Electromagnetic radiation

⚡Relativistic fields: the Jefimenko and Liénard-Wiechert potentials
⚡The Feynman-Heaviside formulas
⚡Simple radiation fields: dipole radiation

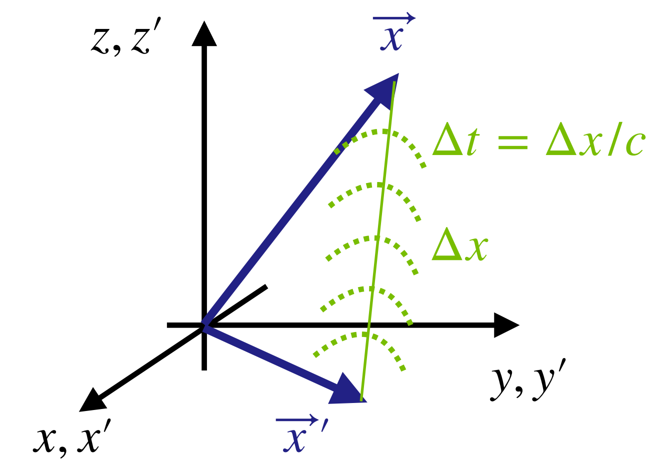
Exact, formal solution to Maxwell's equation

- In our 10th lecture (April 30th) we obtained the ***formal, exact solution*** to the wave equation, $\Box\,\phi = \rho/\epsilon_0$, in terms of the ***retarded Green's function***:

$$G_{Ret}(t,\vec{x};t',\vec{x}') = \frac{1}{4\pi\,\Delta x}\,\delta(t'-t+\Delta x/c_s) \quad , \text{ with } \Delta x = |\vec{x}-\vec{x}'| \text{ , and :}$$

$$\phi(t,\vec{x}) = \int d^3x' \int dt' \frac{\rho(t',\vec{x}')}{\epsilon} G_{Ret}(t,\vec{x};t',\vec{x}') \quad , \text{ resulting in:}$$

$$\phi(t,\vec{x}) = \frac{1}{4\pi\epsilon_0} \int d^3x' \frac{\rho(t_{Ret},\vec{x}')}{|\vec{x}-\vec{x}'|} \quad , \quad \text{where } t_{Ret} = t - \Delta x/c$$



- The interpretation is that, if you move a charge density at some position $\vec{x}'$, the potential at the position $\vec{x}$ will only respond to that change after a time $\Delta x/c$.

- For the vector potential, the equation is exactly the same, and so is the solution:

$$\vec{A}(t,\vec{x}) = \frac{\mu_0}{4\pi} \int d^3x' \frac{\vec{J}(t_{Ret},\vec{x}')}{|\vec{x}-\vec{x}'|}$$

Retarded potentials and fields

- We can now try to compute the electric and magnetic fields from these retarded solutions for the potentials — even if those expressions may not be very useful. These are called the ***Jefimenko fields***.
- By using the Gauss and Ampère laws we can write:

$$\vec{E}(t,\vec{x}) = \frac{1}{4\pi\epsilon_0}\int d^3x' \, \frac{1}{\Delta x}\left[-\vec{\nabla}'\rho(t',\vec{x}') - \frac{1}{c^2}\frac{\partial \vec{J}}{\partial t'}\right]_{t'=t_{Ret}} \quad , \quad \text{and:}$$

$$\vec{B}(t,\vec{x}) = \frac{\mu_0}{4\pi}\int d^3x' \, \frac{1}{\Delta x}\left[\vec{\nabla}'\times\vec{J}\right]_{t'=t_{Ret}}$$

- It is important to notice that we cannot simply evaluate $\vec{\nabla}'f(t'=t_{Ret})$, since $\vec{\nabla}'f(t'=t_{Ret}) \neq [\vec{\nabla}'f]_{t'=t_{Ret}}$. In fact:

$$\vec{\nabla}'[f\big|_{t'=t_{Ret}}] = [\vec{\nabla}'f]_{t'=t_{Ret}} + \left[\frac{\partial f}{\partial t'}\vec{\nabla}'(t'=t_{Ret})\right]_{t'=t_{Ret}} = [\vec{\nabla}'f]_{t'=t_{Ret}} + \left[\frac{\partial f}{\partial t'}\vec{\nabla}'\frac{1}{c}\Delta x\right]_{t'=t_{Ret}} = [\vec{\nabla}'f]_{t'=t_{Ret}} + \frac{\Delta\hat{x}}{c}\left[\frac{\partial f}{\partial t'}\right]_{t'=t_{Ret}} \quad ,$$

where $\Delta\hat{x} = (\vec{x}-\vec{x}')/|\vec{x}-\vec{x}'|$.

