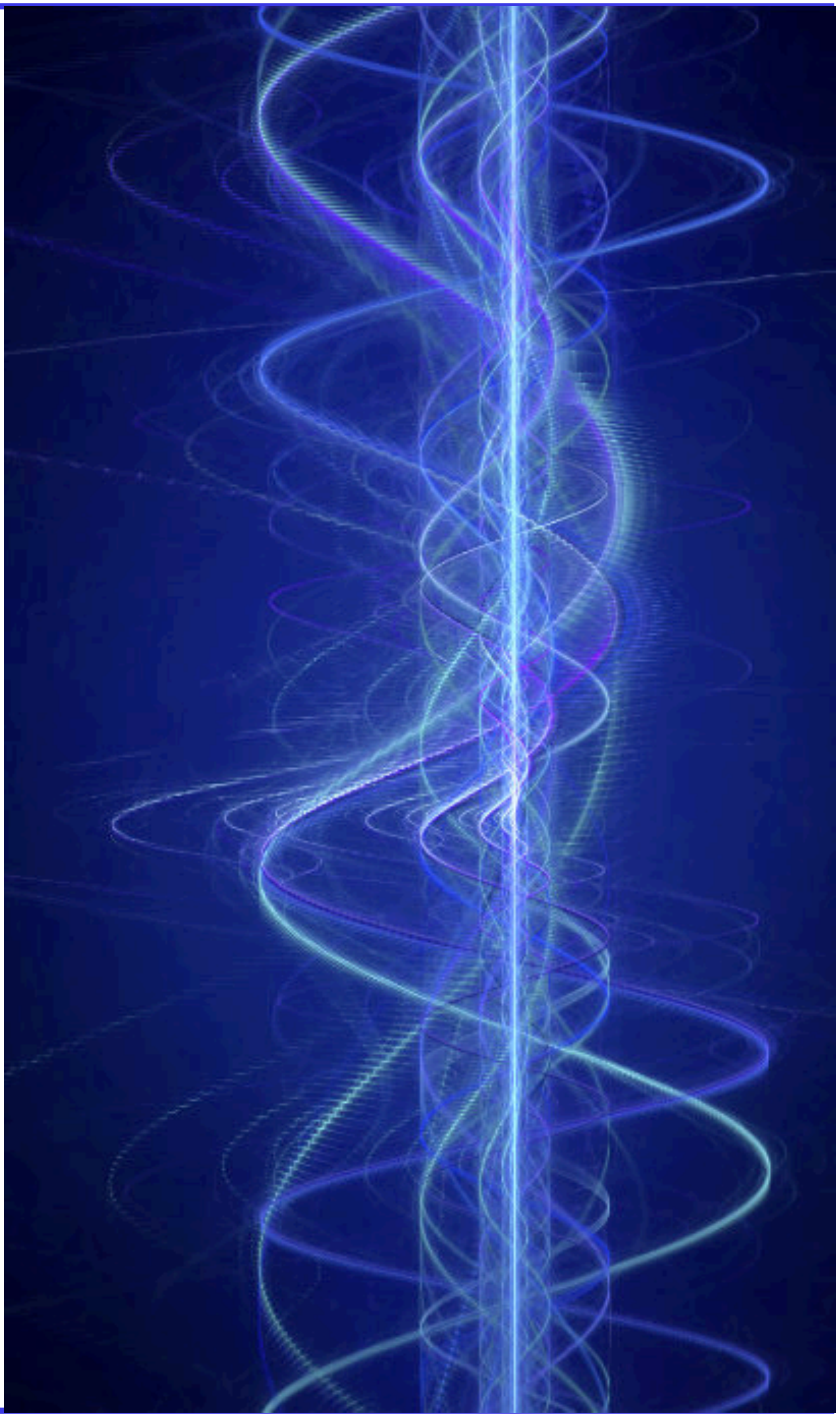
Electrodynamics

⚡ **The wave equation**

⚡ **Green's function**

⚡ **Relativity**



The wave equation

- In the last class we arrived at the Maxwell Equations with sources, which in a general gauge are:

$$-\vec{\nabla}^2\phi - \frac{\partial}{\partial t}\left(\vec{\nabla}\cdot\vec{A}\right) = \frac{\rho}{\epsilon} \quad \text{and}$$

$$\frac{1}{c_s^2}\frac{\partial^2\vec{A}}{\partial t^2} - \vec{\nabla}^2\vec{A} + \vec{\nabla}\left(\vec{\nabla}\cdot\vec{A} + \frac{1}{c_s^2}\frac{\partial\phi}{\partial t}\right) = \mu\,\vec{J}$$

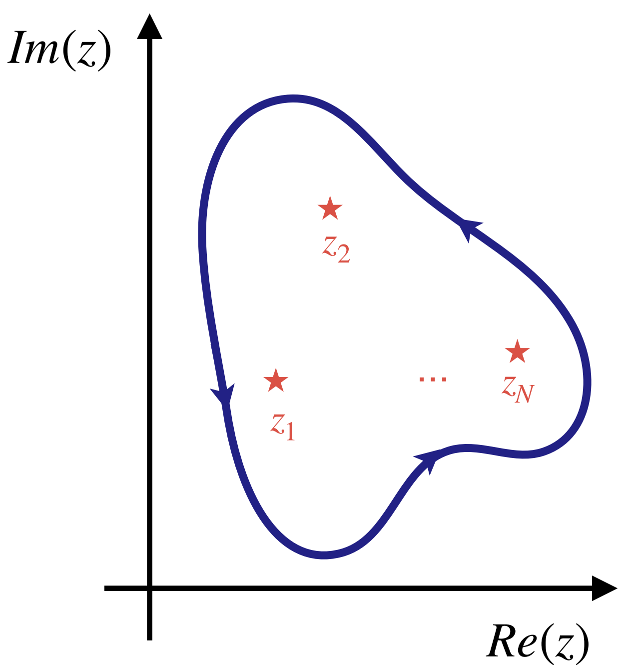
- In the Lorentz gauge, $\vec{\nabla}\cdot\vec{A} + \frac{1}{c_s^2}\frac{\partial\phi}{\partial t} = 0$, and substituting on the Maxwell Eqs. we get the wave equation:

$$\frac{1}{c_s^2}\frac{\partial^2\phi}{\partial t^2} - \vec{\nabla}^2\phi = -\Box\phi = \frac{\rho}{\epsilon} \quad , \quad \text{where the D'Alembertian is} \quad \Box = -\frac{1}{c_s^2}\frac{\partial^2}{\partial t^2} + \nabla^2 \quad , \quad \text{and}$$

$$\frac{1}{c_s^2}\frac{\partial^2\vec{A}}{\partial t^2} - \vec{\nabla}^2\vec{A} = \mu\,\vec{J} \ .$$

