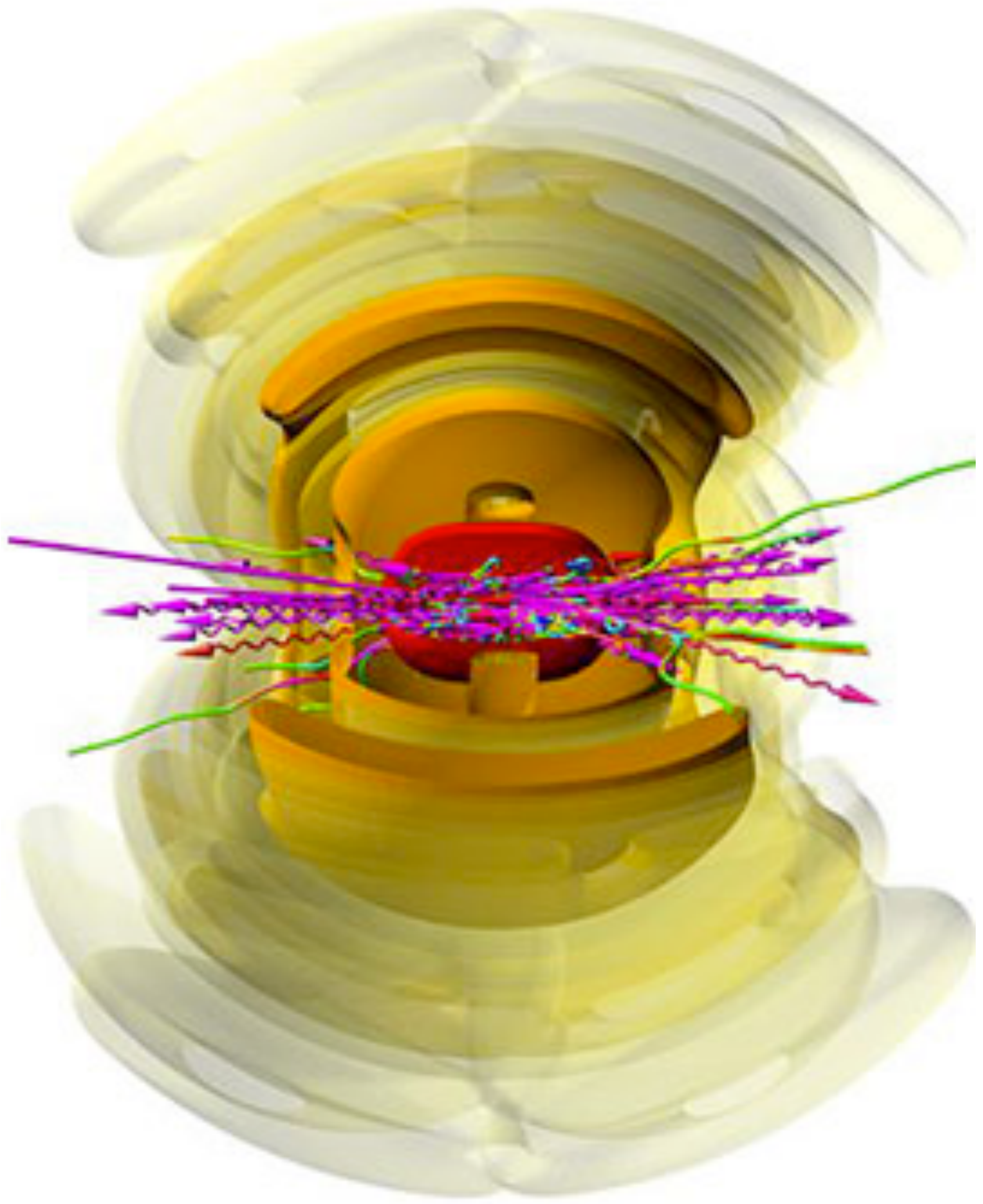
Relativistic Electrodynamics

- ⚡ Relativistic actions/Lagrangians
- ⚡ The Maxwell Lagrangian
- ⚡ Stress-energy tensor: another derivation
- ⚡ Back to Maxwell's equations



Relativistic electrodynamics

- During our exploration of Electrodynamics and its relation to Relativity, we saw that, in **covariant notation**: the Maxwell equation are written as:

$\partial_\nu F^{\mu\nu} = \mu_0 J^\mu$, with $\partial_\mu J^\mu = 0$ being an integrability condition

$\partial_\nu F^{*\mu\nu} = 0$, where $F^{*\mu\nu} = \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}$ is the Hodge dual of $F^{\mu\nu}$.

