**Decision Tree Notes**

Decision trees model sequential decision problems under uncertainty. A decision tree describes graphically the decisions to be made, the events that may occur, and the outcomes associated with combinations of decisions and events. Probabilities are assigned to the events, and values are determined for each outcome. A major goal of the analysis is to determine the best decisions.

Decision tree models include such concepts as nodes, branches, terminal values, strategy, payoff distribution, certainty equivalent, and the rollback method. The following problem illustrates the basic concepts.

**DriveTek Problem**

DriveTek Research Institute discovers that a computer company wants a new tape drive for a proposed new computer system. Since the computer company does not have research people available to develop the new drive, it will subcontract the development to an independent research firm. The computer company has offered a fee of $250,000 for the best proposal for developing the new tape drive. The contract will go to the firm with the best technical plan and the highest reputation for technical competence.

DriveTek Research Institute wants to enter the competition. Management estimates a cost of $50,000 to prepare a proposal with a fifty-fifty chance of winning the contract.

However, DriveTek's engineers are uncertain about how they will develop the tape drive if they are awarded the contract. Three alternative approaches can be tried. The first approach is a mechanical method with a cost of $120,000, and the engineers are certain they can develop a successful model with this approach. A second approach involves electronic components. The engineers estimate that the electronic approach will cost only $50,000 to develop a model of the tape drive, but with only a 50 percent chance of satisfactory results. A third approach uses magnetic components; this costs $80,000, with a 70 percent chance of success.

DriveTek Research can work on only one approach at a time and has time to try only two approaches. If it tries either the magnetic or electronic method and the attempt fails, the second choice must be the mechanical method to guarantee a successful model.

The management of DriveTek Research needs help in incorporating this information into a decision to proceed or not.

**Nodes and Branches**

Decision trees have three kinds of nodes and two kinds of branches. A decision node is a point where a choice must be made; it is shown as a square. The branches extending from a decision node are decision branches, each branch representing one of the possible alternatives or courses of action available at that point. The set of alternatives must be mutually exclusive (if one is chosen, the others cannot be chosen) and collectively exhaustive (all possible alternatives must be included in the set).

There are two major decisions in the DriveTek problem. First, the company must decide whether or not to prepare a proposal. Second, if it prepares a proposal and is awarded the contract, it must decide which of the three approaches to try to satisfy the contract.

An event node is a point where uncertainty is resolved (a point where the decision maker learns about the occurrence of an event). An event node, sometimes called a "chance node," is shown as a circle. The event set consists of the event branches extending from an event node, each branch representing one of the possible events that may occur at that point. The set of events must be mutually exclusive (if one occurs, the others cannot occur) and collectively exhaustive (all possible events must be included in the set). Each event is assigned a subjective probability; the sum of probabilities for the events in a set must equal one.

The three sources of uncertainty in the DriveTek problem are: whether it is awarded the contract or not, whether the electronic approach succeeds or fails, and whether the magnetic approach succeeds or fails.

In general, decision nodes and branches represent the controllable factors in a decision problem; event nodes and branches represent uncontrollable factors.

Decision nodes and event nodes are arranged in order of subjective chronology. For example, the position of an event node corresponds to the time when the decision maker learns the outcome of the event (not necessarily when the event occurs).

The third kind of node is a terminal node, representing the final result of a combination of decisions and events. Terminal nodes are the endpoints of a decision tree, shown as the end of a branch on hand-drawn diagrams and as a triangle on computer-generated diagrams.

The following table shows the three kinds of nodes and two kinds of branches used to represent a decision tree.

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
|  | Type of  Node |  | Written  Symbol |  | Computer  Symbol |  | Node  Successor |
|  | Decision |  | square |  | square |  | decision branches |
|  | Event |  | circle |  | circle |  | event branches |
|  | Terminal |  | endpoint |  | triangle |  | terminal value |

**Terminal Values**

Each terminal node has an associated terminal value, sometimes called a payoff value, outcome value, or endpoint value. Each terminal value measures the result of a *scenario*: the sequence of decisions and events on a unique path leading from the initial decision node to a specific terminal node.

