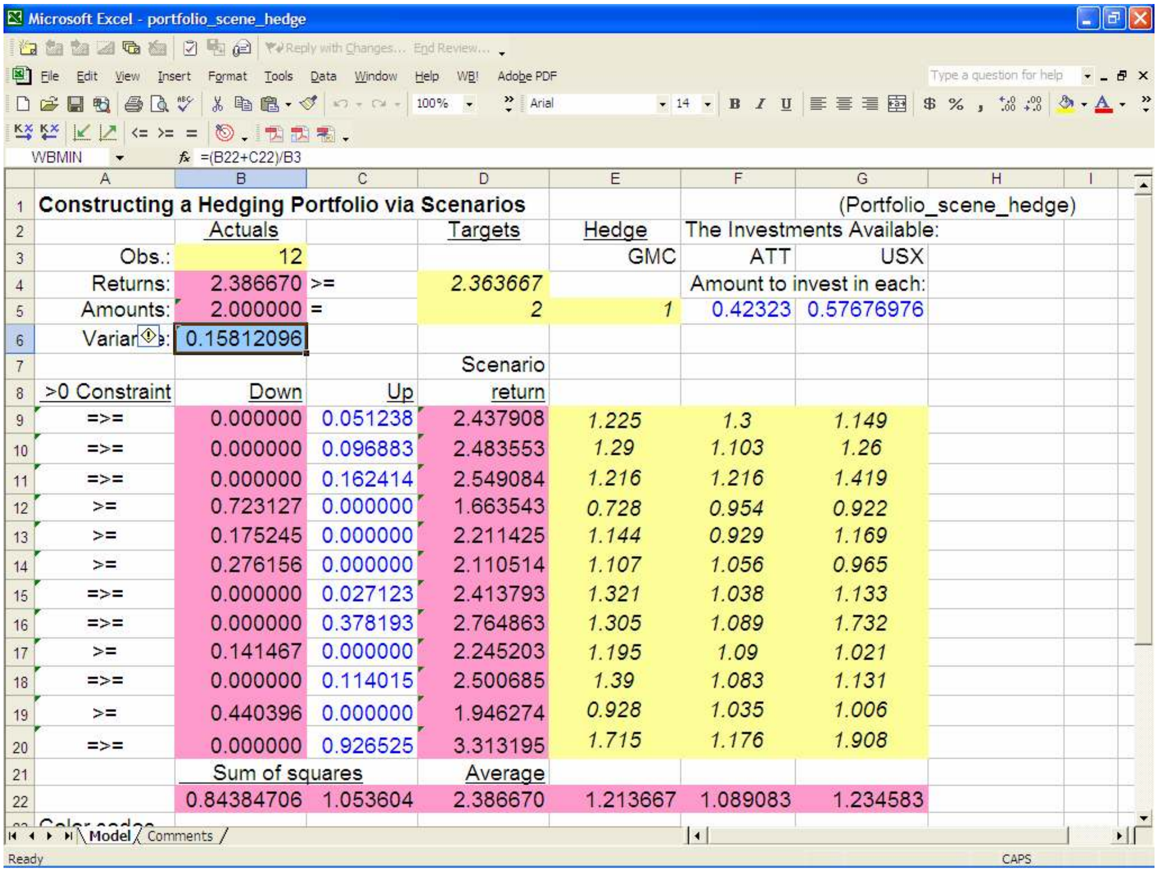
WBMIN fx =WBINNERPRODUCT(B8:B11,E5:H5)

	A	B	C	D	E	F	G	H
1		Markowitz Portfolio with Matching					(Portfolio_match)	
2						The Investments Available:		
3					SP500	ATT	GMC	USX
4		Actuals		Targets		Amount to invest in each:		
5	Amounts:	0.00000	=	0	-1	0.231097	0.454222	0.31468103
6	Returns:	0.00000	=>=	0	1.191458	1.089083	1.213667	1.234583
7	Variance:	0.00526				The Covariance Matrix:		
8	SP500	0.00415			0.028737	0.012665	0.035628	0.0437888
9	ATT	-0.0004			0.012665	0.010808	0.012407	0.0130751
10	GMC	0.0112			0.035628	0.012407	0.058392	0.0554264
11	USX	0.01406			0.043789	0.013075	0.055426	0.0942268
12								
13	Color codes							
14	Input data							
15	Computations							
16	Objective							

The formulae in the matching model are the same as in the hedging model. The only difference is in the data entered.

