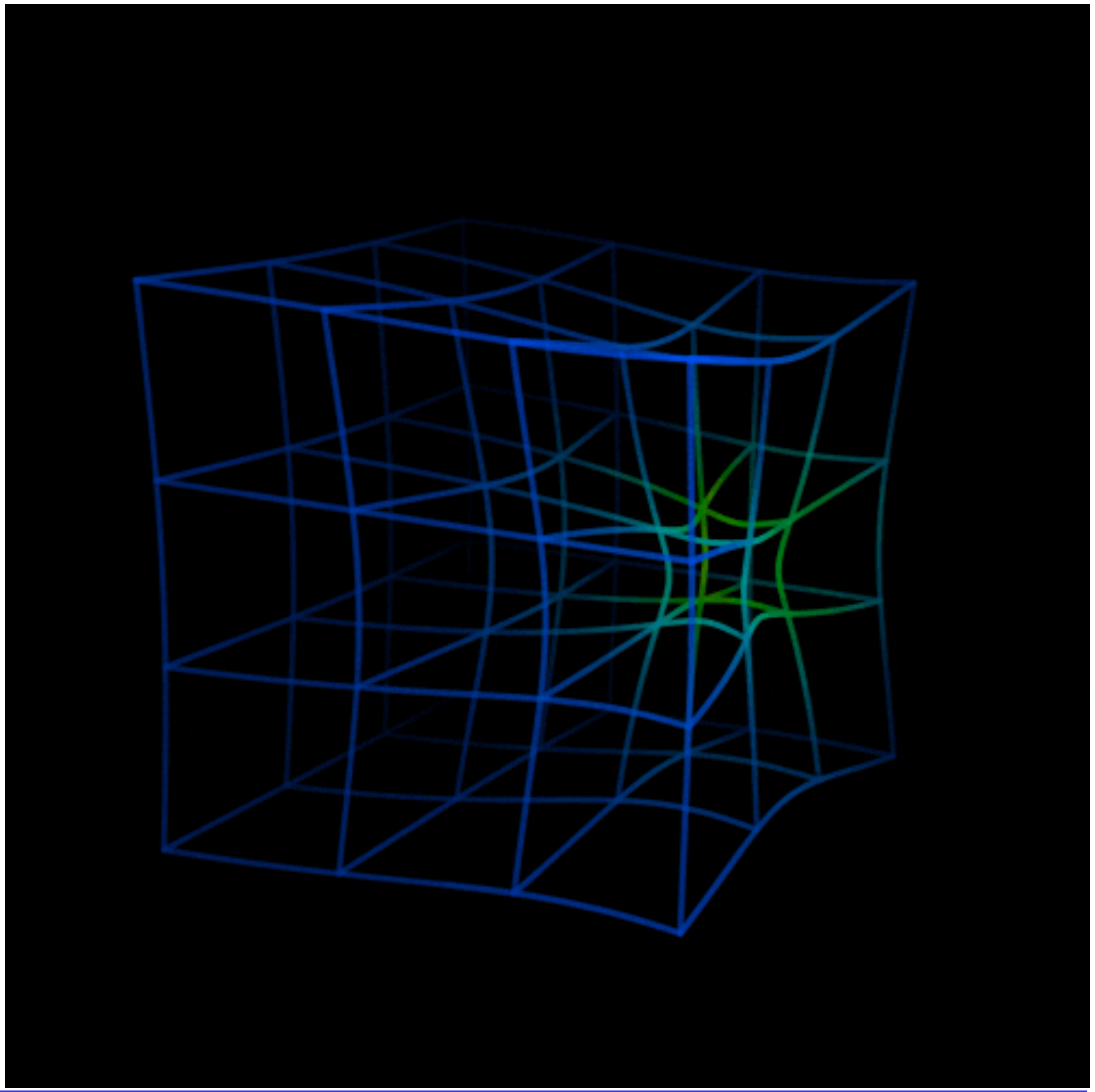
Relativistic Electrodynamics

⚡The Lorentz force
⚡The stress-energy tensor
⚡Energy and momentum conservation



Relativistic electrodynamics

- In our last class we saw that, the Maxwell equations are, in **covariant notation**:

$\partial_\nu F^{\mu\nu} = \mu_0 \, J^\mu$, with $\partial_\mu J^\mu = 0$ being an integrability condition

$\partial_\nu F^{*\mu\nu} = 0$, where $F^{*\mu\nu} = \epsilon^{\mu\nu\alpha\beta} \, F_{\alpha\beta}$ is the Hodge dual of $F^{\mu\nu}$.

