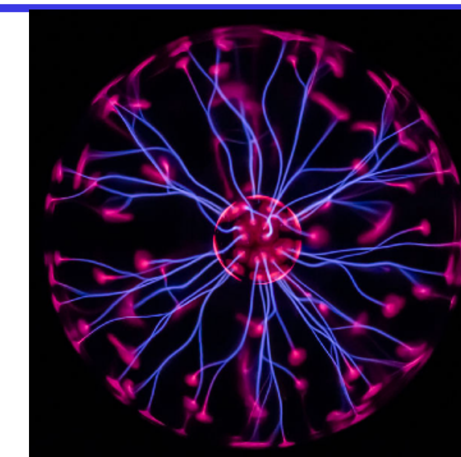
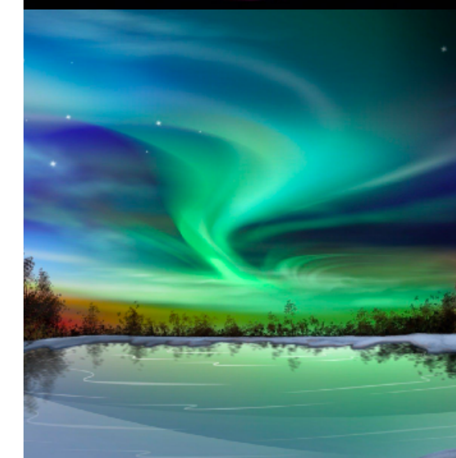
Plasmas

⚡ **What is a plasma, where can we find it?**
⚡ **Dielectric properties of plasmas**
⚡ **Faraday rotation**

Plasma: what is it?

- Plasma is a ***gas of electrically charged particles***. The gas is usually neutral, on average, but its constituents are either totally, or partially charged.
- The ***negative charges*** in plasmas are usually ***electrons*** which are ionized — and the positive charges are the ionized atoms or molecules from which those electrons were pulled off.
- Because the electrons are much lighter than the atoms/molecules, they are the ones that move around, so it is usually a very good approximation to consider only the electrons as the moving parts — but, of course, momentum conservation means that the atoms/molecules also feel the same forces.
- Maintaining the electrons as free particles, and the atoms/molecules ionized, takes a lot of energy, therefore the temperatures of plasmas are typically very high: typically, thousands of degrees Kelvin.
- Plasmas are not only electrically polarized media, they also have magnetic properties. The diffusion and propagation of electric and magnetic fields in plasmas is quite different than what we usually see for typical dielectric and magnetic materials. This will be the subject of the next two lectures.

Motions in a plasma

- Consider what happens to an electron (assumed to be approximately at rest) as an EM wave passes.
- To a very good approximation, it is the electric force that does most of the motion:

$$m_e \frac{d^2\vec{r}}{dt^2} = -\,e\,\vec{E} \quad , \text{ where the charge of the electron is } q = -\,e\,.$$

- For simplicity, let's assume that the field is polarized in the $\hat{x}$ direction, and that the wave is monochromatic. Hence at the electron's position the motion is given by:

$$\frac{d^2x}{dt^2} = -\frac{e}{m_e} E_0 \sin(\omega t) \quad , \text{ which we can immediately solve:}$$

$$x(t) = \frac{e\,E_0}{\omega^2\,m_e} \sin(\omega t) \quad , \text{ where the amplitude of the movement is } x_0 = \frac{e\,E_0}{\omega^2\,m_e}\,.$$

- This movement generates an ***oscillating electric dipole***:

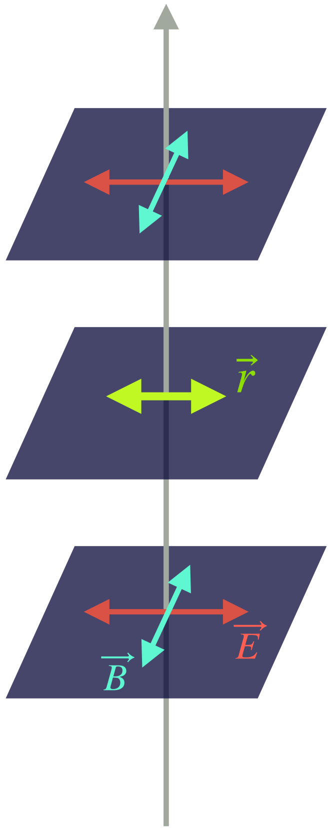
$$\vec{p}(t) = -\,e\,x(t)\,\hat{x} = -\,p_0 \sin(\omega t)\,\hat{x} \quad , \text{ with } \quad p_0 = e^2 E_0/(\omega^2 m_e)$$

- Notice that, in a plasma, the positive charges are ionized atoms, with nuclei that weigh much (thousands of times!) more than the electron, hence their electric dipoles are negligible.
- Now, let's consider that, in our plasma, we have an average ***number density of free electrons*** that we express as:

$$n_e = \frac{dN_e}{d^3x}$$

Hence, if a plasma has n_e free electrons per unit volume, then it has a density of electric dipoles given by:

$$\vec{P} = -\frac{n_e\,e^2}{\omega^2 m_e}\vec{E}$$



Motions in a plasma

- We have seen in Lecture 5 that the dielectric properties of any medium are given in terms of the polarization by:

  $$\epsilon_0 \vec{E} + \vec{P} = \vec{D} = \epsilon \, \vec{E} \quad ,$$

  where the last equality follows from assuming linear media, and we usually write:

  $$\epsilon = (1 + \chi_E) \, \epsilon_0$$

- Therefore, we can say that a medium with plenty of free electrons that are able to move around (i.e., they are not bound to atoms or molecules) has

  $$\vec{P} = - \frac{n_e \, e^2}{\omega^2 m_e} \vec{E} \quad \Rightarrow \quad \chi_E = - \frac{n_e \, e^2}{\omega^2 m_e \epsilon_0}$$

- We usually write this in terms of the plasma frequency:

  $$\omega_p^2 = \frac{n_e \, e^2}{m_e \epsilon_0} \quad \Rightarrow \quad \chi_E = - \frac{\omega_p^2}{\omega^2}$$

- You can see how this expression has a number of problems: the speed of light in a dielectric medium is given by:

  $$c_s^2 = \frac{1}{\sqrt{\mu_0 \epsilon}} = \frac{c}{\sqrt{1 - \frac{\omega_p^2}{\omega^2}}}$$, which would naively imply that light waves propagate faster than the speed of light.