- Therefore, we have that:

$$[\vec{\nabla}'\rho]_{t'=t_{Ret}} = \vec{\nabla}'[\rho(t'=t_{Ret})] - \frac{\Delta\hat{x}}{c}\left[\frac{\partial\rho}{\partial t'}\right]_{t'=t_{Ret}} \quad , \quad \text{and}$$

$$[\vec{\nabla}'\times\vec{J}]_{t'=t_{Ret}} = \vec{\nabla}'\times[\vec{J}(t'=t_{Ret})] - \frac{\Delta\hat{x}}{c}\times\left[\frac{\partial\vec{J}}{\partial t'}\right]_{t'=t_{Ret}}$$

We can now substitute these back into the equations for $\vec{E}$ and $\vec{B}$ at the top of the page.

The Jefimenko fields

- This is the result of making those substitutions, first for the electric field:

$$\begin{aligned}
\vec{E}(t,\vec{x}) &= \frac{1}{4\pi\epsilon_0}\int d^3x' \, \frac{1}{\Delta x}\left[-\vec{\nabla}'[\rho(t'=t_{Ret})] + \frac{\Delta\hat{x}}{c}\left[\frac{\partial \rho}{\partial t'}\right]_{t'=t_{Ret}} - \frac{1}{c^2}\frac{\partial \vec{J}}{\partial t'}\bigg|_{t'=t_{Ret}} \right] \\
&= \frac{1}{4\pi\epsilon_0}\int d^3x' \left[-\vec{\nabla}'\left[\frac{1}{\Delta x}\rho(t'=t_{Ret})\right] + \left[\vec{\nabla}'\frac{1}{\Delta x}\right]\rho(t'=t_{Ret}) + \frac{\Delta\hat{x}}{c\,\Delta x}\left[\frac{\partial \rho}{\partial t'}\right]_{t'=t_{Ret}} - \frac{1}{c^2}\frac{1}{\Delta x}\frac{\partial \vec{J}}{\partial t'}\bigg|_{t'=t_{Ret}} \right] \\
&= 0 + \frac{1}{4\pi\epsilon_0}\int d^3x' \left[\frac{\Delta\hat{x}}{\Delta x^2}\rho(t'=t_{Ret}) + \frac{\Delta\hat{x}}{c\,\Delta x}\left[\frac{\partial \rho}{\partial t'}\right]_{t'=t_{Ret}} - \frac{1}{c^2}\frac{1}{\Delta x}\frac{\partial \vec{J}}{\partial t'}\bigg|_{t'=t_{Ret}} \right]
\end{aligned}$$

where we integrated out the surface term $\rho/\Delta x$.

- Similarly, for the magnetic field we obtain the expression:

$$\vec{B}(t,\vec{x}) = \frac{\mu_0}{4\pi}\int d^3x' \left[\vec{J}(t'=t_{Ret}) \times \frac{\Delta\hat{x}}{\Delta x^2} - \frac{\Delta\hat{x}}{c\,\Delta x} \times \left[\frac{\partial \vec{J}}{\partial t'}\right]_{t'=t_{Ret}} \right]$$

The Jefimenko fields

- Let's collect the two results again here:

$$\vec{E}(t,\vec{x}) = \frac{1}{4\pi\epsilon_0}\int d^3x' \left[\frac{\Delta\hat{x}}{\Delta x^2}\rho(t'=t_{Ret}) + \frac{\Delta\hat{x}}{c\,\Delta x}\left[\frac{\partial\rho}{\partial t'}\right]_{t'=t_{Ret}} - \frac{1}{c^2}\frac{1}{\Delta x}\frac{\partial\vec{J}}{\partial t'}\bigg|_{t'=t_{Ret}} \right]$$

$$\vec{B}(t,\vec{x}) = \frac{\mu_0}{4\pi}\int d^3x' \left[\vec{J}(t'=t_{Ret}) \times \frac{\Delta\hat{x}}{\Delta x^2} - \frac{\Delta\hat{x}}{c\,\Delta x} \times \left[\frac{\partial\vec{J}}{\partial t'}\right]_{t'=t_{Ret}} \right]$$

- We can now see clearly the structure of these fields: there are the ***static parts of the solutions***,

$$\vec{E}_{Stat} = \frac{1}{4\pi\epsilon_0}\int d^3x' \,\rho(t'=t_{Ret}) \left(\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3} \right)$$

$$\vec{B}_{Stat} = \frac{\mu_0}{4\pi}\int d^3x' \,\vec{J}(t'=t_{Ret}) \times \left(\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3} \right)$$

which are just the ***generalizations*** of the ***electrostatic solution*** from Gauss's law (i.e., Coulomb's law), and the ***magnetostactic solution*** from Ampère's law (i.e., the Biot-Savart law)!