2.8.1 Scenario Approach to Benchmark Portfolios

The scenario approach can be used for constructing hedging and matching portfolios in much the same way as the classical Markowitz model was used. The following model tries to construct a hedge relative to GMC from ATT and USX.



Microsoft Excel - portfolio_scene_hedge

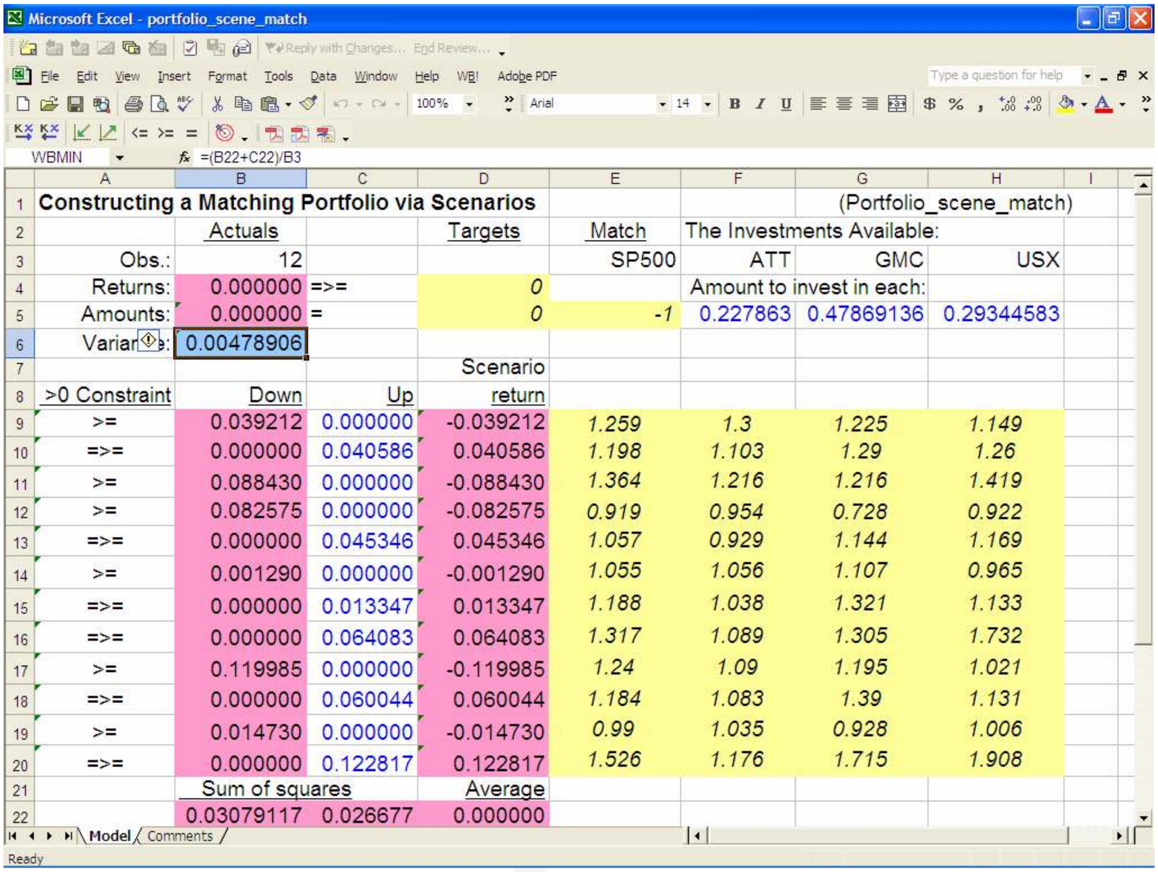
WBMIN =(B22+C22)/B3

	A	B	C	D	E	F	G	H
1	Constructing a Hedging Portfolio via Scenarios						(Portfolio_scene_hedge)	
2		Actuals		Targets	Hedge	The Investments Available:		
3	Obs.:	12			GMC	ATT	USX	
4	Returns:	2.386670	>=	2.363667		Amount to invest in each:		
5	Amounts:	2.000000	=	2	1	0.42323	0.57676976	
6	Variance:	0.15812096						
7				Scenario				
8	>0 Constraint	Down	Up	return				
9	=>=	0.000000	0.051238	2.437908	1.225	1.3	1.149	
10	=>=	0.000000	0.096883	2.483553	1.29	1.103	1.26	
11	=>=	0.000000	0.162414	2.549084	1.216	1.216	1.419	
12	>=	0.723127	0.000000	1.663543	0.728	0.954	0.922	
13	>=	0.175245	0.000000	2.211425	1.144	0.929	1.169	
14	>=	0.276156	0.000000	2.110514	1.107	1.056	0.965	
15	=>=	0.000000	0.027123	2.413793	1.321	1.038	1.133	
16	=>=	0.000000	0.378193	2.764863	1.305	1.089	1.732	
17	>=	0.141467	0.000000	2.245203	1.195	1.09	1.021	
18	=>=	0.000000	0.114015	2.500685	1.39	1.083	1.131	
19	>=	0.440396	0.000000	1.946274	0.928	1.035	1.006	
20	=>=	0.000000	0.926525	3.313195	1.715	1.176	1.908	
21		Sum of squares		Average				
22		0.84384706	1.053604	2.386670	1.213667	1.089083	1.234583	

The crucial formulae are:

B4=D22,
