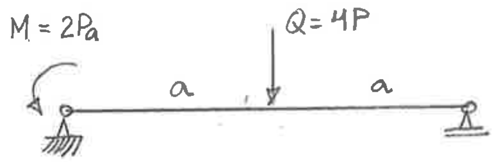
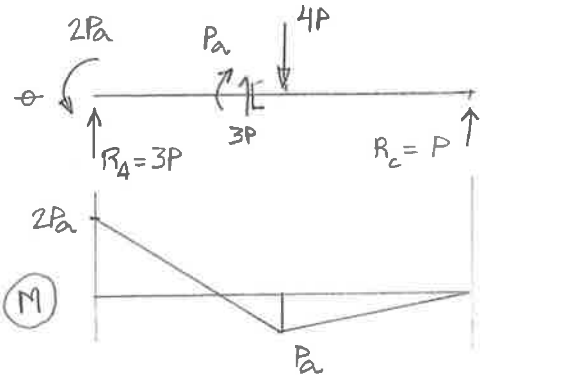


EJ = const.

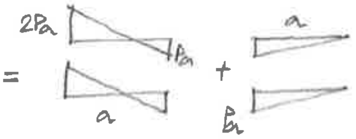
a) Energia de deformação expressa em função da força P, U(P)



$$\circlearrowleft C \left\{ -2Pa + R_A \times 2a - 4P \times a = 0 \right. \Rightarrow R_A = 3P$$

$$\uparrow \left\{ R_A + R_c = 4P \right. \Rightarrow R_c = P$$

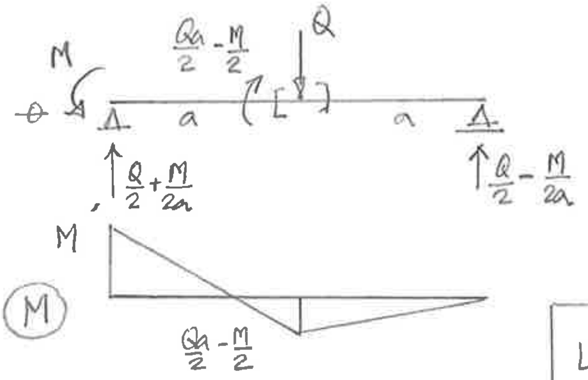
$$U(P) = \sum_{b=1}^{2} \int_0^{l_b} \frac{M^2}{2EI}\, dx =$$



$$= \frac{1}{2EI} \left\{ \frac{a}{6}\left[2(2Pa)^2 + (Pa)^2 - 2(2Pa \times Pa)\right] + \frac{a}{3}(Pa)^2 \right\}$$

$$= \frac{1}{2EI} \left\{ P^2a^3 + \frac{P^2a^3}{3} \right\} \Rightarrow \boxed{U(P) = \frac{2P^2a^3}{3EI}}$$

b) Energia de deformação expressa em função de M e Q, U(M,Q)



$$U(M,Q) = \frac{1}{2EI} \left\{ \frac{a}{6}\left[2\left(M^2 + \left(\frac{Qa}{2} - \frac{M}{2}\right)^2\right] - 2\left[M\left(\frac{Qa}{2} - \frac{M}{2}\right)\right] + \frac{a}{3}\left(\frac{Qa}{2} - \frac{M}{2}\right)^2 \right\}$$

$$= \frac{1}{2EI} \left\{ \frac{a}{6}\left[2M^2 + \frac{Q^2a^2}{2} - QaM + \frac{M^2}{2} - MQa + M^2\right] + \frac{a}{3}\left(\frac{Q^2a^2}{4} - \frac{QaM}{2} - \frac{M^2}{4}\right) \right\}$$

$$\boxed{U(M,Q) = \frac{a}{12EI}\left[4M^2 - 3MQa + Q^2a^2\right]}$$

verif.: $U(M=2Pa, Q=4P) = \frac{a}{12EI}\left[16P^2a^2 - 24P^2a^2 + 16P^2a^2\right] = \frac{2P^2a^3}{3EI}$ OK

c) $$\varphi_A = \left.\frac{\partial U}{\partial M}\right|_{\substack{M=2Pa \\ Q=4P}} = \frac{a}{12EI}\left[8M - 3Qa\right] = \frac{a}{12EI}\left[16Pa - 12Pa\right] \Rightarrow \boxed{\varphi_A = \frac{Pa^2}{3EI} \ (\circlearrowleft \text{ rad})}$$