To determine the terminal value, one approach assigns a cash flow value to each decision branch and event branch and then sum the cash flow values on the branches leading to a terminal node to determine the terminal value. In the DriveTek problem, there are distinct cash flows associated with many of the decision and event branches. Some problems require a more elaborate value model to determine the terminal values.

The following diagram shows the arrangement of branch names, probabilities, and cash flow values on an unsolved tree.



**DriveTek Unsolved Tree**

The following diagram shows the unsolved decision tree model for the DriveTek problem.



For example, the +$30,000 terminal value on the far right of the diagram is associated with the following scenario:

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  | Branch type |  | Branch name |  | Cash flow |
|  | Decision |  | Prepare proposal |  | -$50,000 |
|  | Event |  | Awarded contract |  | +$250,000 |
|  | Decision |  | Try electronic method |  | -$50,000 |
|  | Event |  | Electronic failure |  | $0 |
|  | Decision |  | Use mechanical method |  | -$120,000 |
|  |  |  |  |  |  |
|  |  |  | Terminal value |  | +$30,000 |

**Strategy**

A strategy specifies an initial choice and any subsequent choices to be made by the decision maker. The subsequent choices usually depend upon events. The specification of a strategy must be comprehensive; if the decision maker gives the strategy to a colleague, the colleague must know exactly which choice to make at each decision node.

Most decision problems have many possible strategies, and a goal of the analysis is to determine the optimal strategy, taking into account the decision maker's risk attitude. There are four strategies in the DriveTek problem. One of the strategies is: Prepare the proposal; if not awarded the contract, stop; if awarded the contract, try the magnetic method; if the magnetic method is successful, stop; if the magnetic method fails, use the mechanical method. The four strategies will be discussed in detail below.

**Payoff Distribution**

Each strategy has an associated payoff distribution, sometimes called a risk profile. The payoff distribution of a particular strategy is a probability distribution showing the probability of obtaining each terminal value associated with a particular strategy.

In decision tree models, the payoff distribution can be shown as a list of possible payoff values, x, and the discrete probability of obtaining each value, P(X=x), where X represents the uncertain terminal value. Since a strategy specifies a choice at each decision node, the uncertainty about terminal values depends only on the occurrence of events. The probability of obtaining a specific terminal value equals the product of the probabilities on the event branches on the path leading to the terminal node.

**DriveTek Strategies**

In this section each strategy of the DriveTek problem is described by a shorthand statement and a more detailed statement. The possible branches following a specific strategy are shown in decision tree form, and the payoff distribution is shown in a table with an explanation of the probability calculations.

**Strategy 1 (Mechanical)**: Prepare; if awarded, use mechanical.

Details: Prepare the proposal; if not awarded the contract, stop (payoff = -$50,000); if awarded the contract, use the mechanical method (payoff = +$80,000).



|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  | Value, x |  | Probability  P(X=x) |  |  |
|  | +$80,000 |  | 0.500 |  |  |
|  | -$50,000 |  | 0.500 |  |  |
|  |  |  | 1.000 |  |  |

**Strategy 2 (Electronic)**: Prepare; if awarded, try electronic.

Details: Prepare the proposal; if not awarded the contract, stop (payoff = -$50,000); if awarded the contract, try the electronic method; if the electronic method is successful, stop (payoff = +$150,000); if the electronic method fails, use the mechanical method (payoff = +$30,000).



|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  | Value, x |  | Probability  P(X=x) |  |  |
|  | +$150,000 |  | 0.250 |  | = 0.500 \* 0.500 |
|  | +$30,000 |  | 0.250 |  | = 0.500 \* 0.500 |
|  | -$50,000 |  | 0.500 |  |  |
|  |  |  | 1.000 |  |  |