C4=wb(B4,">=",D4),
B5=SUM(E5:G5),
C5=wb(B5,"=",D5),
B6=(B22+C22)/B3,
B9=C9-D9+$D$22,
D9=SUMPRODUCT(E9:G9,E$5:G$5),
B22=SUMPRODUCT(B9:B20,B9:B20),
C22=SUMPRODUCT(C9:C20,C9:C20),
D22=AVERAGE(D9:D20),
E22=AVERAGE(E9:E20).

The following is a scenario model for constructing a portfolio matching the S&P500:



	A	B	C	D	E	F	G	H
1	Constructing a Matching Portfolio via Scenarios						(Portfolio_scene_match)	
2		Actuals		Targets	Match	The Investments Available:		
3	Obs.:	12			SP500	ATT	GMC	USX
4	Returns:	0.000000	=>=	0		Amount to invest in each:		
5	Amounts:	0.000000	=	0	-1	0.227863	0.47869136	0.29344583
6	Variance:	0.00478906						
7				Scenario				
8	>0 Constraint	Down	Up	return				
9	>=	0.039212	0.000000	-0.039212	1.259	1.3	1.225	1.149
10	=>=	0.000000	0.040586	0.040586	1.198	1.103	1.29	1.26
11	>=	0.088430	0.000000	-0.088430	1.364	1.216	1.216	1.419
12	>=	0.082575	0.000000	-0.082575	0.919	0.954	0.728	0.922
13	=>=	0.000000	0.045346	0.045346	1.057	0.929	1.144	1.169
14	>=	0.001290	0.000000	-0.001290	1.055	1.056	1.107	0.965
15	=>=	0.000000	0.013347	0.013347	1.188	1.038	1.321	1.133
16	=>=	0.000000	0.064083	0.064083	1.317	1.089	1.305	1.732
17	>=	0.119985	0.000000	-0.119985	1.24	1.09	1.195	1.021
18	=>=	0.000000	0.060044	0.060044	1.184	1.083	1.39	1.131
19	>=	0.014730	0.000000	-0.014730	0.99	1.035	0.928	1.006
20	=>=	0.000000	0.122817	0.122817	1.526	1.176	1.715	1.908
21		Sum of squares		Average				
22		0.03079117	0.026677	0.000000				

Notice that we get the same portfolio as with the Markowitz model.