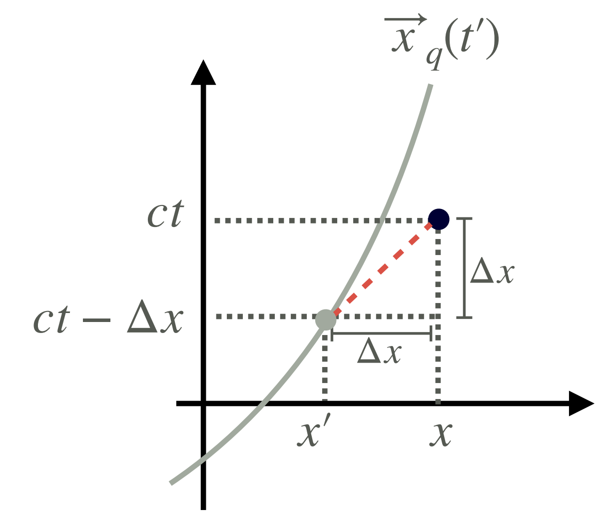
Point charges: Feynman-Heaviside fields

- It is interesting to specialize the previous results to point charges,

$$\rho(t', \vec{x}') = q\delta[\vec{x}' - \vec{x}_q(t')] \quad , \quad \text{and}$$

$$\vec{J}(t', \vec{x}') = q\,\vec{v}_q(t')\,\delta[\vec{x}' - \vec{x}_q(t')] \quad , \quad \text{with } \vec{v}_q(t') = \dot{\vec{x}}_q(t')$$

- Before we proceed, it is important to notice that the Dirac delta-function above carries an ***implicit dependence on the position*** $\vec{x}'$ ***also through the retarded time***, since the position (and velocity) of the point charge is evaluated at $\vec{x}_q\,(t' = t - \Delta x/c)$.



- This means that any integral over d^3x' of that distribution should take into account the position $\vec{x}'$ where the charge ***was*** at the retarded time — see the figure.
- In other words, the ***zero*** of the argument of the delta function is the solution of:

$$\vec{x}' - \vec{x}_q\left(t - |\vec{x}' - \vec{x}|/c\right) \;=\; 0 \quad .$$

- Let's express the argument of the Dirac delta-function using $\Delta\vec{x} = \vec{x} - \vec{x}'$, so:

$$\vec{x}' - \vec{x}_q\left(t - |\vec{x}' - \vec{x}|/c\right) \;=\; \vec{x} - \Delta\vec{x} - \vec{x}_q(t - \Delta x/c) \;=\; 0 \quad .$$

- Changing variables in the DIrac delta function from $\vec{x}'$ to $\Delta\vec{x}$ yields:

$$\delta\left[\vec{x}' - \vec{x}_q\left(t - |\vec{x}' - \vec{x}|/c\right)\right] \;=\; \delta\left[\vec{x} - \Delta\vec{x} - \vec{x}_q(t - \Delta x/c)\right] \times \frac{1}{\left|\vec{\nabla}_{\Delta x}\left[\vec{x} - \Delta\vec{x} - \vec{x}_q(t - \Delta x/c)\right]\right|}$$

$$= \delta\left[\vec{x} - \Delta\vec{x} - \vec{x}_q(t - \Delta x/c)\right] \times \frac{1}{1 - \vec{v}_q(t_{Ret}) \cdot \Delta\hat{x}/c}$$

> Another way to see how this factor appears is to realize that $t' = t - |\vec{x} - \vec{x}_q(t')|/c$, so:
>
> $$\frac{dt}{dt'} = \kappa = 1 - \Delta\hat{x} \cdot \vec{v}_q/c$$

Point charges: Feynman-Heaviside fields

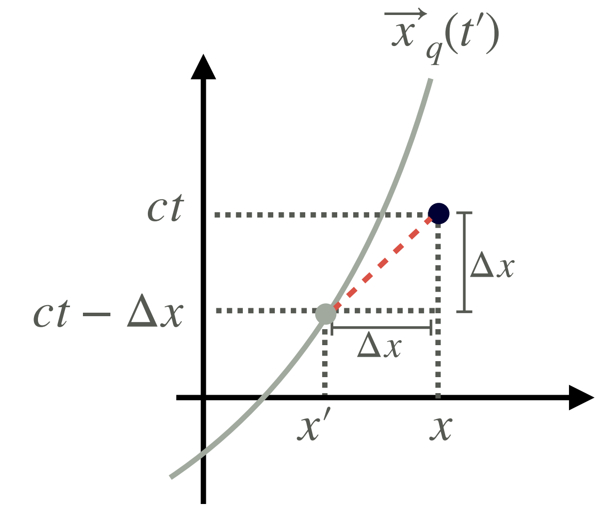
- The bottom line is that an integral of a DIrac delta function including the retarded time leads to an extra factor of

$$\frac{1}{\kappa} = \frac{1}{1 - \vec{v}_q(t_{Ret}) \cdot \Delta\hat{x}/c}$$

- That's basically it for the complication introduced for point charges due to the retardation. We can now write the electric and magnetic fields as:

$$\vec{E}(t,\vec{x}) = \frac{q}{4\pi\epsilon_0}\left\{\left[\frac{\Delta\hat{x}}{\kappa\Delta x^2}\right]_{Ret} + \frac{\partial}{c\,\partial t}\left[\frac{\Delta\hat{x}}{\kappa\Delta x}\right]_{Ret} - \frac{\partial}{c^2\partial t}\left[\frac{\vec{v}_q}{\kappa\,\Delta x}\right]_{Ret}\right\}$$

$$\vec{B}(t,\vec{x}) = \frac{\mu_0 q}{4\pi}\left\{\left[\frac{\vec{v}_q \times \Delta\hat{x}}{\kappa\Delta x^2}\right]_{Ret} + \frac{\partial}{c\,\partial t}\left[\frac{\vec{v}_q \times \Delta\hat{x}}{\kappa\Delta x}\right]_{Ret}\right\}$$