- The Faraday tensor includes both the electric and the magnetic fields:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$F^{\mu\nu} = \begin{array}{cc} & \xrightarrow{\hspace{8em}} \nu \\ \Big\downarrow \mu & \begin{pmatrix} 0 & +E_x/c & +E_y/c & +E_z/c \\ -E_x/c & 0 & +B_z & -B_y \\ -E_y/c & -B_z & 0 & +B_x \\ -E_z/c & +B_y & -B_x & 0 \end{pmatrix} \end{array} \qquad F_{\mu\nu} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & +B_z & -B_y \\ E_y/c & -B_z & 0 & +B_x \\ E_z/c & +B_y & -B_x & 0 \end{pmatrix}$$

The Lorentz force

- We know from non-relativistic Electromagnetism that the force on a point charge is given by:

$$\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right)$$

- But this doesn't seem to be in covariant (relativistic) form: the force here doesn't seem to be part of a 4-vector, and the fields are not organized in terms of the Faraday tensor, $F_{\mu\nu}$.

- We can start to see what is going on by noting that the velocity is part of a 4-vector: $U^\mu = dx^\mu/d\tau = \gamma\{c, \vec{v}\}$.

$$U^\mu = \frac{dx^\mu}{d\tau} = \gamma\{c, \vec{v}\} \quad , \quad \text{where the proper time is } d\tau = dt/\gamma(v)$$

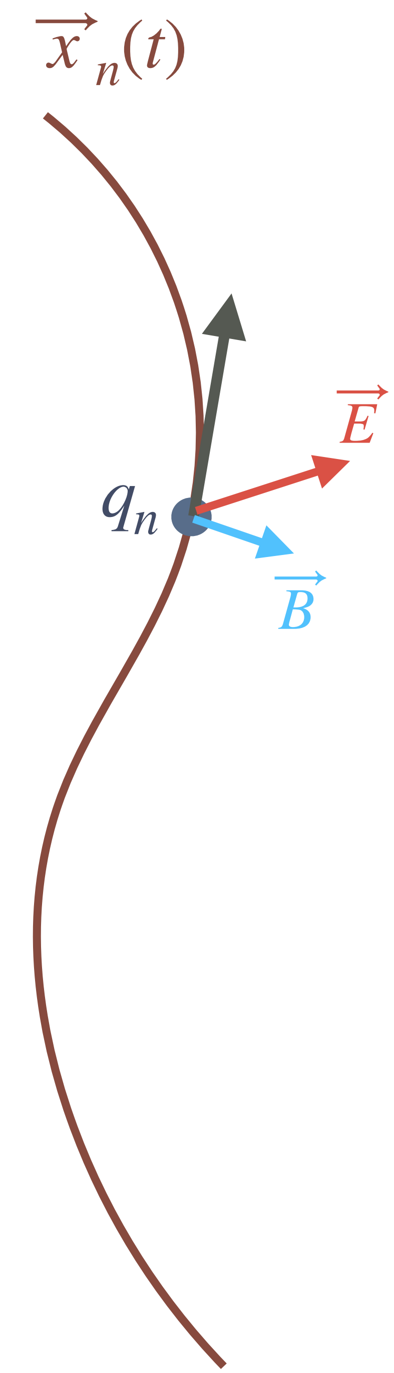
and remember that $||U||^2 = \eta_{\mu\nu}U^\mu U^\nu = U_\mu U^\mu = -c^2$

- This particle has a 4-momentum given by

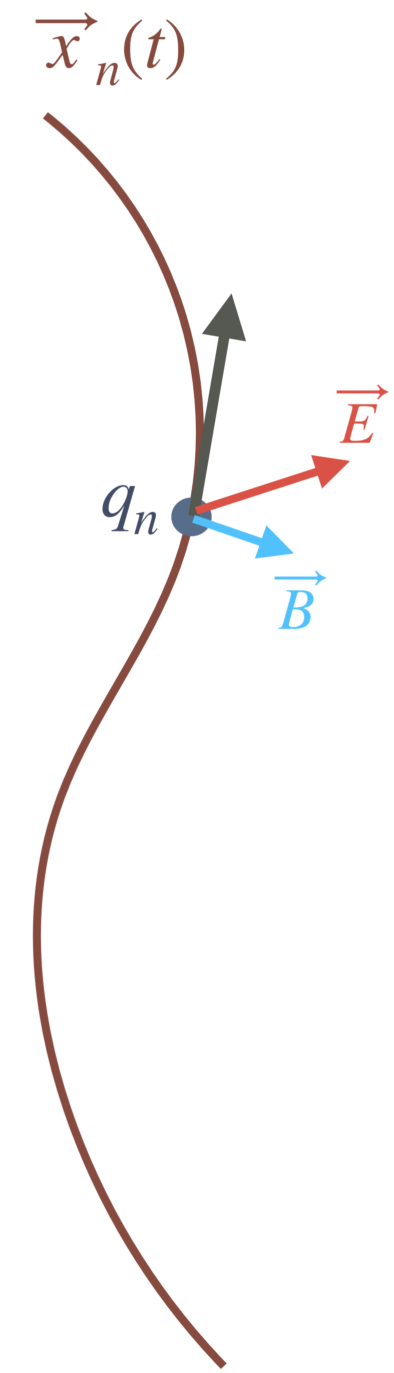
$$p^\mu = mU^\mu = m\,\gamma\{c, \vec{v}\} = \{E/c, \vec{p}\} \quad , \quad \text{which we can also write as:}$$

$$p_\mu = \eta_{\mu\nu}\,p^\nu = \{-E/c, \vec{p}\}$$

- The idea now is to write the force as the rate of change of the 4-momentum.