  Moreover, for $\omega < \omega_p$ we have a negative dielectric constant, and an imaginary refractive index.

- What is going on??

The plasma dispersion relation

- Let’s look at a the electric field of a plane, monochromatic wave that propagates in this plasma, but now let’s write the space-dependent part explicitly (which we neglected to do earlier):

$$\vec{E} = E_0\, \hat{z}\, e^{-i(\omega t - kz)}$$

- In this expression, the phase velocity appears when we write the ***phase*** above as:

$$\varphi = \omega t - kz = k\left(\frac{\omega}{k}t - z\right) = k\left(v_p\, t - z\right)$$

- The results above show that, for a plasma, the frequency is given by:

$$\omega^2 = \frac{k^2c^2}{1 - \frac{\omega_p^2}{\omega^2}} \quad \Rightarrow \quad \omega^2 - \omega_p^2 = k^2c^2$$

- We usually call this type of relation between the wave frequency and its wavenumber (k) a ***dispersion relation***. For plasmas we get:

$$\omega^2(k) = k^2c^2 + \omega_p^2$$

- In terms of the dispersion relation, the phase velocity is:

$$v_p = \frac{\omega}{k} = c\sqrt{1 + \frac{\omega_p^2}{c^2k^2}}$$

- It is important to remember that the ***phase velocity*** can, in fact, be greater than the speed of light in vacuum, since it doesn’t correspond to any motions of particles, or flows of energy or momentum.

The plasma dispersion relation

- As you have probably guessed, what really matters is not so much the phase velocity, but the group velocity. Let's remember what are those two things.
- Take a scalar wave in a medium with dispersion relation $\omega(k)$. A wave packet can then be expresses as:

$$\psi(t,x) = \int dk\, e^{i[kx-\omega(k)t]}\, \tilde{\psi}(k)$$

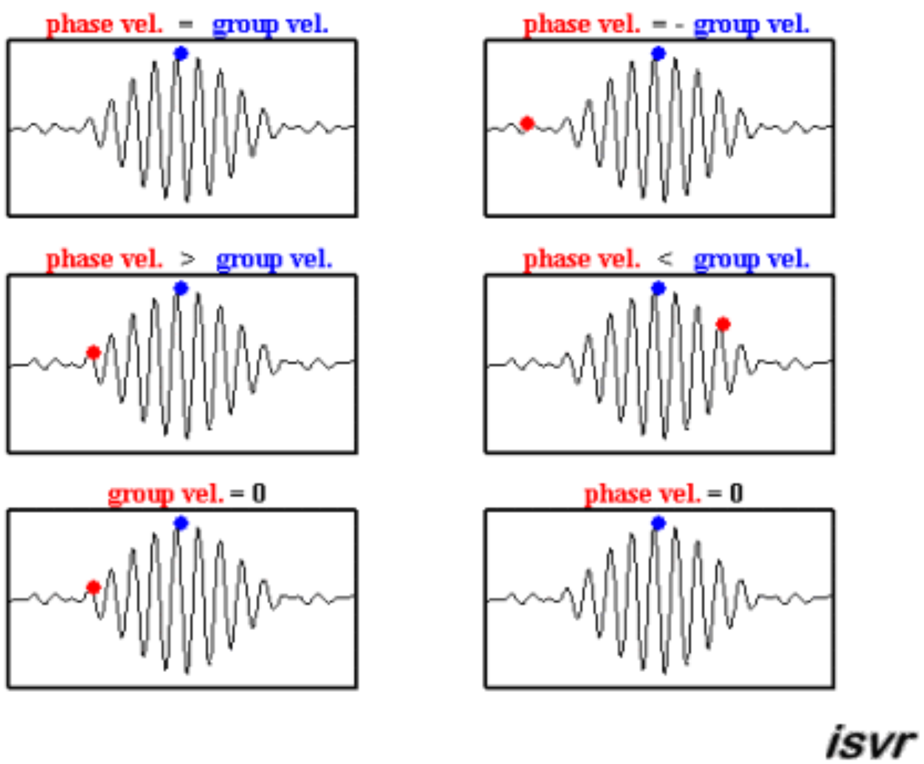
- For the exercise below we will consider each phase $\phi = kx - \omega t$ of this wave.
- Let's first look at a point $x_p(t)$ whose phase is fixed, and ask how fast that point moves. Since we want the phase to be fixed in time, we want $d\phi = 0$. Therefore:

$$k\, dx_p - \omega\, dt = 0 \quad \Rightarrow \quad \frac{dx_p}{dt} = \frac{\omega}{k}$$

- Now, let's ask about a ***feature*** in the wave, like a peak, or a trough, or a kink. How can we find the speed with which that feature moves? What we need to recall is that a feature is something in the ***form of the wave***, which is more or less independent of time. This means that when a feature moves, the phases of all the modes that contribute to that feature move in such a way that the phase around that feature remain the same.
- In other words, if we write the phases for a feature x_f at any time t and we demand that the phases are invariant, we get:

$$dk\, x_f - d\omega\, t = 0 \quad \Rightarrow \quad \frac{d\omega}{dk} = \frac{x_f}{t} \equiv v_g$$

- This is the **group velocity**, which tells us how fast a feature of the wave form moves in space.



