Spherical waves

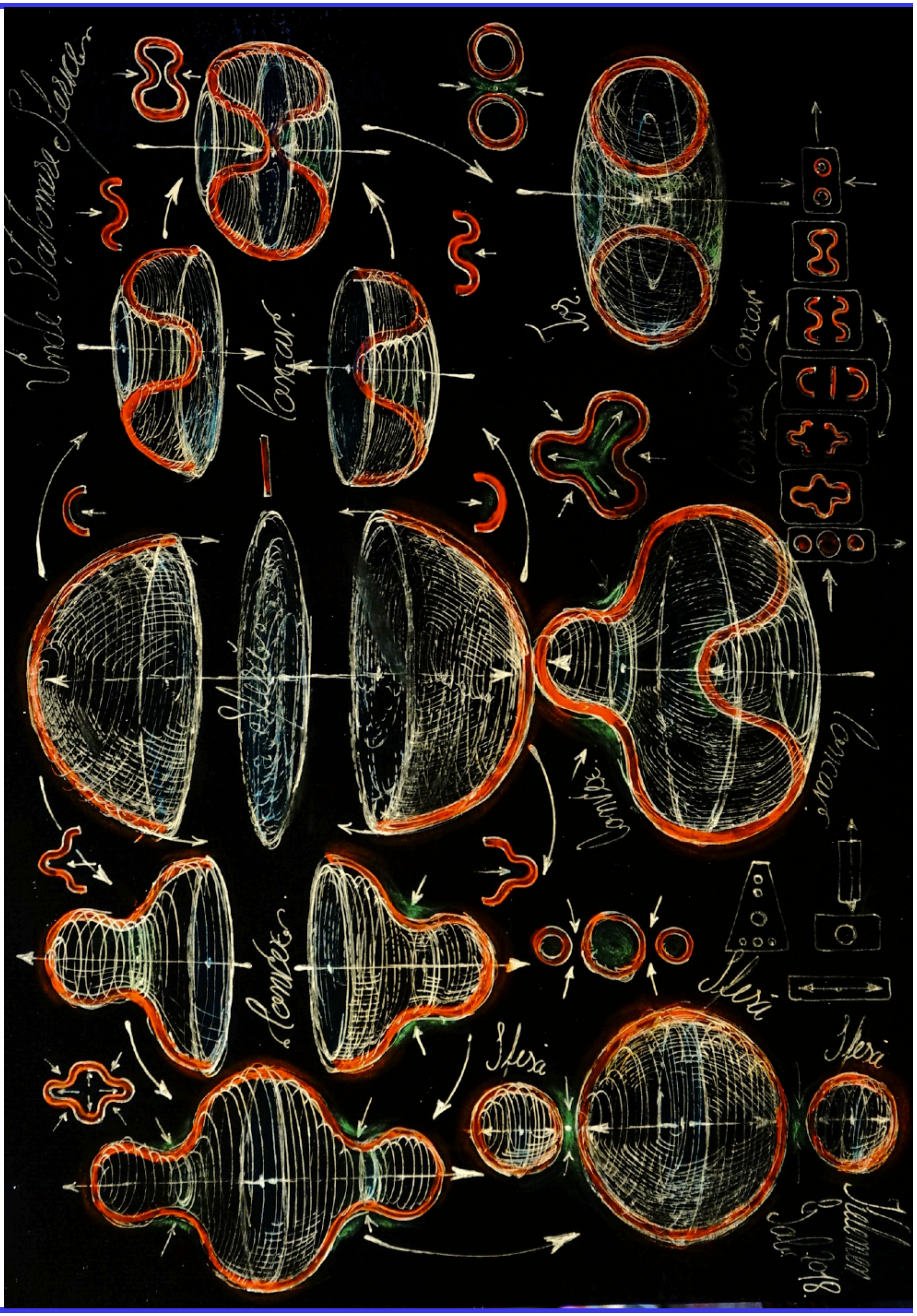
⚡ **Spherical wave solutions**

⚡ **Multipole expansion and partial waves**

⚡ **Multipole expansion of EM fields**

⚡ **Vector spherical harmonics**

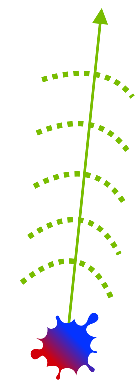
[Jackson, Ch. 9, Zangwill Ch. 20]



Freely propagating waves

- Typically, "radiative" systems inject waves from some center where the "medium" is perturbed. After the fields (the wave) has detached from the sources, it propagates freely.

- For a "scalar wave" ψ, the equation satisfied by the field is:



$$\Box \psi = -\frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2} + \vec{\nabla}^2 \psi = 0$$

- Instead of treating the most general time-dependent problem, what we usually do is to decompose the source and the waves in frequencies:

$$\psi = \sum_{\omega} e^{-i\omega t} \psi_\omega$$

- The wave equation then becomes the ***Helmholtz equation***:

$$\left(\vec{\nabla}^2 + k^2\right) \psi_\omega = 0 \quad , \quad \text{where } k = \omega / c \, .$$

- We will now study this equation assuming that the source of the waves is a small, localized region in space. This means that we should use spherical coordinates — with the source at the origin, of course.

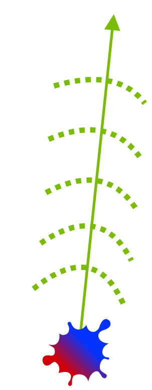
- In spherical coordinates, the angular modes are described by the spherical harmonic functions, $Y_{\ell m}(\theta, \varphi)$:

$$\psi_\omega(r, \theta, \varphi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} f^{\omega}_{\ell m}(r) \, Y_{\ell m}(\theta, \varphi)$$

Spherical waves: radial equation

- We can now write the equation for the radial modes $f^{\omega}_{\ell m}(r)$, which we will simply call $f(r)$ for simplicity. Using the Laplacian in spherical coordinates we obtain:

$$\left[\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + k^2 - \frac{\ell(\ell+1)}{r^2}\right] f(r) = 0$$



- It turns out that this equation is basically the Bessel equation. Writing the radial modes as $f(r) = u(r)/\sqrt{r}$ we obtain:

$$\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + k^2 - \frac{(\ell+1/2)^2}{r^2}\right] u(r) = 0$$, which is the Bessel equation!

- So, the solutions to the radial modes are slightly modified Bessel functions of ***half-integer*** order:

$$f_\ell(r) = \frac{1}{\sqrt{r}}\left[a_\ell J_{\ell+1/2}(kr) + b_\ell N_{\ell+1/2}(kr)\right]$$

- These slightly modified Bessel functions are usually normalized in a standard way, and called Spherical Bessel Functions — or spherical Hankel functions in the complex case:

$$j_\ell(x) = \sqrt{\frac{\pi}{2x}}\, J_{\ell+1/2}(x) \qquad h^{(1)}_\ell(x) \;=\; j_\ell(x) \;+\; i n_\ell(x)$$

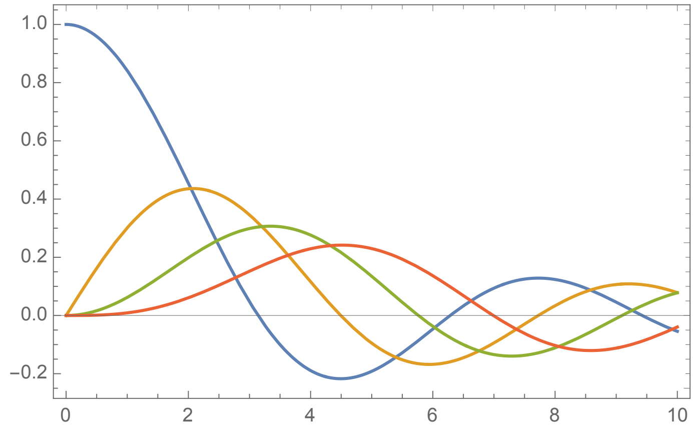
$$n_\ell(x) = \sqrt{\frac{\pi}{2x}}\, N_{\ell+1/2}(x) \qquad h^{(2)}_\ell(x) \;=\; j_\ell(x) \;-\; i n_\ell(x)$$