- With some careful manipulations of these derivatives, these expressions can be cast into the form:

$$\vec{E}(t,\vec{x}) = \frac{q}{4\pi\epsilon_0}\left\{\left[\frac{\Delta\hat{x}}{\Delta x^2}\right]_{Ret} + \Delta x_{Ret}\frac{\partial}{c\,\partial t}\left[\frac{\Delta\hat{x}}{\Delta x^2}\right]_{Ret} - \frac{\partial^2}{c^2\partial t^2}\left[\Delta\hat{x}\right]_{Ret}\right\}$$

$$\vec{B}(t,\vec{x}) = \frac{\mu_0 q}{4\pi}\left\{\left[\frac{\vec{v}_q \times \Delta\hat{x}}{\kappa^2\Delta x^2}\right]_{Ret} + \frac{1}{\Delta x_{Ret}}\frac{\partial}{c\,\partial t}\left[\frac{\vec{v}_q \times \Delta\hat{x}}{\kappa}\right]_{Ret}\right\}$$

- The beauty of these expressions is the fact that you can see now see clearly the ***radiative part of the electric field***, which ***does not fall*** with $1/\Delta x^2$, and is sourced by an ***acceleration*** term, $\partial^2/\partial t^2(\ldots)$. [There is a corresponding radiative term for the magnetic field, but it is harder to see it independently.]

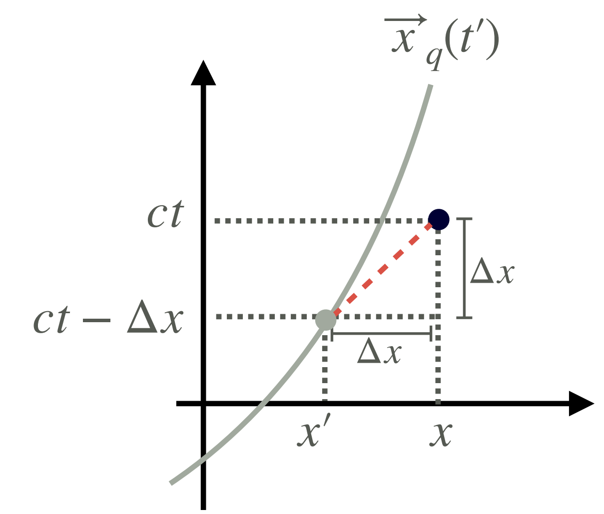
Point charges: Liénard-Wiechert potentials

- The same calculation that we performed above for the fields can be made for the potentials, and the results for the electric potential and vector potential are:

$$\phi(t,\vec{x}) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\Delta x}\frac{1}{\kappa}\right]_{Ret} = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\Delta x - \vec{\beta}_q\cdot\Delta\vec{x}}\right]_{Ret} \quad ,$$

where $\vec{\beta}_q = \vec{v}_q/c$, and

$$\vec{A}(t,\vec{x}) = \frac{q\,\mu_0}{4\pi}\left[\frac{\vec{v}_q}{\Delta x - \vec{\beta}_q\cdot\Delta\vec{x}}\right]_{Ret}$$



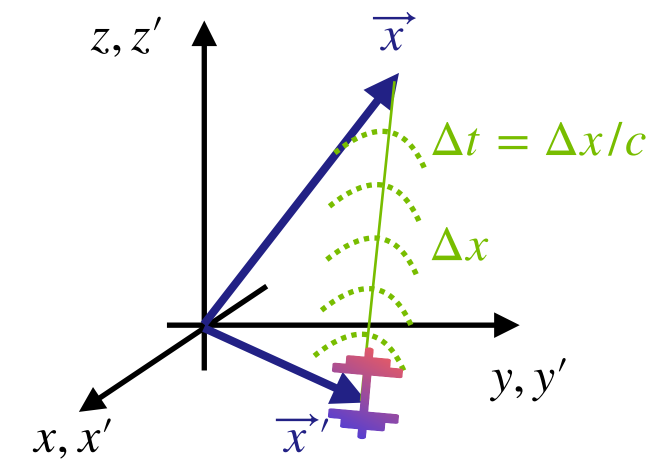
- These are the ***Liénard-Wiechert potentials***, from which it is trivial to get the non-relativistic limits.
- It is also useful to write the Liénard-Wiechert 4-vector potential, which can be written in an elegant and ***manifestly covariant form*** as :

$$A^\mu(x) = \frac{q\,\mu_0\,c}{4\pi}\left[\frac{U^\mu_q}{U_{q\,,\,\alpha}\,\Delta x^\alpha}\right]_{Ret}$$