- The Faraday tensor includes both the electric and the magnetic fields:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$F^{\mu\nu} = \begin{array}{cc} & \xrightarrow{\hspace{6em}} \nu \\ \downarrow \mu & \begin{pmatrix} 0 & +E_x/c & +E_y/c & +E_z/c \\ -E_x/c & 0 & +B_z & -B_y \\ -E_y/c & -B_z & 0 & +B_x \\ -E_z/c & +B_y & -B_x & 0 \end{pmatrix} \end{array} \qquad F_{\mu\nu} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & +B_z & -B_y \\ E_y/c & -B_z & 0 & +B_x \\ E_z/c & +B_y & -B_x & 0 \end{pmatrix}$$

Relativistic electrodynamics

- The stress-energy tensor, which determines the intrinsic properties of the electric and magnetic fields, such as energy, energy flux/momentum, pressure, stresses, etc., is given by:

$$T^{\alpha}_{\ \mu} = -\frac{1}{\mu_0}\left[F^{\alpha\nu}F_{\nu\mu} + \frac{1}{4}\,\delta^{\alpha}_{\mu}\,F^2\right] = \frac{1}{\mu_0}\left[F^{\alpha\nu}F_{\mu\nu} - \frac{1}{4}\,\delta^{\alpha}_{\mu}\,F^2\right] \quad , \quad \text{or}$$

$$T^{\alpha\mu} = \frac{1}{\mu_0}\left[F^{\alpha\nu}F^{\mu}_{\ \nu} - \frac{1}{4}\,\eta^{\alpha\mu}\,F^2\right] \quad , \quad \text{where recall that } F^2 = -2\left(\frac{\vec{E}^2}{c^2} - \vec{B}^2\right)$$

- We also showed that some of the components of this stress-energy tensor are familiar to us: e.g., the electromagnetic energy density is given by:

$$T^{00} = \rho_{EM} = \frac{1}{2}\left(\epsilon_0 \vec{E}^2 + \frac{1}{\mu_0}\vec{B}^2\right)$$

- Therefore, the ***stress-energy tensor has dimensions of energy density***.

- In fact, ***the scalar*** $\frac{1}{\mu_0}F^2$ ***has dimensions of energy density***. Can we construct ***other scalars*** using only **quadratic combinations** of the Faraday tensor? We could try, e.g.,

$$F^{*\mu\nu}F_{\mu\nu} = \epsilon^{\mu\nu\alpha\beta}\,F_{\alpha\beta}F_{\mu\nu} \quad , \quad \text{but this is } \textit{identically zero}\text{! (Show!!)}$$

- So, we are stuck with this quantity F^2/μ_0 as the ***only*** scalar function of $F_{\mu\nu}$, and which has units of energy density

Relativistic actions and Lagrangians

- Our goal now is to derive Maxwell's equations (and the stress-energy tensor) from an action principle, and a Lagrangian.
- So, let's talk about relativistic actions and Lagrangians.
- We can start with a point particle with mass m. In non-relativistic mechanics, we have that the Lagrangian of a point particle is simply:

$$L = \frac{1}{2} m \vec{\dot{x}}^2 - U(\vec{x}) \quad , \quad \text{so the action is}$$

$$S = \int dt\, L$$

- Minimization of this action (assuming fixed start/end points) leads to the dynamical equations:

$$\delta S = 0 \quad \Rightarrow \quad \frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{x}^i} - \frac{\partial L}{\partial x^i} = 0 \quad , \quad \text{which we can also write as:}$$

$$m \vec{\ddot{x}} = - \vec{\nabla} U = \vec{F}$$

- But what about the relativistic particle?

Relativistic actions and Lagrangians

- We can get inspiration from the fact that the action must be an invariant: the equations of motion must have the same form in any reference frame. So, "coordinate time" is out (it is ***not*** invariant), and we can try instead "proper time" (which ***is*** an invariant).

- Therefore, we should try something like this:

$$S = \int dt\, L \quad \longrightarrow \quad \int d\tau\, L$$

- But what is this Lagrangian now? Notice that the action has dimensions of energy × time. But what is the only scalar that one can construct using (mass)×(speed)^2 ?

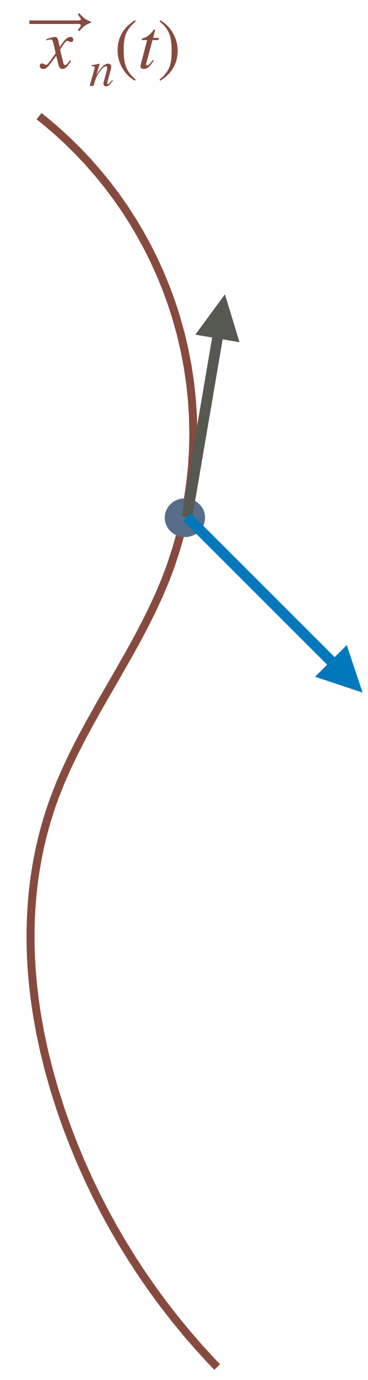
$$\frac{1}{2} m \dot{\vec{x}}^2 \quad \longrightarrow \quad m\, c^2$$