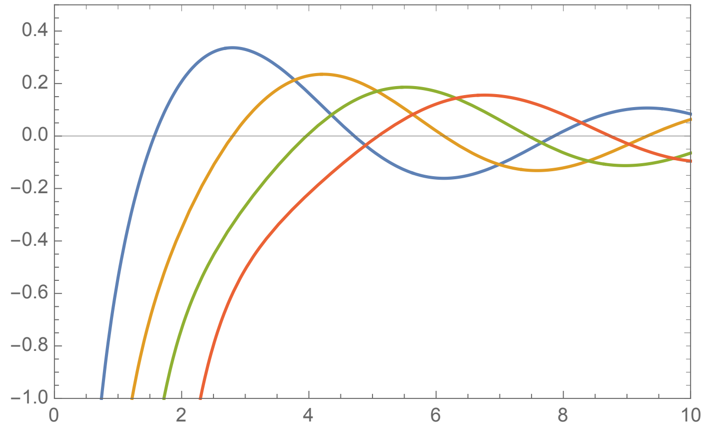
Notice that this is fundamentally different from the solutions to the Laplace equation, for which $k \to 0$. In that case we obtained the radial modes r^ℓ and $r^{-\ell-1}$.

Spherical Bessel functions

- Although the Bessel functions of integer order do not have a closed, simple form, the spherical Bessel functions can in fact be written in terms of elementary function! Here are the first few ones.

ℓ	j_ℓ	n_ℓ
0	$\dfrac{\sin x}{x}$	$-\dfrac{\cos x}{x}$
1	$\dfrac{\sin x - x\cos x}{x^2}$	$\dfrac{-\cos x - x\sin x}{x^2}$
2	$\dfrac{3 - 3x\cos x - x^2\sin x}{x^3}$	$\dfrac{-3 - 3x\sin x + x^2\cos x}{x^3}$





Spherical Bessel functions

- Like the original Bessel functions, these spherical Bessel functions have very different behaviors near the origin ($x \to 0$) and for large arguments ($x \to \infty$).
- In the limit $x \ll Min(1,\ell)$ we have:

$$\lim_{x\to 0} j_\ell(x) = \frac{x^\ell}{(2\ell+1)!!}\left(1 - \frac{x^2}{2(2\ell+3)} + \dots\right)$$

$$\lim_{x\to 0} n_\ell(x) = -\frac{(2\ell-1)!!}{x^{\ell+1}}\left(1 + \frac{x^2}{2(2\ell-1)} + \dots\right)$$

- In the opposite limit, $x \gg Max(1,\ell)$, we have:

$$\lim_{x\to\infty} j_\ell(x) = \frac{1}{x}\sin\left(x - \ell\pi/2\right)$$

$$\lim_{x\to\infty} n_\ell(x) = -\frac{1}{x}\cos\left(x - \ell\pi/2\right)$$

- Clearly, solutions which are ***regular at the origin*** can only include the spherical Bessel functions of the first kind, $j_\ell(kr)$.
- On the other hand, if the wave has an asymptotic behavior such as $\psi \sim e^{\pm ikr}/r$, then we must combine the two solutions in the spherical Hankel functions, such that:

$$\lim_{x\to\infty} h_\ell^{(1)}(x) = (-i)^{\ell+1}\frac{e^{ix}}{x}$$, and the complex conjugate in the case of $h_\ell^{(2)}$.

The Rayleigh expansion of plane waves

- We often use the fact that any wave can be decomposed into a superposition of plane waves — this is, after all, the basis of the Fourier transform.
- Any wave can then be cast in the form:

$$\psi(\vec{x},t) = \int \frac{d^3k}{(2\pi)^3}\, \tilde{\psi}(\vec{k})\, e^{-i(\omega t - \vec{k}\cdot\vec{x})} \quad , \text{ with } k = \omega/c$$

- But the expansion we have just derived is equally appropriate, but now in terms of *spherical waves*:

$$\psi(t,x,\theta,\varphi) = \sum_{\omega} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} e^{-i\omega t} \tilde{f}_{\ell m}(k)\, j_\ell(kx)\, Y_{\ell m}(\theta,\varphi)$$

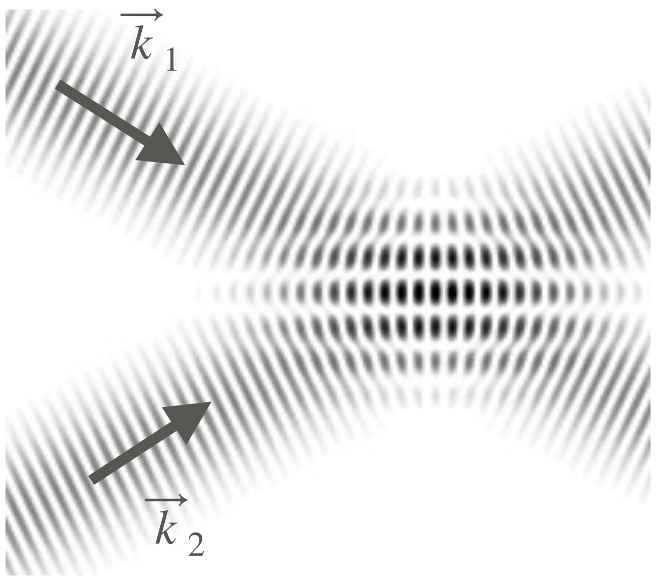
- For reasons that will become clear later, it is useful to define this expansion as:

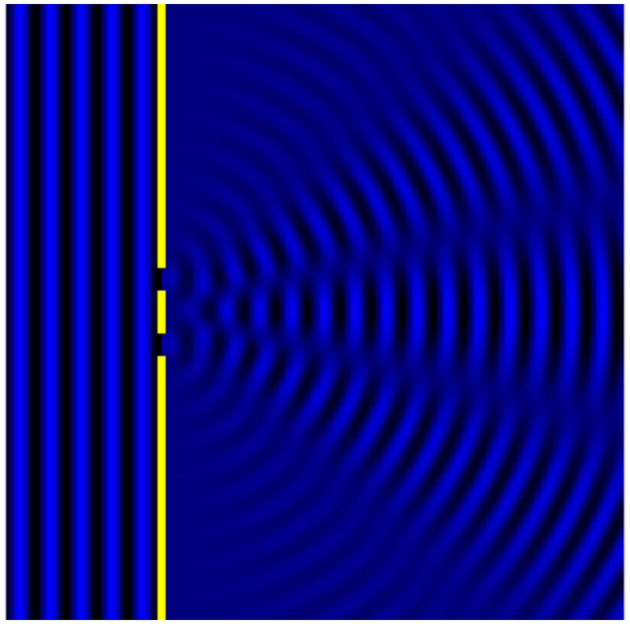
$$\psi(t,x,\hat{x}) = \int_0^\infty dk\, k^2\, e^{-i\omega t} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \tilde{f}_{\ell m}(k)\, j_\ell(kx)\, Y_{\ell m}(\hat{x}) \quad ,$$

- These spherical waves form the basis for much of optics, interference, diffraction etc.
- We can actually make a connection between the two expressions, linking the plane waves to these spherical waves. This was first done by Lord Rayleigh, when he was studying light. The key for this connection is the expression:

$$e^{i\vec{k}\cdot\vec{x}} = 4\pi \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} i^\ell j_\ell(kx)\, Y_{\ell m}(\hat{k})\, Y^*_{\ell m}(\hat{x}) = 4\pi \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} i^\ell j_\ell(kx)\, Y_{\ell m}(\hat{k})^*\, Y_{\ell m}(\hat{x})$$

$$= \sum_{\ell=0}^{\infty} i^\ell\, (2\ell+1)\, j_\ell(kx)\, P_\ell(\hat{k}\cdot\hat{x})$$