The Lorentz force



- We start by noticing that the electric field is the time/space part of the Faraday tensor, while the magnetic field is already in the spatial part, in the form of an axial vector/vector product:

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E^j/c \\ E^i/c & \epsilon^{ijk} B^k \end{pmatrix}$$

, i.e., $F_{i0} = E^i/c$ and $F_{0j} = -E^j/c$

- On the other hand, the 4-velocity and the 4-momentum have pure temporal parts (γc and $m\gamma c$), as well as pure spatial parts, $\vec{v} = \gamma \vec{v}$ and $\vec{p} = m\gamma\vec{v}$.

- The left-hand side of the force law, $\vec{F} = d\vec{p}/dt$, should be promoted to something like:

$$f^\mu = \frac{d\,p^\mu}{d\tau} \quad , \quad \text{or} \quad f_\mu = \frac{d\,p_\mu}{d\tau}$$

- But what about the right-hand side, the electromagnetic force? We can try to combine the temporal and spatial parts of 4-velocity and Faraday tensor, and the only way to accomplish this in a covariant way is to consider something like:

$$F_{\mu\nu}\,U^\nu$$

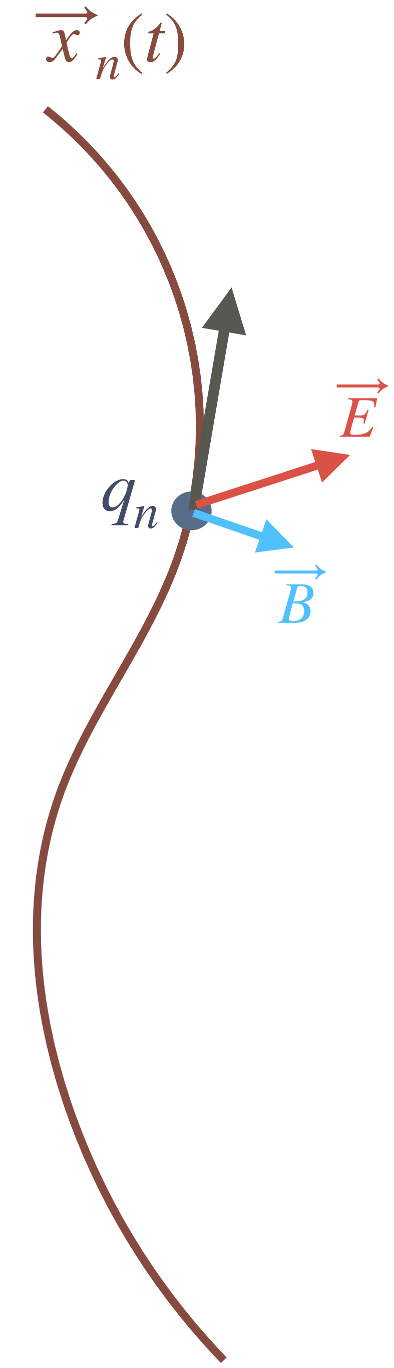
- This combination is a 4-vector, and its components are:

$$F_{0\nu}\,U^\nu = F_{00}\,U^0 + F_{0i}\,U^i = -\frac{\gamma}{c}\vec{E}\cdot\vec{v} \quad , \quad \text{and}$$

$$F_{i\nu}\,U^\nu = F_{i0}\,U^0 + F_{ij}\,U^j = \frac{E^i}{c}\gamma c + \gamma\epsilon^{ijk}B^k v^j \;\to\; \gamma\left(\vec{E} + \vec{v}\times\vec{B}\right)$$

which is exactly what we want, at least in the non-relativistic limit ($v \ll c$, $\gamma \to 1$).

The Lorentz force



- Therefore, we arrive at an ansatz for the covariant form of the Lorentz force:

$$f_\mu = \frac{d\,p_\mu}{d\tau} = q\,F_{\mu\nu}\,U^\nu$$

- The spatial part of this expression is:

$$\frac{d\,\vec{p}}{d\tau} = q\,\gamma\left(\vec{E} + \vec{v}\times\vec{B}\right)$$, and going back to coordinate time using $d\tau = dt/\gamma(v)$, we get:

$$\frac{d\,\vec{p}}{dt} = q\,\left(\vec{E} + \vec{v}\times\vec{B}\right)$$, which is exactly the expression we had for the Lorentz force!

- So, somehow magically the Lorentz force we know since high school is ***already in covariant form***!

- But what about the temporal part of this expression,

$$\frac{d\,p_0}{d\tau} = -\frac{\gamma}{c}\frac{dE}{dt} = q\,F_{0\nu}\,U^\nu = -\frac{q\,\gamma}{c}\vec{E}\cdot\vec{v}$$

$$\Rightarrow \quad \frac{dE}{dt} = q\,\vec{E}\cdot\vec{v}$$, which is simply the ***work done by the force on the particle***!