The plasma dispersion relation

- From the plasma dispersion relation,

  $$\omega^2 = k^2 c^2 + \omega_p^2 \quad ,$$

  we find that

  $$2\omega \frac{d\omega}{dk} = 2kc^2 \quad \Rightarrow \quad \frac{\omega}{k}\frac{d\omega}{dk} = c^2 \quad \Rightarrow \quad v_p \, v_g = c^2$$

- It's perhaps useful to note here that physically it is usually better to think of $k = k(\omega)$ as opposed to $\omega = \omega(k)$, since the frequency ω of the wave is basically fixed, given.
- Explicitly, we have:

  $$v_p = \frac{c}{\sqrt{1 - \omega_p^2/\omega^2}} \geq c \quad , \text{ and}$$

  $$v_g = c\sqrt{1 - \omega_p^2/\omega^2} \leq c$$

- However, there is still the "mystery" about what happens when the frequency drops below the plasma frequency, and ***both*** the group and phase ***velocities become complex***. What is going on in there now?
- In order to see what is going on, it is better to write the phase as:

  $$\psi(t,x) = \int dk \, e^{i[k(\omega)\, x - \omega\, t]} \, \tilde{\psi}(k) \quad , \quad \text{where} \quad k^2 = (\omega^2 - \omega_p^2)/c^2$$

- For $\omega < \omega_p$ we get $k = i/c\sqrt{\omega_p^2 - \omega^2}$, and then:

  $$\psi(t,x) = \int dk \, e^{-|k|\, x - i\omega\, t} \, \tilde{\psi}(k) \quad ,$$ i.e., the wave is ***attenuated*** (it ***decays***).

The plasma dispersion relation

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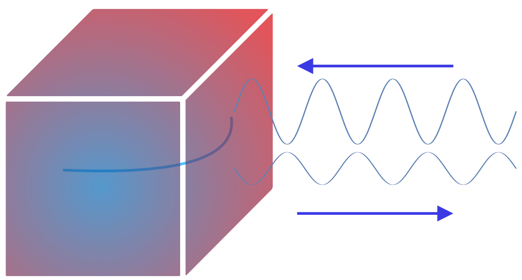


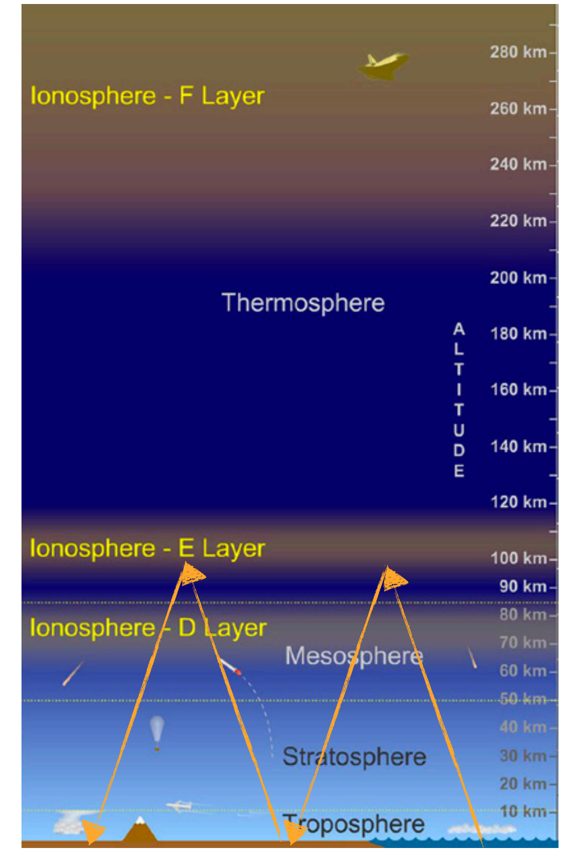
- But what about the other root of the equation, $k = -\,i/c\sqrt{\omega_p^2 - \omega^2}$? Those would be exponentially growing solutions:

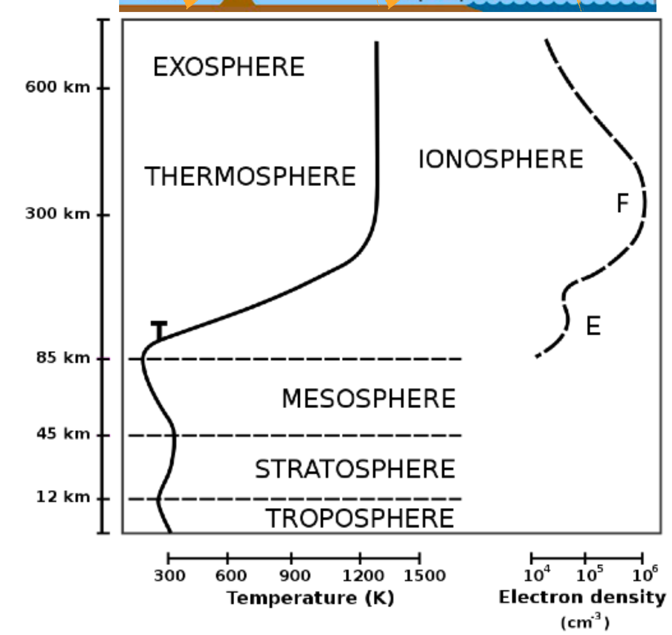
$$\psi(t,x) \;=\; \int dk\, e^{+|k|\,x - i\omega\, t}\,\tilde{\psi}(k)$$

What happens in this case is that, when a low-frequency waves hits the plasma, part of it penetrates the plasma, and part of it is reflected back — just like a wave penetrates a material with a certain "skin depth". The boundary conditions of the electric and magnetic fields guarantee that only the exponentially decaying mode gets inside the dielectric.