The Rayleigh expansion of plane waves

- Let's start by computing the coefficients of the spherical wave expansion:

$$\psi(t,x,\hat{x}) = \int_0^\infty dk\, k^2\, e^{-i\omega t} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \tilde{f}_{\ell m}(k)\, j_\ell(kx)\, Y_{\ell m}(\hat{x})$$

- We can compute these coefficients by using the orthogonality of the Bessel functions:

$$\int_0^\infty dx\, x^2\, j_\ell(kx)\, j_\ell(k'x) = \frac{\pi}{2\,k^2}\,\delta(k-k') \quad ,$$

as well as that for the spherical harmonics,

$$\int d^2\hat{x}\, Y_{\ell m}(\hat{x})\, Y_{\ell' m'}(\hat{x}) = \delta_{\ell\ell'}\,\delta_{mm}$$

- Integrating the left-hand side and the right-hand side of the expression above over the volume with these two basis functions,

$$\int d^3x\, j_\ell(kx)\, Y^*_{\ell m}(\hat{x})\,[\cdots] \quad ,$$ we obtain:

$$\int_0^\infty dx x^2\, j_\ell(kx) \int d^2\hat{x}\, Y^*_{\ell m}(\hat{x})\,\psi(t,x,\hat{x}) = \int_0^\infty dx x^2\, j_\ell(kx) \int d^2\hat{x}\, Y^*_{\ell m}(\hat{x}) \int_0^\infty dk'\, k'^2\, e^{-i\omega' t} \sum_{\ell'=0}^{\infty} \sum_{m'=-\ell'}^{\ell'} \tilde{f}_{\ell' m'}(k')\, j_{\ell'}(k'x)\, Y_{\ell' m'}(\hat{x})$$

$$= \sum_{\ell'=0}^{\infty} \sum_{m'=-\ell'}^{\ell'} \int_0^\infty dk'\, k'^2 \tilde{f}_{\ell' m'}(k')\, e^{-i\omega' t} \left[\int_0^\infty dx x^2\, j_\ell(kx)\, j_{\ell'}(kx)\right] \left[\int d^2\hat{x}\, Y^*_{\ell m}(\hat{x})\, Y_{\ell' m'}(\hat{x})\right]$$

$$= \sum_{\ell'=0}^{\infty} \sum_{m'=-\ell'}^{\ell'} \int_0^\infty dk'\, k'^2 \tilde{f}_{\ell' m'}(k')\, e^{-i\omega' t} \left[\frac{\pi}{2\,k^2}\delta(k-k')\right] \left[\delta_{\ell\ell'}\,\delta_{mm'}\right]$$

$$= \frac{\pi}{2}\,\tilde{f}_{\ell m}(k)\, e^{-i\omega t}$$

The Rayleigh expansion of plane waves

- Therefore, we found that any solution of the wave equation that is valid in all space (including the origin) can be written as:

$$\psi(t,x,\hat{x}) = \int_0^\infty dk\, k^2\, e^{-i\omega t} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \tilde{f}_{\ell m}(k)\, j_\ell(kx)\, Y_{\ell m}(\hat{x}) \quad , \text{ where}$$

$$\tilde{f}_{\ell m}(k) = \frac{2}{\pi} \int_0^\infty dx\, x^2\, j_\ell(kx) \int d^2\hat{x}\, Y^*_{\ell m}(\hat{x})\, \psi(x,\hat{x}) = \frac{2}{\pi} \int d^3x\, j_\ell(kx) Y^*_{\ell m}(\hat{x})\, \psi(x,\hat{x})$$

- Notice how this is incredibly similar to the Fourier expansion in plane waves. In fact, as we will see now, the two are closely related. Let's get back to the expression:

$$e^{i\vec{k}\cdot\vec{x}} = 4\pi \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} i^\ell\, j_\ell(kx)\, Y^*_{\ell m}(\hat{k})\, Y_{\ell m}(\hat{x}) \quad , \text{ and substitute that into the plane wave expansion:}$$

$$\psi(t,\vec{x}) = \int \frac{d^3k}{(2\pi)^3}\, \tilde{\psi}(\vec{k})\, e^{-i(\omega t - \vec{k}\cdot\vec{x})}$$

$$= \int \frac{d^3k}{(2\pi)^3}\, \tilde{\psi}(\vec{k})\, e^{-i\omega t} \times 4\pi \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} i^\ell\, j_\ell(kx)\, Y^*_{\ell m}(\hat{k})\, Y_{\ell m}(\hat{x})$$

$$= \frac{1}{2\pi^2} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} i^\ell\, Y_{\ell m}(\hat{x}) \int_0^\infty k^2 dk\, e^{-i\omega t} \int d^2\hat{k}\, \tilde{\psi}(\vec{k})\, j_\ell(kx)\, Y^*_{\ell m}(\hat{k})$$

- You can now recognize the coefficients in the two expansions. We find that:

$$\tilde{f}_{\ell m}(k) = \frac{1}{2\pi^2} i^\ell \int d^2\hat{k}\, \tilde{\psi}(\vec{k})\, Y^*_{\ell m}(\hat{k})$$

The Rayleigh expansion of plane waves

- We can arrive at a more enlightening expression by defining the expansions in real (configuration) and Fourier space in a symmetric way — and below, for simplicity, we have written $\tilde{\psi}(t, \overrightarrow{k}) = e^{-i\omega t}\tilde{\psi}(\overrightarrow{k})$:

$$\psi(t,x,\hat{x}) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} f_{\ell m}(t,r) Y_{\ell m}(\hat{x}) \quad , \text{ where } \quad f_{\ell m}(t,x) = \int d^2\hat{x}\, Y^*_{\ell m}(\hat{x})\, \psi(t,x,\hat{x})$$

$$\tilde{\psi}(t,k,\hat{k}) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} \tilde{f}_{\ell m}(t,k) Y_{\ell m}(\hat{k}) \quad , \text{ where } \quad \tilde{f}_{\ell m}(t,k) = \int d^2\hat{k}\, Y^*_{\ell m}(\hat{k})\, \tilde{\psi}(t,k,\hat{k}).$$

- The connection between the two is obtained by using the Rayleigh expansion, resulting in:

$$\tilde{f}_{\ell m}(t,k) = 4\pi\, i^{\ell} \int_0^{\infty} dx\, x^2 f_{\ell m}(t,x)\, j_{\ell}(kx) \quad ,$$

  The converse relation is identical, but recall that the Fourier transforms are such that $\int d^3x(\ldots) \leftrightarrow \int d^3k/(2\pi)^3(\ldots)$, so we get:

$$f_{\ell m}(t,x) = \frac{4\pi\, i^{\ell}}{(2\pi)^3} \int_0^{\infty} dk\, k^2\, e^{-i\omega t} \tilde{f}_{\ell m}(t,k)\, j_{\ell}(kx)$$

- You can switch between the two descriptions by using the orthogonality of the spherical Bessel functions:

$$\int_0^{\infty} dx\, x^2 j_{\ell}(kx)\, j_{\ell}(k'x) = \frac{\pi}{2\,k^2}\,\delta(k-k') \quad , \quad \int_0^{\infty} dk\, k^2 j_{\ell}(kx)\, j_{\ell}(kx') = \frac{\pi}{2\,x^2}\,\delta(x-x')$$