(for details , see Jackson, Ch. 14.1)

Electromagnetic radiation

- Let's now compute some actual radiation fields from moving charges.
- The first, simplest case is known as ***electric dipole radiation***: we basically "shake" a charge up and down, and obtain electromagnetic radiation. So, in the expression:



$$\phi(t, \vec{x}) = \frac{1}{4\pi\epsilon_0} \int d^3x' \, \frac{\rho(t_{Ret}, \vec{x}')}{\Delta x} \quad , \quad t_{Ret} = t - \Delta x/c \quad ,$$

we use the charge and current density distributions:

$$\rho(t', \vec{x}') = \sum_\omega \rho_\omega(\vec{x}')\, e^{-i\omega t'} \quad , \text{ and}$$

$$\vec{J}(t', \vec{x}') = \sum_\omega \vec{J}_\omega(\vec{x}')\, e^{-i\omega t'} \quad , \text{ with the two related by the continuity equation:}$$

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0 \quad \Rightarrow \quad -i\omega\rho_\omega + \vec{\nabla} \cdot \vec{J}_\omega = 0$$

- So, e.g., the vector potential becomes:

$$\vec{A}(t, \vec{x}) = \sum_\omega \frac{\mu_0}{4\pi} \int d^3x' \, \frac{\vec{J}_\omega(\vec{x}')}{\Delta x} \, e^{-i\omega(t - \Delta x/c)}$$

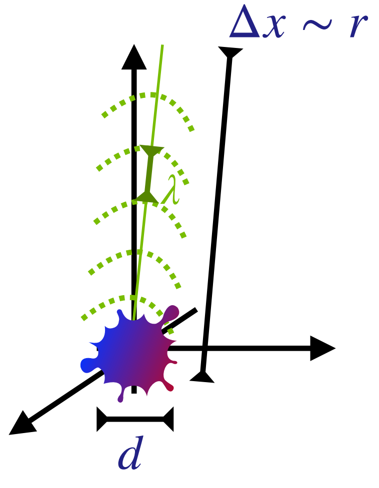
$$= \sum_\omega \frac{\mu_0}{4\pi} \, e^{-i\omega t} \int d^3x' \, \frac{\vec{J}_\omega(\vec{x}')}{\Delta x} \, e^{i\omega \Delta x/c}$$

- Therefore, our integral is now only in terms of expressions involving $\Delta x = |\vec{x}' - \vec{x}|$.

Near, intermediate and far zones

- Integrals such as this as very hard to compute exactly, but they are ideally suited for approximation schemes. The basic idea is to look at the "hard" part of the integrand, assuming that our sources are near the origin:

$$\frac{e^{ik\Delta x}}{\Delta x} \quad , \qquad \text{where } k = \omega/c = 2\pi/\lambda \ .$$



- We can imagine three "regions" of decreasing difficulty (here I write $|\vec{x}| = r$):

Near zone: $d \ll r \quad , \quad r \ll \lambda$

Intermediate zone: $d \ll r \quad , \quad r \sim \lambda$

Radiaton zone: $d \ll r \quad , \quad r \gg \lambda$

- The main approximation that we can do is to assume that we are in the radiation (or "far" zone), such that $kx \gg 1$, and $x' \ll x$. In that case we can write:

$$\Delta x = |\vec{x} - \vec{x}'| = r\sqrt{1 - (x'/r)^2 - 2(x'/r)\,\hat{x}\cdot\hat{x}'}$$

$$\simeq r\left(1 + \frac{1}{2}\frac{x'^2}{r^2} - \frac{x'}{r}\,\hat{x}\cdot\hat{x}' + \ldots\right)$$

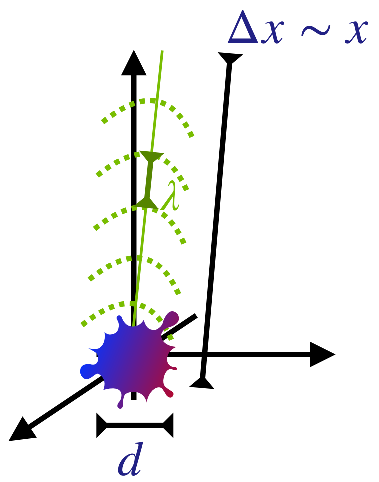
$$\simeq r - x'(\hat{x}\cdot\hat{x}') + \ldots$$

Near, intermediate and far zones

- This means that we can now write each Fourier component of the fields as $\vec{A} = \sum_\omega \vec{A}_\omega e^{-i\omega t}$, in the radiation zone, as:

$$\vec{A}_\omega \simeq \frac{\mu_0}{4\pi} e^{ikr} \int d^3x' \frac{e^{-i\vec{k}\cdot\vec{x}'}}{r - \hat{x}\cdot\vec{x}'} \vec{J}_\omega(\vec{x}') \quad , \quad \text{where } \vec{k} \equiv k\hat{x}$$

[Notice this import definition!]