- Another nice feature of the covariant form of the Lorentz form is that $q\,U^\mu \to j^\mu$.

- Finally, you can check that the Lorentz force (in fact, ***any*** force!) is orthogonal to the 4-velocity:

$f_\mu U^\mu = 0$ [Exercise: prove this using the covariant form and the fact that $U_\mu U^\mu = -c^2$.]

The Lorentz force

- It is useful to write also the Lorentz force in the case of a charge distribution, $q\, U^\mu \to J^\mu$. Let's first go back to the non-relativistic expression:

$$\vec{F} = \int d^3x \left(\rho \vec{E} + \vec{J} \times \vec{B} \right) \quad ,$$ or even better:

$$\frac{d\,\vec{F}}{d^3x} = \frac{d\,\vec{p}}{dt\, d^3x} = \rho \vec{E} + \vec{J} \times \vec{B}$$ (force per unit volume)

- The left-hand side is basically the derivative with respect to the 4-volume, $d^4x = c dt\, d^3x$. Now, it is interesting to note that the 4-volume element is invariant:

$$x^\mu \to x'^\mu = \Lambda^\mu_{\ \nu}\, x^\nu \quad \Rightarrow \quad d^4x \to d^4x' = d^4x$$

- In fact, for any *manifold* such as the Minkowski spacetime, endowed with a metric $g_{\mu\nu}$, the volume element transforms with the determinant of the Jacobian:

$$x^\mu \to x'^\mu = \Lambda^\mu_{\ \nu}\, x^\nu \quad \Rightarrow \quad d^4x \to d^4x' = \det \Lambda\; d^4x \quad ,$$

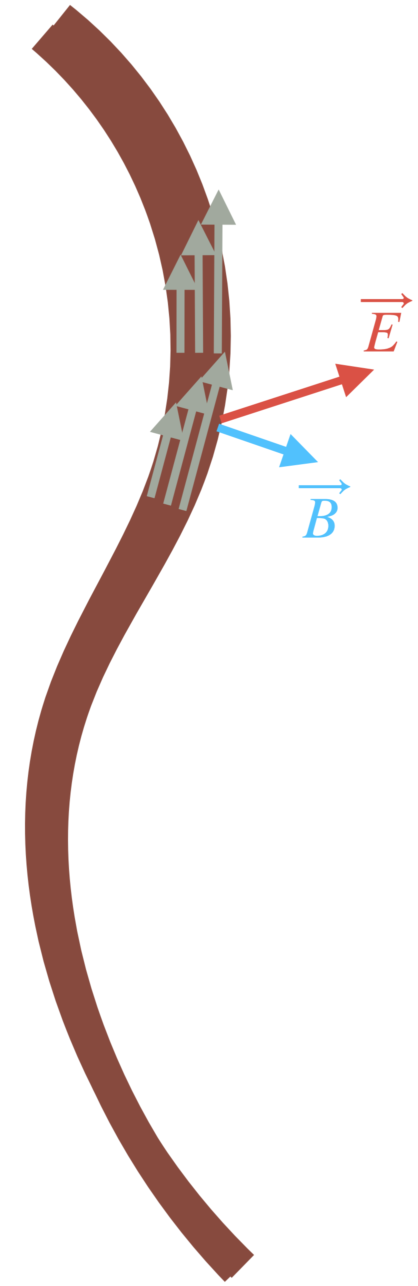
and since $dx'^\mu = \Lambda^\mu_{\ \nu}\, dx^\nu$, the invariance of $ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu \ = \ g'_{\mu\nu}\, dx'^\mu dx'^\nu = ds'^2$ means that

$$\sqrt{|\det g_{\mu\nu}|}\; d^4x = \sqrt{|\det g'_{\mu\nu}|}\; d^4x'$$ is an invariant .

- For the Minkowski spacetime, $g_{\mu\nu} = \eta_{\mu\nu} = \mathrm{diag}\{-1,1,1,1\}$, hence $|\det \eta_{\mu\nu}| = 1$, and therefore $d^4x' = d^4x$.

- The discussion above implies that we can write the Lorentz force for a charge distribution as:

$$\frac{d\, p_\mu}{d^4x} = F_{\mu\nu}\, J^\nu$$



The stress-energy tensor

- Since we are talking about forces, momentum and energy, let's go back to the issue of the dynamical properties of the electromagnetic field. We can start by thinking like Maxwell, in the sense that the fields also reflect "mechanical" properties of a "fluid", and therefore are endowed with energy, momentum, pressure, fluxes, stresses, etc.

- If we think of a fluid, and we capture an element of that fluid, that element has a series of properties:
  - energy density, ϵ (to not confuse with charge density ρ)
  - momentum, $\vec{p}$
  - pressure, P
  - stress, Π
  - angular momentum, $\vec{L}$

- Obviously, all these properties are inter-connected, so they cannot be described by a 4-vector, or even by a combination of 4-vectors. We need an object that is ***more general***, more complete.