The Fourier-Bessel expansion

- Now, stop for a moment to absorb this amazing result. Consider that any field configuration can be given in real space, $\psi(\vec{x})$, or in Fourier space, $\tilde{\psi}(\vec{k})$.
- We can now decompose the field in "spherical shells" (like the layers of an onion), in **both descriptions**:

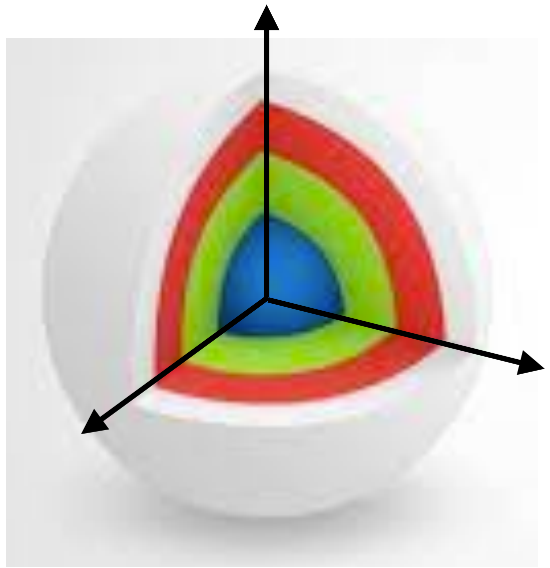
$$\psi(t,x,\hat{x}) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} f_{\ell m}(t,r) Y_{\ell m}(\hat{x}) \quad , \quad \text{or} \quad \tilde{\psi}(t,k,\hat{k}) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} \tilde{f}_{\ell m}(t,k) Y_{\ell m}(\hat{k})$$

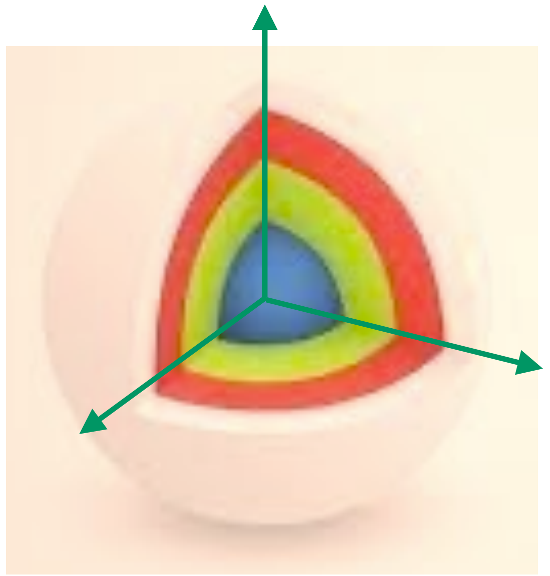
- Incredibly, the angular modes $f_{\ell m}$ in ***real space*** are basically the same as the angular modes in ***Fourier space***, $\tilde{f}_{\ell m}$, up to the ***Fourier-Bessel transformation***, that takes place only along the ***radial direction***:

$$\tilde{f}_{\ell m}(t,k) = 4\pi\, i^{\ell} \int_0^{\infty} dx\, x^2 f_{\ell m}(t,x)\, j_{\ell}(kx) \quad ,$$

$$f_{\ell m}(t,x) = \frac{4\pi\, i^{\ell}}{(2\pi)^3} \int_0^{\infty} dk\, k^2\, e^{-i\omega t} \tilde{f}_{\ell m}(t,k)\, j_{\ell}(kx)$$

- Therefore, the angular properties of the real-space configuration are completely preserved in the Fourier configuration: if a given mode $f_{\ell_0 m_0}$ vanishes, then the Fourier mode $\tilde{f}_{\ell_0 m_0}$ also vanishes.





The Fourier-Bessel expansion

- This result connects back to something we derived in our last class, when we computed the potentials from sources using the approximation that we are evaluating the fields ***in the radiation zone***:

$$\vec{A}_\omega \simeq \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \int d^3x' \, e^{-i\vec{k}\cdot\vec{x}'} \vec{J}_\omega(\vec{x}') \quad , \text{ and}$$

$$\phi_\omega \simeq \frac{1}{4\pi\epsilon_0} \frac{e^{ikr}}{r} \int d^3x' \, e^{-i\vec{k}\cdot\vec{x}'} \rho_\omega(\vec{x}') \quad , \text{ where we identified } \vec{k} = k\,\hat{x} = (\omega/c)\,\hat{x}$$

- The right-hand side are in fact simply the Fourier transforms of the sources:

$$\vec{A}_\omega \simeq \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \tilde{\vec{J}}_\omega(\vec{k} = k\hat{x}) \quad , \text{ and}$$

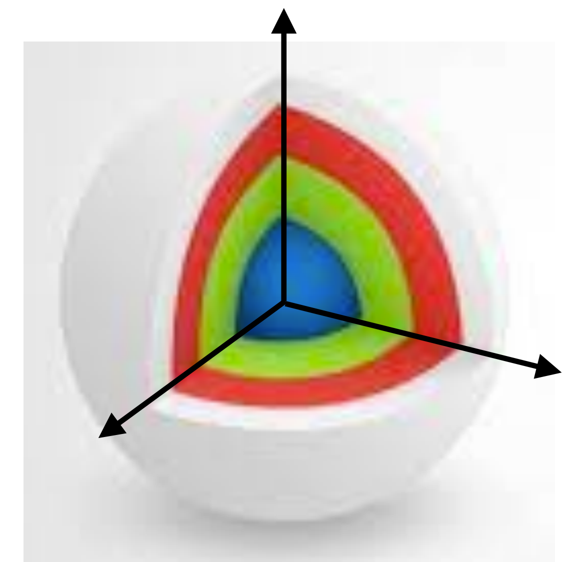
$$\phi_\omega \simeq \frac{1}{4\pi\epsilon_0} \frac{e^{ikr}}{r} \tilde{\rho}_\omega(\vec{k} = k\hat{x})$$

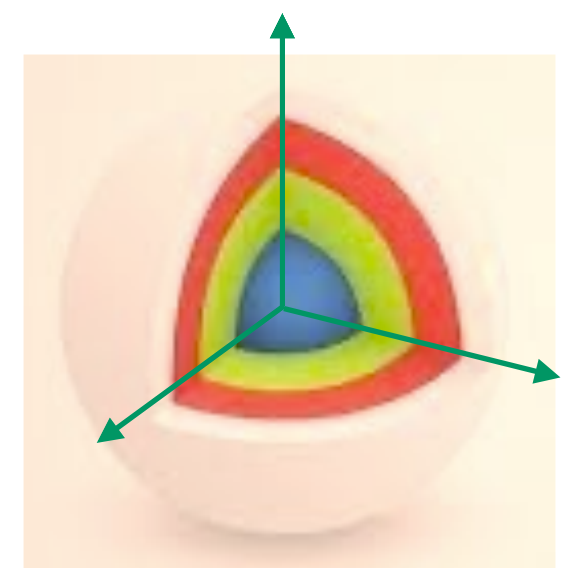
- This result implies that the potentials are given by the ***spherical wave expansion***:

$$\phi(t, \vec{x}) = \sum_{\ell m} \phi_{\ell m}(t, r) Y_{\ell m}(\hat{x}) \quad , \text{ with}$$