**Strategy 3 (Magnetic)**: Prepare; if awarded, try magnetic.

Details: Prepare the proposal; if not awarded the contract, stop (payoff = -$50,000); if awarded the contract, try the magnetic method; if the magnetic method is successful, stop (payoff = +$120,000); if the magnetic method fails, use the mechanical method (payoff = $0).



|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  | Value, x |  | Probability  P(X=x) |  |  |
|  | +$120,000 |  | 0.350 |  | = 0.500 \* 0.700 |
|  | $0 |  | 0.150 |  | = 0.500 \* 0.300 |
|  | -$50,000 |  | 0.500 |  |  |
|  |  |  | 1.000 |  |  |

**Strategy 4 (Don't)**: Don't.

Details: Don't prepare the proposal (payoff = $0).



|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  | Value, x |  | Probability  P(X=x) |  |  |
|  | $0 |  | 1.000 |  |  |
|  |  |  | 1.000 |  |  |

**Strategy Choice**

Since each strategy can be characterized completely by its payoff distribution, selecting the best strategy becomes a problem of choosing the best payoff distribution.

One approach is to make a choice by direct comparison of the payoff distributions.

|  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | **Strategy 1 (Mechanical)** | | |  |  |  | **Strategy 2 (Electronic)** | | |
|  | Value, x |  | Probability  P(X=x) |  | | | Value, x |  | Probability  P(X=x) |
|  | +$80,000 |  | 0.500 |  | | | +$150,000 |  | 0.250 |
|  | -$50,000 |  | 0.500 |  | | | +$30,000 |  | 0.250 |
|  |  |  | 1.000 |  | | | -$50,000 |  | 0.500 |
|  |  |  |  |  | | |  |  | 1.000 |

|  |  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | **Strategy 3 (Magnetic)** | | | |  |  |  | | **Strategy 4 (Don't)** | | | |
|  | Value, x |  | Probability  P(X=x) |  | | | | Value, x | |  | Probability  P(X=x) |
|  | +$120,000 |  | 0.350 |  | | | | $0 | |  | 1.000 |
|  | $0 |  | 0.150 |  | | | |  | |  | 1.000 |
|  | -$50,000 |  | 0.500 |  | | | |  | |  |  |
|  |  |  | 1.000 |  | | | |  | |  |  |

Another approach for making choices involves certainty equivalents.

**Certainty Equivalent**

A certainty equivalent is a certain payoff value which is equivalent, for the decision maker, to a particular payoff distribution. If the decision maker can determine his or her certainty equivalent for the payoff distribution of each strategy, then the optimal strategy is the one with the highest certainty equivalent.

The certainty equivalent is the minimum selling price for a payoff distribution; it depends on the decision maker's personal attitude toward risk. A decision maker may be risk preferring, risk neutral, or risk avoiding.

If the terminal values are not regarded as extreme (relative to the decision maker's total assets), if the decision maker will encounter other decision problems with similar payoffs, and if the decision maker has the attitude that he or she will "win some and lose some," then the decision maker's attitude toward risk may be described as risk neutral.

If the decision maker is risk neutral, the expected value is the appropriate certainty equivalent for choosing among the strategies. Thus, for a risk neutral decision maker, the optimal strategy is the one with the highest expected value.

The expected value of a payoff distribution is calculated by multiplying each terminal value by its probability and summing the products. The expected value calculations for each of the four strategies of the DriveTek problem are shown below.

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
|  | **Strategy 1 (Mechanical)** | | |  |  |  |  |
|  | Value, x |  | Probability  P(X=x) |  | x \* P(X=x) | | | |
|  | +$80,000 |  | 0.500 |  | +$40,000 | | | |
|  | -$50,000 |  | 0.500 |  | -$25,000 | | | |
|  |  |  |  |  | **+$15,000** | | | |