- When we combine the factor $1/\gamma(v)$ from $d\tau = dt/\gamma(v)$ we get:

$$\frac{1}{\gamma(v)} mc^2 = \sqrt{1 - \frac{\vec{v}^2}{c^2}}\; mc^2 \simeq mc^2 - \frac{1}{2} m \vec{v}^2 + \ldots$$

$$\Rightarrow \quad S_{free} = - \int d\tau\, m\, c^2$$

Relativistic actions and Lagrangians



- Ok, but how do we include the ***interactions*** of these particles?
- Usually, we think of potentials a scalar, and the force as $\vec{F} = -\vec{\nabla} U$.
- However, for the EM fields, the potential is a 4-vector! Moreover, the force is the Lorentz force, which, as we saw earlier, can be expressed in covariant notation as:

$$f_\mu = \frac{d\,p_\mu}{d\tau} = q\,F_{\mu\nu}\,U^\nu \quad \text{for a point charge, and}$$

$$\frac{d\,p_\mu}{d^4x} = F_{\mu\nu}\,J^\nu \qquad \text{for a distribution of charge/current densities}$$

- So, in some sense the Lorentz force (F) must come out of something like a derivative of the potential (A) in a way that involves also the 4-velocity. Moreover, that term must be linear in the action!
- Now, what is the only scalar that we can construct using the 4-potential, that can go into some relativistic action? Clearly, the only option is that, for a point charge q, we have something like:

$$q\int A_\mu\,dx^\mu$$

- Notice that $q\,A$ has dimensions of energy (potential energy), and dx has dimensions of length.
- But the action has dimensions of (energy) × (time) , so we must somehow write (energy) × (space) × (time/space). Obviously, the only universal constant which has units of (time/space) is the $1/c$! Therefore, we get that the potential energy must be something like:

$$\frac{q}{c}\int A_\mu\,dx^\mu \qquad , \qquad \text{and we will show soon that this is exactly right!}$$

Action, Lagrangian and Hamiltonian

- So, collecting everything now we obtain:

$$S_{free} + S_{int} = -\int d\tau\, m\, c^2 + \frac{q}{c}\int A_\mu\, dx^\mu = -\int d\tau\, m\, c^2 + \frac{q}{c}\int \left(-\phi\, cdt + \vec{A}\cdot d\vec{x}\right)$$

$$= \int dt \left[-\frac{1}{\gamma(v)} m\, c^2 - q\phi + q\vec{A}\cdot\frac{\vec{v}}{c}\right]$$

- It is interesting that, from the point of view of the dynamical equations, the ***conjugate momentum*** is now given by:

$$P_i = \frac{\partial L}{\partial \dot{x}^i} = \frac{\partial}{\partial \dot{x}^i}\left[-\frac{1}{\gamma(v)} m\, c^2 - q\phi + q\vec{A}\cdot\frac{\vec{v}}{c}\right]$$

$$= \gamma(v) m\, \dot{x}^i + \frac{q}{c} A_i = p_i + \frac{q}{c} A_i$$

- This means that we can write the Hamiltonian for the particle in an electromagnetic field as:

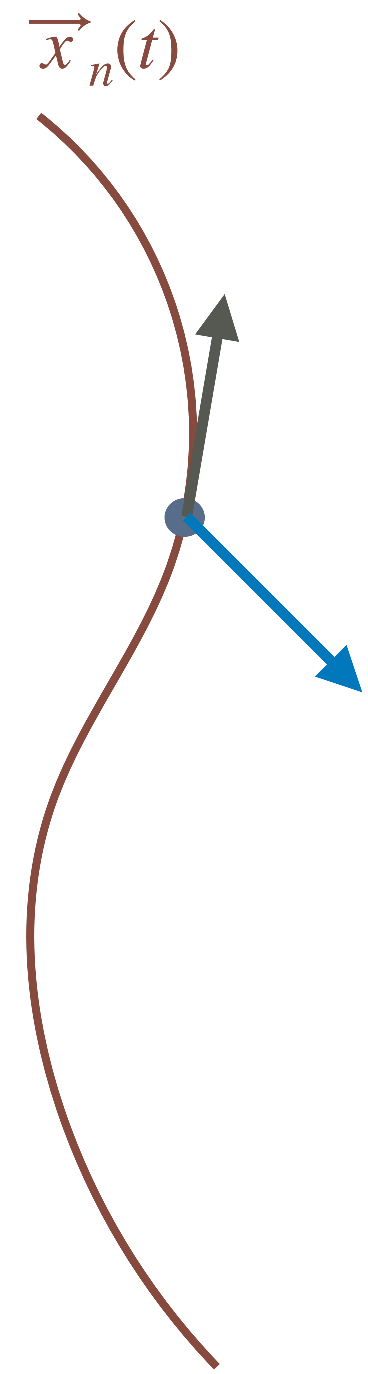
$$H = v^i \frac{\partial L}{\partial v^i} - L = \frac{m c^2}{\sqrt{1 - \frac{v^2}{c^2}}} + q\phi \quad ,$$