Plasmas and radio communications

- The Earth's ***ionosphere*** (at 50-300 km altitude above sea level) is a series of layers of the atmosphere that received direct ultra-violet (UV) light from the Sun. This radiation ionizes some of the electrons of the gases and dust that make up the atmosphere, creating an environment where one can find number densities of electrons of $10^4 - 10^6$ electrons/cm^3. It is a plasma!
- The resulting plasma frequency of the ionosphere is approximately 1 MHz at the lower altitudes (~100 km), raising to 10 MHz or more at higher altitudes — remember that $\omega_p^2 = n_e e^2/(m_e \epsilon_0)$.
- Long- and medium-wave radio communications (that use frequencies < 1 MHz) rely on the ionosphere as a "mirror" that reflects those waves: those radio frequencies fall below the plasma frequency of the ionosphere, hence they cannot propagate in that plasma!
- The Earth's surface is also a half-decent conductor, so it also acts as a mirror to those waves. The result is that low/medium-frequency radio waves bounce up the ionosphere and down on the surface of the Earth multiple times, often reaching the other side of the planet!
- On the other hand, short-wave (high-frequency) radio, including FM and TV, use bands of frequencies above 10 MHz, so their frequencies are larger than the plasma frequency of the ionosphere and they just propagate away. Therefore, short-wave radio only work for relatively short distances: as soon as they get reflected towards the sky, they are "lost in space".
- An interesting phenomenon takes place at night: with the Sun absent, the production of ionized electrons stops, and the number density of electrons in the ionosphere starts to drop. As a result, the plasma frequency also starts to drop, making communications harder and harder, and limiting long-range communications to lower frequencies. During the day, the Sun replenishes the stock of free electrons and the radio waves start bouncing on the ionosphere again.





Faraday rotation

- We will now study how free electrons in a plasma move when subjected to electric and magnetic fields. However, we will now suppose that there is an ***external magnetic field*** which is more or less constant.

- This situation can arise when there are magnetic fields "around" — from, e.g., Earth's magnetism, or in a cyclotron, or even in galaxies such as the Milky Way.

- In the non-relativistic limit, the equations of motion are:

$$m_e \frac{d^2 \vec{r}}{dt^2} = - e \left(\vec{E} + \frac{d\vec{r}}{dt} \times \vec{B} \right)$$

- Just as before, we will assume that the ***electric field*** is varying as $e^{-i\omega t}$, just like a plane monochromatic wave — and we can neglect the wave's magnetic field. However, we leave the ***two polarization degrees of freedom*** to be completely free.

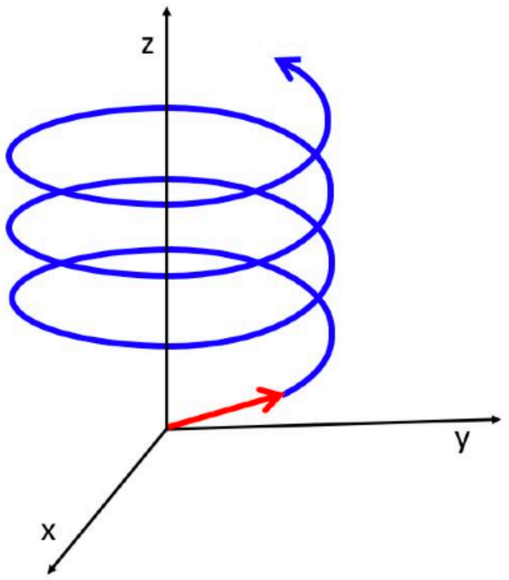
- For simplicity, we assume that the magnetic field is aligned in the z direction, so $\vec{B} = B_z \hat{z}$, with $B_z \simeq$ const.

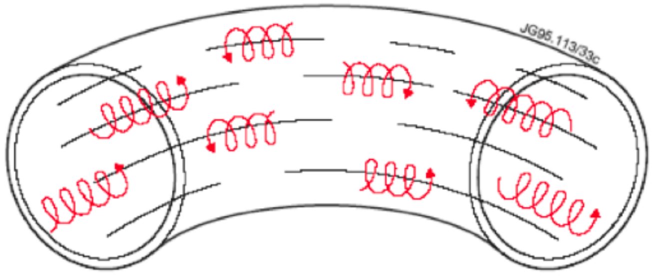
- Therefore, we get the equations:

$$\ddot{x} = - \frac{e}{m_e} \left(E_x + v_y B_z \right)$$

$$\ddot{y} = - \frac{e}{m_e} \left(E_y - v_x B_z \right)$$

$$\ddot{z} = - \frac{e}{m_e} \left(E_z + 0 \right)$$





Faraday rotation

- The motion along the z axis is trivial, and was solved in earlier slides.
- The motion on the plane $x - y$, on the other hand, is far more interesting. Since the only "dynamical" part is the factor $e^{-i\omega t}$ of the electric field, it is clear that $x(t) \sim e^{-i\omega t}$ and $y(t) \sim e^{-i\omega t}$, so we get:

$$-\omega^2 x = -\frac{e}{m_e}\left(E_x - i\omega\, y\, B_z\right)$$

$$-\omega^2 y = -\frac{e}{m_e}\left(E_y + i\omega\, x\, B_z\right)$$

- This linear system can be easily solved for x and y. However, it is far more convenient here to replace the linear polarizations E_x and E_y by the ***circular polarizations***:

$E_+ = E_x + iE_y$ ("clockwise") and $E_- = E_x - iE_y$ ("anti-clockwise") , oriented in the directions:

$$\hat{r}_+ = \frac{1}{2}\left(\hat{x} + i\hat{y}\right) \quad \text{and} \quad \hat{r}_- = \frac{1}{2}\left(\hat{x} - i\hat{y}\right)$$

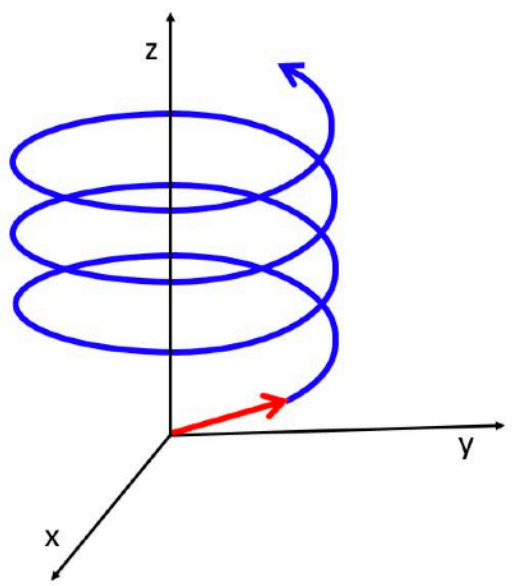
- In a similar fashion, instead of using x and y we use:

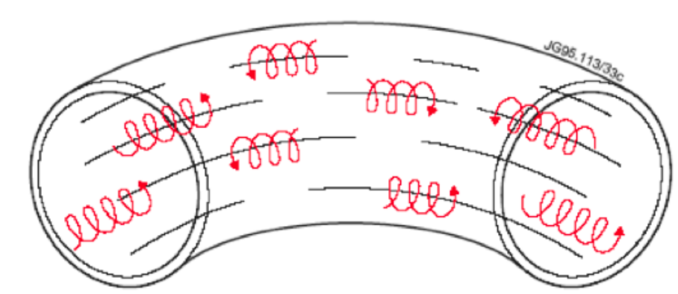
$$r_+ = x + iy \quad \text{and} \quad r_- = x - iy$$

- The resulting equations are:

$$-\omega^2 r_+ = -\frac{e}{m_e}\left(E_+ - \omega\, r_+ B_z\right)$$

$$-\omega^2 r_- = -\frac{e}{m_e}\left(E_- - \omega\, r_- B_z\right)$$





Faraday rotation

- In terms of the circular polarizations, the solutions for the motion on the $x - y$ plane are:

$$r_+ = \frac{1}{1 + \frac{eB_z}{m\omega}} \frac{e\,E_+}{m_e\,\omega^2} \quad , \quad \text{i.e.} \quad , \quad x + iy = \frac{1}{1 + \frac{eB_z}{m\omega}} \frac{e\,(E_x + iE_y)}{m_e\,\omega^2}$$

$$r_- = \frac{1}{1 - \frac{eB_z}{m\omega}} \frac{e\,E_-}{m_e\,\omega^2} \quad , \quad \text{i.e.} \quad , \quad x - iy = \frac{1}{1 - \frac{eB_z}{m\omega}} \frac{e\,(E_x - iE_y)}{m_e\,\omega^2}$$

- These solutions indicate that the magnetic field B_z creates a kind of ***frequency*** $\Omega_B = eB_z/m_e$, which is called the ***cyclotron frequency*** for the electron's motion in the magnetic field B_z. Therefore, we have:

$$r_\pm = \frac{e}{m_e\,\omega^2} \frac{1}{1 \pm \frac{\Omega_B}{\omega}} E_\pm$$

- Now, these two types of motions mean that the two circular polarizations generate ***two different electric polarizations*** in the plasma medium:

$$\vec{p}_+(t) = -\,e\,r_+(t)\,\hat{r}_+ \quad , \quad \hat{r}_+ = \frac{1}{2}\left(\hat{x} + i\hat{y}\right)$$

$$\vec{p}_-(t) = -\,e\,r_-(t)\,\hat{r}_- \quad , \quad \hat{r}_- = \frac{1}{2}\left(\hat{x} - i\hat{y}\right)$$

Faraday rotation

- In other words: the two circular polarization modes have different dielectric properties:

$$\epsilon_+ = \epsilon_0 \left[1 - \frac{n_e e^2}{m_e\, \epsilon_0\, \omega^2 (1 + \Omega_B/\omega)} \right] = \epsilon_0 \left[1 - \frac{\omega_p^2}{\omega(\omega + \Omega_B)} \right] \quad , \text{ and } \quad \epsilon_- = \epsilon_0 \left[1 - \frac{\omega_p^2}{\omega(\omega - \Omega_B)} \right]$$

- This means that ***the two circular polarizations propagate in different ways in a plasma***!
- Recall that the relationship between the frequency and the wavenumber is given by:

$$\frac{k^2}{\omega^2} = \mu\epsilon \to \frac{1}{c^2}(1 + \chi_E) \quad , \text{ therefore we get that } \quad k_\pm^2 = \frac{\omega^2}{c^2} \left[1 - \frac{\omega_p^2}{\omega(\omega \pm \Omega_B)} \right]$$