- We will now find an exact, formal solution for this equation. In the derivation we will use the Cauchy Theorem, which tells us that:

$$\oint dz\, f(z) = 2\pi\, i \sum_j \mathrm{Res}[f(z_j)]$$



The wave and the Helmoltz equations

- The first step is to go from real ("configuration") space to Fourier space, both for the sources and for the fields themselves. For the spatial part we have:

$$\tilde{f}(\vec{k}) = \int d^3x \, e^{i\vec{k}\cdot\vec{x}} f(\vec{x}) \quad \leftrightarrow \quad f(\vec{x}) = \int \frac{d^3k}{(2\pi)^3} \, e^{-i\vec{k}\vec{x}} \tilde{f}(\vec{k}) \quad ,$$

  and for the time dependence we have:

$$\tilde{f}(\omega) = \int dt \, e^{-i\omega t} f(t) \quad \leftrightarrow \quad f(t) = \int \frac{d\omega}{2\pi} \, e^{i\omega t} \tilde{f}(\omega)$$

- One of the nicest things about Fourier transforms is that derivatives become:

$$\vec{\nabla} \to -i\vec{k} \quad , \quad \text{and} \quad \frac{\partial}{\partial t} \to i\omega$$

- So, in general a function of space and time (like the fields and charge densities/currents) is:

$$\tilde{f}(\omega, \vec{k}) = \int d^3x \int dt \; e^{i(\vec{k}\cdot\vec{x}-\omega t)} f(t, \vec{x}) \quad \leftrightarrow \quad f(t, \vec{x}) = \int \frac{d^3k}{(2\pi)^3} \int \frac{d\omega}{2\pi} \, e^{-i(\vec{k}\vec{x}-\omega t)} \tilde{f}(\omega, \vec{k})$$

- So, let's now Fourier-transform the equation:

$$\frac{1}{c_s^2}\frac{\partial^2 \phi}{\partial t^2} - \vec{\nabla}^2 \phi \;=\; \frac{\rho}{\epsilon} \quad \Rightarrow \quad -\frac{\omega^2}{c_s^2}\tilde{\phi} + \vec{k}^2 \tilde{\phi} \;=\; \frac{\tilde{\rho}}{\epsilon} \quad , \quad \text{with the immediate solution:} \quad \tilde{\phi} \;=\; \frac{\tilde{\rho}}{\epsilon} \frac{1}{\vec{k}^2 - \frac{\omega^2}{c_s^2}}$$

If we had only Fourier-transformed time, and no sources, this would be the ***Helmholtz equation***:

$$\frac{\omega^2}{c_s^2}\frac{\partial^2 \phi}{\partial t^2} + \vec{\nabla}^2 \phi \;=\; 0$$

The wave equation with sources

- Well, ok, so that's not a "real" solution, what we actually want are the potentials (and fields) in real space, i.e.,

$$\phi(t,\vec{x}) \;=\; \int \frac{d^3k}{(2\pi)^3}\int \frac{d\omega}{2\pi}\, e^{-i(\vec{k}\vec{x}-\omega t)}\,\tilde{\phi}(\omega,\vec{k}) \;=\; \int \frac{d^3k}{(2\pi)^3}\int \frac{d\omega}{2\pi}\, e^{-i(\vec{k}\vec{x}-\omega t)}\,\frac{\tilde{\rho}}{\epsilon}\frac{1}{\vec{k}^2-\frac{\omega^2}{c_s^2}}$$

- Now, the idea is that we substitute the Fourier transform of the charge density back into this equation, and work it out. We have:

$$\tilde{\rho}(\omega,\vec{k}) = \int d^3x' \int dt'\; e^{i(\vec{k}\cdot\vec{x}'-\omega t')}\,\rho(t',\vec{x}') \qquad ,$$

  where we should be careful not to confuse the integration variables $\{t',\vec{x}'\}$ with the time and position where we evaluate the potential $\phi(t,\vec{x})$.

- We have, therefore:

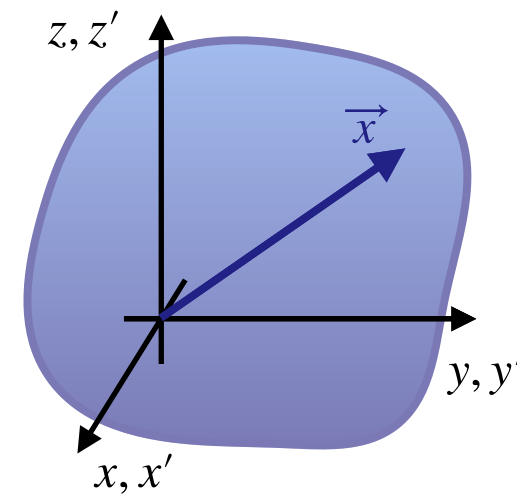
$$\phi(t,\vec{x}) \;=\; \frac{1}{\epsilon}\int \frac{d^3k}{(2\pi)^3}\int \frac{d\omega}{2\pi}\, e^{-i(\vec{k}\vec{x}-\omega t)}\,\frac{1}{\vec{k}^2-\frac{\omega^2}{c_s^2}}\int d^3x' \int dt'\; e^{i(\vec{k}\cdot\vec{x}'-\omega t')}\,\rho(t',\vec{x}')$$

  which we can rewrite as:

$$\phi(t,\vec{x}) \;=\; \frac{1}{\epsilon}\int d^3x' \int dt'\,\rho(t',\vec{x}')\boxed{\int \frac{d^3k}{(2\pi)^3}\int \frac{d\omega}{2\pi}\,\frac{e^{-i\left[\vec{k}(\vec{x}-\vec{x}')-\omega(t-t')\right]}}{\vec{k}^2-\frac{\omega^2}{c_s^2}}}$$