- This object is the ***stress-energy tensor*** (also known as ***energy-momentum tensor***). A generic non-relativistic fluid has a stress-energy tensor given by something like:

$$T^{\mu\nu} = \begin{pmatrix} \epsilon & cp^i \\ cp^i & P\delta_{ij} + \Pi_{ij} \end{pmatrix} \quad , \qquad \text{where } \mathrm{Tr}\,\Pi = \Pi_{ii} = 0$$

The stress-energy tensor is symmetric

The stress-energy tensor

- For a perfect relativistic fluid the stress-energy tensor can be expressed in terms of the mass density $\rho_m = \epsilon / c^2$:

$$T^{\mu\nu} = \left(\rho_m + \frac{P}{c^2} \right) U^\mu U^\nu + P \eta^{\mu\nu} \qquad \text{or, equivalently,} \qquad T^\mu{}_\nu = \left(\rho_m + \frac{P}{c^2} \right) U^\mu U_\nu + P \delta^\mu_\nu$$

- From this expression, or from the expression in the previous page, we can obtain two very important equations from the 4-divergence of the ***matter*** stress-energy tensor:

$\partial_\mu T^{\mu\nu} = f^\nu$ where f^ν expresses the ***external forces*** on the matter system (the fluid)

- The non-relativistic limits is more familiar to us, so let's use the approximation $U^\mu \simeq \{c, \vec{v}\}$ in the expression above.

- Taking first $\nu = 0$ we obtain:

$$\partial_\mu T^{\mu 0} = \partial_0 T^{00} + \partial_i T^{i0} = \frac{\partial}{c \partial t} \left[\frac{1}{c^2} (\epsilon + P) c^2 - P \right] + \partial_i \left[\frac{1}{c^2} (\epsilon + P) v^i c \right]$$

- Typically, pressure is sub-dominant compared with energy — after all, energy has a factor of c^2, $\epsilon = \rho_m c^2$. Hence, a good approximation to this expression is:

$$\partial_\mu T^{\mu 0} \simeq \frac{1}{c} \left[\dot{\epsilon} + \epsilon \vec{\nabla} \cdot \vec{v} + (\vec{\nabla} \epsilon) \cdot \vec{v} \right]$$

- Finally, notice that the gradient of the energy density is $\vec{\nabla} \epsilon = \vec{\nabla} (\kappa + u)$, where κ is the density of kinetic energy, and u is the potential energy. Since κ is not a function of position, and $\vec{\nabla} u = - \vec{f}$ is the ***external force*** per unit volume, we obtain:

$\partial_\mu T^{\mu 0} = f^0 \quad \Rightarrow \quad \dot{\epsilon} + \epsilon \vec{\nabla} \cdot \vec{v} - \vec{f} \cdot \vec{v} = 0$, and therefore, in the absence of external forces we get

$\dot{\rho}_m + \rho_m \vec{\nabla} \cdot \vec{v} = 0$ which is simply the statement of energy continuity/conservation!

The stress-energy tensor

- So, the 0-component of the 4-divergence of the stress-energy tensor gives the "continuity equation" for energy (i.e., energy conservation).

- Now, if we take $\nu = j$ (spatial indices) we obtain:

$$\partial_\mu T^{\mu j} = \partial_0 T^{0j} + \partial_i T^{ij}$$

$$= \frac{\partial}{c\partial t}\left[\frac{1}{c^2}(\epsilon + P)\, v^i c\right] + \partial_i \left[\frac{1}{c^2}(\epsilon + P)\, v^i v^j\right]$$

- This equation takes a bit more algebra, but after we take the non-relativistic limit, the result is the ***Euler equation*** that determines the evolution of momentum and pressure in a fluid:

$$\epsilon\left(\frac{\partial v^j}{\partial t} + v^i\,\partial_i\, v^j\right) + \partial_i P = \vec{f}$$

- Finally, a corollary of the stress-energy conservation is the fact that in the absence of external forces we also have ***angular momentum conservation***. One can in fact show that:

$$\partial_\mu T^{\mu\nu} = 0 \quad \Rightarrow \quad \partial_\nu\left(x^\alpha T^{\mu\nu} - x^\mu T^{\alpha\nu}\right) = 0 \qquad .$$

  This anti-symmetric combination is in fact 6 equations, three of which yield the statement that $d\vec{L}/dt = \vec{\tau}$ (the extra 3 equations are valid identically with the Euler and continuity equations)

The Maxwell stress-energy tensor

- Ok, but what we really want to know is the ***stress-energy tensor for the Electromagnetic fields***, and in particular the the expression for $T^{\mu\nu}$ in terms of the Faraday tensor $F^{\mu\nu}$.

- But first, let's think about the ***whole system*** — both the ***matter*** (particles with charge, mass, current, etc.) and the ***fields***. They are interacting with each other, of course (through, e.g., Maxwell's equations), but each have their own energy, momenta etc.