which, after using the equation above to write $\vec{v}$ in terms of $\vec{P}$, yields:

$$H = c^2 \sqrt{m^2 c^2 + \left(\vec{P} - \frac{q}{c}\vec{A}\right)^2} + q\phi$$

- The non-relativistic limit of this expression is:

$$H = \frac{1}{2m}\left(\vec{P} - \frac{q}{c}\vec{A}\right)^2 + q\phi$$



Action, Lagrangian and Hamiltonian

- Now let's go back to our previous expression for the action of a point particle in the presence of Electromagnetic fields:

$$S_{free} + S_{int} \;=\; \int dt \left[-\frac{1}{\gamma(v)} m\, c^2 - q\phi + q\vec{A}\cdot\frac{\vec{v}}{c} \right]$$

- We would now like to write this in spacetime parlance, and for a charge/current distribution. First, notice that the interaction Lagrangian can be written as:

$$S_{int} \;=\; \int d\tau \left[q\,(-\phi)\,\gamma c + q\vec{A}\cdot(\gamma\vec{v}) \right]$$

- Now, notice that $A^\mu = \{\phi, \vec{A}\}$, and $A_\mu = \{-\phi, \vec{A}\}$, so we can make the identification

$$S_{int} \;=\; \int d\tau\, A_\mu\, q\, \frac{dx^\mu}{d\tau} \;=\; \int d\tau\, A_\mu(x_q)\, q\, U^\mu(x_q)$$

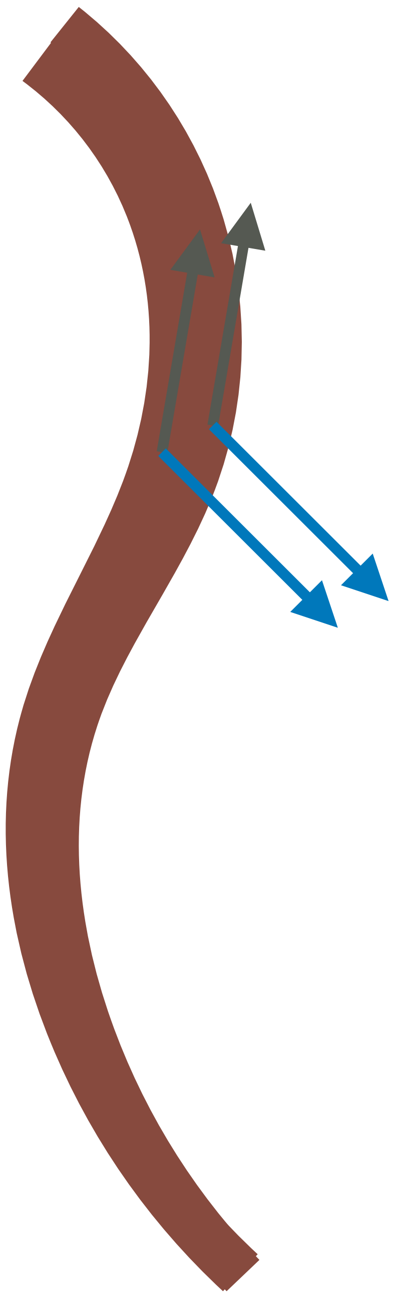
- Finally, we have to generalize this "current" qU^μ to a distribution, J^μ .

$$S_{int} \;=\; \int d(ct) \left[A_0(t,\vec{x}_q)\, q + \vec{A}(t,\vec{x}_q)\cdot\left(q\frac{\vec{v}}{c}\right)\right] \;=\; \frac{1}{c}\int d^4x \left[A_0(t,\vec{x})\, qc + \vec{A}(t,\vec{x})\cdot(q\vec{v})\right]\delta[\vec{x}-\vec{x}_q(t)]$$

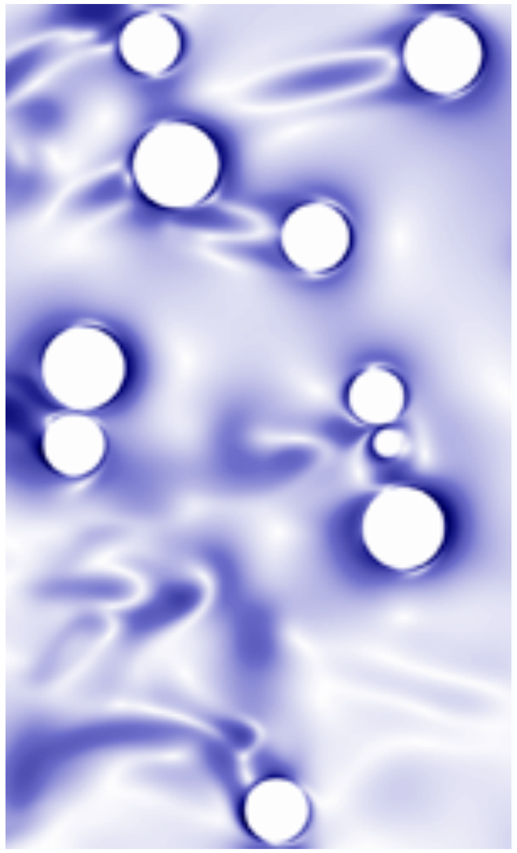
And now we can associate $q\{c,\vec{v}\}\delta(\vec{x}-\vec{x}_q) \to J^\mu$, and write the ***invariant action***:

$$\Rightarrow \quad S_{int} \quad \to \quad \frac{1}{c}\int d^4x\, A_\mu\, J^\mu$$

It is trivial to show that this reduces to the expression for a point particle.



The electromagnetic Lagrangian



- Finally, we would like to put all this together and write the full Lagrangian for matter + fields + interactions:

$$S_{Mat} + S_{Int} + S_{EM} = -\sum_n \int dt \, \frac{m_{(n)}}{\gamma_{(n)}} c^2 + \frac{1}{c} \int d^4x \, A_\mu J^\mu + S_{EM} \quad , \quad \text{or}$$

$$S_{Mat} + S_{Int} + S_{EM} = \frac{1}{c} \int d^4x \, L_{Mat} + \frac{1}{c} \int d^4x \, A_\mu J^\mu + S_{EM}$$

where the factor $1/c$ appears here just because $d^4x = c \, dt \, d^3x$.

- The only part which is missing here is the expression for the action (or Lagrangian) of the "free" electromagnetic field. From our previous discussions, the only scalar that we can construct with the fields, and which has dimensions of energy density, is:

$L_{EM} \sim F^{\mu\nu} F_{\mu\nu} = F^2$, and if we pay attention to the signs and dimensions we can write:

$$L_{EM} = \frac{1}{4 \mu_0 c} F^2 = -\frac{1}{\mu_0 c} \frac{1}{2} \left(\frac{\vec{E}^2}{c^2} - \vec{B}^2 \right)$$

- So, we arrive finally at what is really an ansatz, that we must verify through the Euler-Lagrange equations:

$$S_{Mat} + S_{Int} + S_{EM} = \frac{1}{c} \int d^4x \, L_{Mat} + \frac{1}{c} \int d^4x \, A_\mu J^\mu + \frac{1}{c} \int d^4x \, \frac{1}{4 \mu_0} F^2$$

The Euler-Lagrange equations for EM fields

- The very first thing we should verify is that this action gives us the Maxwell equations.
- We can derive those equations from the Euler-Lagrange equations for the total Lagrangian:

$$L_{Tot} \;=\; L_{Mat} \;+\; A_\mu J^\mu \;+\; \frac{1}{4\mu_0} F^{\mu\nu} F_{\mu\nu} \quad \text{and taking the derivatives:} \quad \frac{\partial L_{Tot}}{\partial A_\mu} - \partial_\nu \frac{\partial L_{Tot}}{\partial(\partial_\nu A_\mu)} = 0$$

- Clearly, the matter Lagrangian does not depend on the 4-potential or its derivatives, so only the interaction and free EM parts contribute. We have:

$$\frac{\partial L_{Tot}}{\partial A_\mu} = J^\mu$$

$$\frac{\partial L_{Tot}}{\partial(\partial_\nu A_\mu)} = \frac{1}{4\mu_0} \frac{\partial}{\partial(\partial_\nu A_\mu)} \left[\left(\partial^\alpha A^\beta - \partial^\beta A^\alpha\right) \left(\partial_\alpha A_\beta - \partial_\beta A_\alpha\right) \right] = \frac{1}{4\mu_0} \frac{\partial}{\partial(\partial_\nu A_\mu)} \eta^{\alpha\eta} \eta^{\beta\lambda} \left[\left(\partial_\eta A_\lambda - \partial_\lambda A_\eta\right) \left(\partial_\alpha A_\beta - \partial_\beta A_\alpha\right) \right]$$

$$= \frac{1}{4\mu_0} \eta^{\alpha\eta} \eta^{\beta\lambda} \left[\left(\delta^\nu_\eta \delta^\mu_\lambda - \delta^\nu_\lambda \delta^\mu_\eta\right) \left(\partial_\alpha A_\beta - \partial_\beta A_\alpha\right) + \left(\partial_\eta A_\lambda - \partial_\lambda A_\eta\right) \left(\delta^\nu_\alpha \delta^\mu_\beta - \delta^\nu_\beta \delta^\mu_\alpha\right) \right]$$

$$= \frac{1}{2\mu_0} \left[\left(\partial^\nu A^\mu - \partial^\mu A^\nu\right) + \left(\partial^\nu A^\mu - \partial^\mu A^\nu\right) \right] \;=\; \frac{1}{\mu_0} F^{\mu\nu} \qquad !!! \qquad \left[\text{ i.e., } \frac{\partial F^2}{\partial(\partial_\nu A_\mu)} = 4F^{\mu\nu} \right]$$

- Therefore, we obtain that the Euler-Lagrange equations give us:

$$J^\mu - \partial_\nu \left(\frac{1}{\mu_0} F^{\mu\nu} \right) = 0 \quad , \quad \text{which is the same as } \partial_\nu F^{\mu\nu} = \mu_0 J^\mu \quad \text{— also know as the Maxwell equations !!!}$$

Stress-energy tensor (reload)

- I will leave as an exercise for you to show that the Euler-Lagrange equations for matter (i.e., the Lorentz force for the point particles) follow from the same action and Lagrangian we derived above — all you need to do is to vary the Lagrangian with respect to the matter degrees of freedom.