|  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | **Strategy 2 (Electronic)** | | |  |  | |  |  |
|  | Value, x |  | Probability  P(X=x) |  | | x \* P(X=x) | | | |
|  | +$150,000 |  | 0.250 |  | | +$37,500 | | | |
|  | +$30,000 |  | 0.250 |  | | +7,500 | | | |
|  | -$50,000 |  | 0.500 |  | | -$25,000 | | | |
|  |  |  |  |  | | **+$20,000** | | | |

|  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | **Strategy 3 (Magnetic)** | | | |  | |  |  |  |
|  | Value, x |  | Probability  P(X=x) |  | | x \* P(X=x) | | | | |
|  | +$120,000 |  | 0.350 |  | | +$42,000 | | | | |
|  | $0 |  | 0.150 |  | | $0 | | | | |
|  | -$50,000 |  | 0.500 |  | | -$25,000 | | | | |
|  |  |  |  |  | | **+$17,000** | | | | |

|  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | **Strategy 4 (Don't)** | | |  | |  |  |  | |
|  | Value, x |  | Probability  P(X=x) |  | x \* P(X=x) | | | |
|  | $0 |  | 1.000 |  | $0 | | | |
|  |  |  |  |  | **$0** | | | |

The four strategies of the DriveTek problem have expected values of +$15,000, +$20,000, +$17,000, and $0. Strategy 2 (Electronic) is the optimal strategy with expected value +$20,000.

A risk neutral decision maker's choice is based on the expected value. However, note that if strategy 2 (Electronic) is chosen, the decision maker does not receive +$20,000. The actual payoff will be +$150,000, +$30,000, or -$50,000, with probabilities shown in the payoff distribution.

**Rollback Method**

If we have a method for determining certainty equivalents (expected values for a risk neutral decision maker), we don't need to examine every possible strategy explicitly. Instead, the method known as rollback determines the single best strategy.

The rollback algorithm, sometimes called backward induction or "average out and fold back," starts at the terminal nodes of the tree and works backward to the initial decision node, determining the certainty equivalent rollback values for each node. Rollback values are determined as follows:

At a terminal node, the rollback value equals the terminal value.

At an event node, the rollback value for a risk neutral decision maker is determined using expected value; the branch probability is multiplied times the successor rollback value, and the products are summed.

At a decision node, the rollback value is set equal to the highest rollback value on the immediate successor nodes.

Rollback values are located to the left and below each decision, event, and terminal node as shown below.



Rollback values for the DriveTek problem are shown below.



**Optimal Strategy**

After the rollback method has determined certainty equivalents for each node, the optimal strategy can be identified by working forward through the tree. At the initial decision node, the +$20,000 rollback value equals the rollback value of the "Prepare proposal" branch, indicating the alternative that should be chosen. DriveTek will either be awarded the contract or not; there is a subsequent decision only if DriveTek obtains the contract. (In a more complicated decision tree, the optimal strategy must include decision choices for all decision nodes that might be encountered.) At the decision node following "Awarded contract," the +$90,000 rollback value equals the rollback value of the "Try electronic method" branch, indicating the alternative that should be chosen. Subsequently, if the electronic method fails, DriveTek must use the mechanical method to satisfy the contract.

The pairs of rollback values at the relevant decision nodes (+$20,000, +$90,000, and +$30,000) and the preferred decision branches are shown below in bold.



Taking into account event branches with subsequent terminal nodes, all branches associated with the optimal risk neutral strategy are shown below.



The rollback method has identified strategy 2 (Electronic) as optimal. The rollback value on the initial branch of the optimal strategy is +$20,000, which must be the same as the expected value for the payoff distribution of strategy 2. Some of the intermediate calculations for the rollback method differ from the calculations for the payoff distributions, but both approaches identify the same optimal strategy with the same initial expected value. For decision trees with a large number of strategies, the rollback method is more efficient.

These notes were prepared using Excel 4.0a, TreePlan decision tree add-in for Excel, and Word for Windows 2.0c.

Michael R. Middleton

McLaren School of Business

University of San Francisco

2130 Fulton Street

San Francisco, CA 94117-1080

Internet: middleton@usfca.edu

June 1994