- Notice that the factor in the exponential $e^{-i\vec{k}\cdot\vec{x}'}$ can vary quite rapidly, while the small term in the denominator is sub-dominant. Therefore, in the radiation zone we can further approximate the integral above to:

$$\vec{A}_\omega \simeq \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \int d^3x' \, e^{-i\vec{k}\cdot\vec{x}'} \vec{J}_\omega(\vec{x}')$$

, and the same for the potential:

$$\phi_\omega \simeq \frac{1}{4\pi\epsilon_0} \frac{e^{ikr}}{r} \int d^3x' \, e^{-i\vec{k}\cdot\vec{x}'} \rho_\omega(\vec{x}')$$

- We will in fact see this type of solution again when we deal with the theory of scattering and diffraction of light.
- But for now we arrive at our main result here, which is that in the radiation zone the potentials are just the Fourier transforms of the sources:

$$\vec{A}_\omega \simeq \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \tilde{\vec{J}}_\omega(\vec{x}') \quad , \text{ and}$$

$$\phi_\omega \simeq \frac{1}{4\pi\epsilon_0} \frac{e^{ikr}}{r} \tilde{\rho}_\omega(\vec{x}')$$

Solutions in the radiation (far) zone

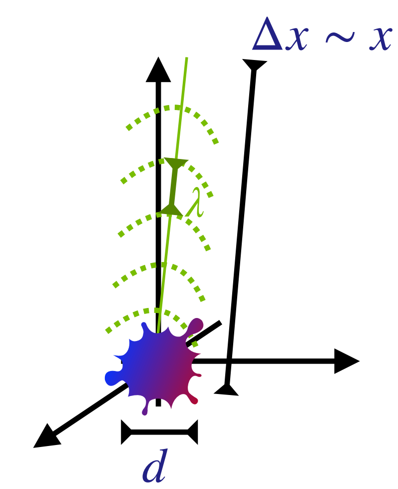
- Now we can go even further, and simplify this expression even more. Let's assume that the phase in the Fourier integral is always small, i.e., that

$$\vec{k} \cdot \vec{x}' = \frac{\omega}{c} x' \hat{x} \cdot \hat{x}' \quad \rightarrow \quad \frac{\omega\, d}{c} \cos\theta \ll 1$$

i.e., that the typical ***dimensions of the source*** are ***smaller than the wavelengths*** of the radiation, $\lambda = 2\pi c/\omega$.

This will happen if we move the source slowly: e.g., for a frequency of $10^3\, s^{-1} = 10^3\,\mathrm{Hz}$ we get a wavelength of thousands of kilometers. For light on the GHz scale, the wavelengths are still of the order of cm — much larger than the sizes of the atoms/molecules that radiate that light!

In other words, this is a damn good approximation!



- In that case, we have that the radiation is well described by:

$$\vec{A}_\omega \simeq \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \int d^3x' \left(1 - i\vec{k} \cdot \vec{x}' + \ldots \right) \vec{J}_\omega(\vec{x}') \quad , \text{ and}$$

$$\phi_\omega \simeq \frac{1}{4\pi\epsilon_0} \frac{e^{ikr}}{r} \int d^3x' \left(1 - i\vec{k} \cdot \vec{x}' + \ldots \right) \rho_\omega(\vec{x}')$$

- Let's look at all these terms now, and see what we get for each of these four terms.

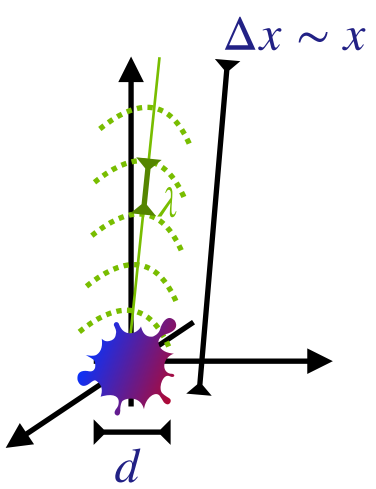
Solutions in the radiation (far) zone

- The very first term in the electric potential is the simple integral:

$$\phi_\omega \simeq \frac{1}{4\pi\epsilon_0}\frac{e^{ikr}}{r}\int d^3x'\,\rho_\omega(\vec{x}') = \frac{1}{4\pi\epsilon_0}\frac{e^{ikr}}{r}Q_\omega$$

But this is simply the total charge, which for any source must be ***constant,*** $\omega = k = 0$!

This solution therefore belongs to the realm of electrostatics — it is simply the potential ~ Q/r for a "point" charge as seen from "far away"!