- What we will do now is to obtain the stress-energy (or energy-momentum) tensor for the EM fields in a different way than we did in our previous class. The idea is to start with the action and Lagrangian for the "free" EM fields:

$$L_{EM} \;=\; \frac{1}{4\mu_0} F^{\mu\nu} F_{\mu\nu}$$

- In a completely general sense, if we have a system with a Lagrangian $L(q, \partial_\mu q)$ we can ask what are the ***conserved currents*** associated with that Lagrangian (the Noether theorem). By varying the action and obtaining the Euler-Lagrange equations we have, as before:

$$\frac{\partial L}{\partial q} - \partial_\nu \frac{\partial L_{Tot}}{\partial(\partial_\nu q)} = 0$$

- Now, notice that:

$$\partial_\mu L = \frac{\partial L}{\partial q} \partial_\mu q + \frac{\partial L_{Tot}}{\partial(\partial_\nu q)} \partial_\mu (\partial_\nu q)$$

- Substituting $\partial L/\partial q$ from the Euler-Lagrange equation we obtain:

$$\partial_\mu L \;=\; \left[\partial_\nu \frac{\partial L_{Tot}}{\partial(\partial_\nu q)} \right] \partial_\mu q + \left[\frac{\partial L_{Tot}}{\partial(\partial_\nu q)} \right] \partial_\mu (\partial_\nu q) \;=\; \partial_\nu \left[\frac{\partial L_{Tot}}{\partial(\partial_\nu q)} \partial_\mu q \right]$$

Stress-energy tensor (reload)

- Therefore, for any system with a Lagrangian we can write that:

$$\partial_\mu L = \partial_\nu \left[\frac{\partial L_{Tot}}{\partial(\partial_\nu q)} \partial_\mu q \right]$$, so that it is trivial to see that:

$$\partial_\nu \left[\delta^\nu_\mu L - \frac{\partial L_{Tot}}{\partial(\partial_\nu q)} \partial_\mu q \right] = 0$$

- Now, let's write the object inside these brackets as our "conserved current" (exchancing $\nu \leftrightarrow \alpha$):

$$T^\alpha{}_\mu = \frac{\partial L_{Tot}}{\partial(\partial_\alpha q)} \partial_\mu q - \delta^\alpha_\mu L$$, so that

$$\partial_\alpha T^\alpha{}_\mu = 0$$, i.e., conservation of the stress-energy tensor $T^\alpha{}_\mu$!

- Now, let's show that this calculation, in the case of the EM fields, lead to:

$$T^\alpha{}_\mu = -\frac{1}{\mu_0} \left[F^{\alpha\nu} F_{\nu\mu} + \frac{1}{4} \delta^\alpha_\mu F^2 \right] = \frac{1}{\mu_0} \left[F^{\alpha\nu} F_{\mu\nu} - \frac{1}{4} \delta^\alpha_\mu F^2 \right]$$

- Clearly, the generalization now is $q \to A_\nu$ and $\partial_\alpha q \to \partial_\alpha A_\nu$, with $L_{EM} = \frac{1}{4\mu_0} F^{\mu\nu} F_{\mu\nu} = \frac{1}{4\mu_0} F^2$.

Stress-energy tensor (reload)

- We write, therefore:

$$T^{\alpha}{}_{\mu} = \frac{\partial L_{Tot}}{\partial(\partial_\alpha q)}\partial_\mu q - \delta^\alpha_\mu L \quad \rightarrow \quad \frac{\partial L_{EM}}{\partial(\partial_\alpha A_\nu)}\partial_\mu A_\nu - \delta^\alpha_\mu L_{EM}$$

- Using the expression for the EM Lagrangian,

$$L_{EM} = \frac{1}{4\mu_0}F^{\mu\nu}F_{\mu\nu} = \frac{1}{4\mu_0}F^2$$, and our previous result that:

$$\frac{\partial F^2}{\partial(\partial_\mu A_\nu)} = 4F^{\mu\nu}$$, we arrive at $T^{\alpha}{}_{\mu} = \frac{1}{\mu_0}F^{\alpha\nu}\partial_\mu A_\nu - \delta^\alpha_\mu \frac{1}{4\,\mu_0}F^2$

- Notice that this tensor is not quite right! The expression we derived earlier was:

$$T^{\alpha}{}_{\mu} = \frac{1}{\mu_0}F^{\alpha\nu}F_{\mu\nu} - \delta^\alpha_\mu \frac{1}{4\,\mu_0}F^2$$