Green's function for the wave equation: $G(t,\vec{x};t',\vec{x}')$

Green's function for the wave equation



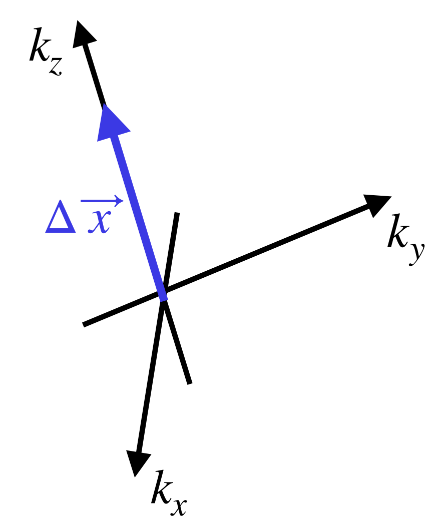
- We have then transformed our problem into a more general one: computing the Green's function for the wave equation:

$$G(t, \vec{x}; t', \vec{x}') = \int \frac{d^3k}{(2\pi)^3} \int \frac{d\omega}{2\pi} \frac{e^{-i\left[\vec{k}(\vec{x} - \vec{x}') - \omega(t - t')\right]}}{\vec{k}^2 - \frac{\omega^2}{c_s^2}} \quad , \quad \text{and}$$

$$\phi(t, \vec{x}) = \int d^3x' \int dt' \frac{\rho(t', \vec{x}')}{\epsilon} G(t, \vec{x}; t', \vec{x}')$$

- In the Green function is best to start with the "spatial" integration. Defining, for simplicity, $\Delta t = t - t'$ and $\Delta \vec{x} = \vec{x} - \vec{x}'$ we have:

$$G(t, \vec{x}; t', \vec{x}') = \int \frac{d\omega}{2\pi} \int \frac{d^3k}{(2\pi)^3} \frac{e^{-i\left(\vec{k} \cdot \Delta\vec{x} - \omega \Delta t\right)}}{\vec{k}^2 - \frac{\omega^2}{c_s^2}}$$



- In this spatial integral over $\vec{k}$ we are completely free to choose the coordinates, so we align the k_z axis with the direction of $\Delta \vec{x}$, with the result that:

$$G(t, \vec{x}; t', \vec{x}') = \int \frac{d\omega}{2\pi} e^{i\omega\Delta t} \int \frac{k^2 dk\, d(\cos\theta_k)\, d\varphi_k}{(2\pi)^3} \frac{e^{-ik\Delta x \cos\theta_k}}{k^2 - \frac{\omega^2}{c_s^2}}$$

which is not much more easy to do!

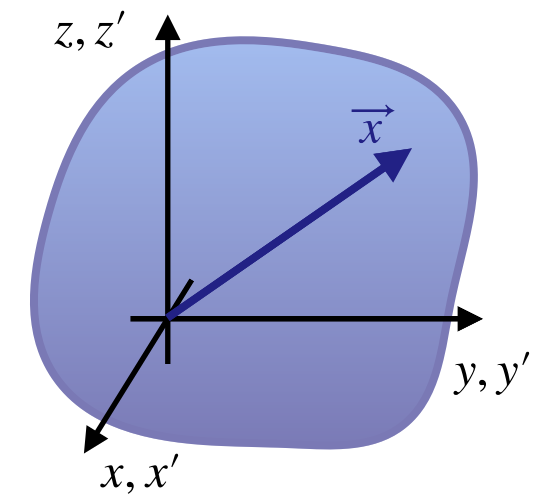
Green's function for the wave equation

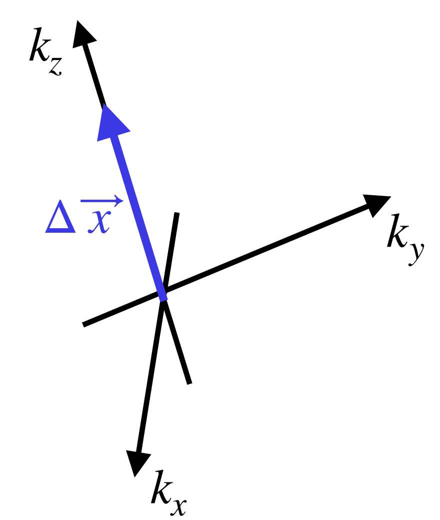
- In this rotated frame the integral becomes (with $\cos\theta_k = \mu$):

$$G(t,\vec{x};t',\vec{x}') = \frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\,e^{i\omega\Delta t}\int_0^\infty k^2dk\int_{-1}^{1}d\mu\,\frac{e^{-ik\Delta x\mu}}{k^2-\frac{\omega^2}{c_s^2}}\int_0^{2\pi}d\varphi_k$$

$$= \frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty}d\omega\,e^{i\omega\Delta t}\int_0^\infty k^2dk\,\frac{1}{k^2-\frac{\omega^2}{c_s^2}}\int_{-1}^{1}d\mu\,e^{-ik\Delta x\mu}$$

$$= \frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty}d\omega\,e^{i\omega\Delta t}\int_0^\infty k^2dk\,\frac{1}{k^2-\frac{\omega^2}{c_s^2}}\,\frac{2\,\sin k\Delta x}{k\Delta x}$$





- We now rewrite this expression as:

$$G(t,\vec{x};t',\vec{x}') = \frac{2}{(2\pi)^3}\int_0^\infty k^2dk\,\frac{\sin k\Delta x}{k\Delta x}\int_{-\infty}^{+\infty}d\omega\,\frac{e^{i\omega\Delta t}}{k^2-\frac{\omega^2}{c_s^2}}$$

- The idea now is that we compute the last integral in the complex ω plane, and use the Cauchy Theorem.