- Let's define then the ***total stress-energy for matter + fields***, and assume that all the interactions can do is to exchange energy and momentum between these two parts.

- Since the particles and fields are ***everything***, we must have ***conservation of the total stress-energy tensor***:

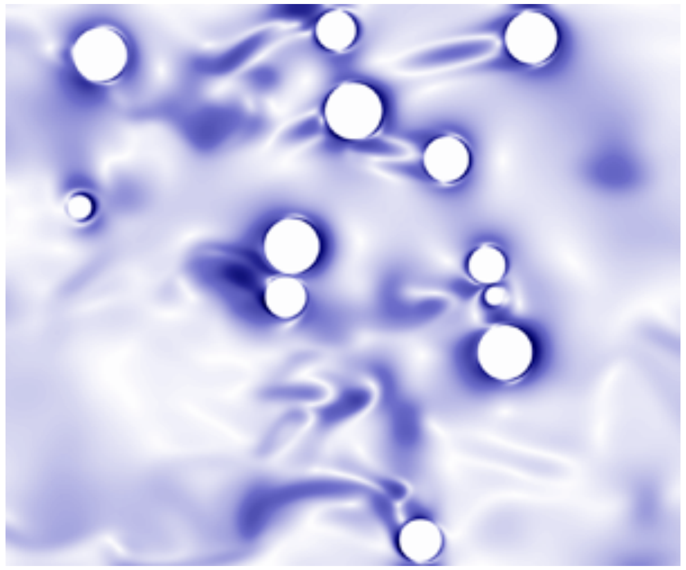
$$T^{\mu\nu}_{Tot} = T^{\mu\nu}_{Mat} + T^{\mu\nu}_{EM} \quad \Rightarrow \quad \partial_\mu T^{\mu\nu}_{Tot} = 0$$

- However, each part of this stress-energy tensor is not necessarily ***conserved separately***, because of the ***forces***: matter generates fields, pumping energy into the field configurations; and the fields affect the motion of the matter, doing work on the particles. Therefore:

$$\partial_\mu T^{\mu\nu}_{Mat} = - \partial_\mu T^{\mu\nu}_{EM}$$

- But, like we saw on the last few slides, the exchange of energy/momentum between matter and is given exactly by the ***forces***, so:

$$\partial_\mu T^{\mu\nu}_{Mat} = f^\nu \quad \Longleftrightarrow \quad \partial_\mu T^{\mu\nu}_{EM} = - f^\nu$$



The Maxwell stress-energy tensor

- Let's search for an expression for $T^{\mu\nu}$ in terms of the Faraday tensor $F^{\mu\nu}$. Clearly they cannot be one and the same thing, because the Faraday tensor is ***linear*** in the fields, while the energy, momentum, momentum flux, etc., are ***quadratic functions*** of the fields.

- Let's start with the fact that we know that we should have stress-energy conservation, i.e.:

$$\partial_\mu T^{\mu\nu}_{EM} = -f^\nu \quad , \quad \text{or} \quad \partial_\mu T^{\mu}_{EM\ ,\,\nu} = -f_\nu$$

[We will suppress the subscript "EM" from now on, to relieve the notation.]

- Consider also the fact that the Faraday tensor itself also obeys a kind of "conservation equation", in the sense that:

$$\partial_\nu F^{\mu\nu} = \mu_0 \, J^\mu$$

But the Lorentz force is given by $f_\mu = F_{\mu\nu} \, J^\nu$, so we can write:

$$f_\mu = F_{\mu\nu} \, J^\nu = F_{\mu\nu} \left(\frac{1}{\mu_0} \partial_\alpha F^{\nu\alpha} \right) = \frac{1}{\mu_0} \left[\partial_\alpha \left(F_{\mu\nu} F^{\nu\alpha} \right) - \left(\partial_\alpha F_{\mu\nu} \right) F^{\nu\alpha} \right]$$

$$\left(\partial_\alpha F_{\mu\nu} \right) F^{\nu\alpha} = \left(\partial_\nu F_{\mu\alpha} \right) F^{\alpha\nu} = \left(\partial_\nu F_{\alpha\mu} \right) F^{\nu\alpha}$$

- I will now manipulate this expression using the anti-symmetry $F^{\mu\nu} = -F^{\nu\mu}$, as well as the Jacobi identity, $\partial_\alpha F_{\mu\nu} + \partial_\nu F_{\alpha\mu} + \partial_\mu F_{\nu\alpha} = 0$:

$$f_\mu = \frac{1}{\mu_0} \left[\partial_\alpha \left(F_{\mu\nu} F^{\nu\alpha} \right) - \frac{1}{2} \left(\partial_\alpha F_{\mu\nu} + \partial_\nu F_{\alpha\mu} \right) F^{\nu\alpha} \right] = \frac{1}{\mu_0} \left[\partial_\alpha \left(F_{\mu\nu} F^{\nu\alpha} \right) + \frac{1}{2} \left(\partial_\mu F_{\nu\alpha} \right) F^{\nu\alpha} \right]$$