- What is happening here is that we are ***missing a piece of the puzzle*** — a derivate of the action with respect to the A_μ from the interaction Lagrangian, $L_{Int} \rightarrow A_\mu J^\mu$.
- The point is that this "exchange" of energy and momentum mean that the individual stress-energy tensor of each species is not conserved anymore: only the total energy-momentum tensor is (see the discussion in our last class).
- In order to compensate for this, we must add a term to the stress-energy tensor which turns out to be (for details, see Landau, Ch. 32-33)

$$T^{\alpha}{}_{\mu} \quad \rightarrow \quad T^{\alpha}{}_{\mu} - \frac{1}{\mu_0}F^{\alpha\nu}\partial_\nu A_\mu$$ in such a way that the 4-divergence of that term gives us (in the absence of charges/currents):

$$\partial_\alpha T^{\alpha}{}_{\mu} \quad \rightarrow \quad \partial_\alpha T^{\alpha}{}_{\mu} - \partial_\alpha\left(\frac{1}{\mu_0}F^{\alpha\nu}\partial_\nu A_\mu\right) = \partial_\alpha T^{\alpha}{}_{\mu} - \frac{1}{\mu_0}\left[\left(\partial_\alpha F^{\alpha\nu}\right)\partial_\nu A_\mu + F^{\alpha\nu}\left(\partial_\alpha\partial_\nu A_\mu\right)\right] = \partial_\alpha T^{\alpha}{}_{\mu}$$

Stress-energy tensor (reload)

- So we have, finally, the stress-energy tensor for the free Electromagnetic field:

$$T^{\alpha}_{(EM)\ \mu} \;=\; \frac{1}{\mu_0} F^{\alpha\nu}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right) - \delta^{\alpha}_{\mu}\frac{1}{4\,\mu_0}F^2 \qquad ,$$

which is now identical to the one we derived before:

$$T^{\alpha}_{(EM)\ \mu} \;=\; \frac{1}{\mu_0} F^{\alpha\nu} F_{\mu\nu} - \;\delta^{\alpha}_{\mu}\,\frac{1}{4\,\mu_0}F^2$$

- Before we move on, let's obtain one final result here: we will show that the ***total*** stress-energy tensor is indeed conserved, as a result of the equations of motion.

- We know from our last class that the EM stress-energy tensor above actually observes the conservation law:

$$\partial_\alpha T^{\alpha}{}_{\mu} = \partial_\alpha\left(T^{\alpha}_{(Mat)\ \mu} + T^{\alpha}_{(EM)\ \mu}\right) = 0 \quad , \quad \text{or} \quad \partial_\alpha T^{\alpha}_{(Mat)\ \mu} = f_\mu = -\,\partial_\alpha T^{\alpha}_{(EM)\ \mu} \qquad , \qquad \text{with} \quad f_\mu = F_{\mu\beta}\,J^\beta$$

- Let's write the matter stress-energy tensor in a simplified form here, in terms of point masses. The action for a point particle is:

$$S_m = -\int dt\,\frac{m}{\gamma}\,c^2 \qquad ,$$

so the Lagrangean for a distribution of point particles is:

$$S_{Mat} \;=\; \int d^4x \sum_n \frac{m_n\,c}{\gamma_n}\delta(\vec{x} - \vec{x}_n) \;=\; \int d^4x \frac{c}{\gamma}\sum_n m_n\,\delta(\vec{x} - \vec{x}_n) \;=\; \int d^4x \frac{c}{\gamma}\rho_m$$

Stress-energy tensor (reload)

- The stress-energy tensor for this distribution of point particles is, therefore:

$$T^{\alpha}_{(Mat)\;\mu} \;=\; \frac{c}{\gamma}\rho_m U^\alpha U_\mu \;=\; c\rho_m U^\alpha U_\mu \frac{\partial\tau}{\partial t} \;=\; c\rho_m \frac{\partial x^\alpha}{\partial t} U_\mu$$

- Taking the stress-energy conservation for the matter component we have:

$$\partial_\alpha T^{\alpha}_{(Mat)\;\mu} \;=\; c\partial_\alpha \left(\rho_m \frac{\partial x^\alpha}{\partial t} U_\mu\right) \;=\; c\left[\rho_m U_\mu\, \partial_\alpha \left(\frac{\partial x^\alpha}{\partial t}\right) + \frac{\partial x^\alpha}{\partial t}\, \partial_\alpha \left(\rho_m U_\mu\right)\right]$$

- The first term above is identically zero, but the second term remains, and can be expressed as:

$$\partial_\alpha T^{\alpha}_{(Mat)\;\mu} \;=\; c\frac{\partial x^\alpha}{\partial t}\, \partial_\alpha \left(\rho_m U_\mu\right) \;=\; c\frac{\partial}{\partial t} \left(\rho_m U_\mu\right)$$