- Now let's look at the first term in the vector potential:

$$\vec{A}_\omega \simeq \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int d^3x'\,\vec{J}_\omega(\vec{x}')$$

We can rewrite this integral by means of partial integrations, as:

$$A_{\omega,i} \simeq \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int d^3x'\,J_{\omega,i}(\vec{x}') = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int d^3x'\left[\partial'_i\left(x'_jJ_{\omega,i}\right) - x'_j\left(\partial'_iJ_{\omega,i}\right)\right]$$

$$= \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\left[\int d^2S'^i\left(x'_jJ_{\omega,i}\right) - \int d^3x'\,x'_j\left(i\omega\rho_\omega\right)\right]$$

$$\Rightarrow \quad \vec{A}_\omega = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}(-i\omega)\int d^3x'\,\vec{x}'\rho_\omega(\vec{x}') = -\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}i\omega\vec{p}_\omega$$

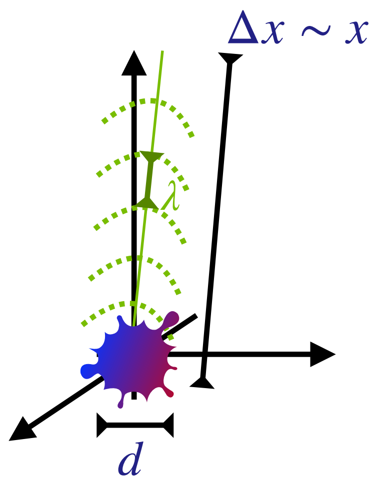
- So, this term corresponds to an ***electric dipole*** $\vec{p}_\omega$, which ***oscillates*** with a period $T = 2\pi/\omega$!

Dipole radiation

- The ***first non-vanishing term in the electric potential*** is in fact the counterpart in ϕ for the same oscillating dipole that we have just found for the vector potential:

$$\phi_\omega \simeq \frac{1}{4\pi\epsilon_0}\frac{e^{ikr}}{r}\int d^3x'\,(-i\vec{k}\cdot\vec{x}')\,\rho_\omega(\vec{x}') = \frac{c^2\mu_0}{4\pi}\frac{e^{ikr}}{r}\left(-i\frac{\omega}{c}\hat{x}\right)\cdot\vec{p}_\omega$$

$$\Rightarrow \quad \phi_\omega = c\hat{x}\cdot\vec{A}_\omega$$

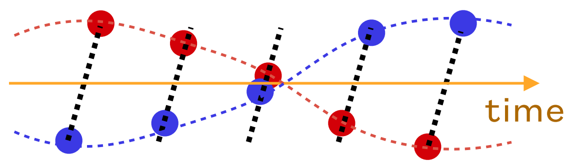


- These first non-vanishing terms for ϕ and $\vec{A}$ are called ***electric dipole radiation*** potentials:

$$\vec{A}^{E-dip}_\omega = -\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\,i\omega\vec{p}_\omega \quad , \text{ and}$$

$$\phi^{E-dip}_\omega = c\hat{x}\cdot\vec{A}_\omega$$

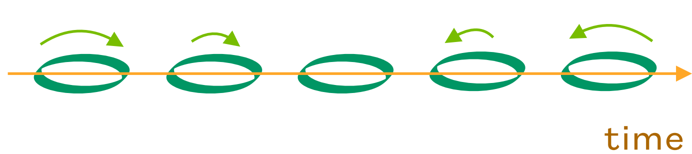
They correspond to the simplest possible situation: a neutral pair of charge that moves oscillates around a center.



- The next order term in the vector potential is called ***magnetic dipole radiation***:

$$\vec{A}_\omega \simeq \vec{A}^{E-dip}_\omega + \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int d^3x'\,(-i\vec{k}\cdot\vec{x}')\,\vec{J}_\omega(\vec{x}')$$

$$\Rightarrow \quad A^{M-dip}_{\omega\,,i} = -\,i\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}k_j\int d^3x'\;x'_j\,J_{\omega\,,i}$$



Magnetic dipole radiation can be pictured as a loop with a current that oscillates clockwise/counterclockwise.

Radiation potentials and radiation fields

- Let's now compute the actual electric and magnetic fields generated by these radiation potentials For this we will use the following notation:

$$\vec{H}_\omega = \frac{1}{\mu_0} \vec{\nabla} \times \vec{A}_\omega \quad , \text{ and}$$

$$\vec{E}_\omega = \frac{i}{\omega} \frac{\mu_0}{\epsilon_0} \vec{\nabla} \times \vec{H}_\omega = \frac{i}{\omega} Z_0 \vec{\nabla} \times \vec{H}_\omega$$

where Z_0 is the *vacuum impedance*.