Green's function for the wave equation

- Just to recap: we found that the solution to the wave equation is

$$\phi(t,\vec{x}) \;=\; \int d^3x' \int dt'\, \frac{\rho(t',\vec{x}')}{\epsilon} G(t,\vec{x};t',\vec{x}') \quad , \quad \text{with}$$

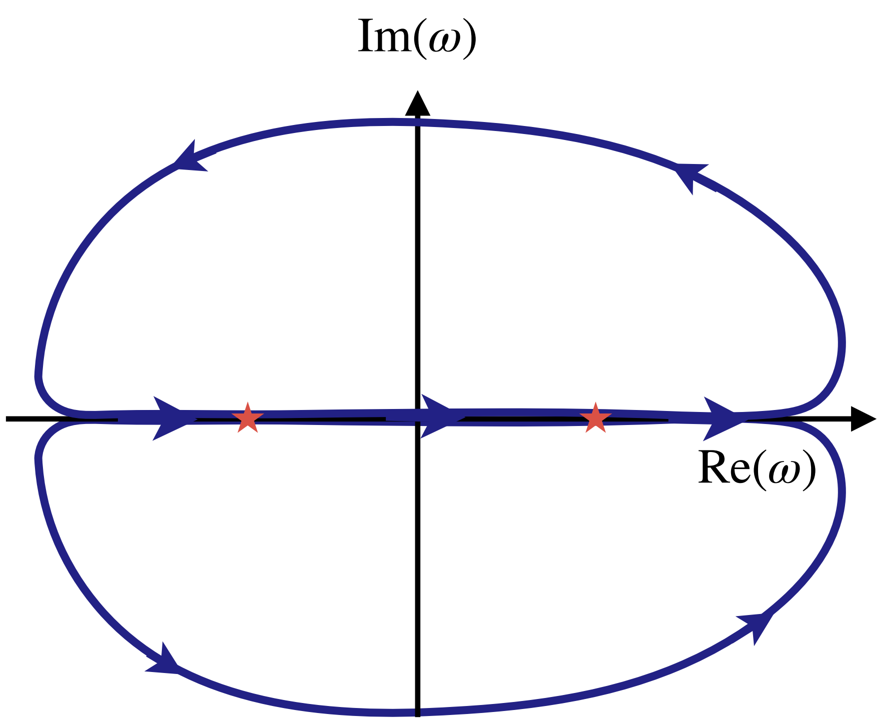
$$G(t,\vec{x};t',\vec{x}') \;=\; \frac{2}{(2\pi)^3}\int_0^\infty dk\,k^2\,\frac{\sin k\Delta x}{k\Delta x}\int_{-\infty}^{+\infty} d\omega\,\frac{e^{i\omega\Delta t}}{k^2-\frac{\omega^2}{c_s^2}}$$

- The integrand of the last integral is the Fourier transform of the wave operator (the D'Alembertian), which is also usually called the **propagator** — since it is responsible for propagating the signal from the source.
- The propagator has **poles** at $\omega = \pm\, c_s\, k$, and these are simple (first order) poles:

$$\frac{1}{k^2-\frac{\omega^2}{c_s^2}} = \frac{1}{\left(k-\frac{\omega}{c_s}\right)\left(k+\frac{\omega}{c_s}\right)}$$

- From the sign of the exponential, $e^{i\omega\Delta t}$, we see that in order to close the contour we should come back on the upper side, $\mathrm{Im}(\omega)\to\infty$, if $\Delta t > 0$, and on the lower side, $\mathrm{Im}(\omega)\to-\infty$, if $\Delta t < 0$. This is in fact connected to the question about where do we put the poles: inside or outside the contour? (If they go on the outside, then the result is zero!)
- But look at the expression for ϕ: it gives us the field at time t that is generated by a source at time t'. Therefore, it is clear that we should choose $\Delta t = t - t' > 0$, so that the "effect" (the potential) appears ***after*** the "cause" (the sources/charges).
- Therefore, we pick the **upper** side of the contour, which gives us the standard sign for the Cauchy Theorem:

$$\oint dz\, f(z) = +\,2\pi\, i \sum_j \mathrm{Res}[f(z_j)]$$



Green's function for the wave equation

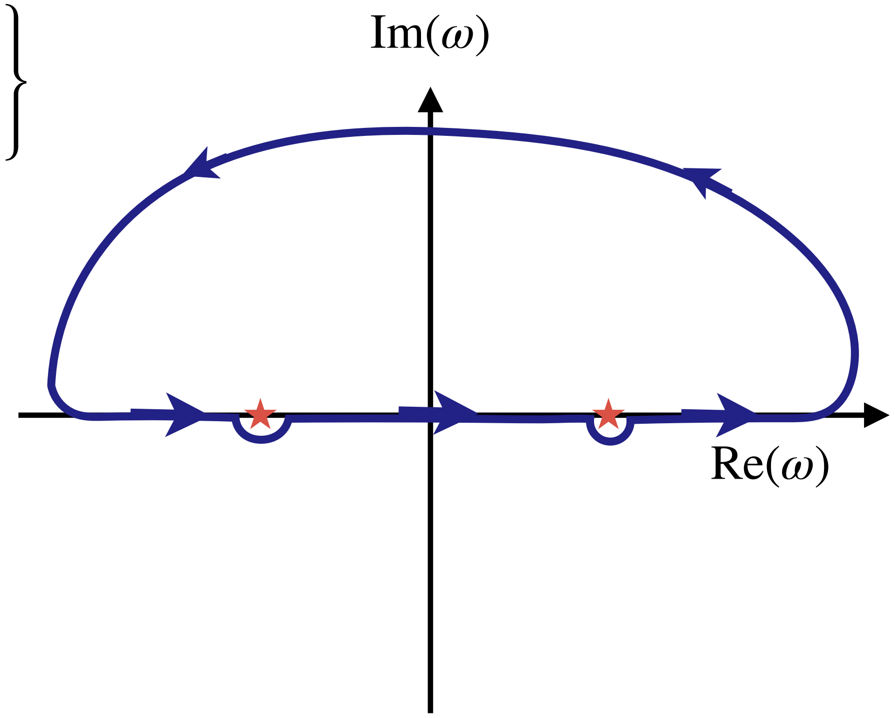
- Closing the circuit from above we obtain that the integral:

$$\int_{-\infty}^{+\infty} d\omega \frac{e^{i\omega\Delta t}}{\left(k-\frac{\omega}{c_s}\right)\left(k+\frac{\omega}{c_s}\right)} = -c_s^2 \int_{-\infty}^{+\infty} d\omega \frac{e^{i\omega\Delta t}}{(\omega - kc_s)(\omega + kc_s)}$$

$$= -c_s^2\, 2\pi i \left\{ \left[\frac{e^{i\omega\Delta t}}{(\omega - kc_s)(\omega + kc_s)} (\omega - kc_s) \right]_{\omega \to kc_s} + \left[\frac{e^{i\omega\Delta t}}{(\omega - kc_s)(\omega + kc_s)} (\omega + kc_s) \right]_{\omega \to -kc_s} \right\}$$

$$= -c_s^2\, 2\pi i \left\{ \frac{e^{ikc_s\Delta t}}{2kc_s} + \frac{e^{-ikc_s\Delta t}}{-2kc_s} \right\}$$

$$= -c_s^2\, 2\pi i \times i \frac{\sin kc_s\Delta t}{kc_s} = 2\pi\, c_s^2 \frac{\sin kc_s \Delta t}{kc_s}$$