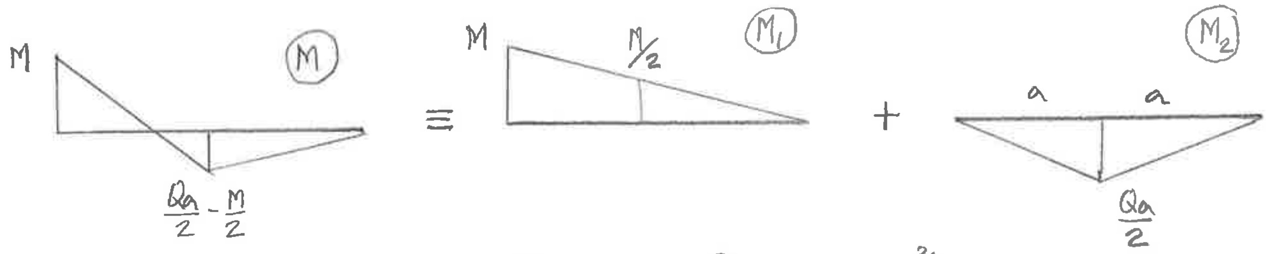
$$v_B = \left.\frac{\partial U}{\partial Q}\right|_{\substack{M=2Pa \\ Q=4P}} = \frac{a}{12EI}\left[-3Ma + 2Qa^2\right] = \frac{a}{12EI}\left[-6Pa^2 + 8Pa^2\right] \Rightarrow \boxed{v_B = \frac{Pa^3}{6EI} \ (\downarrow)}$$

d)

Obs:

1. Outras formas de calcular a integral em b)

- Decomposição do diagrama de momentos.



$$\int_0^{2a}\frac{M^2}{2EI}\,dx=\int_0^{2a}\frac{(M_1+M_2)^2}{2EI}\,dx=\int_0^{2a}\frac{M_1^2}{2EI}\,dx+2\int_0^{2a}\frac{M_1M_2}{2EI}\,dx+\int_0^{2a}\frac{M_2^2}{2EI}\,dx$$

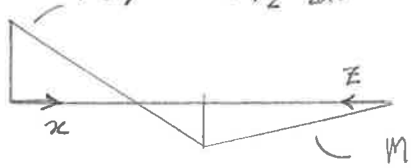
$$=\frac{1}{2EI}\left\{\frac{2a}{3}M^2+2\left[\frac{a}{6}\left(-\frac{MQa}{2}-2\frac{M}{2}\frac{Qa}{2}\right)+\frac{a}{3}\left(-\frac{M}{2}\frac{Qa}{2}\right)\right]+2\left[\frac{a}{3}\left(\frac{Qa}{2}\right)^2\right]\right.$$

$$=\frac{a}{2EI}\left\{\frac{2}{3}M^2+2\left[-\frac{MQa}{6}-\frac{MQa}{12}\right]+\frac{Qa^2}{6}\right\}$$

$$=\frac{a}{12EI}\left(4M^2-3MQa+Q^2a^2\right)$$

- Integração analítica (sem usar as fórmulas de produto)

$M(x)=-M+\left(\frac{Q}{2}+\frac{M}{2a}\right)x \qquad (0\le x\le a)$



$M(z)=\left(\frac{Q}{2}-\frac{M}{2a}\right)z \qquad (0\le z\le a)$

$$U(M,Q)=\int_0^a\frac{M(x)^2}{2EI}\,dx+\int_0^a\frac{M(z)^2}{2EI}\,dz=\frac{1}{2EI}\left\{\int_0^a\left[M^2-M\left(Q+\frac{M}{a}\right)x+\left(\frac{Q}{2}+\frac{M}{2a}\right)^2x^2\right]dx\right.$$
$$\left.+\int_0^a\left(\frac{Q}{2}-\frac{M}{2a}\right)^2z^2\,dz\right\}$$

$$=\frac{1}{2EI}\left\{\left[M^2x-\frac{M}{2}\left(Q+\frac{M}{a}\right)x^2+\frac{1}{12}\left(Q^2+\frac{2QM}{a}+\frac{M^2}{a^2}\right)x^3\right]_0^a+\left[\frac{1}{12}\left(Q^2-\frac{2QM}{a}+\frac{M^2}{a^2}\right)z^3\right]_0^a\right\}$$

$$=\frac{1}{2EI}\left\{\left[M^2a-\frac{M^2a}{2}-\frac{MQa^2}{2}+\frac{Q^2a^3}{12}+\frac{QMa^2}{6}+\frac{M^2a}{12}+\frac{Q^2a^3}{12}-\frac{QMa^2}{6}+\frac{M^2a}{12}\right]\right\}$$

$$=\frac{1}{2EI}\left\{\frac{2}{3}M^2a-\frac{MQa^2}{2}+\frac{Q^2a^3}{6}\right\}=\frac{a}{12EI}\left(4M^2-3MQa+Q^2a^2\right)$$

—||—

2. Teorema de Maxwell

↳ M e Q unitários aplicados alternadamente.

$$\left.\begin{aligned} \bar{\varphi}_A &= \left.\frac{\partial U}{\partial M}\right|_{\substack{M=0 \\ Q=1}} = -\frac{a^2}{3EI} \\ \bar{v}_B &= \left.\frac{\partial U}{\partial Q}\right|_{\substack{M=1 \\ Q=0}} = -\frac{a^2}{3EI} \end{aligned}\right\} \text{OK}$$