$$= \frac{1}{\mu_0} \left[\partial_\alpha \left(F_{\mu\nu} F^{\nu\alpha} \right) + \frac{1}{4} \partial_\mu \left(F_{\nu\alpha} F^{\nu\alpha} \right) \right] \quad ,$$

where in the last line we used the fact that $\partial_\mu \left(F_{\nu\alpha} F^{\nu\alpha} \right) = \left(\partial_\mu F_{\nu\alpha} \right) F^{\nu\alpha} + F_{\nu\alpha} \left(\partial_\mu F^{\nu\alpha} \right) = 2 F_{\nu\alpha} \left(\partial_\mu F^{\nu\alpha} \right) = 2 F^{\nu\alpha} \left(\partial_\mu F_{\nu\alpha} \right)$

The Maxwell stress-energy tensor

- We can further simplify this expression using $F^2 = F^{\mu\nu} F_{\mu\nu}$, and the result is that:

$$f_\mu = \frac{1}{\mu_0} \partial_\alpha \left[F_{\mu\nu} F^{\nu\alpha} + \frac{1}{4} \delta^\alpha_\mu F^2 \right]$$

- Therefore, we should be very tempted to remember that:

$\partial_\alpha T^\alpha{}_\mu = - f_\mu$, and therefore we could write:

$$\boxed{T^\alpha{}_\mu = - \frac{1}{\mu_0} \left[F^{\alpha\nu} F_{\nu\mu} + \frac{1}{4} \delta^\alpha_\mu F^2 \right] = \frac{1}{\mu_0} \left[F^{\alpha\nu} F_{\mu\nu} - \frac{1}{4} \delta^\alpha_\mu F^2 \right]}$$

- And ***this is indeed the stress-energy tensor for the Electromagnetic field!*** This result was first shown in this way by Minkowski himself, in 1908.

- Let's look at some of the components to see if we recognize some "old acquaintances". But first, let's compute the invariant:

$$F^2 = F^{\mu\nu} F_{\mu\nu} = - F^{\mu\nu} F_{\nu\mu} = - 2 \left(\frac{\vec{E}^2}{c^2} - \vec{B}^2 \right)$$

- Now let's compute the first component (time-time, 0-0) of the stress-energy tensor:

$$T^0{}_0 = - \frac{1}{\mu_0} \left[F^{0\nu} F_{\nu 0} + \frac{1}{4} F^2 \right] = - \frac{1}{\mu_0} \left[\frac{\vec{E}^2}{c^2} + \frac{1}{4} F^2 \right] = - \frac{1}{\mu_0} \left[\frac{\vec{E}^2}{c^2} - \frac{1}{2} \left(\frac{\vec{E}^2}{c^2} - \vec{B}^2 \right) \right]$$

$$= - \frac{1}{\mu_0} \left[\frac{1}{2} \frac{\vec{E}^2}{c^2} + \frac{1}{2} \vec{B}^2 \right] = - \left(\frac{1}{2} \epsilon_0 \vec{E}^2 + \frac{1}{2} \frac{1}{\mu_0} \vec{B}^2 \right) = - \rho_{EM}$$

, which is exactly what we expect since $T^0{}_0 = \eta_{0\nu} T^{0\nu} = - T^{00} = - \epsilon$

The Maxwell stress-energy tensor

- Now let's compute the time-space components:

$$T^0{}_i = -\frac{1}{\mu_0}\left[F^{0\nu}F_{\nu i} + 0 \times F^2\right] = -\frac{1}{\mu_0}\left[F^{0j}F_{ji}F^2\right] = \frac{1}{\mu_0}\frac{\vec{E}\times\vec{B}}{c}$$

But this is exactly the Poynting vector, $\vec{S} = \vec{E}\times\vec{B}/\mu_0$, which gives the flux of energy, also known as momentum — in fact, the momentum density is $\vec{p} = \vec{S}/c^2$.

- Finally, we can compute the purely spatial components, which are:

$$T^i{}_j = -\frac{1}{\mu_0}\left[F^{i\nu}F_{\nu j} + \frac{1}{4}\delta^i_j F^2\right]$$

After some algebra you can show that:

$$T^i{}_j = \frac{1}{\mu_0}\left[\frac{1}{c^2}E^iE^j + B^iB^j - \frac{1}{2}\delta_{ij}\left(\frac{\vec{E}^2}{c^2} + \vec{B}^2\right)\right]$$

- It is useful to brake this object into a trace component (which we associate with the isotropic pressure) and a traceless component, which is the anisotropic stress. Taking the trace of this 3x3 matrix we get:

$$T^i{}_i = \frac{1}{\mu_0}\left[\frac{\vec{E}^2}{c^2} + \vec{B}^2 - \frac{3}{2}\left(\frac{\vec{E}^2}{c^2} + \vec{B}^2\right)\right] = -\rho_{EM}$$

