Eletro: mini-seminários, datas e P1

Proposta:

* Mini-seminários dos alunos sobre temas de livre escolha: dia 7/Maio (terça-feira); caso haja demanda, fazemos uma outra sessão dia 11/Maio (sexta-feira)

Formulário para coletar propostas de mini-seminários: https://forms.gle/AnPxGQoejAPJvQMT6

* Apresentações podem valer nota (?)

* Entrega da 3a Lista dia 11/Maio (terça-feira)

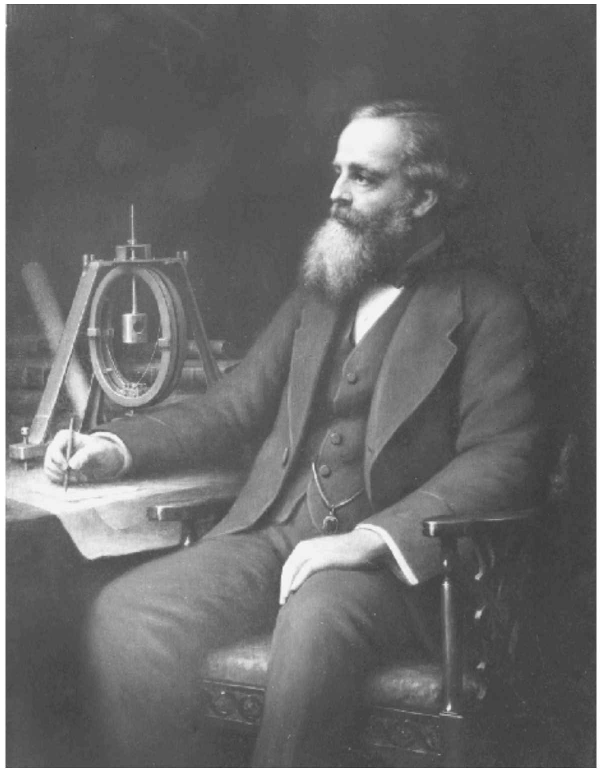
* Prova P1 dia 14/Maio (sexta-feira)

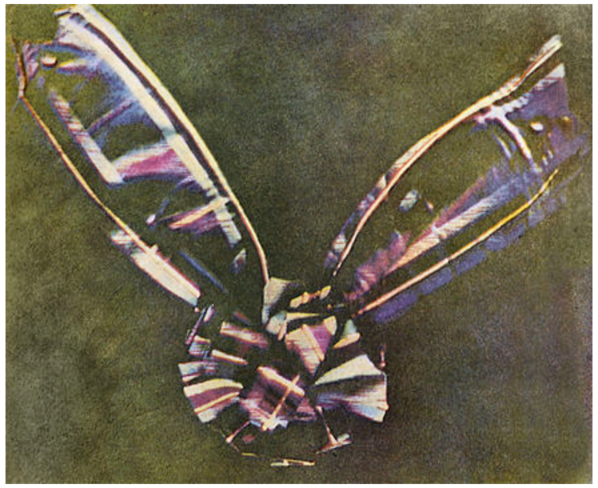
Electrodynamics!

- ⚡ Maxwell's equations
- ⚡ Electromagnetic potentials
- ⚡ Gauge invariance
- ⚡ Averaged equations in materials

Maxwell's Equations

- James Clerk Maxwell (1831-1879) was a Scottish mathematical physicist. At age 19 he was already an accomplished mathematician in Edinburgh, and came to Cambridge on a scholarship, graduating in 1854.
- He stayed two more years at Cambridge, then moved to Aberdeen (Marischal College, 1856-1860), after which he was offered a job at King's College (London).
- It was at King's College that Maxwell started to get an interest in electricity and magnetism, by way of his life-long interest in color, light and photography — he was in fact one of the first to develop color photography! (He was also a theologist, a poet, and made seminal contributions to fields such as control engineering!)
- During his tenure at King's College Maxwell also started to get an interest in hydrodynamics, and was introduced to Michael Faraday — who at the time was already quite old.
- It was around this time that Faraday introduced the concept of a field to Maxwell, and presented to him the challenge of understanding the nature of the laws of electricity and magnetism, which were incomplete at that time. This was decisive for Maxwell to turn his attention to the area.
- Maxwell's first paper on the subject (1855) was about "Faraday's lines of force". In 1861 he described Faraday's law of induction in terms of magnetic flux, and around 1862 he discussed the displacement current.
- In 1862 he also calculated the speed of an electromagnetic wave, coming to the conclusion that it was too close to the measured speed of light to be a coincidence. This led him to write the paper "A Dynamical Theory of the Electromagnetic Field". Those works were later collected in his book, "A Treatise on Electricity and Magnetism".
- It is interesting to point out that Maxwell originally described Electrodynamics using 20 equations in the language of *quaternions* (for t,x,y,z) . It was Oliver Heaviside that reduced them to the four equations that we are familiar with!





The equations of electrodynamics

- From ~1830 until ~1861, the laws of electricity and magnetism were:

$$\vec{\nabla} \cdot \vec{D} = \rho$$ (Gauss's law for the electric field)

$$\vec{\nabla} \cdot \vec{B} = 0$$ (Gauss's law for the magnetic field)

$$\vec{\nabla} \times \vec{H} = \vec{J}$$ (Ampère's law)

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$ (Faraday's law)

- All these equations had been derived using stationary charges and/or currents (including Faraday's law, which was mostly tested using magnets).
- It was a bit of a mystery why these laws would not still be valid when there is a time dependence of charges, currents and the fields themselves.
- The main problem with the existing equations was in Ampère's law. It is a mathematical identity that:

$$\vec{\nabla} \cdot \left(\vec{\nabla} \times \vec{H} \right) = 0$$, so Ampère's law can **only** be valid if:

$$\vec{\nabla} \cdot \vec{J} = -\frac{\partial \rho}{\partial t} \rightarrow 0$$

The equations of electrodynamics

- However, we can rewrite the continuity equation as:

$$\vec{\nabla} \cdot \vec{J} = -\frac{\partial \rho}{\partial t} = -\frac{\partial \left(\vec{\nabla} \cdot \vec{D} \right)}{\partial t} \quad , \qquad \text{so we have:}$$

$$\vec{\nabla} \cdot \left(\vec{J} + \frac{\partial \vec{D}}{\partial t} \right) = 0 \quad ,$$

where $\partial \vec{D} / \partial t$ is called Maxwell's **displacement current**.