The two scenario models both used variance for the measure of risk relative to the benchmark. It is easy to modify them, so more asymmetric risk measures, such as downside risk, could be used.

The formulae in this model are the same as in the previous.

2.8.2 Efficient Benchmark Portfolios

We say a portfolio is on the efficient frontier if there is no other portfolio that has both higher expected return and lower risk.

Let:

r_i = expected return on asset i,
t = an arbitrary target return for the portfolio.

A portfolio, with weight m_i on asset i, is efficient if there exists some target t for which the portfolio is a solution to the problem:

Minimize risk
subject to

$$\sum_{i=0}^{n} m_i = 1 \qquad \text{(budget constraint)}$$

$$\sum_{i=0}^{m} r_i\, m_i \geq t \qquad \text{(return target constraint).}$$

Portfolio managers are frequently evaluated on their performance relative to some benchmark portfolio. Let b_i = the weight on asset i in the benchmark portfolio. If the benchmark portfolio is not on the efficient frontier, then an interesting question is: What are the weights of the portfolio on the efficient frontier that is closest to the benchmark portfolio in the sense that the risk of the efficient portfolio relative to the benchmark is minimized?

There is a particularly simple answer when the measure of risk is portfolio variance, there is a risk-free asset, borrowing is allowed at the risk-free rate, and short sales are permitted. Let m_0 = the weight on the risk-free asset. An elegant result, in this case, is that there is a so-called "market" portfolio with weights mi on asset i, such that effectively only m_0 varies as the return target varies. Specifically, there are constants m_i, for i = 1, 2, . . . , n, such that the weight on asset i is simply $(1 - m_0) \times m_i$. Define:

$q = 1 - m_0$ = weight to put on the market portfolio,
R_i = random return on asset i.

Then the variance of any efficient portfolio relative to the benchmark portfolio can be written as:

$$\text{var}\Big(\sum_{i=1}^{n} R_i[q*m_i - b_i]\Big)$$

$$= \quad \sum_{i=1}^{n} (q*m_i - b_i)^2 \text{ var}(R_i) + 2 \sum_{j>i} (q*m_i - b_i)(q*m_j - b_j) \text{ Cov}(R_i, R_j).$$

Setting the derivative of this expression with respect to q equal to zero gives the result:

$$q = \frac{\sum_{i=1}^{n} m_i * b_i \text{ var}(R_i) + \sum_{j>i} (m_i*b_j*m_j*b_i) \text{ Cov}(R_i, R_j)}{\sum_{i=1}^{n} m_i^2 \text{ var}(R_i) + 2 \sum_{i>i} m_i\, m_j \text{ Cov}(R_i, R_j)}$$

For example, if the benchmark portfolio is on the efficient frontier with weight b_0 on the risk-free asset, then $b_i = (1 - b_0)m_i$ and therefore $q = 1 - b_0$. Thus, a manager who is told to outperform the benchmark portfolio $\{b_0, b_1, \ldots, b_n\}$ should perhaps, in fact, be compensated according to his performance relative to the efficient portfolio given by q above.

2.9 Project Portfolios

Some organizations use a yearly budgeting process to select which projects to pursue in the coming year. Examples of projects might be: which crude oil fields to develop for a petroleum exploration firm, which drugs to develop for a pharmaceutical firm, and which types of markets and technologies to pursue for a telecommunications firm. Many of the ideas underlying the portfolio models considered thus far also apply to the project selection portfolio problem. For example, an overall budget may be set at the beginning of the planning exercise for how much can be invested in new projects this year. The major differences distinguishing the project portfolio problem are: a) the investment variables are 0/1, "go/no go" decision variables, b) it is much less obvious how one develops the covariance or correlation matrix describing the project and interproject risks, and c) there may be logical constraints among the projects, typically of an "either-or" nature or an "if we do project A we must do project B" flavor. Consider the following.

Example

The BTT communications company has six projects it is considering for the coming year.