- But this is basically $\partial p_\mu/\partial t$ — the change of 4-momentum per unit volume, i.e., the force! The right-hand side can be written for a point charge (momentum P_μ) and for a distribution of charges/currents (momentum density p_μ) as:

$$\frac{dP_\mu}{dt} = qF_{\mu\nu}\, U^\nu \quad\leftrightarrow\quad \frac{dp_\mu}{dt} = F_{\mu\nu}\, J^\nu$$

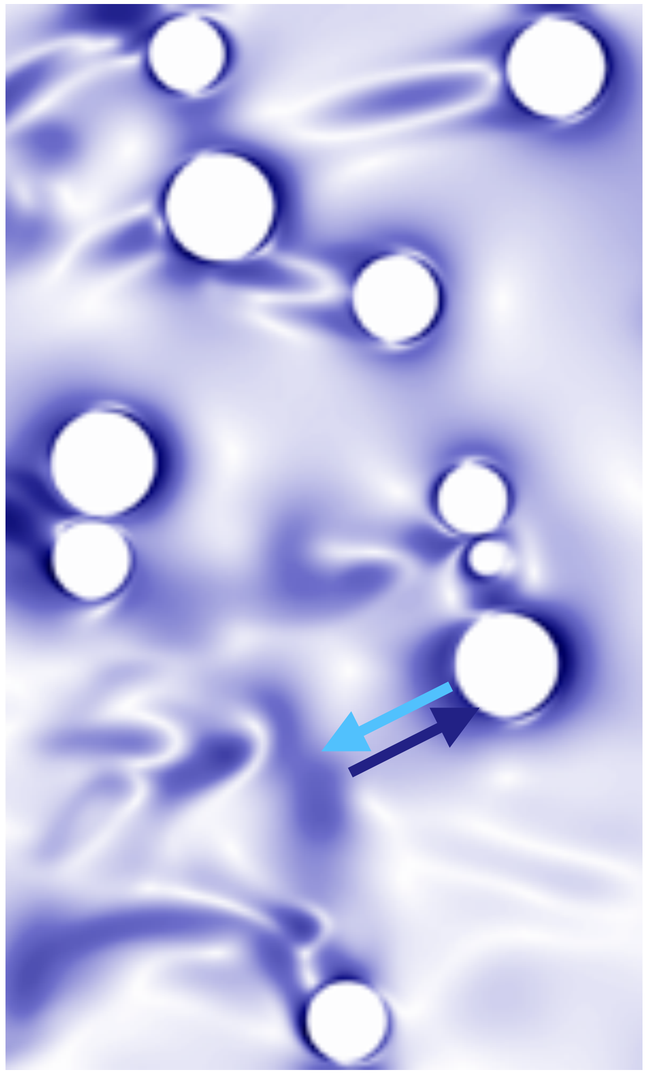
- So, we finally arrive at the expression for the force for a distribution of point charges/masses in an EM field:

$$\partial_\alpha T^{\alpha}_{(Mat)\;\mu} \;=\; F_{\mu\nu}\, J^\mu \qquad \text{which is exactly the Lorentz force!}$$

Relativistic Lagrangian and stress-energy tensor: summary

- The total Lagrangian for matter + EM fields is therefore given by:

$$S_{Mat} + S_{Int} + S_{EM} \; = \; \frac{1}{c}\int d^4x \, L_{Mat} \; + \; \frac{1}{c}\int d^4x \, A_\mu \, J^\mu \; + \; \frac{1}{c}\int d^4x \, \frac{1}{4\,\mu_0} F^2$$



- The stress-energy tensor describes the energy density, energy flux (momentum) as well as ***all relevant dynamical properties of the matter and fields*** in the system.

- The ***conservation of the stress-energy tensor*** is simply the conservation of the ***Noether currents*** associated with invariance under space-time translations (energy/momentum) and rotations (angular momentum):

$$\partial_\alpha T^{\alpha\mu} = 0$$

- If any two parts of the system are interacting, it means that there can be exchange of energy, momentum, and/or angular momentum between the two parts, and we have:

$$\partial_\alpha T^{\alpha\mu}_{(1)} + \partial_\alpha T^{\alpha\mu}_{(2)} \; = \; 0 \quad \Longrightarrow \quad \partial_\alpha T^{\alpha\mu}_{(1)} \; = \; f^\mu_{(1,2)} \quad , \quad \partial_\alpha T^{\alpha\mu}_{(2)} \; = \; f^\mu_{(2,1)}$$

where $f^\mu_{(1,2)} = - f^\mu_{(2,1)}$ is simply "Newton's 3rd Law"

- For the Electromagnetic field, the stress-energy tensor which describes all properties of the free fields is given by:

$$T^{\alpha\mu}_{(EM)} \; = \; \frac{1}{\mu_0} F^{\alpha\nu} F^\mu{}_\nu - \; \delta^\alpha_\mu \, \frac{1}{4\,\mu_0} F^2 \; = \; \frac{1}{\mu_0} F^\alpha{}_\nu F^{\mu\nu} - \; \delta^\alpha_\mu \, \frac{1}{4\,\mu_0} F^2$$

Next class:

- Radiation from accelerated charges
- Dipole radiation
- Radiation from accelerated charges — fully relativistic calculation
- Jackson, Ch. 9 & 12