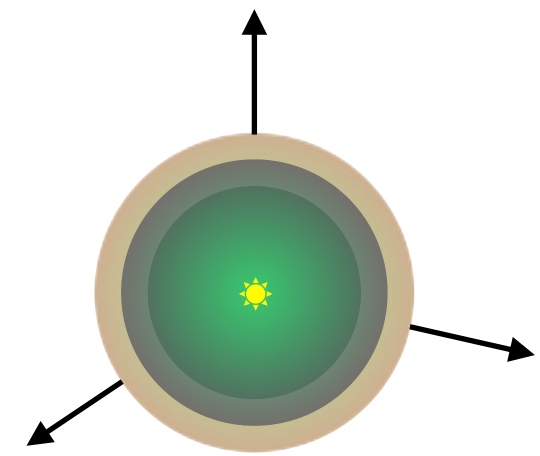
$$\phi_{\ell m}(t, r) = \frac{1}{4\pi\epsilon_0} \frac{e^{-i(\omega t - kr)}}{r} \tilde{\rho}_{\ell m} \quad , \qquad \text{where the } \textbf{\textit{fixed coefficients}} \text{ are:}$$

$$\tilde{\rho}_{\ell m} = \int d^2\hat{k}\, \tilde{\rho}(\vec{k})\, Y^*_{\ell m}(\hat{k}) = 4\pi\, i^\ell \int_0^\infty dx\, x^2 j_\ell(kx)\, \rho_{\ell m}(x)$$





The spherical (*partial*) wave expansion



- Notice that all these "partial waves" have exactly the same time and radial dependence:

$$\phi_{\ell m}(t,r) = \frac{1}{4\pi\epsilon_0}\,\frac{e^{-i(\omega t - kr)}}{r}\,\tilde{\rho}_{\ell m}$$

, i.e., and ***out-going spherical wave***!

- In particular, this means that the amplitude of the multipole radiation fields decay as $\sim 1/r^2$, e.g.:

$$|\phi|^2(t,r) = \int d^2\hat{x}\,\phi(t,\vec{x})\,\phi^*(t,\vec{x})$$

$$= \int d^2\hat{x}\left[\sum_{\ell m}\frac{1}{4\pi\epsilon_0}\frac{e^{-i(\omega t-kr)}}{r}\tilde{\rho}_{\ell m}Y_{\ell m}(\hat{x})\right]\left[\sum_{\ell' m'}\frac{1}{4\pi\epsilon_0}\frac{e^{+i(\omega t-kr)}}{r}\tilde{\rho}^*_{\ell' m'}Y^*_{\ell' m'}(\hat{x})\right]$$

$$= \left(\frac{1}{4\pi\epsilon_0}\frac{1}{r}\right)^2\sum_{\ell m}\sum_{\ell' m'}\tilde{\rho}_{\ell m}\,\tilde{\rho}^*_{\ell' m'}\times\delta_{\ell\ell'}\,\delta_{mm'}$$

$$= \left(\frac{1}{4\pi\epsilon_0}\frac{1}{r}\right)^2\sum_{\ell m}|\tilde{\rho}_{\ell m}|^2$$

- Although this calculation doesn't really mean anything in terms of the fields or, e.g. energy, it is basically identical to the calculation that we perform for the Poynting vector, and indicates that the power emitted by any radiating source decays exactly as $\sim 1/r^2$, with an amplitude that is a sum over the multipoles.

The multipole expansion of EM fields

- In the discussion above we made use of the fact that the spherical harmonics and the spherical Bessel functions are the ***eigenfunctions*** of the angular and radial parts of the wave/Helmholtz equation operator in spherical coordinates.
- Namely, for the angular dependence we used the angular momentum operator,

  $\vec{L} = -i\,\vec{x} \times \vec{\nabla}$, and the spherical harmonics are eigenvectors of this operator in the sense of:

  $L^2\, Y_{\ell m}(\hat{x}) = \ell(\ell+1)\, Y_{\ell m}(\hat{x})$, and

  $L_z\, Y_{\ell m}(\hat{x}) = m\, Y_{\ell m}(\hat{x})$.

  Remember also that $\nabla^2(\cdots) = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\ \cdots) - \frac{L^2}{r^2}(\cdots)$

- We then used these basis functions, $Y_{\ell m}(\hat{x})$ to expand the sources (ρ and $\vec{J}$) as well as the potentials (ϕ and $\vec{A}$).
- However, the electric and magnetic fields are obtained ***from the potentials*** , in terms of operators involving $\vec{\nabla}$, and these are in turn related to the angular momentum operator $\vec{L}$. This means that the electric and magnetic fields cannot be decomposed into regular spherical harmonic functions, but by objects such as $\vec{L}\, Y_{\ell m}$.
- But the spherical harmonic functions are ***not*** eigenfunctions of $\vec{L}$: they are eigenfunctions of $\vec{L}^2$ and L_z only! In fact:

  $$L_\pm \equiv L_x \pm i\, L_y \quad \Rightarrow \quad L_\pm\, Y_{\ell m} = \sqrt{(\ell \mp m)(\ell \pm m + 1)}\, Y_{\ell, m\pm 1}$$

- This means that the action of $\vec{L}$ on $Y_{\ell m}$ result in a different set of functions — called the ***vector spherical harmonics***.

The multipole expansion of EM fields

- Let's go back to the basic equations that determine the free propagation of fields in empty space, assuming a fundamental frequency ω and therefore $k = \omega/c$.
- Using $\vec{H}$ instead of $\vec{B}$, and the definition of vacuum impedance $Z_0 = \sqrt{\epsilon_0/\mu_0} = 376.7\,\Omega$, we have:

$$\vec{\nabla} \times \vec{E} = i\,k\,Z_0\,\vec{H} \quad , \quad \vec{\nabla} \times \vec{H} = -\,i\,k\,\vec{E}/Z_0$$

$$\vec{\nabla} \cdot \vec{E} = \vec{\nabla} \cdot \vec{H} = 0$$

- The Helmholtz equations for these fields follows from taking the curl from either one of the two equations above:

$$\left(\nabla^2 + k^2\right)\vec{E} = 0 \quad , \quad \text{with} \quad \vec{H} = -\frac{i}{kZ_0}\vec{\nabla} \times \vec{E} \quad , \text{ and}$$

$$\left(\nabla^2 + k^2\right)\vec{H} = 0 \quad , \quad \text{with} \quad \vec{E} = \frac{i\,Z_0}{k}\vec{\nabla} \times \vec{H} \quad .$$

- Notice the meaning of these two equations: we can imagine that, on one hand, we could have electric multipoles generating electric fields, which then induce magnetic fields through $\vec{H} = -\frac{i}{kZ_0}\vec{\nabla} \times \vec{E}$.
- OR, we could have on the other hand magnetic multipoles generating magnetic fields, which then induce electric fields through $\vec{E} = \frac{i\,Z_0}{k}\vec{\nabla} \times \vec{H}$.

The multipole expansion of EM fields

- We should now formalize this "guess" that there are two types of modes of electromagnetic radiation. For that, start with the Maxwell equations:

$$\vec{\nabla} \times \vec{E} = i\,k\,Z_0\,\vec{H} \quad , \quad \vec{\nabla} \times \vec{H} = -\,i\,k\,\vec{E}/Z_0 \quad , \text{ and}$$

$$\vec{\nabla} \cdot \vec{E} = \vec{\nabla} \cdot \vec{H} = 0 \quad ,$$

from where we derived

$$\left(\nabla^2 + k^2\right) \vec{E} = 0 \quad , \quad \left(\nabla^2 + k^2\right) \vec{H} = 0 \quad .$$

- It is a trivial matter to show that, as a consequence of these equations, the longitudinal components of the fields also obey the Helmholtz equation, namely:

$$\left(\nabla^2 + k^2\right) \left(\vec{r} \cdot \vec{E}\right) = 0 \quad , \quad \left(\nabla^2 + k^2\right) \left(\vec{r} \cdot \vec{H}\right) = 0 \quad .$$

- Here it is important to clear out a common misunderstanding: it is not true that, in general, radiating electric and magnetic fields are purely transversal, in the sense that $\vec{r} \cdot \vec{E} \to 0$ and/or $\vec{r} \cdot \vec{H} \to 0$. This is only true for plane waves! So, let's keep in mind that, in general, we have in principle all degrees of freedom for the electric and magnetic fields, minus two constraints (their divergence being zero).