- So, in a plasma, electromagnetic waves with left- and right-handed circular polarizations have different wavelengths, and ***travel with different velocities***.
- Using the vacuum wavenumber $k_0 = \omega/c$ and the vacuum wavelength $\lambda_0 = 2\pi c/\omega$, let's define the plasma wavenumber and the plasma wavelength as:

$$k_p = k_0 \sqrt{1 - \frac{\omega_p^2}{\omega^2}} \quad , \quad \lambda_p = \lambda_0 / \sqrt{1 - \frac{\omega_p^2}{\omega^2}}$$

- Now, assuming that $\omega \gg \omega_p, \Omega_B$, the two circular polarizations have wavenumbers and wavelengths given by:

$$k_\pm \simeq k_p \left(1 \pm \frac{1}{2} \frac{\omega_p^2\, \Omega_B}{\omega^3} \right) \quad , \quad \lambda_\pm \simeq \lambda_p \left(1 \mp \frac{1}{2} \frac{\omega_p^2\, \Omega_B}{\omega^3} \right)$$

Faraday rotation

- What this all means is that one polarization propagates a bit faster than the other !
- Let's see what happens with an EM wave as it enters a plasma. Consider a wave with a linear polarization E_x^0 , which in terms of the circular polarizations is:

$$E_\pm = E_x \pm i E_y \quad \Rightarrow \quad E_x^0 = \frac{1}{2}\left(E_+^0 + E_-^0\right) \quad , \quad E_y^0 = \frac{1}{2}\left(E_+^0 - E_-^0\right) = 0 \quad ,$$

so it is clear that $E_+^0 = E_-^0 = E_x^0/2$.

- Therefore, as this wave enters the plasma (at $z = 0$, $t = 0$) , it propagates in space and time as:

$$\overrightarrow{E}(t,z) = E_+^0 \, \hat{r}_+ \, e^{i(k_+ z - \omega t)} + E_-^0 \, \hat{r}_- \, e^{i(k_- z - \omega t)}$$

- Let's check that initially the wave is linearly polarized:

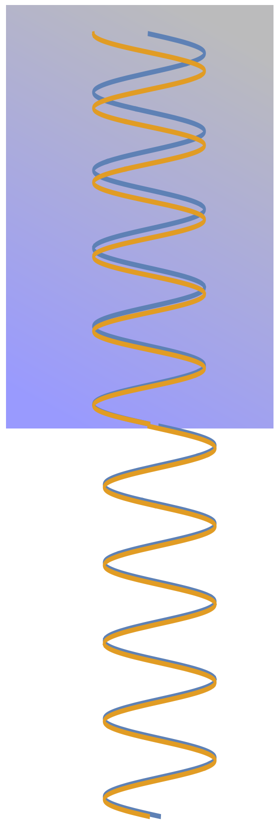
$$\overrightarrow{E}(t, z=0) = e^{-i\omega t}\left(E_+^0 \, \hat{r}_+ + E_-^0 \, \hat{r}_-\right) = e^{-i\omega t}\left[\frac{1}{2}E_+^0\,(\hat{x} + i\hat{y}) + \frac{1}{2}E_-^0\,(\hat{x} - i\hat{y})\right]$$

$$= e^{-i\omega t}\left[\frac{1}{2}(E_+^0 + E_-^0)\hat{x} \; + \; i\frac{1}{2}(E_+^0 - E_-^0)\,\hat{y}\right] = e^{-i\omega t}\, E_x^0 \, \hat{x}$$

- But notice what happens as the wave moves inside the plasma: since $k_\pm \simeq k_p \left(1 \pm \frac{1}{2}\frac{\omega_p^2\,\Omega_B}{\omega^3}\right)$ we can write:

$$\overrightarrow{E}(t,z) = \frac{1}{2}E_x^0 e^{i(k_p z - \omega t)}\left[\hat{r}_+ \, e^{i\Delta k z} + \hat{r}_- \, e^{-i\Delta k z}\right] \quad , \quad \text{where} \quad \Delta k \simeq k_p \, \omega_p^2 \, \Omega_B/(2\,\omega^3)$$

[Incidentally, notice that this expression shows that the direction of the electric field is "real", since the second term inside the square brackets is the complex conjugate of the first term!]