Project Tech1 is a technology development project that requires an initial investment of $1.9M and has an expected value of $2.36M after one year. The standard deviation in the value after one year is $.37M

Project Tech2 is an alternative to Tech1. It requires an initial investment of $2.5M and has an expected value of $3.1M after one year. The standard deviation in the value after one year is $.39M.

Project Ads is an advertising campaign for a certain metropolitan area for a new kind of call handling service. This service has already been introduced on a trial basis in some regions of the city. It requires an initial investment of $1.7M and has an expected value of $1.5M after one year. The standard deviation in the value after one year is $.3M. Note that its incremental return is negative, so that it does not appear worthwhile until we consider projects Regn1, Regn2, and Regn3.

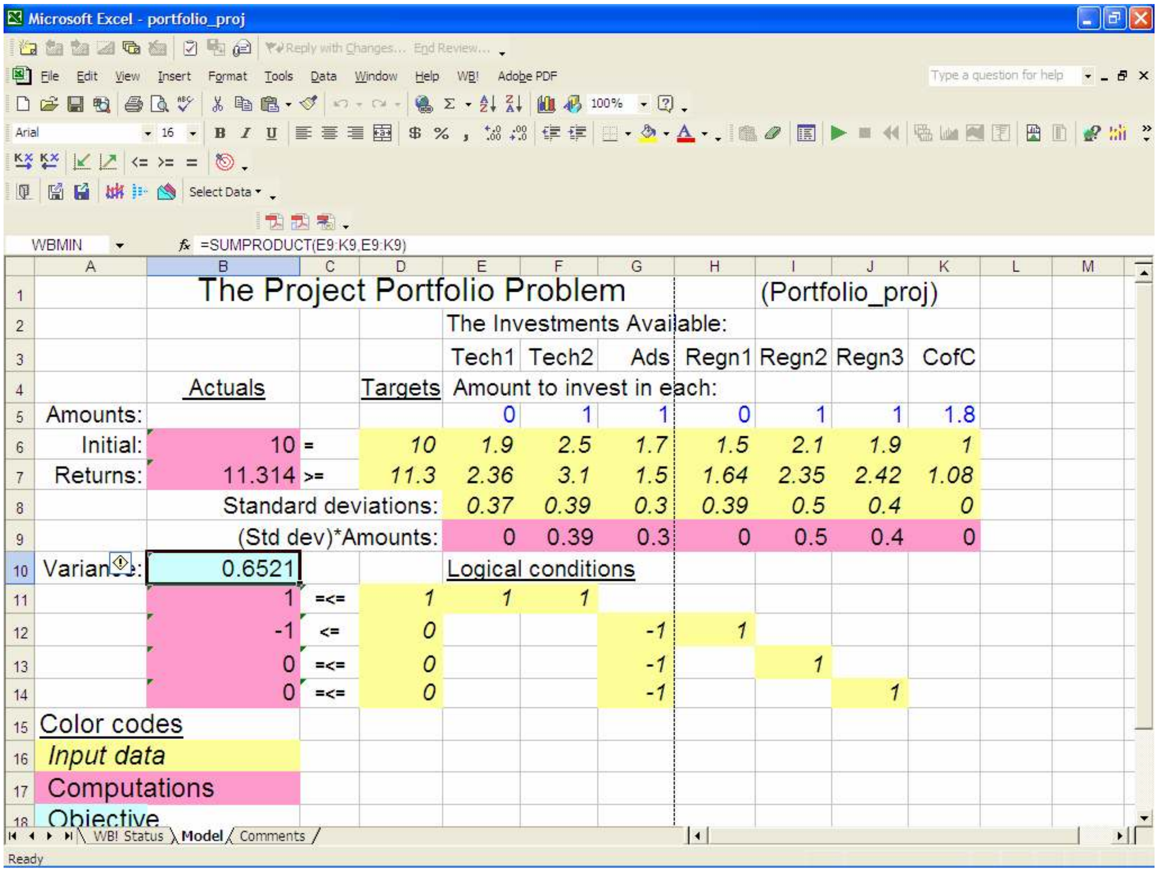
Project Regn1 is the project to install the new call handling capability into Region 1. It requires an initial investment of $1.5M and has an expected value of $1.64M after one year. The standard deviation in the value after one year is $.39M. Note, this expected return for Regn1 is based on the assumption that the major metropolitan advertising campaign, project Ads above, for the call handling service will be undertaken, else project Regn1 will not be worthwhile.