- This motivates us to study the two possible situations:

(i) $\vec{r} \cdot \vec{E} = 0$, which we call "transverse <u>magnetic</u>" (TM)

(ii) $\vec{r} \cdot \vec{H} = 0$, which we call "transverse <u>electric</u>" (TE)

Transverse electric and magnetic modes

- Let's start with the transverse magnetic (™). Our multipole expansion means that we can write:

$$\vec{r} \cdot \vec{E}^{(\mathscr{M})} = 0$$

$$\vec{r} \cdot \vec{H}^{(\mathscr{M})} = \sum_{\ell m} \frac{\ell(\ell+1)}{k} \mathscr{M}_{\ell m}(kr)\, Y_{\ell m}(\hat{x}) \quad , \quad \text{with} \quad \frac{\ell(\ell+1)}{k} \mathscr{M}_{\ell m}(kr) = \int d^2\hat{x} \left[\vec{r} \cdot \vec{H}^{(\mathscr{M})}\right] Y^*_{\ell m}(\hat{x})$$

where the pre-factor $\ell(\ell+1)/k$ was introduced for later convenience.

- Now, the constraint between the magnetic and electric fields imply that:

$$\vec{H}^{(\mathscr{M})} = -\frac{i}{Z_0 k} \vec{\nabla} \times \vec{E}^{(\mathscr{M})} \quad \Rightarrow \quad \vec{r} \cdot \vec{H}^{(\mathscr{M})} = -\frac{i}{k\, Z_0} \vec{r} \cdot \vec{\nabla} \times \vec{E}^{(\mathscr{M})} = -\frac{i}{k\, Z_0} (\vec{r} \times \vec{\nabla}) \cdot \vec{E}^{(\mathscr{M})}$$

$$\Rightarrow \quad \vec{r} \cdot \vec{H}^{(\mathscr{M})} = \frac{1}{k\, Z_0} \vec{L} \cdot \vec{E}^{(\mathscr{M})}$$

- Therefore, for these ™ modes we have that:

$$\Rightarrow \quad \vec{L} \cdot \vec{E}^{(\mathscr{M})} = Z_0 \sum_{\ell m} \ell(\ell+1)\, \mathscr{M}_{\ell m}(kr)\, Y_{\ell m}(\hat{x}) = Z_0 \sum_{\ell m} \mathscr{M}_{\ell m}(kr)\, \vec{L}^2\, Y_{\ell m}(\hat{x})$$

- From this it is obvious that the solution for the electric field in this case is given by:

$$\vec{E}^{(\mathscr{M})} = Z_0 \sum_{\ell m} \mathscr{M}_{\ell m}(kr)\, \vec{L}\, Y_{\ell m}(\hat{x})$$

Transverse electric and magnetic modes

- The exact same argument applies for the TE modes. Our multipole expansion means that we can write:

$$\vec{r} \cdot \vec{H}^{(\mathscr{E})} = 0$$

$$\vec{r} \cdot \vec{E}^{(\mathscr{M})} = -Z_0 \sum_{\ell m} \frac{\ell(\ell+1)}{k} \mathscr{E}_{\ell m}(kr)\, Y_{\ell m}(\hat{x}) \qquad .$$

- Now, the constraint between the magnetic and electric fields imply that:

$$\vec{E}^{(\mathscr{E})} = \frac{i\,Z_0}{k} \vec{\nabla} \times \vec{H}^{(\mathscr{E})} \quad \Rightarrow \quad \vec{r} \cdot \vec{E}^{(\mathscr{E})} = \frac{i\,Z_0}{k} \vec{r} \cdot \vec{\nabla} \times \vec{H}^{(\mathscr{E})} = \frac{i\,Z_0}{k} (\vec{r} \times \vec{\nabla}) \cdot \vec{H}^{(\mathscr{E})}$$

$$\Rightarrow \quad \vec{r} \cdot \vec{E}^{(\mathscr{E})} = -\frac{Z_0}{k} \vec{L} \cdot \vec{H}^{(\mathscr{E})}$$

- Using the definition above and the same argument as before we obtain that:

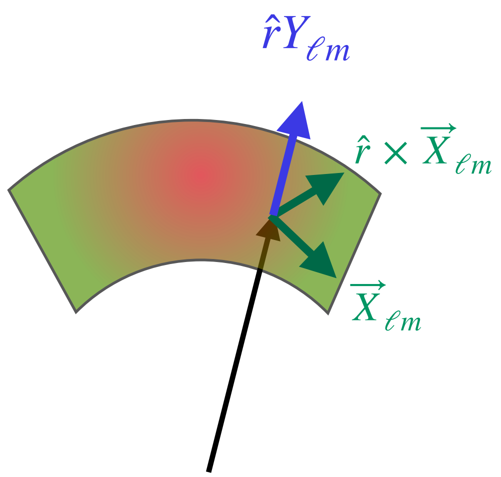
$$\vec{H}^{(\mathscr{E})} = \sum_{\ell m} \mathscr{E}_{\ell m}(kr)\, \vec{L}\, Y_{\ell m}(\hat{x})$$

Vector spherical harmonics

- It is now completely clear that the vector fields $\vec{L} Y_{\ell m}$ play a key role in the multipole expansion of the electric and magnetic fields. We in fact define the ***vector spherical harmonics*** as:

$$\vec{X}_{\ell m}(\hat{x}) \equiv \frac{1}{\sqrt{\ell(\ell+1)}} \vec{L}\, Y_{\ell m}(\hat{x}) \quad .$$

  [Notice that $\vec{X}_{00} = 0$ identically — which should be obvious, since there is no spherically symmetric vector on the 2D unit sphere!]



- These functions obey orthogonality relations and are normalized just as the regular (scalar) spherical harmonics:

$$\int d^2\hat{x}\, \vec{X}_{\ell m} \cdot \vec{X}^*_{\ell' m'} = \delta_{\ell \ell'} \delta_{mm'}$$