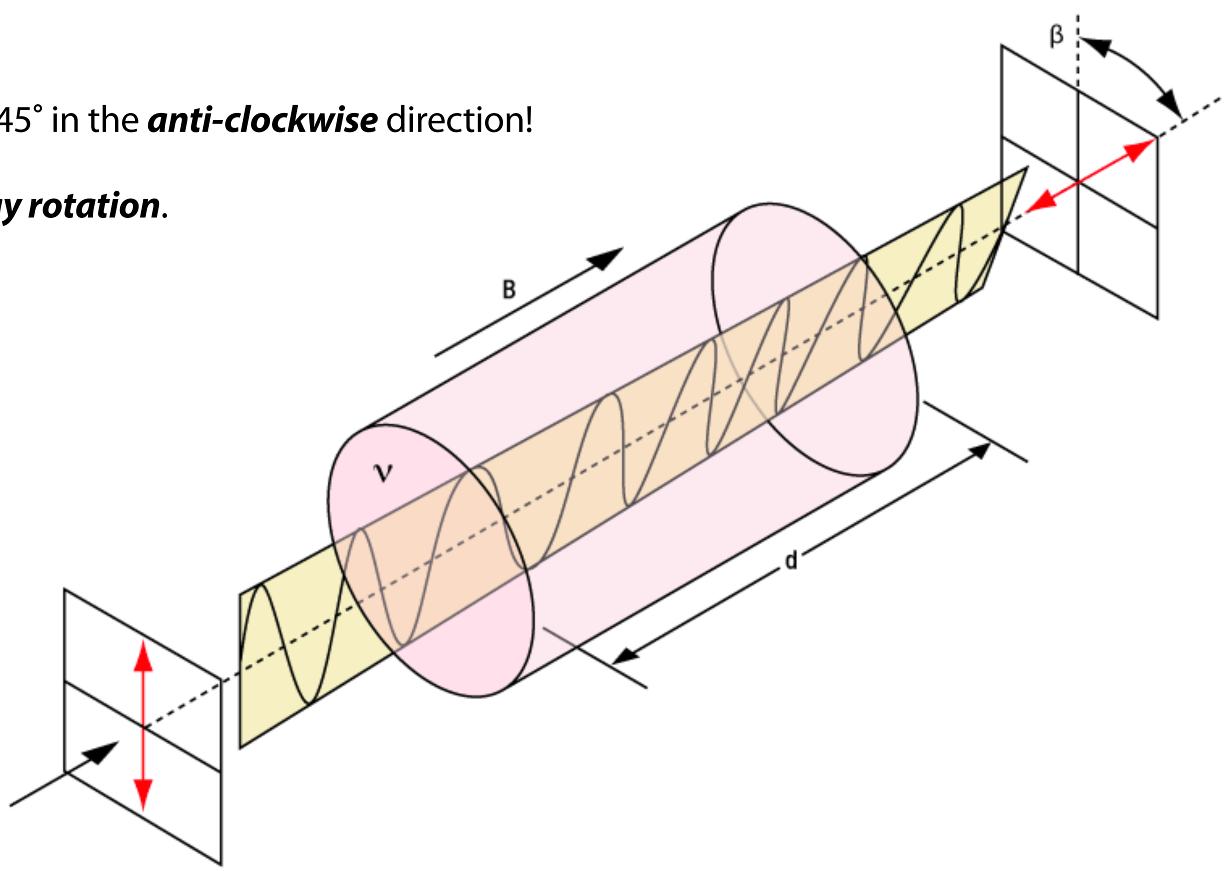
Faraday rotation

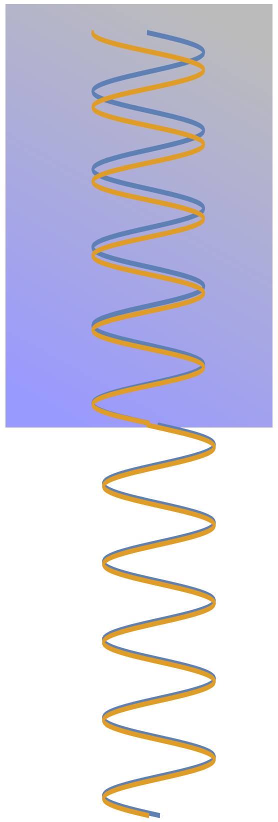
- Therefore, if we look at this wave at some point deeper into the plasma, the direction of the electric field will have rotated !
- Suppose we go enough inside the plasma that $\Delta k\, z = \pi/4$. We will find that the field is now oriented in the direction:

$$\vec{E} \sim \hat{r}_+ e^{i\pi/4} + \hat{r}_- e^{-i\pi/4} = \frac{1}{2}(\hat{x} + i\hat{y})\frac{1+i}{\sqrt{2}} + \frac{1}{2}(\hat{x} - i\hat{y})\frac{1-i}{\sqrt{2}}$$

$$\Rightarrow \quad \vec{E} \sim \frac{\hat{x} - \hat{y}}{\sqrt{2}}$$

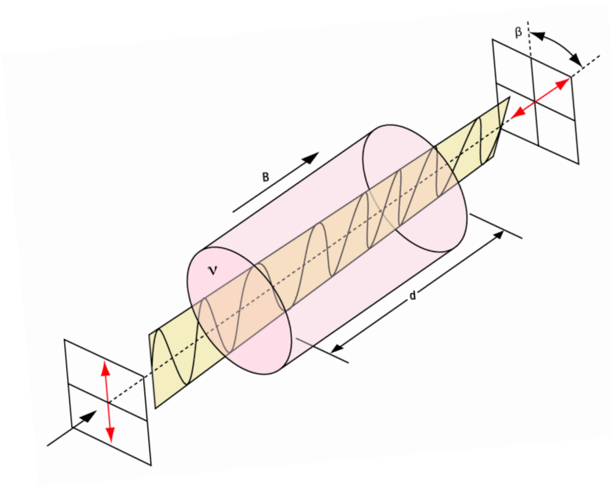
- Therefore, the polarization rotated by 45˚ in the ***anti-clockwise*** direction!
- This phenomenon is known as ***Faraday rotation***.





Faraday rotation

- The phase change between the two circular polarizations grows linearly with the length that the wave travels inside the plasma. In other words, the phase change is given by:



$$\frac{d\varphi}{dz} = \Delta k = \frac{k_p\,\omega_p^2\,\Omega_B}{2\,\omega^3} \simeq \frac{\omega_p^2\,\Omega_B}{2\,\omega^2\,c} = \frac{n_e\,e^3\,B_z}{2\,\epsilon_0\,\omega^2\,m_e^2\,c}$$

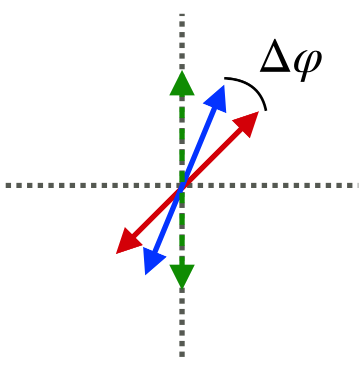
- Since typically there are no physical mechanisms that change the frequency ω of a wave , we find that the rotation of the wave's polarization is given by:

$$\varphi(z) = \frac{e^3}{2\,\epsilon_0\,\omega^2\,m_e^2\,c}\int dz\; n_e(z)\,B_z(z)$$

- Therefore, if we ***know the magnetic field*** strength, we can use a measurement of the Faraday rotation to ***measure the density of electrons in a medium***; or (more realistically), ***knowing the number density of free electrons***, we can use Faraday rotation to ***measure the strength of the magnetic field***.