- Therefore, we can “fix" Ampère's law by substituting:

$$\vec{\nabla} \times \vec{H} = \vec{J} \quad \rightarrow \quad \vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

- The meaning of this equation is that time-varying electric fields can also induce magnetic fields, even in the absence of a current. In other words, a “displacement" of the electric field with time is equivalent to a current.

The equations of electrodynamics

- It is interesting that Maxwell arrived at these equations and conclusions in a way that is much less direct than we showed above.
- Maxwell was thinking about electric and magnetic fields in terms of "molecular vortices", and about fields as their mechanical manifestations in materials. In his words:

  "I propose to examine magnetic phenomena from a mechanical point of view, and to determine which tensions or movements in the medium will produce the mechanical phenomena that we observe"

- He use fluid dynamics analogies:

  "I do not wish to explain the origin of the observed forces through the effects of those forces and tensions in elastic materials, but to take advantage of the mathematical analogies between the two problems"

  He was referring to the notion of continuous fields that describe charge densities, currents, as analogous to fluids, gases, and elastic media.

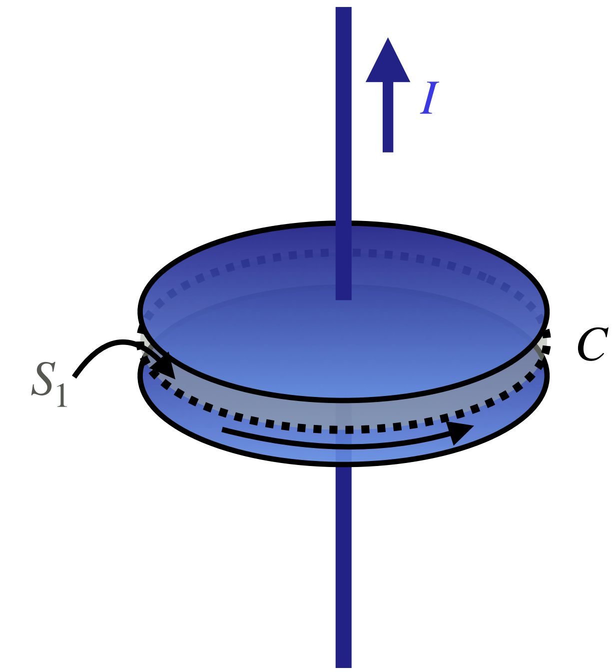
- It is apparent that Maxwell was trying to understand these "molecular vortices" (the magnetic fields) in a way similar to the linear polarization of dielectric media, using a mechanical model.
- In the end, all that matters is the field description: continuous distributions of charges and currents, and the fields that fill up the space in a continuous way.

The equations of electrodynamics

- Going back to the displacement current, the classic example that shows the need for it is that of a capacitor that is being charged up.
- From the original Ampère law we have that:

$$\vec{\nabla} \times \vec{H} = \vec{J}$$

$$\Rightarrow \quad \int_{S(C)} d\vec{S} \left(\vec{\nabla} \times \vec{H} \right) = \oint_C d\vec{l} \cdot \vec{H} = \int_{S(C)} d\vec{S} \cdot \vec{J}$$

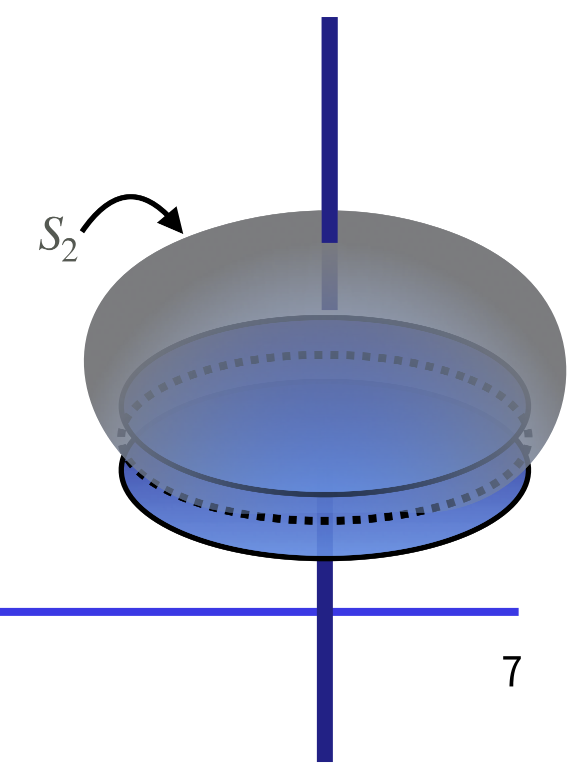


- However, our choice of the surface circumscribed by the circuit C is completely arbitrary: on one hand, we could choose the flat disk described by the circle, and we would get:

$$\int_{S_1(C)} d\vec{S} \cdot \vec{J} = 0$$

- On the other hand, we could equally well choose the surface to be closed on the top of the figure, in such a way that the current I crosses that surface. In that case we have:

$$\int_{S_2(C)} d\vec{S} \cdot \vec{J} = I$$



- The problem is, of course, that the integral on the first surface should include the displacement current, in which case:

$$\int_{S_1(C)} d\vec{S} \cdot \left(\vec{J} + \frac{\partial \vec{D}}{\partial t} \right) = \frac{\partial}{\partial t} \int_{S_1(C)} d\vec{S} \cdot \vec{D} = \frac{\partial Q}{\partial t} = I$$ (to see this, consider the surface right next to a plate)