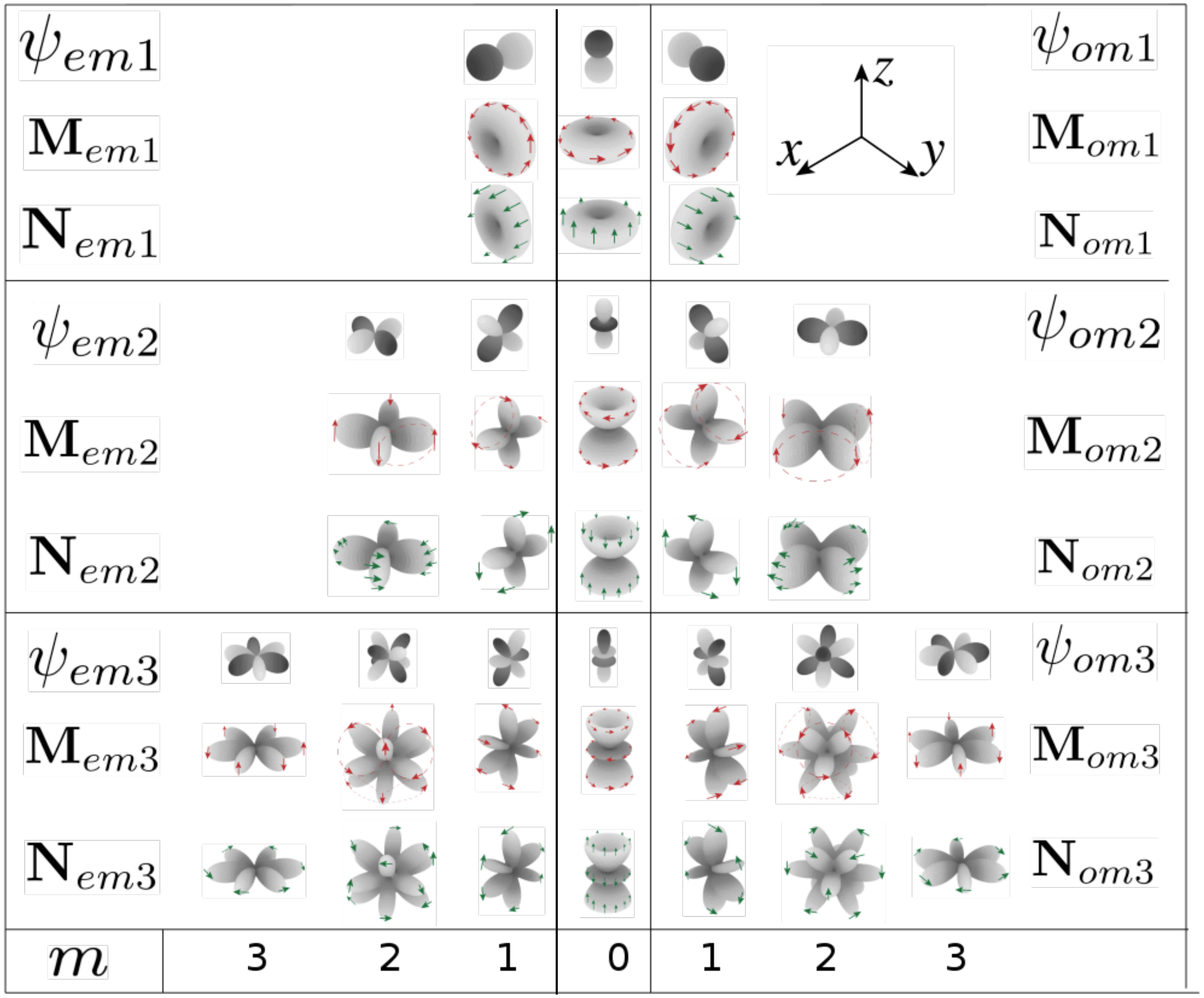
- Moreover, because of the fact that $\vec{L} = -i\vec{r} \times \vec{\nabla}$, it is clear that $\vec{r} \cdot \vec{X}_{\ell m} = 0$, so the vector harmonics should form a basis over the 2D unit sphere.

- However, a vector basis in the 2D surface of the needs two components, like $\hat{\theta}$ and $\hat{\varphi}$. Therefore, in order to complete the description of vectors we need to complement the basis functions $\vec{X}_{\ell m}$ with functions such as $\hat{r} \times \vec{X}_{\ell m}$. In fact,

$$\vec{X}_{\ell m} \cdot (\hat{r} \times \vec{X}^*_{\ell' m'}) = 0$$

Vector spherical harmonics

Spherical multipoles



Wikipedia

Multipole expansion of EM fields

- We can now collect the TM and TE fields and write the complete solution as a superposition of the two modes.
- In general the radial modes can include both the regular spherical Bessel functions $j_\ell(kr)$ as well as the functions of the second kind, $n_\ell(kr)$ — but if the origin is included in the description, then only the functions of the first kind appear. In full generality we write then:

$$\mathscr{M}_{\ell m}(kr) \to a^{\mathscr{M}}_{\ell m} \left[\alpha_\ell j_\ell(kr) + \beta_\ell n_\ell(kr)\right] \quad , \text{ and analogously for the TE modes .}$$

- It is almost always the case that the radiation is produced in terms of spherical waves that propagate outwards from the source, $e^{-i(\omega t - kr)}/r$, which means that

$$\mathscr{M}_{\ell m}(kr) \to a^{\mathscr{M}}_{\ell m} h^{(1)}_\ell(kr) \quad , \text{ and } \quad \mathscr{E}_{\ell m}(kr) \to a^{\mathscr{E}}_{\ell m} h^{(1)}_\ell(kr)$$

- It is better to redefine the coefficients to allow for the normalizations of the vector spherical harmonics, and in this way we write the multipole expansions:

$$\begin{aligned}
\vec{E}^{(\mathscr{M})} &= e^{-i\omega t} Z_0 \sum_{\ell m} a^{\mathscr{M}}_{\ell m} h^{(1)}_\ell(kr)\, \vec{X}_{\ell m}(\hat{x}) \\
\vec{H}^{(\mathscr{M})} &= e^{-i\omega t} \left(-\frac{i}{k} \vec{\nabla} \times \right) \sum_{\ell m} a^{\mathscr{M}}_{\ell m} h^{(1)}_\ell(kr)\, \vec{X}_{\ell m}(\hat{x}) \;=\; -\, e^{-i\omega t} \frac{i}{k} \sum_{\ell m} a^{\mathscr{M}}_{\ell m} \vec{\nabla} \times \left[h^{(1)}_\ell(kr)\, \vec{X}_{\ell m}(\hat{x})\right] \\
\vec{H}^{(\mathscr{E})} &= e^{-i\omega t} \sum_{\ell m} a^{\mathscr{E}}_{\ell m} h^{(1)}_\ell(kr)\, \vec{X}_{\ell m}(\hat{x}) \\
\vec{E}^{(\mathscr{E})} &= e^{-i\omega t} \left(\frac{i}{k} \vec{\nabla} \times \right) \sum_{\ell m} a^{\mathscr{E}}_{\ell m} h^{(1)}_\ell(kr)\, \vec{X}_{\ell m}(\hat{x}) \;=\; e^{-i\omega t} \frac{i}{k} \sum_{\ell m} a^{\mathscr{E}}_{\ell m} \vec{\nabla} \times \left[h^{(1)}_\ell(kr)\, \vec{X}_{\ell m}(\hat{x})\right]
\end{aligned}$$

- The multipole expansions of the total electric and magnetic fields are, therefore:

$$\vec{E} = \vec{E}^{\mathscr{E}} + \vec{E}^{\mathscr{M}}$$

$$\vec{H} = \vec{H}^{\mathscr{E}} + \vec{H}^{\mathscr{M}}$$

Multipole expansion of EM fields

- Finally, we can go back to our multipole expansion of the sources, and try to write the coefficients of the multipole expansion of the EM fields in terms of the sources.
- The exercise is not too hard (see Jackson, Ch. 9.10) to show that:

$$a^{\mathscr{M}}_{\ell m} = -i \frac{k^2 c}{\ell(\ell+1)} \int d^3x \, Y^*_{\ell m}(\hat{x}) j_\ell(kr) \vec{\nabla} \cdot \left[\vec{r} \times \vec{J}(\vec{x}) \right]$$

$$a^{\mathscr{E}}_{\ell m} = -i \frac{k^2 c}{\ell(\ell+1)} \int d^3x \, Y^*_{\ell m}(\hat{x}) \left\{ \frac{\partial \left[r j_\ell(kr) \right]}{\partial r} \rho(\vec{x}) + i \omega j_\ell(kr) \vec{r} \cdot \vec{J}(\vec{x}) \right\}$$

- Obs: in order to derive this result we need an expansion similar to the Rayleigh expansion of plane waves:

$$\frac{e^{ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|} = 4\pi i \sum_{\ell=0}^{\infty} h^{(1)}_\ell(kx) j_\ell(kx) P_\ell(\hat{x} \cdot \hat{x}')$$

(In fact, this expression can be obtained from the Rayleigh expression.)

- Now this essentially closes the issue of the multipole expansion: how to go from the sources to the fields, and vice-versa. All that is left for us to do is to explore a bit more the power of this spherical wave expansion.