- Substituting back on the Green's function we get:

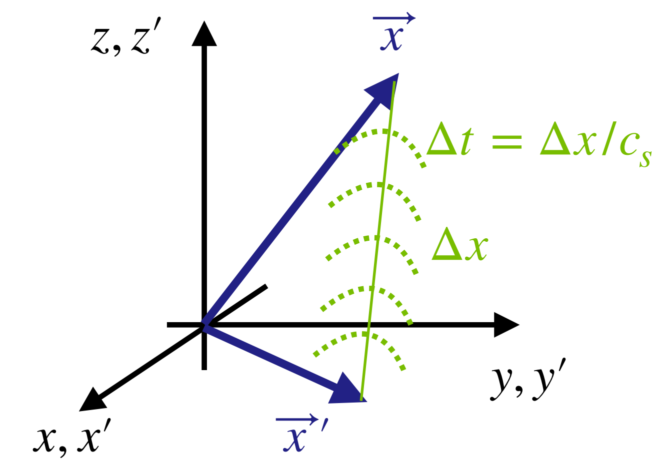
$$G(t, \vec{x}; t', \vec{x}') = \frac{2\, c_s^2}{(2\pi)^2} \int_0^\infty dk\, k^2 \frac{\sin k\Delta x}{k\Delta x} \frac{\sin kc_s\Delta t}{kc_s}$$

$$= \frac{c_s}{2\pi^2\, \Delta x} \int_0^\infty dk \frac{\cos(k\Delta x - kc_s\Delta t) - \cos(k\Delta x + kc_s\Delta t)}{2}$$

- We are almost done here. All we need to do now is to notice that the integrand above is even (symmetric) under $k \leftrightarrow -k$, which means that we can write:

$$G(t, \vec{x}; t', \vec{x}') = \frac{c_s}{4\pi^2\, \Delta x} \frac{1}{2} \int_{-\infty}^{\infty} dk \left[\cos(k\Delta x - kc_s\Delta t) - \cos(k\Delta x + kc_s\Delta t)\right]$$

Green's function for the wave equation



- Finally, remember that $\cos \alpha = \mathrm{Re}[e^{i\alpha}]$, so we have that:

$$G(t, \vec{x}; t', \vec{x}\,') = \frac{c_s}{8\pi^2 \,\Delta x} \,\mathrm{Re} \int_{-\infty}^{\infty} dk \,\left[e^{k(\Delta x - c_s \Delta t)} - e^{k(\Delta x + c_s \Delta t)}\right]$$

$$= \frac{c_s}{8\pi^2 \,\Delta x} \,\mathrm{Re}\, 2\pi \left[\delta(\Delta x - c_s \Delta t) - \delta(\Delta x + c_s \Delta t)\right]$$

- Now, recall that we assumed that $\Delta t > 0$, and since $\Delta x = |\vec{x} - \vec{x}\,'| > 0$, the second delta function is identically zero, always. This leaves us then with the final result:

$$G(t, \vec{x}; t', \vec{x}\,') = \frac{c_s}{4\pi \,\Delta x} \,\delta(\Delta x - c_s \Delta t) \qquad \text{or better still,} \qquad G(t, \vec{x}; t', \vec{x}\,') = \frac{c_s \,\theta(\Delta t)}{4\pi \,\Delta x} \,\delta(\Delta x - c_s \Delta t)$$

- This is called the **Retarded Green's Function** for the wave equation (and for Electromagnetism!).
- The argument of the delta function means that there will only be a response to the source at $\{t', \vec{x}\,'\}$ at the position $\vec{x}$ after some time $\Delta t = \Delta x / c_s$, i.e., at $t = t' + \Delta x / c_s$. This is called **retarded time**.
- Notice that we can use the transformation properties of Dirac delta functions to write:

$$\delta(\Delta x - c_s \Delta t) = \frac{1}{c_s} \delta\left(\frac{\Delta x}{c_s} - \Delta t\right) = \frac{1}{c_s} \delta\left[\frac{\Delta x}{c_s} - (t - t')\right] = \frac{1}{c_s} \delta\left[t' - \left(t - \frac{\Delta x}{c_s}\right)\right]$$

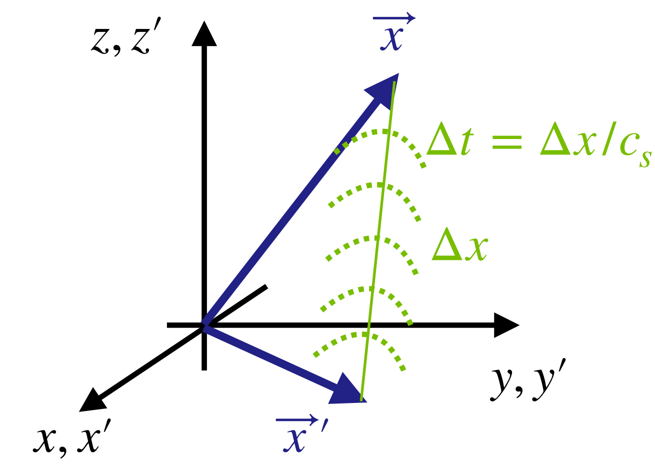
Green's function for the wave equation

- We can finally plug the retarded Green's function in this final form,

$$G(t,\vec{x};t',\vec{x}') = \frac{1}{4\pi\,\Delta x}\delta(t'-t+\Delta x/c_s) \quad , \text{ into the integral for the potential:}$$

$$\phi(t,\vec{x}) = \int d^3x' \int dt' \frac{\rho(t',\vec{x}')}{\epsilon} G(t,\vec{x};t',\vec{x}') \quad , \text{ resulting in:}$$

$$\phi(t,\vec{x}) = \frac{1}{4\pi\epsilon}\int d^3x' \frac{\rho(t_{Ret},\vec{x}')}{|\vec{x}-\vec{x}'|} \quad , \quad \text{where} \quad t_{Ret} = t - \Delta x/c_s$$



- The interpretation is clear: if you move a charge density at some position $\vec{x}'$, the potential at the position $\vec{x}$ will only respond to that change after a time $\Delta x/c_s$.