Maxwell equations for the potentials

- It is useful to phrase Maxwell's equations in terms of the potentials. In Electrostatics and Magnetostatics we had:

$$\vec{E} = -\vec{\nabla}\phi \quad , \quad \vec{B} = \vec{\nabla} \times \vec{A}$$

- From Faraday's law we get:

$$\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = \vec{\nabla} \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t} \right) = 0$$

- But the equation above implies that $\vec{E} + \partial \vec{A} / \partial t$ is a gradient. In fact, in Electrostatics we defined $\vec{E} \to -\vec{\nabla}\phi$, but now, given Faraday's law the correct identification is to assign:

$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}$$

- Since Gauss's law for the magnetic field is unchanged, we don't need to change the definition $\vec{B} = \vec{\nabla} \times \vec{A}$.
- Let's see now what this generalization of the potentials mean in Electrodynamics, when we look at Gauss's law for the electric field, and Ampère's law.

Maxwell equations for the potentials

- From our redefinition of the electric field, Gauss's law is now expressed as a modified Poisson equation:

$$\vec{\nabla} \cdot \vec{E} = \vec{\nabla} \cdot \left(-\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t} \right) = \frac{\rho}{\epsilon}$$

$$\Rightarrow \quad -\vec{\nabla}^2\phi - \frac{\partial}{\partial t}\left(\vec{\nabla} \cdot \vec{A}\right) = \frac{\rho}{\epsilon}$$

- On the other hand, Ampère's law becomes:

$$\vec{\nabla} \times \vec{B} \ = \ \mu \left(\vec{J} + \epsilon \frac{\partial \vec{E}}{\partial t} \right)$$

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{A} \right) \ = \ \mu \left[\vec{J} + \epsilon \frac{\partial}{\partial t} \left(-\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t} \right) \right]$$

- Defining the "sound speed" $1/c_s^2 = \mu\epsilon$ as the term multiplying $\partial^2 \vec{A}/\partial t^2$ we obtain the equation:

$$\frac{1}{c_s^2}\frac{\partial^2 \vec{A}}{\partial t^2} - \vec{\nabla}^2\vec{A} + \vec{\nabla}\left(\vec{\nabla} \cdot \vec{A} + \frac{1}{c_s^2}\frac{\partial \phi}{\partial t} \right) \ = \ \mu \, \vec{J}$$

Maxwell equations for the potentials

- These two equations for the potentials,

$$-\vec{\nabla}^2\phi - \frac{\partial}{\partial t}\left(\vec{\nabla}\cdot\vec{A}\right) = \frac{\rho}{\epsilon} \qquad \text{and}$$

$$\frac{1}{c_s^2}\frac{\partial^2\vec{A}}{\partial t^2} - \vec{\nabla}^2\vec{A} + \vec{\nabla}\left(\vec{\nabla}\cdot\vec{A} + \frac{1}{c_s^2}\frac{\partial\phi}{\partial t}\right) = \mu\,\vec{J}$$

  may look a bit more complex than Maxwell's equations, but they are the basis upon which we construct methods to solve those equations.

- One of the reasons why these equations are so useful is the fact that we have the **freedom to choose the gauge** that is most convenient for a given problem. This gauge invariance is the fact that Maxwell's equations are invariant under the gauge transformations:

$$\vec{A} \;\to\; \vec{A} + \vec{\nabla}\Lambda$$

$$\phi \;\to\; \phi - \frac{\partial\Lambda}{\partial t}$$

  from which it is obvious that both fields, $\vec{B} = \vec{\nabla}\times\vec{A}$ and $\vec{E} = -\vec{\nabla}\phi - \dfrac{\partial\vec{A}}{\partial t}$, are **invariant** under those changes.

- This was the first time that this fundamental concept of **gauge invariance** entered physics. It is now a cornerstone of the **Standard Model** of particles and fields (and, therefore, Quantum Field Theory), as well as **General Relativity**. It is a generalization of the idea of **invariance under coordinate transformations**: you can, in effect, think of a gauge as a choice for representing your fields, which **does not affect the physical observables**.

Gauge transformations

- The function $\Lambda(t, \vec{x})$ is an unobservable scalar function that can be chosen in a completely arbitrary way. It doesn't have to obey any symmetry or constraint: it is an entirely **free choice** of the user (of Maxwell's equations!).
- When we say that we have fixed a gauge, we mean that we have chosen the function Λ in such a way that we are able to impose some constraint. For example, we can use the **Lorentz gauge** by setting:

$$\vec{\nabla} \cdot \vec{A} + \frac{1}{c_s^2} \frac{\partial \phi}{\partial t} \rightarrow \vec{\nabla} \cdot \left(\vec{A} + \vec{\nabla} \Lambda \right) + \frac{1}{c_s^2} \frac{\partial}{\partial t} \left(\phi - \frac{\partial \Lambda}{\partial t} \right) = 0$$

- Let's say that in an unspecified gauge we had initially the result that:

$$\vec{\nabla} \cdot \vec{A}\,' + \frac{1}{c_s^2} \frac{\partial \phi'}{\partial t} = f(t, \vec{x}) \rightarrow \vec{\nabla} \cdot \left(\vec{A} + \vec{\nabla} \Lambda \right) + \frac{1}{c_s^2} \frac{\partial}{\partial t} \left(\phi - \frac{\partial \Lambda}{\partial t} \right) = f(t, \vec{x})$$

Here $\vec{A}\,'$, ϕ' are the "old" gauge; $\vec{A}$, ϕ are the new (Lorentz) gauge