Project Regn2 is similar to Regn1, except it applies to region 2. Regn2 requires an initial investment of $2.1M and has an expected value of $2.35M after one year. The standard deviation in the value after one year is $.5M. This expected return for Regn2 is based on the assumption that the major metropolitan advertising campaign for the call handling service will be undertaken, else project Regn2 will not be worthwhile.

Project Regn3 is similar to Regn1, except it applies to region 3. Regn3 requires an initial investment of $1.9M and has an expected value of $2.42M after one year. The standard deviation in the value after one year is $.4M. This expected return for Regn3 is based on the assumption that the major metropolitan advertising campaign for the call handling service will be undertaken, else project Regn3 will not be worthwhile.

BTT has available a budget of $10M to invest in these projects. Either because of the lumpiness of the project, or perhaps for other reasons, we may not wish to use exactly $10M. How should we treat any left over funds? If we are borrowing the money, then we should simply apply the borrowing rate to these left over funds because we avoid the interest payment. Alternatively, we may have other standard investments with fairly reliable returns in which left over funds are invested. For BTT, this "Cost of Capital" rate is 8%. It is represented in the model as the investment "CofC". Suppose that after one year, BTT would like its investment to have an expected return of 13%. This means would like the $10M budget to grow to a value $11.3 after one year.

Which projects should be undertaken? The following spreadsheet illustrates the model and the suggested solution.



	A	B	C	D	E	F	G	H	I	J	K
1		The Project Portfolio Problem							(Portfolio_proj)		
2					The Investments Available:						
3					Tech1	Tech2	Ads	Regn1	Regn2	Regn3	CofC
4		Actuals		Targets	Amount to invest in each:						
5	Amounts:				0	1	1	0	1	1	1.8
6	Initial:	10	=	10	1.9	2.5	1.7	1.5	2.1	1.9	1
7	Returns:	11.314	>=	11.3	2.36	3.1	1.5	1.64	2.35	2.42	1.08
8		Standard deviations:			0.37	0.39	0.3	0.39	0.5	0.4	0
9		(Std dev)*Amounts:			0	0.39	0.3	0	0.5	0.4	0
10	Variance:	0.6521			Logical conditions						
11		1	=<=	1	1	1					
12		-1	<=	0			-1	1			
13		0	=<=	0			-1		1		
14		0	=<=	0			-1			1	
15	Color codes										
16	Input data										
17	Computations										
18	Objective										

The solution suggests that we should invest in projects Tech2, Ads, Regn2, and Regn3 and leave 1.8 million in the Cost of Capital fund. This solution has some interesting features. For example, the rate of return for Tech1 is (2.36 – 1.9)/ 1.9 = .2421, whereas the return on Tech2 is (3.1 – 2.5)/2.5 = 1.24. So Tech1 has a slightly higher return, and Tech1 has lower risk, .37, than Tech2, .39. Nevertheless, Tech2 was chosen over the alternative Tech1. Why? The key is that Tech2 allows us to invest more money at a very good rate. If we invested in Tech1 rather than Tech2, where would we invest the 2.5 – 1.9 million dollars that would become available? The obvious place would be in the CofC fund. But there it only earns an incremental return of .08, vs. the .24 return it would earn in Tech2.

The "ABC's of Optimization" for this model are:

A) The adjustable cells in this model are E5:K5. Cells E5:J5 are declared to be 0/1 or binary variables, whereas the investment of surplus funds in CofC is left as a continuous variable.

B) The "Best" or objective cell, to be minimized, is the variance computed in cell B10 by the formula: =SUMPRODUCT(E9:K9,E9:K9).