- Notice that we could only really measure Faraday rotation for an individual wave if we knew the ***original polarization state*** of the wave, ***before*** it entered the plasma. But that can be very difficult, especially in Astrophysics, where we can't tell how the sources are oriented. What do we do then? Think a little bit…

- The answer is that we can look at the ***spectrum*** of waves of ***different frequencies***. Since $\varphi(z) \sim 1/\omega^2$, ***each frequency rotates by a different phase:***



$$\varphi(z;\omega) - \varphi(z;\omega') = \left(\frac{1}{\omega^2} - \frac{1}{\omega'^2}\right)\frac{e^3}{2\,\epsilon_0\,m_e^2\,c}\int dz\; n_e(z)\,B_z(z)$$

[Lower frequencies rotate faster!]

- It is by comparing these ***phase differences*** that we can tell what the Faraday rotation is — and from them, we can compute either the magnetic field or the number of ionized electrons.

Collisional plasmas

- We have been discussing plasma as a gas of free electrons, as if there were no interactions between them. This is not entirely true…
- For a typical conductor, the current is roughly proportional to the electric field, and we express this as Ohm's law:

$$\vec{J} = \sigma \vec{E} \quad , \quad \text{where } \sigma \text{ is the conductivity of the medium.}$$

- Let's recall what happens in a wave inside a conducting medium. Using the current above in Maxwell's equations, assuming stationary currents and neglecting charge densities, $\vec{\nabla} \cdot \vec{E} = \vec{\nabla} \cdot \vec{J} = 0$, we obtain:

$$\vec{\nabla} \times \vec{E} = \frac{\partial \vec{B}}{\partial t} \quad , \quad \text{and}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \epsilon\mu_0 \frac{\partial \vec{E}}{\partial t}$$

- Taking the curl of one equation, and substituting the other, we obtain two identical wave equations, which for the electric field read:

$$\epsilon\mu_0 \frac{\partial^2 \vec{E}}{\partial t^2} + \sigma\mu_0 \frac{\partial \vec{E}}{\partial t} - \nabla^2 \vec{E} = 0$$

- The solution is the usual one, $\vec{E} \sim e^{i(\vec{k}\cdot\vec{x}-\omega t)}$, only that now we obtain:

$$-\epsilon\mu_0\omega^2 - i\omega\sigma\mu_0 + k^2 = 0 \quad \Rightarrow \quad k^2 = \mu_0\omega(\epsilon\omega + i\sigma)$$

- This dispersion relation means that the wave suffers an exponential attenuation as it enters the conductor — this is typically what happens when we get a complex dispersion relation.

Collisional plasmas

- Let's see now what happens in a plasma, which is not quite a conductor, but it acts a bit like one.
- Let's look at the equations of motion for the electrons in that medium, but let's assume that the electrons can now interact ("collide") with the other charged particles, such as ions, in the medium. We have then:

  $m_e \ddot{\vec{r}} + m_e \gamma \dot{\vec{x}} = - e \vec{E}$, where γ gives the ***rate of collisions*** of the electrons (it has dimensions of 1/time).

- Assuming the usual $e^{-i\omega t}$ dependence for all the time-dependent quantities, we obtain a solution for the trajectory:

$$\vec{r} = \frac{i}{\omega} \vec{v} = \frac{e \vec{E}}{m_e \omega (\omega + i\gamma)}$$

- For a plasma with a number density of electrons given by n_e , the current is therefore:

$$\vec{J} = (-e)\, n_e \vec{v} = \frac{i\, n_e e^2 \vec{E}}{m_e(\omega + i\gamma)}$$

- From this we can find the conductivity:

$$\sigma = \frac{J}{E} = \frac{i\, n_e e^2}{m_e(\omega + i\gamma)} = \frac{n_e e^2}{m_e(\gamma - i\omega)}$$

  Hence, in the limit $\omega \ll \gamma$ the conductivity is (mostly) real, and the plasma acts as a conducting medium, dissipating the wave.

- In the presence of these collisions the dispersion relation becomes:

$$k^2 = \mu_0 \omega (\epsilon\omega + i\sigma) \quad \Rightarrow \quad k^2 = \mu_0 \omega \left[\epsilon\omega - \frac{n_e e^2}{m_e(\omega + i\gamma)} \right] = \frac{1}{c^2} \left[\omega^2 - \frac{\omega_p^2\, \omega}{\omega + i\gamma} \right]$$

  It is clear that for $\omega \gg \gamma$ this reduces to the dispersion relation in a plasma, $c^2 k^2 = \omega^2 - \omega_p^2$, and in the opposite limit we go back to the dispersion relation for a conductor, which leads to the exponential attenuation of the wave in the medium. In other words, when there are lots of collisions going on, a plasma works like a conductor; and conversely, in the high-frequency limit a conductor acts like a collisionless plasma.

That’s all folks!

- This is as much as I can say about plasma phenomena.
- You can learn more about it in Electrodynamics II, with Prof. Galvão.

- This concludes the main topics of this graduate course. We will reconvene next Friday for the presentations about topics.
- P2 is scheduled for June 25th. It will be a “take-home” exam again (hand out Friday, send in Monday), and we will cover all of Relativistic/Covariant Electrodynamics, as well as radiation.