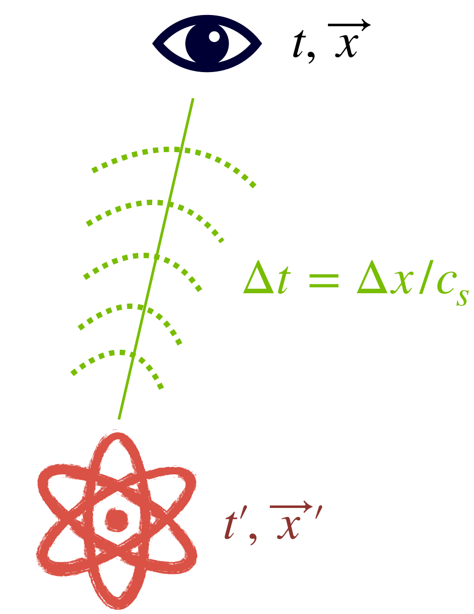
- In other words: the information from the source propagates with a velocity c_s, which in vacuum is $c = \sqrt{1/\mu_0\epsilon_0}$=299,792,458 m/s .

- But what about the vector potential? Well, the equation (at least in Cartesian coordinates) is exactly the same, so the solution is exactly the same:

$$\vec{A}(t,\vec{x}) = \frac{\mu}{4\pi}\int d^3x' \frac{\vec{J}(t_{Ret},\vec{x}')}{|\vec{x}-\vec{x}'|}$$

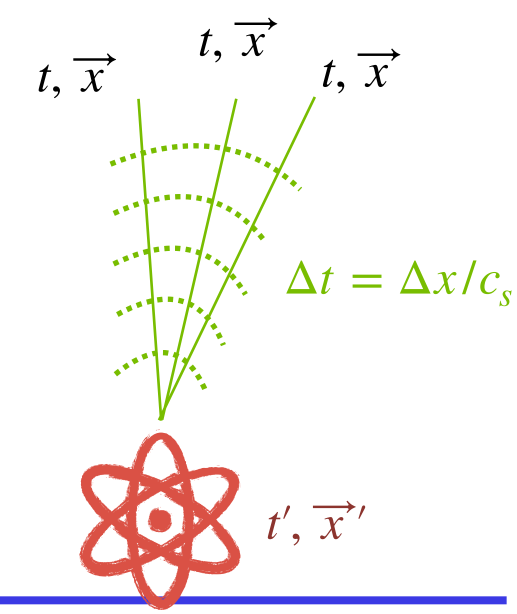
Retarded and advanced Green's functions

- We should now revisit our calculation and try to understand better the nature of our choices.
- For instance, our choice to impose that $\Delta t > 0$ assumes that the field is caused by the charges, and not vice-versa. This is true almost always, and the picture is more or less what we show in the figure on the right: some charges or currents move, and the signal propagates in space and in time, changing the potential at a distant point, and at a later time.



- But imagine now that we invert the arrow of time, in such a way that a configuration of the fields are created such that they propagate a signal towards a place where we have charges, and that these fields, when they get there, move the charges in exactly the right way to mimick the motion that we described earlier. What then??
- In that case we would be motivated to do the opposite: chose $\Delta t < 0$, and carry out a slightly different calculation that, instead of finding a retarded time in our final Green's function, we would find an advanced time:

$$t' = t_{Adv} = t + \Delta x / c_s \quad \text{and} \quad \phi(t, \vec{x}) = \frac{1}{4\pi\epsilon} \int d^3x' \, \frac{\rho(t_{Adv}, \vec{x}')}{|\vec{x} - \vec{x}'|}$$



- But notice that in this case the fields are the cause, and the charge/current motions are the effect! So, after all, what emerges is, again, a causal picture — except that in this case is something that "smells" funny, almost like it violates the second law of thermodynamics!

The Klein-Gordon equation

- Since we came this far, it is interesting to look at similar physical theories and extend our results. In particular, the classical field theory associated with a **scalar field Φ with a mass m** results in the **Klein-Gordon equation** for the field:

$$-\Box\Phi + \frac{m^2c^2}{\hbar^2}\Phi \;=\; \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} - \vec{\nabla}^2\Phi + \frac{m^2c^2}{\hbar^2}\Phi \;=\; s$$

(We say that the field is **free** is the source term is zero.)

- Performing a calculation that is identical to the one we just did, we find that the Green's function is given in terms of the following integral:

$$G(t,\vec{x};t',\vec{x}') \;=\; \int\frac{d^3k}{(2\pi)^3}\int\frac{d\omega}{2\pi}\;\frac{e^{-i\left[\vec{k}(\vec{x}-\vec{x}')-\omega(t-t')\right]}}{\vec{k}^2-\frac{\omega^2}{c^2}+\frac{m^2c^2}{\hbar^2}}$$

- At this point we should simplify things. It is quite ridiculous that we keep writing the space-time distance in the exponent in *that* way, and the norm of the 4-momentum of the denominator in *that* way. Furthermore, let's use "reasonable" units here , and set $\hbar = c = 1$. We then have:

$$G(x^\mu;x'^\mu) \;=\; \int\frac{d^4k}{(2\pi)^4}\,\frac{e^{-i\,k_\mu\,\Delta x^\mu}}{||k||^2+m^2} \qquad ,$$

where $x^\mu = \{t,\vec{x}\}$, $k^\mu = \{w,\vec{k}\}$, and $\Delta x^\mu = x^\mu - x'^\mu$ $(\mu = 0,1,2,3)$.