C) The constraints are computed essentially by the formulae in column B, e.g.
B6= SUMPRODUCT(E6:K6,$E$5:$K$5),
B7= SUMPRODUCT(E7:K7,$E$5:$K$5),
B11= SUMPRODUCT(E11:K11,$E$5:$K$5),

2.9.1 Implementation Issues

The above simple model requires the estimation of three data for each project: a) initial investment, b) expected value after one period, and c) standard deviation in value after one period. Typically, each project in an organization will have a "champion" or supporter. This person may be the best informed person for estimating the above data. The "champion" of a project, however, has an incentive to try to get his project funded this year and worry later about justifying the project if things do not turn out well. Thus, the "champion" will tend to underestimate the initial investment required, overestimate the expected return, and underestimate the expected risk. Thus, you also need an auditor, referee, or arbitrator who can examine the submitted data and try to keep it as unbiased as possible.

The above model approximates the risk only by a standard deviation for each project. It does not include any covariance risk among projects. Our reasoning in this regard is that it is difficult enough to provide an estimate of the standard deviation of a random variable for which we have no historical data. One way of trying to elicit the an estimate of the standard deviation is to assume returns are Normal distributed, in which case, the probability that a return is one standard deviation below the expected value is about one chance in six. Thus, one could ask someone who is knowledgeable about a project: "How much worse the could the value of the project be, so that there is one chance in six of the project doing this poorly?". Treat this difference as one standard deviation.

2.10 Problems

1. You are considering three stocks, IBM, GM, and Georgia-Pacific (GP), for your stock portfolio. The covariance matrix of the yearly percentage returns on these stocks is estimated to be:

	IBM	GM	GP
IBM	10	2.5	1
GM	2.5	4	1.5
GP	1	1.5	9

Thus, if equal amounts were invested in each, the variance would be proportional to 10 + 4 + 9 + 2 (2.5 + 1 + 1.5). The predicted yearly percentage returns for *IBM*, *GM*, and *GP* are 9, 6 and 5, respectively. Find a minimum variance portfolio of these three stocks for which the yearly return is at least 7, at most 80% of the portfolio is invested in *IBM*, and at least 10% is invested in *GP*.

2. Modify your formulation of problem 1 to incorporate the fact that your current portfolio is 50% IBM and 50% GP. Further, transaction costs on a buy/sell transaction are 1% of the amount traded.

3. The manager of an investment fund hypothesizes that three different scenarios might characterize the economy one year hence. These scenarios are denoted Green, Yellow and Red and subjective probabilities 0.7, 0.1, and 0.2 are associated with them. The manager wishes to decide how a model portfolio should be allocated among stocks, bonds, real estate and gold in the face of these possible scenarios. His estimated returns in percent per year as a function of asset and scenario are given in the table below:

	Stocks	Bonds	Real Estate	Gold
Green	9	7	8	-2
Yellow	–1	5	10	12
Red	10	4	-1	15

Formulate and solve the asset allocation problem of minimizing the variance in return subject to having an expected return of at least 6.5.

4. Consider the ATT/GMC/USX portfolio problem discussed earlier. The desired or target rate of return in the solved model was 15%.

   a) Suppose we desire a 16% rate of return. Using just the solution report, what can you predict about the standard deviation in portfolio return of the new portfolio?

   b) We illustrated the situation where the opportunity to invest money risk-free at 5% per year becomes available. That is, this fourth option has zero variance and zero covariance. Now, suppose the risk-free rate is 4% per year rather than 5%. As before, there is no limit on how much can be invested at 4%. Based on only the solution report available for the original version of the problem (where the desired rate of return is 15% per year), discuss whether this new option is attractive when the desired return for the portfolio is 15%.

   c) You have $100,000 to invest. What modifications would need to be made to the original ATT/GMC/USX model, so the answers in the solution report would come in the appropriate units (e.g., no multiplying of the numbers in the solution by 100,000)?

   d) What is the estimated standard deviation in the value of your end-of-period portfolio in (c) if invested as the solution recommends

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