- All we would need to do then would be to solve the equation for Λ:

$$\vec{\nabla}^2 \Lambda - \frac{1}{c_s^2} \frac{\partial^2 \Lambda}{\partial t^2} = f(t, \vec{x}) \qquad \text{[a wave equation!]}$$

- Substituting this into the condition above we get that:

$$\vec{\nabla} \cdot \vec{A} + \frac{1}{c_s^2} \frac{\partial \phi}{\partial t} = 0$$

Gauge transformations

- Lorentz gauge is extremely useful in Electrodynamics because it yields simple, intuitive equations for the potentials.
- So, let's impose the constraint $\vec{\nabla} \cdot \vec{A} + \frac{1}{c_s^2}\frac{\partial \phi}{\partial t} = 0$ on the Maxwell equations:

$$-\vec{\nabla}^2 \phi - \frac{\partial}{\partial t}\left(\vec{\nabla} \cdot \vec{A}\right) = \frac{\rho}{\epsilon} \quad \text{and}$$

$$\frac{1}{c_s^2}\frac{\partial^2 \vec{A}}{\partial t^2} - \vec{\nabla}^2 \vec{A} + \vec{\nabla}\left(\vec{\nabla} \cdot \vec{A} + \frac{1}{c_s^2}\frac{\partial \phi}{\partial t}\right) = \mu \vec{J}$$

- Substituting the vector potential in the first equation we get:

$$-\vec{\nabla}^2 \phi - \frac{\partial}{\partial t}\left(-\frac{1}{c_s^2}\frac{\partial \phi}{\partial t}\right) = \frac{\rho}{\epsilon} \quad \Rightarrow \quad \frac{1}{c_s^2}\frac{\partial^2 \phi}{\partial t^2} - \vec{\nabla}^2 \phi = \frac{\rho}{\epsilon}$$

- Same thing for the electric potential term in Ampère's law:

$$\frac{1}{c_s^2}\frac{\partial^2 \vec{A}}{\partial t^2} - \vec{\nabla}^2 \vec{A} + \vec{\nabla}\left(\vec{\nabla} \cdot \vec{A} - \vec{\nabla} \cdot \vec{A}\right) = \mu \vec{J} \quad \Rightarrow \quad \frac{1}{c_s^2}\frac{\partial^2 \vec{A}}{\partial t^2} - \vec{\nabla}^2 \vec{A} = \mu \vec{J}$$

- We will come back to these equations later, of course.

Gauge transformations

- Another popular gauge choice is the **Coulomb gauge** (sometimes also called "radiation gauge", or "transverse gauge").
- In that case we use the gauge function to impose the constraint that $\vec{\nabla} \cdot \vec{A} = 0$. In that case the equations read:

$$-\vec{\nabla}^2 \phi = \frac{\rho}{\epsilon} \quad \text{, hence} \quad \phi = \frac{1}{4\pi\epsilon} \vec{\nabla} \int dV' \frac{\rho(t, \vec{x}')}{|\vec{x} - \vec{x}'|} \quad , \quad \text{and}$$

$$\frac{1}{c_s^2} \frac{\partial^2 \vec{A}}{\partial t^2} - \vec{\nabla}^2 \vec{A} + \frac{1}{c_s^2} \frac{\partial \vec{\nabla} \phi}{\partial t} = \mu \vec{J}$$

- If we take $\vec{\nabla} \cdot (\ldots)$ above, we simply verify the continuity equation (an integrability condition for Maxwell's equations).
- The last equation can be further simplified if we observe that taking its rotational, $\vec{\nabla} \times (\ldots)$, gets rid of the potential. But what about the current density?
- We can decompose the current density into a pure gradient ("longitudinal") term, and a pure rotational ("transverse") term using the Helmholtz theorem:

$$\vec{J}_l = -\frac{1}{4\pi} \vec{\nabla} \int dV' \frac{\vec{\nabla}' \cdot \vec{J}}{|\vec{x} - \vec{x}'|} \quad \text{and} \quad \vec{J}_t = \frac{1}{4\pi} \vec{\nabla} \times \vec{\nabla} \times \int dV' \frac{\vec{J}}{|\vec{x} - \vec{x}'|}$$

- Using the continuity equation we find that Gauss's law can now be written as:

$$\epsilon \vec{\nabla} \frac{\partial \phi}{\partial t} = \vec{J}_l \quad \Rightarrow \quad \frac{1}{c_s^2} \vec{\nabla} \frac{\partial \phi}{\partial t} = \mu \vec{J}_l$$

- The second equation now becomes simply:

$$\frac{1}{c_s^2} \frac{\partial^2 \vec{A}}{\partial t^2} - \vec{\nabla}^2 \vec{A} = \mu \vec{J}_t$$

Gauge transformations

- It is important to stress that in this gauge the electric potential **does not** obey a wave-like equation.

- You can, however, relate the potentials in Coulomb gauge $(\vec{A}_C, \phi_C)$ to those in the Lorentz gauge $(\vec{A}_L, \phi_L)$ by determining the function which relates the two. Let's say that we start on Lorentz gauge and we want to obtain the potentials in the Coulomb gauge:

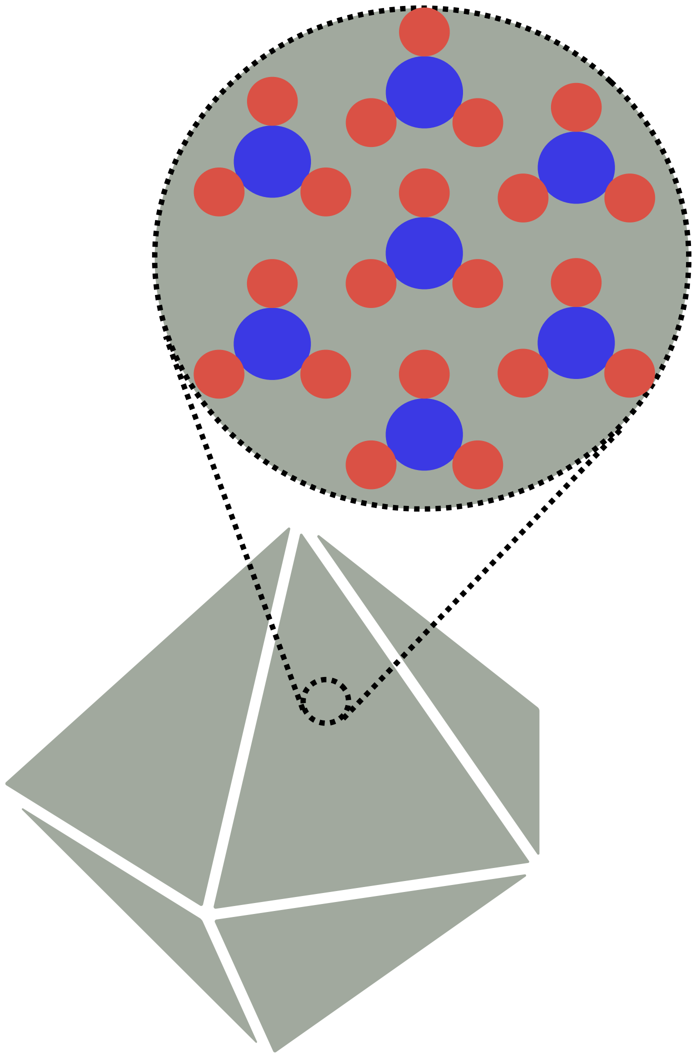
$$\vec{A}_C = \vec{A}_L + \vec{\nabla}\Lambda \quad , \quad \phi_C = \phi_L - \frac{\partial \Lambda}{\partial t}$$

- Now, the function Λ needs to be such that the constraint:

$$\vec{\nabla} \cdot \vec{A}_C = \vec{\nabla} \cdot \left(\vec{A}_L + \vec{\nabla}\Lambda\right) = 0 \quad , \quad \text{therefore} \quad \vec{\nabla}\Lambda = -\vec{A}_L$$

- Therefore, once you compute $\vec{A}_L$ in the Lorentz gauge, you solve for Λ using $\vec{\nabla}\Lambda = -\vec{A}_L$, and then substitute that into $\phi_C = \phi_L - \dfrac{\partial \Lambda}{\partial t}$.

Averaged Maxwell equations



- It is interesting to derive in a different way the Maxwell equations when the fields exist in a region where we find atoms and molecules, which are made of charged "moving parts".
- The typical sizes of molecules and atoms range from 10^{-15} to 10^{-10} m , and timescales are measured by the vibrational models, which are anything from 10^{-12} s and smaller (higher frequencies).
- These are, of course, scales too small (in space and time) for us to measure anything. What we are interested in are the averaged quantities, both in space and in time.
- So, let's define our averages as taken with respect to some interval, or window:

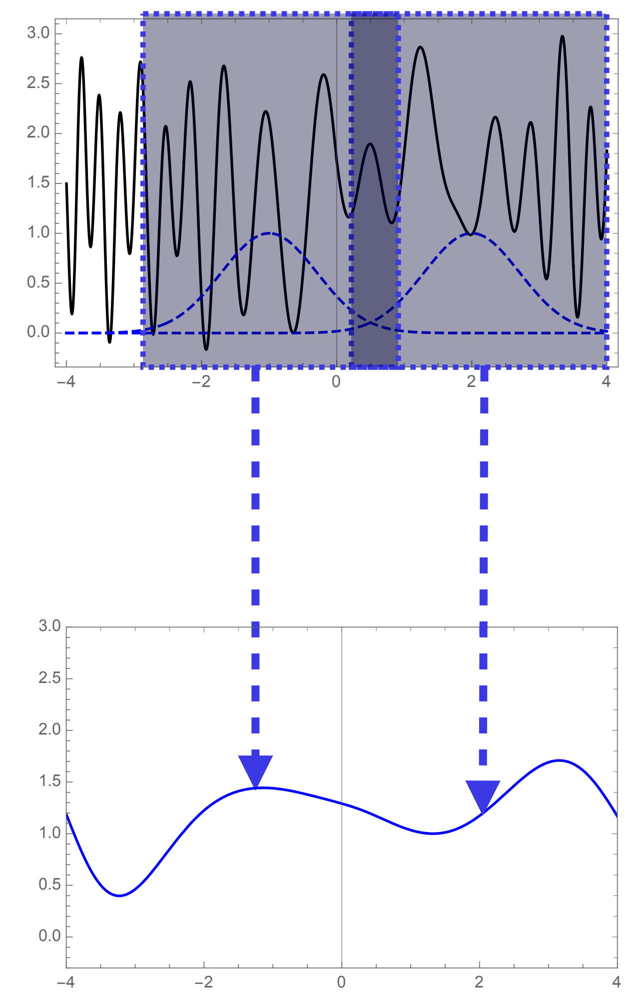
$$\langle f \rangle_V = \int dV' \, W_V(\vec{x}, \vec{x}') f(t, \vec{x}') \qquad \text{and}$$

$$\langle f \rangle_T = \int dt' \, W_T(t, t') f(t', \vec{x})$$

- Here, the window functions in space (W_V) and time (W_T) define intervals inside which we average out these quantities. We usually assume that they are normalized:

$$\int dV' \, W_V(\vec{x}, \vec{x}') = 1 \quad , \quad \int dt' \, W_T(t, t') = 1$$

Averaged Maxwell equations



- Here is a representation of these window functions, and how they act on the function that we want to average.
- On each position, we compute the weighted sum given by the integrals above, and the result is a “smoothed" field.
- An easy function to play with is the Gaussian window, which we define as:

$$W_T(t,t') = \frac{1}{\sqrt{2\pi\Delta T}} e^{-(t-t')^2/2\Delta T^2} \quad ,$$

and the generalization to 3D:

$$W_V(\vec{x},\vec{x}\,') = \frac{1}{(\pi R^2)^{3/2}} e^{-|\vec{x}-\vec{x}\,'|^2/R^2}$$