We have also used the Minkowski metric, $\eta_{\mu\nu} = \mathrm{diag}\{-1,1,1,1\}$, to expressed the 4-norm of the momentum as $||k||^2 = -\,\omega^2 + \vec{k}^2 = \sum_{\mu\nu}\eta_{\mu\nu}k^\mu k^\nu = \eta_{\mu\nu}k^\mu k^\nu$, and the phase as $k_\mu\Delta x^\mu = \eta_{\mu\nu}k^\mu\Delta x^\nu = -\,\omega\Delta t + \vec{k}\cdot\Delta\vec{x}$.

Einstein sum convention: repeated indices are assumed to be summed $\sum_\mu A_{\mu\nu}B^{\mu\alpha} \to A_{\mu\nu}B^{\mu\alpha}$

- The D'Alembertian is in fact the 3D+1 Laplacian, $\Box = -\,\partial^2/\partial t^2 + \nabla^2 = \eta_{\mu\nu}\,\partial_\mu\,\partial_\nu$!

The Klein-Gordon equation

- The same question appears now as we had before: what do we do with the poles of the propagator ?

$$\omega^\star = \pm\sqrt{\vec{k}^2 + m^2}$$

- A useful way to express the choice of where we include the poles of the propagator is to introduce the limit:

$$G(x^\mu; x'^\mu) \;=\; \int \frac{d^4k}{(2\pi)^4} \frac{e^{-i\,k_\mu\,x^\mu}}{\vec{k}^2 - \omega^2 + m^2 \pm i\epsilon} \quad , \text{ or also:}$$

$$G(x^\mu; x'^\mu) \;=\; \int \frac{d^4k}{(2\pi)^4} \frac{e^{-i\,k_\mu\,x^\mu}}{\vec{k}^2 - (\omega \pm i\epsilon)^2 + m^2}$$

- If we choose $t > t'$ (so, $\Delta t = \Delta x^0 > 0$), we choose ϵ in such a way that we include the poles in the circuit, we obtain that :

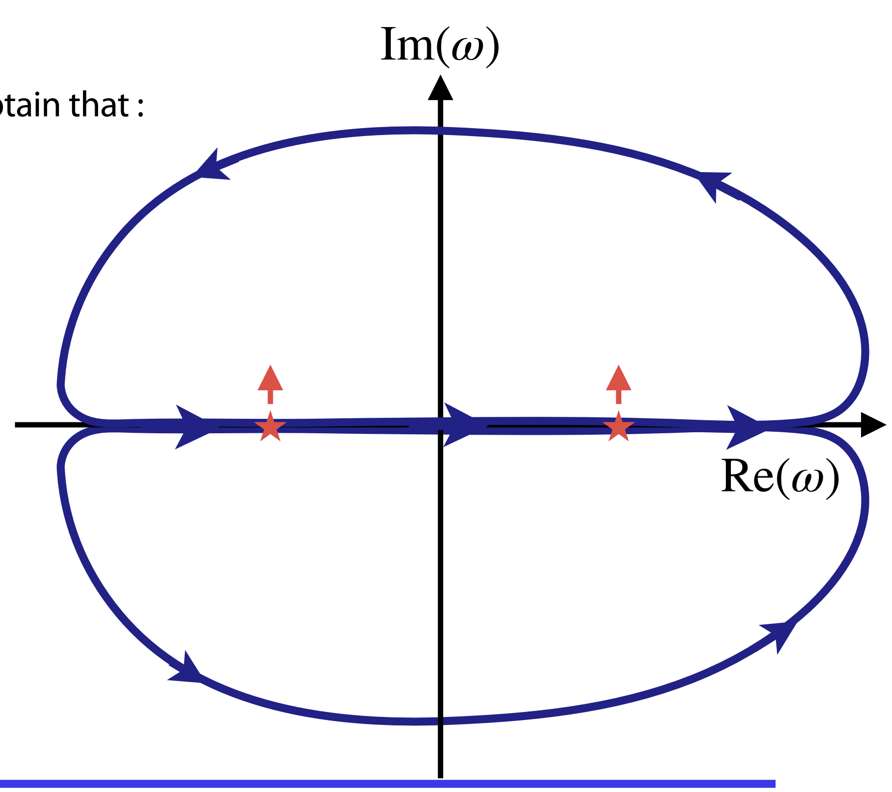
$$G_{Ret} \;=\; \theta(\Delta x^0)\left[\frac{1}{4\pi\,\Delta x}\,\delta(\Delta x - \Delta t) - \theta(\Delta)\frac{m\,J_1(m\Delta)}{4\pi\Delta}\right] \quad ,$$

where $\Delta\tau^2 = \Delta t^2 - \Delta\vec{x}^2$. In fact, we can use rewrite the Dirac delta function above to express :

$$G_{Ret} \;=\; \theta(\Delta x^0)\left[\frac{1}{2\pi}\,\delta(\Delta\tau^2) - \theta(\Delta\tau)\frac{m\,J_1(m\Delta\tau)}{4\pi\Delta\tau}\right]$$

- In a similar fashion the advanced Green's function can be written as:

$$G_{Adv} \;=\; \theta(-\Delta x^0)\left[\frac{1}{2\pi}\,\delta(\Delta\tau^2) - \theta(\Delta\tau)\frac{m\,J_1(m\Delta\tau)}{4\pi\Delta\tau}\right]$$



The Klein-Gordon equation

- Finally, it is interesting to connect these concepts to the quantization of fields. We define the commutator of the fields (which should be regarded as operators) as:

$$\left[\Phi(x), \Phi(x')\right] = \Phi(x)\,\Phi(x') - \Phi(x')\,\Phi(x)$$

- The vacuum expectation value of this commutator can be computed, and it turns out to be:

$$\langle 0 | \left[\Phi(x), \Phi(x')\right] | 0 \rangle \theta(\Delta t) = i G_{Ret}(x, x')$$

- The converse is valid if we invert the time ordering:

$$\langle 0 | \left[\Phi(x), \Phi(x')\right] | 0 \rangle \theta(-\Delta t) = -\, i G_{Adv}(x, x')$$

- Finally, a fundamental object that we encounter in quantum field theory is the Feynman propagator, or Feynman Green's function, which is the time-ordered expectation value:

$$\langle 0 | T \left[\Phi(x)\Phi(x')\right] | 0 \rangle = \langle 0 | \Phi(x)\Phi(x')\theta(\Delta t) + \Phi(x')\Phi(x)\theta(-\Delta t) | 0 \rangle = i G_{Feyn}(x, x')$$