Polarization of EM radiation

- The discussion above means that we have two "modes" of radiation, which we called TE and TM .
- Let's look again at the electric field from the two modes:

$$\vec{E}^{(\mathscr{M})} = e^{-i\omega t} Z_0 \sum_{\ell m} a^{\mathscr{M}}_{\ell m} h^{(1)}_\ell(kr)\, \vec{X}_{\ell m}(\hat{x})$$

$$\vec{E}^{(\mathscr{E})} = e^{-i\omega t} \frac{i}{k} \sum_{\ell m} a^{\mathscr{E}}_{\ell m} \vec{\nabla} \times \left[h^{(1)}_\ell(kr)\, \vec{X}_{\ell m}(\hat{x}) \right]$$

- We already know that the vector spherical harmonics $\vec{X}_{\ell m} \sim \vec{L}\, Y_{\ell m}$ is orthogonal to the direction of propagation, $\hat{x} \cdot \vec{X}_{\ell m} = 0$, so it is obvious that the TM mode is transverse.
- Now consider the orientation of the TE term. In the radiation zone we have that:

$$\vec{\nabla} \times \left[h^{(1)}_\ell(kr)\, \vec{X}_{\ell m}(\hat{x}) \right] \sim \vec{\nabla} \times \left[\frac{e^{ikr}}{kr} \vec{X}_{\ell m}(\hat{x}) \right] \sim \vec{\nabla} \times \left[\frac{e^{ikr}}{kr} \vec{L}\, Y_{\ell m}(\hat{x}) \right]$$

- Opening up this last expression we get:

$$\vec{\nabla} \times \left[\frac{e^{ikr}}{kr} \vec{L}\, Y_{\ell m}(\hat{x}) \right] = \vec{\nabla} \left(\frac{e^{ikr}}{kr} \right) \times \left[\vec{L}\, Y_{\ell m}(\hat{x}) \right] + \frac{e^{ikr}}{kr} \left[\vec{\nabla} \times \vec{L}\, Y_{\ell m}(\hat{x}) \right]$$

- But remember that $\vec{L} = -i\vec{r} \times \vec{\nabla}$ is basically an operator like $\vec{\nabla}_{\hat{x}}$, so $\vec{\nabla} \times \vec{L}(\cdots)$ is basically an operator like $(1/r)\, \vec{\nabla}^2_{\hat{x}}$. Therefore,

$$\vec{\nabla} \times \left[\frac{e^{ikr}}{kr} \vec{L}\, Y_{\ell m}(\hat{x}) \right] = \left(ik\hat{x} \frac{e^{ikr}}{kr} \right) \times \left[\vec{L}\, Y_{\ell m}(\hat{x}) \right] + \mathcal{O}(r^{-2})$$

- Therefore, in the radiation zone the two modes are **orthogonal to $\hat{x}$:** the TM modes are oriented according to $\vec{X}_{\ell m}$, and the TE modes are oriented according to $\hat{x} \times \vec{X}_{\ell m}$.

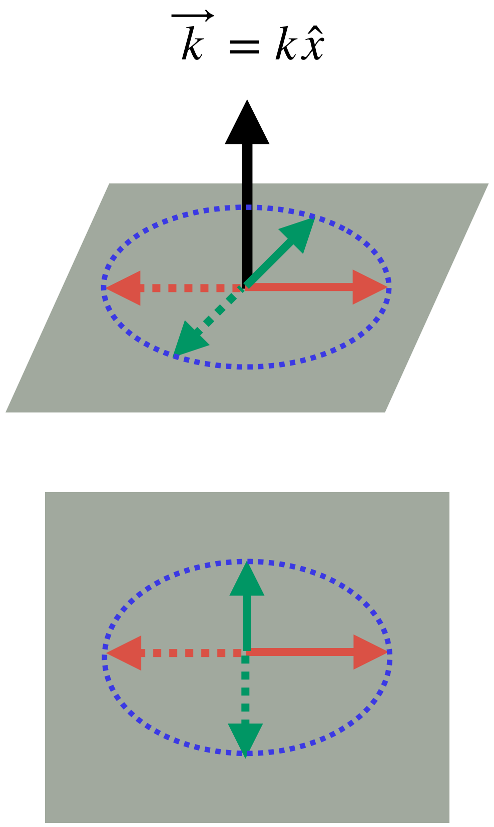
Polarization of EM radiation

- But what are these two independent modes? These are the two independent degrees of freedom of the polarization of the waves!
- The polarization is given by the directions of the electric field (the magnetic field is given implicitly, by the condition that $\vec{E} \times \vec{H} \sim \hat{x}$).
- These expressions are then telling us that in general we can have any superposition of the two independent ("linear") polarizations, each one with their own phases, amplitudes, etc.
- In the radiation zone we can choose any orthogonal basis in the 2D unit sphere to describe the polarization — the vector spherical harmonics may be too complicated! E.g., we can use the Cartesian base as represented in this figure.
- The two modes propagate as:

$$\vec{E} \sim \frac{e^{-i(\omega t - kr)}}{r}\mathscr{E}\,\hat{\epsilon}_1 + \frac{e^{-i(\omega t - kr)}}{r}\mathscr{M}\,\hat{\epsilon}_2$$

$$= \frac{e^{ikr}}{r}\left[e^{-i(\omega t-\psi_1)}A^{(1)}\,\hat{\epsilon}_1 + e^{-i(\omega t-\psi_2)}A^{(2)}\,\hat{\epsilon}_2\right]$$

- This is in general what we call "***elliptic polarization***". Particular cases are ***linear polarization*** (only one of the components is non-zero), and ***circular polarization*** (the two components have equal amplitudes and a phase difference of $\pi/2$).



Energy of the multipole fields

- We are now in a position where we can compute quantities such as the energy in the EM fields in terms of the multipole expansion. In particular, the energy density is:

$$\rho_{EM} = \frac{\epsilon_0}{2} |\vec{E}|^2 + \frac{\mu_0}{2} |\vec{H}|^2 = \frac{\epsilon_0}{4} \left[\vec{E} \cdot \vec{E}* + Z_0^2 \vec{H} \cdot \vec{H}* \right]$$

- In the radiation zone the two terms above are equal, and they fall as $\sim 1/r$.
- Consider first just the transverse electric radiation modes. The energy density in a shell of radius r is given by:

$$4\pi r^2 \rho^{\mathscr{E}} = \frac{dU^{\mathscr{E}}}{dr} = \frac{\mu_0}{2k^2} \int d^2\hat{x} \left[\sum_{\ell m} a^{\mathscr{E}}_{\ell m} \vec{X}_{\ell m} \right] \left[\sum_{\ell' m'} a^{*\mathscr{E}}_{\ell m} \vec{X}^*_{\ell' m'} \right] = \frac{\mu_0}{2k^2} \sum_{\ell m} \left| a^{\mathscr{E}}_{\ell m} \right|^2$$

The same for the energy in the transverse magnetic modes:

$$4\pi r^2 \rho^{\mathscr{M}} = \frac{dU^{\mathscr{M}}}{dr} = \frac{\mu_0}{2k^2} \sum_{\ell m} \left| a^{\mathscr{M}}_{\ell m} \right|^2$$

- The cross-terms are zero because of the orthogonality of the vector spherical harmonics, so the total energy is simply the sum of the two:

$$\frac{dU}{dr} = \frac{\mu_0}{2k^2} \sum_{\ell m} \left(\left| a^{\mathscr{E}}_{\ell m} \right|^2 + \left| a^{\mathscr{M}}_{\ell m} \right|^2 \right)$$

But this is basically the same result we obtained before for the potential of the multipole radiation fields!

$$r^2 |\phi|^2 = \left(\frac{1}{4\pi\epsilon_0} \right)^2 \sum_{\ell m} |\tilde{\rho}_{\ell m}|^2$$

So, we have come "full circle": basically, the multipole expansion allows us to write something like the Parseval Theorem.

In practice, it means that we can expresses the ***power of a source of radiation*** in terms of the ***sum of the powers of each multipole***.

Next class:

- More on radiation
- Field lines and reconnection
- Antennas
- Angular momentum
- Zangwill, Ch. 20