- The idea now is that we define microscopic and macroscopic (averaged) quantities, for the fields and for the charges/currents. Therefore:

$$\vec{E} = \langle \vec{e} \rangle_{V,T} \quad , \quad \rho = \langle \eta \rangle_{V,T} \quad ,$$

$$\vec{B} = \langle \vec{b} \rangle_{V,T} \quad , \quad \vec{J} = \langle \vec{j} \rangle_{V,T} \quad .$$

- You can also average out the equations themselves. So, for instance, Gauss's law becomes:

$$\epsilon_0 \vec{\nabla} \cdot \vec{e} = \eta \quad \longrightarrow \quad \langle \epsilon_0 \vec{\nabla} \cdot \vec{e} \rangle = \langle \eta \rangle = \rho$$

But the divergence of the average is the average of the divergence (prove it!):

$$\langle \vec{\nabla} \cdot \vec{e} \rangle_V = \int dt'\, W_T(t,t') \int dV'\, W_V(\vec{x},\vec{x}\,')\, \vec{\nabla}' \cdot \vec{e}(t',\vec{x}\,') = \vec{\nabla} \cdot \int dt'\, W_T(t,t') \int dV'\, W_V(\vec{x},\vec{x}\,')\, \vec{e}(t',\vec{x}\,')$$

$$= \vec{\nabla} \cdot \langle \vec{e} \rangle = \vec{\nabla} \cdot \vec{E}$$

Averaged Maxwell equations

- The differential operators in fact commute with the averaging procedure, so we end up with the "macroscopic" equations:

$$\epsilon_0 \vec{\nabla} \cdot \vec{E} = \langle \eta \rangle \quad , \quad \text{and}$$

$$\frac{1}{\mu_0} \vec{\nabla} \times \vec{B} - \epsilon_0 \frac{\partial \vec{E}}{\partial t} = \langle \vec{j} \rangle$$

- I will now show to you that this averaging of the sources gives rise to exactly the types of terms that we saw in earlier lectures: the electric dipoles as "polarization charges", and the magnetic dipoles as "magnetization currents".

- The simplest way to accomplish this is to use the **convolution theorem**. This theorem states that, given two functions $f(x)$ and $g(x)$ in $\mathbb{R}$, with Fourier transforms:

$$\tilde{f}(k) = \int dx\, e^{ikx} f(x) \quad \leftrightarrow \quad f(x) = \int \frac{dk}{2\pi} e^{-ikx} \tilde{f}(k) \quad , \quad \text{(and the same for } g\text{)},$$

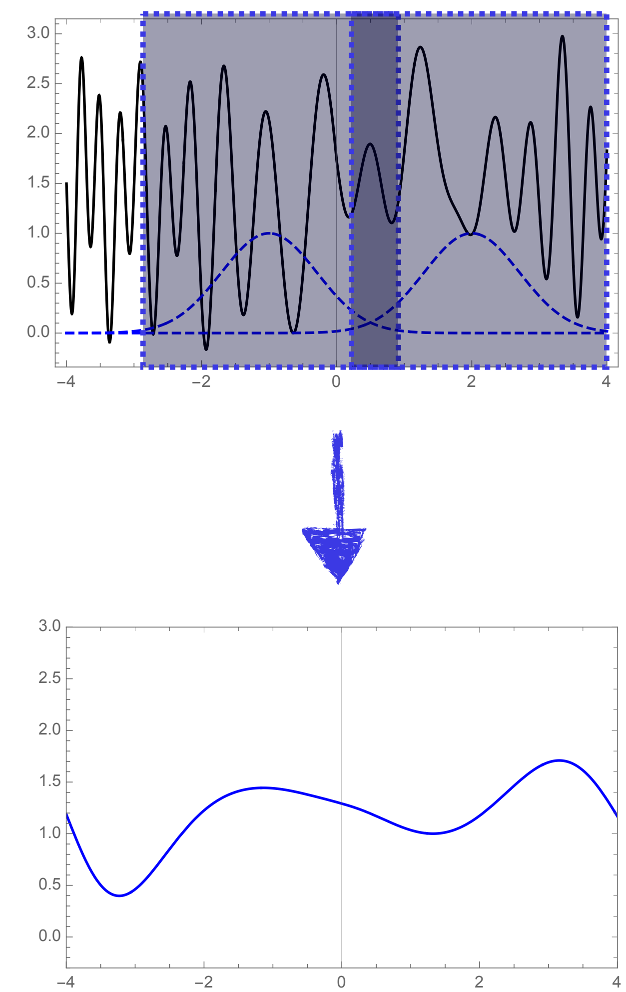
then, the integral:

$$\int dx\, f(x)\, g(x) \;=\; \int dx \left(\int \frac{dk}{2\pi} e^{-ikx} \tilde{f}(k) \right) \left(\int \frac{dk'}{2\pi} e^{-ik'x} \tilde{g}(k') \right)$$

$$= \int \frac{dk}{2\pi} \int \frac{dk'}{2\pi}\, \tilde{f}(k)\, \tilde{g}(k') \int dx\, e^{-i(k+k')x} \;=\; \int \frac{dk}{2\pi} \int \frac{dk'}{2\pi}\, \tilde{f}(k)\, \tilde{g}(k')\, 2\pi \delta(k+k')$$

$$= \int \frac{dk}{2\pi} \tilde{f}(k)\, \tilde{g}(-k) \;=\; \int \frac{dk}{2\pi} \tilde{f}(k)\, \tilde{g}^*(k)$$

where in the last equality we assumed that the function g is real, so $\tilde{f}^*(k) = \tilde{f}(-k)$.