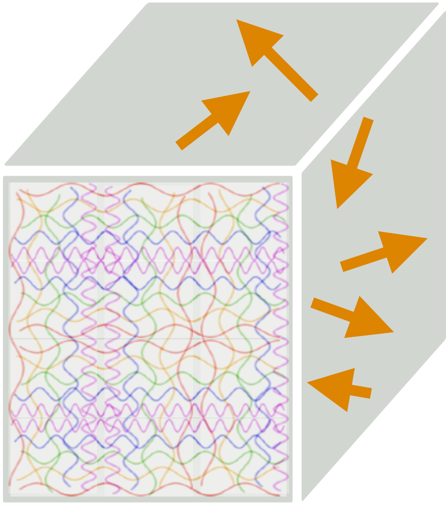
- Therefore, we can write this expression in the form known as the ***Maxwell stress tensor***:

$$T^i{}_j = T^i{}_j - \frac{1}{3}\delta_{ij}T^i{}_j + \frac{1}{3}\delta_{ij}T^i{}_j = \Pi^i{}_j + \frac{1}{3}\delta_{ij}\rho_{EM}$$

where Π is the anisotropic stress, and the pressure is $P_{EM} = \frac{1}{3}\rho_{EM}$

The pressure of the EM field

- We should make several remarks now. First, ***why*** do you think it is that $P_{EM} = \frac{1}{3}\rho_{EM}$? What is the reason for this?



- Think about free electromagnetic fields — ***waves***. For simplicity, think of these waves as bouncing inside a box. This equality means that freely propagating EM waves have not only an ***energy density***, but also a ***pressure*** that is ***exactly*** 1/3 of the energy density.

- If you still can't "see the light", think of light as ***particles*** — ***photons***. What is the main property of the 4-momentum of photons? Let's recall here:

  $$p^\mu = \{E/c, \vec{p}\} = \{p, \vec{p}\} \quad \text{such that} \quad p^\mu p_\mu = 0 \quad ,$$

  so the ***energy*** of a single photon is equal to its ***momentum*** (up to a factor of c).

- Now, consider that for an ***ensemble of photons*** with random (isotropic) momenta, this energy is ***equally distributed*** in all directions, so… 1/3 for each direction…

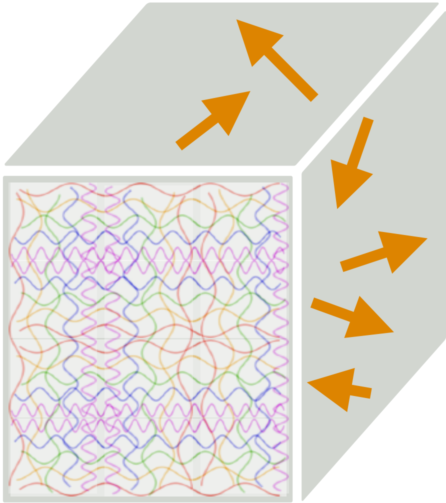
- Yes! The fact that $P_{EM} = \frac{1}{3}\rho_{EM}$ follows directly from the fact that the ***electromagnetic field (the photon) has "zero mass"*** — something which is also manifested in the wave equation and associated propagator:

  $$\Box\psi - m^2\psi = 0 \quad \text{but for Electromagnetism we have} \quad \Box A^\mu = 0 \text{ , so } m = 0 \text{ !}$$

EM stress-energy conservation

- Let's now make a "sanity check" and see what equations follow from the conservation of the electromagnetic stress-energy tensor, i.e.:

$$\partial_\alpha T^{\alpha\mu} + f^\alpha = 0$$



- Consider first the component $\alpha = 0$, and the expressions that we found above. You can show that:

$$\partial_\alpha T^{0\mu} + f^0 = 0 \quad \Rightarrow \quad \frac{\partial \rho_{EM}}{\partial t} + \vec{\nabla} \cdot \vec{S} + \vec{J} \cdot \vec{E} = 0 \quad ,$$

which describes the ***rate of change of energy*** in terms of the ***flux of momentum*** ($\vec{\nabla} \cdot \vec{S}$) and the ***work done to the system*** ($\vec{J} \cdot \vec{E}$).

- The spatial component, on the other hand, again takes a bit more algebra, but you can show that it yields:

$$\partial_\alpha T^{i\mu} + f^i = 0 \quad \Rightarrow \quad \frac{1}{c^2}\frac{\partial S^i}{\partial t} - \partial_j T^{ij} + f^i = 0 \quad ,$$

where $\vec{f}$ is the Lorentz force. This last equation describes ***momentum flux*** (since the density of momentum is $\vec{p} = \vec{S}/c^2$), including the force that is exerted on the system ($\vec{f} = \rho \vec{E} + \vec{J} \times \vec{B}$). The 3-divergence $\partial_i T^{ij}$ includes both the ***pressure gradient*** ($\vec{\nabla} P$) and the ***shear*** ($\partial_i \Pi^{ij}$).

Next class:

- The Maxwell action and Lagrangian
- Another derivation of the stress-energy tensor
- Back to Maxwell's equations
- L. Landau, "The Classical Theory of Fields", Ch. 4