- Obviously, the "monopole" term of the electric potential is not included in the relation above, since it just gives us the electrostatic field, $\vec{E} = Q/(4\pi\epsilon_0)\, \vec{r}/r^3$.
- The electric dipole radiation fields follow from a simple calculation from the formulae above:

$$\vec{H}_\omega^{E-dip} = \frac{1}{\mu_0} \vec{\nabla} \times \vec{A}_\omega^{E-dip} = \frac{1}{\mu_0} \vec{\nabla} \times \left[-\frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} i\omega \vec{p}_\omega \right] \quad , \text{ from which we obtain:}$$

$$\vec{H}_\omega^{E-dip} = \frac{k^2 c}{4\pi} \frac{e^{ikr}}{r} \left(1 + \frac{i}{kr} \right) \hat{x} \times \vec{p}_\omega \quad , \text{ and}$$

$$\vec{E}_\omega^{E-dip} = \frac{1}{4\pi\epsilon_0} \frac{e^{ikr}}{r} \left[\left(3\hat{x}\, (\hat{x} \cdot \vec{p}_\omega) - \vec{p}_\omega \right) \frac{1 - ikr}{r^2} - k^2\, \hat{x} \times (\hat{x} \times \vec{p}_\omega) \right]$$

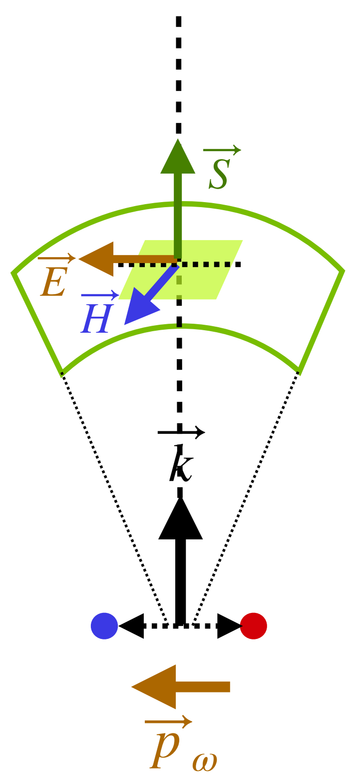
Radiation potentials and radiation fields

- But let's remember that we have been assuming that we are in the radiation zone, i.e., $r \gg \lambda$, $kr \gg 1$. Therefore, coming back to our definition $\vec{k} = (\omega/c)\hat{x}$, we obtain that:

$$\vec{H}_\omega^{E-dip} \simeq \frac{k\,c}{4\pi}\frac{e^{ikr}}{r}\,\vec{k}\times\vec{p}_\omega \quad , \text{ and}$$

$$\vec{E}_\omega^{E-dip} \simeq -\frac{1}{4\pi\epsilon_0}\frac{e^{ikr}}{r}\,\vec{k}\times(\vec{k}\times\vec{p}_\omega) \quad , \text{ i.e.,}$$

$$\vec{E}_\omega^{E-dip} = -\frac{Z_0}{k}\,\vec{H}_\omega^{E-dip}\times\vec{k}$$

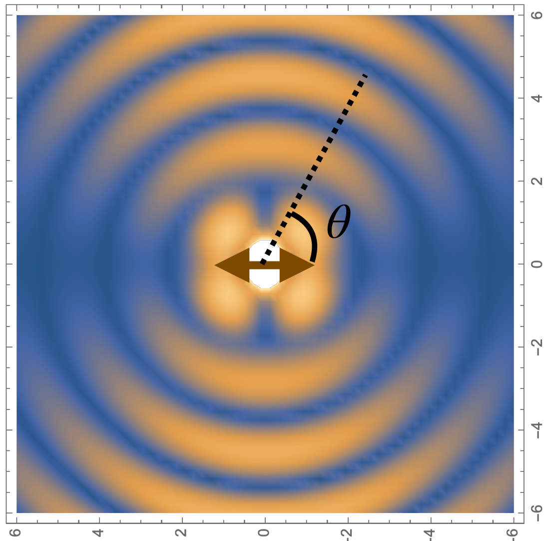


- So, it is quite amazing that the orthogonality of the magnetic and electric radiation fields arise ***naturally*** from the equations! In fact, with these results we can already write the ***Poynting vector*** corresponding to this dipole radiation:

$$\vec{S}_\omega^{E-dip} = \vec{E}_\omega^{E-dip}\times\vec{H}_\omega^{E-dip} = Z_0\,|\,\vec{H}_\omega^{E-dip}|^2\,\hat{k}$$

- The ***radiated power*** over some surface is:

$$P_\omega = \int d\vec{A}\cdot\vec{S}_\omega^{E-dip} \quad\Rightarrow\quad \langle P_\omega\rangle_t = \frac{Z_0\,k^4\,c^2}{32\pi^2}p_\omega^2\,\sin^2\theta$$



(See Mathematica notebook)

Next class:

- Radiation from accelerated charges — fully relativistic calculation
- Spherical wave solutions, partial waves and the spherical Bessel functions
- Multipole expansion for freely propagating electromagnetic fields
- Jackson, Ch. 9