Averaged Maxwell equations

- The nice way that comes out of this expression is that, because the window functions are "smooth", they typically "kill" the large Fourier modes and allows us to approximate the expression above as:

$$\langle f \rangle = \int dx\, f(x)\, W(x) \;=\; \int \frac{dk}{2\pi}\, \tilde{f}(k)\, \tilde{W}^*(k) \;\simeq\; \int \frac{dk}{2\pi} \left[\tilde{f}(k=0) + k \left. \frac{d\tilde{f}}{dk} \right|_{k=0} + \dots \right] \tilde{W}^*(k)$$

- E.g., the Gaussian windows in 1D and 3D have Fourier transforms:

$$W_T(t,t') = \frac{1}{\sqrt{2\pi\Delta T}} e^{-(t-t')^2/2\Delta T^2} \quad \longleftrightarrow \quad \tilde{W}_T(\omega) = e^{-\omega^2 T^2/2}$$

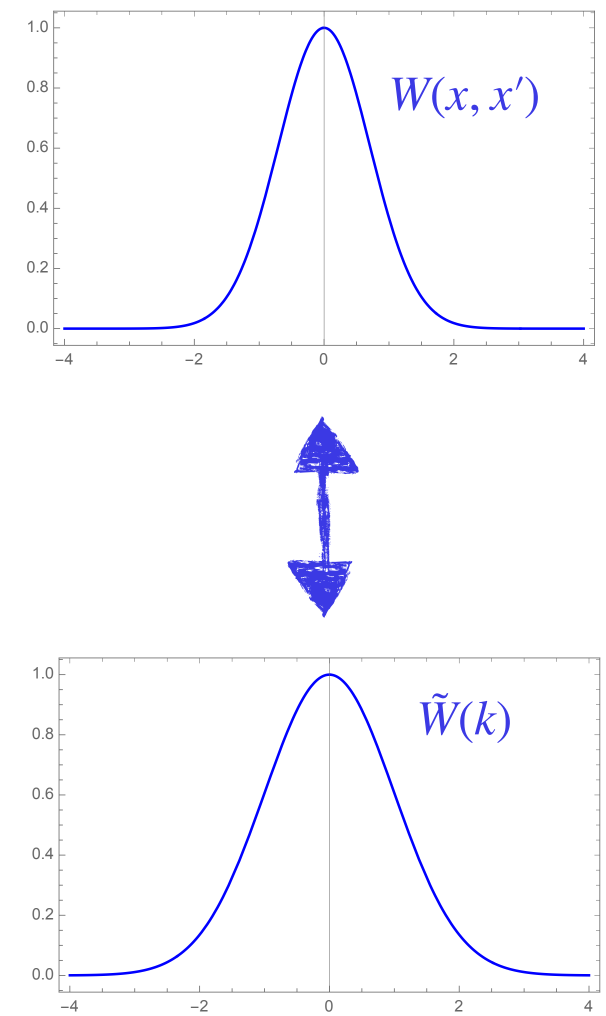
$$W_V(\vec{x}, \vec{x}\,') = \frac{1}{(\pi R^2)^{3/2}} e^{-|\vec{x}-\vec{x}\,'|^2/R^2} \quad \longleftrightarrow \quad \tilde{W}_V(\vec{k}) = e^{-k^2 R^2/4}$$

- Typically, spatial averaging is sufficient, so let's compute the volume average of the charge density:

$$\langle \eta \rangle_V(\vec{x}) \;=\; \int d^3x'\, \eta(\vec{x}\,')\, W_V(\vec{x}, \vec{x}\,') \;=\; \int d^3x' \left[\int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}\,'} \tilde{\eta}(\vec{k}) \right] \left[\int \frac{d^3k'}{(2\pi)^3} e^{i\vec{k}\,'\cdot(\vec{x}\,'-\vec{x})} \tilde{W}_V(\vec{k}\,') \right]$$

$$\langle \eta \rangle_V(\vec{x}) \;=\; \int \frac{d^3k}{(2\pi)^3}\, e^{-i\vec{k}\cdot\vec{x}}\, \tilde{\eta}(\vec{k})\, \tilde{W}^*(\vec{k})$$

- The idea now is to use the fact that the window function cuts off all high values of k, and somehow expand the expression above in k. However, we have to be careful.



Averaged Maxwell equations

- You have to be careful with the expansion in the expression above. Notice, in particular, that the Fourier transform:

$$F[f(x+\Delta x)](k) = \int dx\, e^{ikx} f(x+\Delta x) = \int dx\, e^{ikx} \int \frac{dk'}{2\pi} e^{-ik'(x+\Delta x)} \tilde{f}(k') = e^{-ik\Delta x} \tilde{f}(k)$$

- So, in some sense this is an expansion for the **region around the origin** (which is itself, of course, generic). We then have:

$$\langle \eta \rangle_V(\vec{x}) \simeq \int \frac{d^3k}{(2\pi)^3} \left(1 - i\vec{k}\cdot\vec{x} + \ldots\right) \tilde{\eta}(\vec{k})\, \tilde{W}^*(\vec{k})$$

$$\simeq \langle \eta \rangle_V(0) + \int \frac{d^3k}{(2\pi)^3} \left(-i\vec{k}\cdot\vec{x}\right) \tilde{\eta}(\vec{k})\, \tilde{W}^*(\vec{k}) + \ldots$$

$$\simeq \langle \eta \rangle_V(0) - i\vec{x}\cdot \int \frac{d^3k}{(2\pi)^3}\, \vec{k}\, \tilde{\eta}(\vec{k})\, \tilde{W}^*(\vec{k}) + \ldots$$

Now we go back to “real" space, noticing that $\vec{k} \leftrightarrow -i\vec{\nabla}_x$, so:

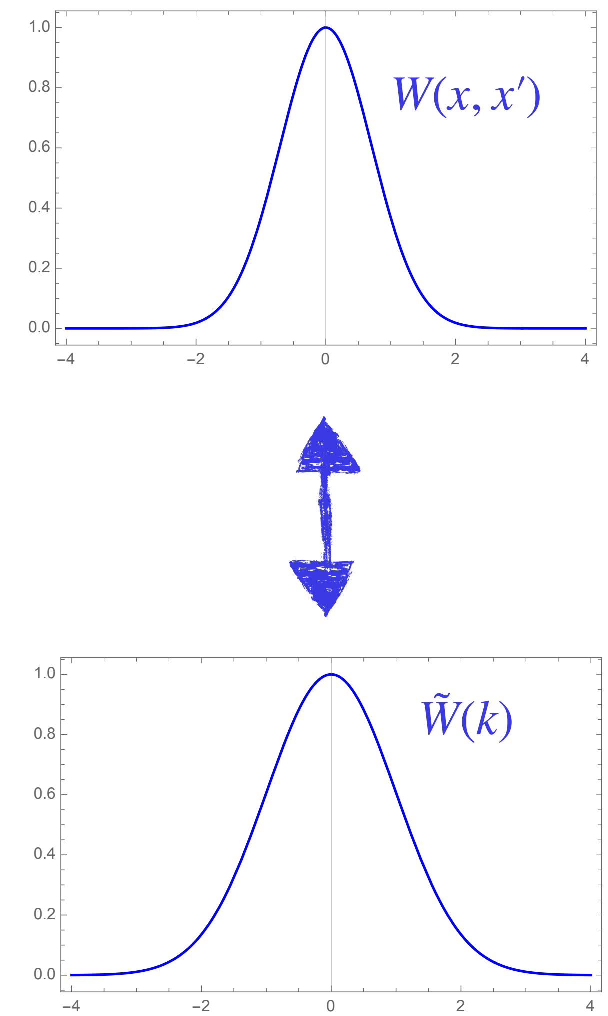
$$\langle \eta \rangle_V \simeq \langle \eta \rangle_V(0) - \vec{x}\cdot \int d^3x' \left[\vec{\nabla}'\eta(\vec{x}\,')\right] W(\vec{x}\,') + \ldots$$

- To simplify this expression, compute the following average:

$\langle \vec{x}\cdot\vec{\nabla}\eta\rangle = \langle \vec{x}\cdot\vec{\nabla}(\eta-\eta_0)\rangle$, where $\eta_0 = \langle \eta \rangle$ is the mean density

$$\Rightarrow \quad \langle \vec{x}\cdot\vec{\nabla}\eta\rangle = \langle \vec{\nabla}[\vec{x}(\eta-\eta_0)] - 3(\eta-\eta_0)\rangle = \vec{\nabla}\langle \vec{x}(\eta-\eta_0)\rangle \;,$$

where we now recognize the dipole , $\langle \vec{x}(\eta-\eta_0)\rangle$.



Averaged Maxwell equations

- Therefore, you can see that the mean microscopic density can be expressed as:

$$\langle \eta \rangle_V(\vec{x}) \simeq \langle \eta \rangle_V - \langle \vec{\nabla}_x \cdot \vec{p} \rangle_V + \dots \quad ,$$

where $\vec{p}$ is the dipole of the microscopic charge distribution, and both expressions are expressed at $\vec{x} = 0$.

- But the previous discussion means that we can define, for **any position** (not only near the origin):

$$\langle \eta \rangle_V(\vec{x}) = \rho(\vec{x}) - \vec{\nabla}_x \cdot \left[\langle \vec{p} \rangle_V(\vec{x}) \right] + \dots$$

where now we defined the **macroscopic charge density** ρ as this "low-frequency" limit of the microscopic charge density; the first correction is the **gradient** of the averaged microscopic **dipole**. [For more details, see Jackson, Ch. 6.6.]

- This is the underlying reason for defining, in Gauss's law,

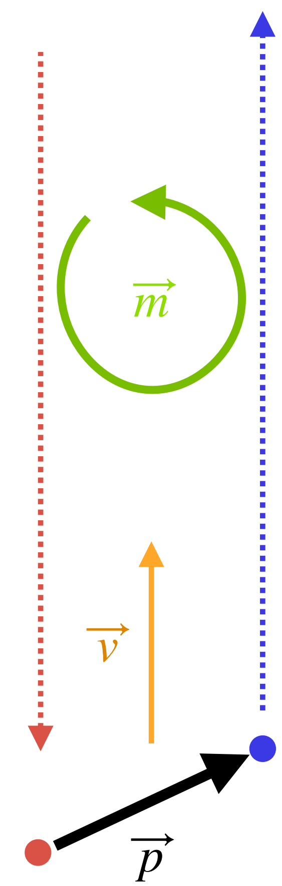
$$\epsilon_0 \vec{\nabla} \cdot \vec{E} = \langle \eta \rangle = \rho(\vec{x}) - \vec{\nabla}_x \cdot \vec{P}(\vec{x}) + \dots$$

$$\Rightarrow \quad \vec{\nabla} \cdot (\epsilon_0 \vec{E} + \vec{P}) = \rho(\vec{x}) + \dots$$

- A similar, but more complex calculation, leads to an expression for the macroscopic Ampère's law as:

$$\frac{1}{\mu_0} \vec{B} - \vec{H} = \vec{M} + (\vec{D} - \epsilon_0 \vec{E}) \times \vec{v} + \dots$$

Besides the averaged magnetization, the last term is extremely interesting: it tells us that if the material is in motion, then the polarization itself can generate the equivalent of an extra magnetic dipole. But this is exactly what you expect if for a dipole that is moving: it does generate a magnetic dipole!



Next class:

- The wave equation
- Green's function for the Wave/Helmholtz equation
- Relativity!
- Jackson, Ch. 6