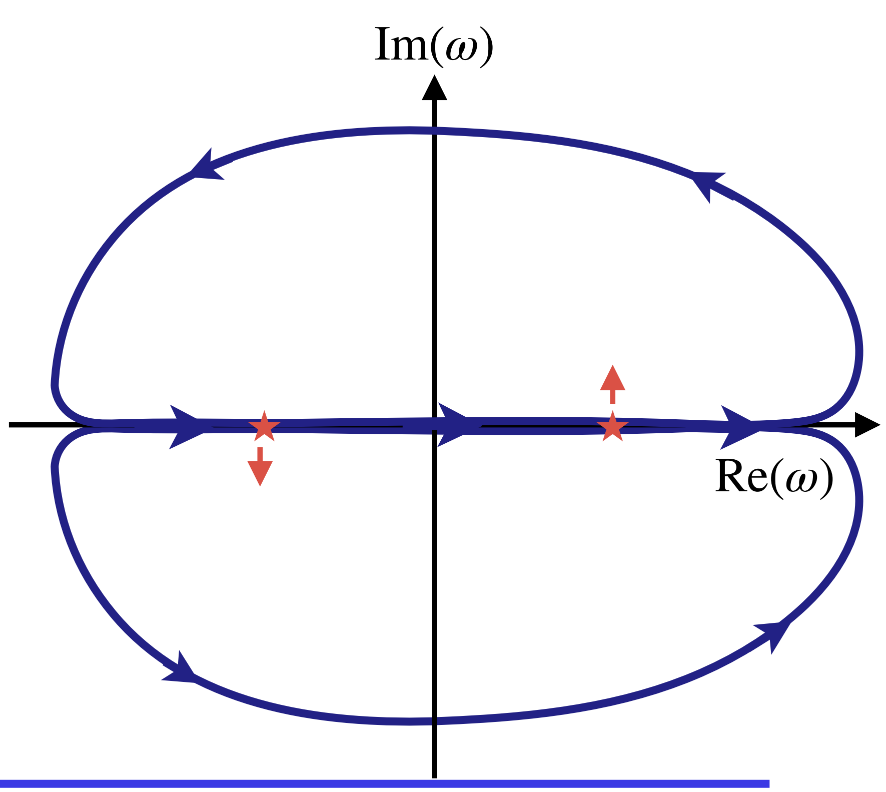
- This type of object appears all the time when we compute amplitudes (**Feynman diagrams**) in quantum field theory, and arises naturally in the **path integral formalism**.

- As it turns out, this propagator is found by **mixing** the poles: we include one, but not the other:

$$G_{Feyn} = \lim_{\epsilon \to 0} \int \frac{d^4 k}{(2\pi)^4} \frac{e^{-i k_\mu x^\mu}}{\vec{k}^2 - \omega^2 + m^2 - i\epsilon}$$

$$\Rightarrow \quad G_{Feyn} = \theta(\Delta\tau^2)\left[\frac{1}{4\pi}\,\delta(\Delta\tau^2) - \frac{m\, H_1(m\Delta)}{8\pi\Delta\tau}\right] + \theta(-\Delta\tau^2)\frac{i\, m\, K_1(-im\Delta\tau)}{4\pi^2\Delta\tau}$$

- For a full discussion of propagators and their interpretation in terms of causality v. non-locality, see Peskin & Schroeder, Ch. 2

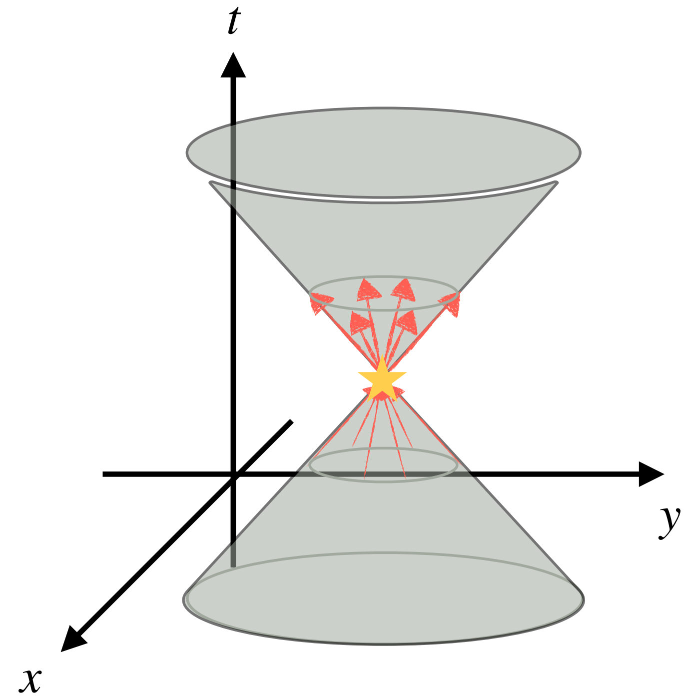


The light cone

- Today's results have introduced to us a concept that is central not only in Electrodynamics, but in all relativistic field theories: the light cone, which we define in terms of:

$$\Delta\tau^2 = \Delta t^2 - \Delta\vec{x}^2 = -\,\eta_{\mu\nu}\,\Delta x^\mu\,\Delta x^\nu$$

- Think of this diagram in terms of the Green's function: the star is some source of the field (at t', $\vec{x}'$), while the observer is in some generic position t, $\vec{x}$.
- The **light cone** is defined as $\Delta\tau = 0$, which we can visualize on the time-space diagram on the right.
- The region of the light cone with $\Delta t > 0\ (t > t')$ is called the **future light cone** of t' . The retarded Green's function tells us that signals from the source can only travel along the future light cone.
- On the other hand the advanced Green's function tells us that only fields propagating over the past light cone can affect the source.
- Objects outside the light cone cannot affect the source, or be affected by it. Now, ***that's*** causality!



Next class:

- Relativity and Electrodynamics
- Lorentz transformations
- The electromagnetic field and the Faraday tensor
- Jackson, Ch. 11; Zangwill, Ch. 22; your favorite Special Relativity book.
- (Another good reference: Bo Thidé's book, http://docente.unife.it/guido.zavattini/allegati/251023059-electromagnetic-field-theory-bo-thide.pdf , Ch. 5)